DGFEM. Theory and Applications to CFD *

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*Presented at Humboldt-Universität, July 2009

DGFEM for a model scalar nonlinear nonstationary convection-diffusion problem

Goal: to work out a sufficiently accurate, robust and theoretically based method for the numerical solution of compressible flow with a wide range of Mach numbers and Reynolds numbers

Difficulties:

nonlinear convection dominating over diffusion $\Longrightarrow$

– boundary layers, wakes for large Reynolds numbers

– shock waves, contact discontinuities for large Mach numbers

– instabilities caused by acoustic effects for low Mach numbers

One of promising, efficient methods for the solution of compressible flow is the discontinuous Galerkin finite element method (DGFEM) using piecewise polynomial approximation of a sought solution without any requirement on the continuity between neighbouring elements.

– Reed&Hill 1973, LeSaint&Raviart 1974, Johnson&Pitkäranta 1986
– Cockburn&Shu 1989, Bassi&Rebay, Baumann&Oden 1997, ... Hartmann, Houston, ... van der Vegt, ... M.F., Dolejší, Kučera, Prokopová, Česenek
– theory for elliptic or parabolic problems: Arnold, Brezzi, Marini, et al, Schwab, Suli,..., Wheeler, Girault, Riviere, ...
– theory for nonstationary (nonlinear) convection-diffusion problems: M.F., Dolejší, Schwab, Sobotíková, Švadlenka, Hájek, Kučera, Prokopová

Continuous model problem

Let us consider the problem to find $u : Q_T = \Omega \times (0,T) \to \mathbb{R}$ such that

$$\text{a)}\ \frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon \Delta u + g \quad \text{in } Q_T, \tag{1}$$

$$\text{b)}\ u|_{\Gamma_D \times (0,T)} = u_D, \qquad \text{c)}\ \varepsilon \frac{\partial u}{\partial n}|_{\Gamma_N \times (0,T)} = g_N,$$

$$\text{d)}\ u(x,0) = u^0(x),\ \ x \in \Omega.$$

We assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded polygonal (if $d = 2$) or polyhedral (if $d = 3$) domain with Lipschitz-continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $T > 0$. The diffusion coefficient $\varepsilon > 0$ is a given constant, $g : Q_T \to \mathbb{R}$, $u_D : \Gamma_D \times (0,T) \to \mathbb{R}$, $g_N : \Gamma_N \times (0,T) \to \mathbb{R}$, and $u^0 : \Omega \to \mathbb{R}$ are given functions, $f_s \in C^1(\mathbb{R})$, $s = 1, \ldots, d$, are prescribed fluxes.

DG space semidiscretization

Let $\mathcal{T}_h$ $(h > 0)$ be a *partition* of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles $(d = 2)$ or tetrahedra $(d = 3)$ K with mutually disjoint interiors such that

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K. \tag{2}$$

We call $\mathcal{T}_h$ a *triangulation* of Ω and do not require the standard conforming properties from the finite element method.

$h_K = \ \text{diam}\,(K)$, $\quad h = \ \max_K \in \mathcal{T}_h$, $\quad \rho_K =$ largest ball inscribed into K

Let $K, K' \in \mathcal{T}_h$. We say that K and K' are *neighbours*, if the set $\partial K \cap \partial K'$ has positive $(d-1)$-dimensional measure. We say that $\Gamma \subset K$ is a *face* of K, if it is a maximal connected open subset either of $\partial K \cap \partial K'$, where K' is a neighbour of K, or of $\partial K \cap \partial \Omega$.

$\mathcal{F}_h$= the system of all faces of all elements $K \in \mathcal{T}_h$, the set of all innner faces:

$$\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\}, \tag{3}$$

the set of all “Dirichlet” boundary faces:

$$\mathcal{F}_h^D = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega_D\}, \tag{4}$$

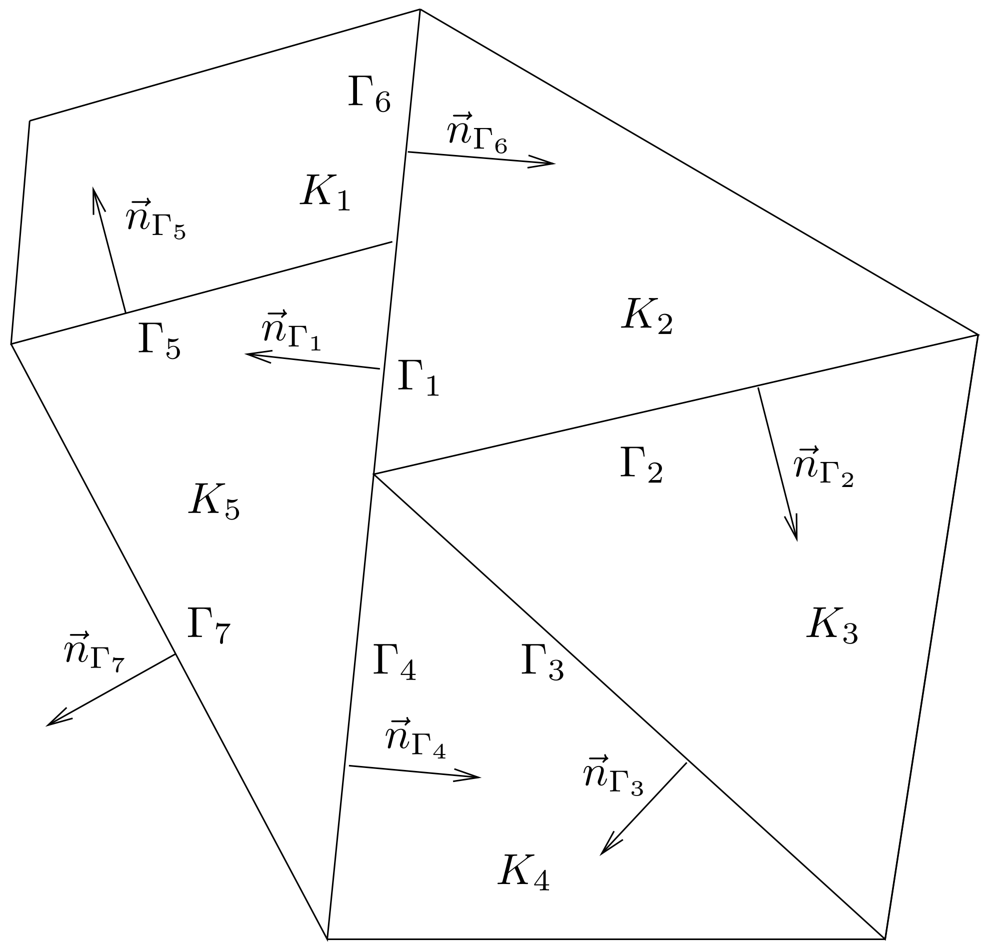
the set of all “Neumann” boundary faces:

$$\mathcal{F}_h^N = \{\Gamma \in \mathcal{F}_h,\ \Gamma \subset \partial\Omega_N\}. \tag{5}$$

Obviously, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D \cup \mathcal{F}_h^N$. For a shorter notation we put

$$\mathcal{F}_h^{ID} = \mathcal{F}_h^I \cup \mathcal{F}_h^D, \qquad \mathcal{F}_h^{DN} = \mathcal{F}_h^D \cup \mathcal{F}_h^N. \tag{6}$$

For each $\Gamma \in \mathcal{F}_h$ we define a *unit normal vector* $\boldsymbol{n}_\Gamma$. We assume that for $\Gamma \in \mathcal{F}_h^{DN}$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma \in \mathcal{F}_h^I$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed. See Figure .



Elements with hanging nodes

$d(\Gamma) =$ **diameter of** $\Gamma \in \mathcal{F}_h$.

Broken Sobolev spaces

Over a triangulation $\mathcal{T}_h$ we define the so-called *broken Sobolev space*

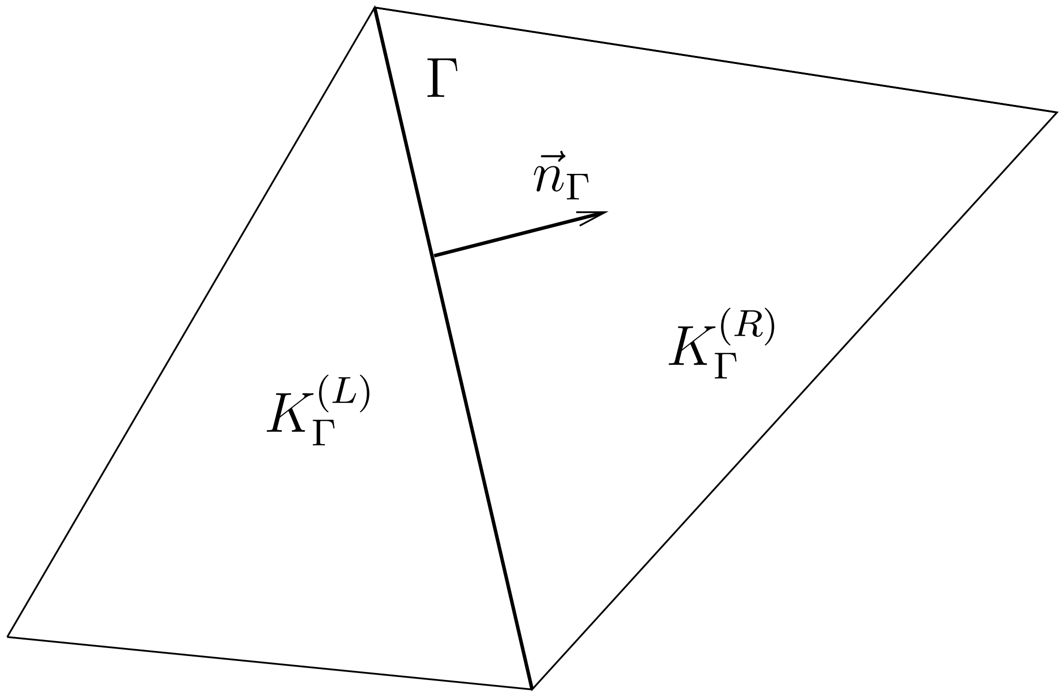
$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K) \ \forall K \in \mathcal{T}_h\} \tag{7}$$

with the norm

$$\|v\|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} \|v\|^2_{H^k(K)} \right)^{1/2} \tag{8}$$

and the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} |v|^2_{H^k(K)} \right)^{1/2}. \tag{9}$$



Neighbouring elements

For each $\Gamma \in \mathcal{F}_h^I$ there exist two neighbouring elements $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use a convention that $K_\Gamma^{(R)}$ lies in the direction of n_Γ and $K_\Gamma^{(L)}$ lies in the opposite direction to n_Γ, see Figure . ($K_\Gamma^{(L)}$, $K_\Gamma^{(R)}$ are *neighbours*.)

For $v \in H^1(\Omega, \mathcal{T}_h)$ and $\Gamma \in \mathcal{F}_h^I$, we introduce the following notation:

$$\begin{aligned} v|_\Gamma^{(L)} &= \textbf{the trace of } v|_{K_\Gamma^{(L)}} \textbf{ on } \Gamma, \qquad (10) \\ v|_\Gamma^{(R)} &= \textbf{the trace of } v|_{K_\Gamma^{(R)}} \textbf{ on } \Gamma, \\ \langle v \rangle_\Gamma &= \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)} \right), \\ [v]_\Gamma &= v|_\Gamma^{(L)} - v|_\Gamma^{(R)}. \end{aligned}$$

The value $[v]_\Gamma$ depends on the orientation of $\boldsymbol{n}_\Gamma$, but the value $[v]_\Gamma \boldsymbol{n}_\Gamma$ is independent of this orientation.

For $\Gamma \in \mathcal{F}_h^{DN}$ there exists element $K_\Gamma^{(L)} \in \mathcal{T}_h$ such that $\Gamma \subset K_\Gamma^{(L)} \cap \partial\Omega$. For $v \in H^1(\Omega, \mathcal{T}_h)$, we set

$$v|_\Gamma^{(L)} = \textbf{the trace of } v|_{K_\Gamma^{(L)}} \textbf{ on } \Gamma, \qquad (11)$$

For $\Gamma \in \mathcal{F}_h^{DN}$ by $v|_\Gamma^{(R)}$ we formally denote the exterior trace of v on Γ given either by a Dirichlet boundary condition or by an extrapolation from the interior of Ω.

The approximate solution – sought in the space of discontinuous piecewise polynomial functions

$$S_h = S_h^{p,-1} = \{v; v|_K \in P^p(K) \ \forall K \in \mathcal{T}_h\},$$

$p > 0$ – integer, $P^p(K)$ – the space of all polynomials on K of degree at most p.

Derivation of the discrete problem

Assume that u – sufficiently regular exact solution

– multiply equation (1), a) by any $\varphi \in H^2(\Omega, \mathcal{T}_h)$

– integrate over $K \in \mathcal{T}_h$

– apply Green's theorem

– sum over all $K \in \mathcal{T}_h$

After some manipulation we obtain the identity

$$
\begin{aligned}
&\int_\Omega \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}x \\
&+ \sum_{K\in\mathcal{T}_h} \sum_{\substack{\Gamma\in\mathcal{F}_h \\ \Gamma\subset\partial K}} \int_\Gamma \sum_{s=1}^{d} f_s(u)\,(n_{\partial K})_s\,\varphi|_\Gamma\,\mathrm{d}S \\
&\quad - \sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{d} f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \\
&+ \sum_{K\in\mathcal{T}_h} \int_K \varepsilon\,\nabla u\cdot\nabla\varphi\,\mathrm{d}x \\
&\quad - \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \varepsilon\langle\nabla u\rangle\cdot\boldsymbol{n}_\Gamma[\varphi]\,\mathrm{d}S \\
&\quad - \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \varepsilon\,\nabla u\cdot\boldsymbol{n}_\Gamma\,\varphi\,\mathrm{d}S \\
= &\int_\Omega g\,\varphi\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^N} \int_\Gamma \varepsilon\,\nabla u\cdot\boldsymbol{n}_\Gamma\,\varphi\,\mathrm{d}S.
\end{aligned}
\tag{12}
$$

To the left-hand side of (12) we add now the terms

$$-\theta \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \varepsilon \langle \nabla \varphi \rangle \cdot \boldsymbol{n}_\Gamma [u] \, \mathrm{d}S \quad (= 0). \tag{13}$$

Further, to the left-hand side and the right-hand side of (12) we add the terms

$$-\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \varepsilon \, \nabla \varphi \cdot \boldsymbol{n}_\Gamma \, u \, \mathrm{d}S \tag{14}$$

and

$$-\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \varepsilon \, \nabla \varphi \cdot \boldsymbol{n}_\Gamma \, u_D \, \mathrm{d}S,$$

respectively, which are identical due to the Dirichlet condition
We consider tho following possibilities:

$\theta = -1$ nonsymmetric discretization of diffusion terms (NIPG) (15)

$\theta = 1$ symmetric discretization of diffusion terms (SIPG)

$\theta = 0$ incomplete discretization of diffusion terms (IIPG)

In view of the Neumann condition, we replace the second term on the right-hand side of (12) by

$$\sum_{\Gamma \in \mathcal{F}_h^N} \int_\Gamma g_N \, \varphi \, \mathrm{d}S. \tag{16}$$

Because of the stabilization of the scheme we introduce the *interior penalty*

$$\varepsilon \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \sigma [u] \, [\varphi] \, \mathrm{d}S \quad (= 0) \tag{17}$$

and the *boundary penalty*

$$\varepsilon \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \sigma \, u \, \varphi \, \mathrm{d}S = \varepsilon \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \sigma u_D \varphi \mathrm{d}S, \tag{18}$$

where σ is a suitable *weight***.**

On the basis of above considerations we introduce the following forms defined for $u,\, \varphi \in H^2(\Omega, \mathcal{T}_h)$:
$(\cdot,\cdot)$ – $L^2(\Omega)$**-scalar product,**

$$\begin{aligned} a_h(u,\varphi) &= \sum_{K\in\mathcal{T}_h} \int_K \varepsilon\, \nabla u \cdot \nabla\varphi \,\mathrm{d}x \\ &- \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \varepsilon \langle\nabla u\rangle \cdot \boldsymbol{n}_\Gamma [\varphi] \,\mathrm{d}S \\ &-\theta \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \varepsilon \langle\nabla \varphi\rangle \cdot \boldsymbol{n}_\Gamma [u] \,\mathrm{d}S \\ &- \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \varepsilon\, \nabla u \cdot \boldsymbol{n}_\Gamma\, \varphi \,\mathrm{d}S \\ &-\theta \sum_{\Gamma\in\mathcal{F}_h^D} \int_\Gamma \varepsilon\, \nabla \varphi \cdot \boldsymbol{n}_\Gamma\, u \,\mathrm{d}S \end{aligned} \tag{19}$$

diffusion form

$\theta = -1$ nonsymmetric discretization of diffusion terms (NIPG)
$\theta = 1$ symmetric discretization of diffusion terms (SIPG)
$\theta = 0$ incomplete discretization of diffusion terms (IIPG)

$$J_h^\sigma(u,\varphi) = \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \sigma [u]\, [\varphi]\, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \sigma\, u\, \varphi\, \mathrm{d}S \qquad (20)$$

interior and boundary penalty

$$\begin{aligned} \ell_h(\varphi)(t) &= \int_\Omega g(t)\, \varphi\, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h^N} \int_\Gamma g_N(t)\, \varphi\, \mathrm{d}S \qquad (21) \\ &\quad -\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \varepsilon\, \nabla \varphi \cdot \boldsymbol{n}_\Gamma\, u_D(t)\, \mathrm{d}S \\ &\quad + \varepsilon \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \sigma\, u_D(t)\, \varphi\, \mathrm{d}S \end{aligned}$$

right-hand side form

Finally, the convective terms are approximated with the aid of a *numerical flux* $H = H(u, v, \boldsymbol{n})$ **by the form**

$$
\begin{aligned}
b_h(u, \varphi) = & - \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{d} f_s(u) \frac{\partial \varphi}{\partial x_s} \, \mathrm{d}x \\
& + \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma H\left(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\right) [\varphi]|_\Gamma \, \mathrm{d}S \\
& + \sum_{\Gamma \in \mathcal{F}_h^{DN}} \int_\Gamma H\left(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma\right) \varphi|_\Gamma^{(L)} \, \mathrm{d}S
\end{aligned}
\tag{22}
$$

convective form

H − **numerical flux**

Definition of the boundary state $u|_\Gamma^{(R)}$ **for** $\Gamma \subset \partial\Omega : u|_\Gamma^{(R)} := u|_\Gamma^{(L)}$ **(extrapolation)**

Assumptions (H):

1. $H(u, v, \boldsymbol{n})$ **is defined in** $I\!R^2 \times B_1$**, where** $B_1 = \{\boldsymbol{n} \in I\!R^d; |\boldsymbol{n}| = 1\}$**, and** *Lipschitz-continuous* **with respect to** u, v**:**

$$|H(u, v, \boldsymbol{n}) - H(u^*, v^*, \boldsymbol{n})| \le C_L(|u - u^*| + |v - v^*|),$$
$$u,\ v,\ u^*,\ v^* \in I\!R,\ \ \boldsymbol{n} \in B_1.$$

2. $H(u, v, \boldsymbol{n})$ **is** *consistent***:**

$$H(u, u, \boldsymbol{n}) = \sum_{s=1}^{d} f_s(u)\, n_s, \qquad u \in I\!R,\ \ \boldsymbol{n} = (n_1, \dots, n_d) \in B_1.$$

3. $H(u, v, \boldsymbol{n})$ **is** *conservative***:**

$$H(u, v, \boldsymbol{n}) = -H(v, u, -\boldsymbol{n}), \qquad u,\ v \in I\!R,\ \ \boldsymbol{n} \in B_1.$$

The exact sufficiently regular solution u satisfies the identity

$$\left(\frac{\partial u(t)}{\partial t}, \varphi_h\right) + b_h(u(t), \varphi_h) + a_h(u(t), \varphi_h) + \varepsilon J_h^\sigma(u(t), \varphi_h)$$
$$= \ell_h(\varphi_h)\,(t) \quad \textbf{for all } \varphi_h \in S_h \textbf{ and for a.a. } t \in (0, T).$$

Discrete problem

We say that u_h is a DGFE approximate solution of the convection-diffusion problem (1), if

a) $u_h \in C^1([0, T]; S_h),$ (23)

b) $$\left(\frac{\partial u_h(t)}{\partial t}, \varphi_h\right) + a_h(u_h(t), \varphi_h) + b_h(u_h(t), \varphi_h) + J_h^\sigma(u_h(t), \varphi_h)$$
$$= \ell_h(\varphi_h)\,(t) \qquad \forall\, \varphi_h \in S_h, \ \forall\, t \in (0, T),$$

c) $u_h(0) = u_h^0 = S_h-$approximation of u^0.

The **discrete problem is equivalent to a large system of nonlinear ordinary differential equations**.

In practical computations: suitable *time discretization* is applied, e.g.
– Euler forward or backward scheme,
– Runge–Kutta methods,
– discontinuous Galerkin time discretization

The forward Euler and Runge-Kutta schemes are *conditionally stable* – time step is strongly restricted by the *CFL-stability condition*.

Suitable: *semi-implicit scheme* - leads to a linear algebraic system on each time level

Integrals are evaluated with the aid of *numerical integration*.

Error analysis

Assumptions

– **Assumptions (H)**

– **The weak solution u of problem (1) is regular, namely**

$$\frac{\partial u}{\partial t} \in L^2(0,T; H^{p+1}(\Omega)). \tag{24}$$

Then

$$\frac{d}{dt}(u(t),\varphi_h) + a_h(u(t),\varphi_h) + \varepsilon J_h^\sigma(u(t),\varphi_h) \tag{25}$$
$$+ b_h(u(t),\varphi_h) = \ell_h(\varphi_h)(t),$$
$$\forall \varphi_h \in S_h, \ \text{for a.e. } t \in (0,T).$$

– $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, - **regular system of triangulations of the domain Ω: there exists $C_T > 0$ such that**

$$\frac{h_K}{\rho_K} \le C_T \quad \forall K \in \mathcal{T}_h \quad \forall h \in (0,h_0). \tag{26}$$

Some auxiliary results

Multiplicative trace inequality:

There exists a constant $C_M > 0$ independent of v, h and K such that

$$
\begin{aligned}
&\|v\|^2_{L^2(\partial K)} \\
&\leq C_M \left(\|v\|_{L^2(K)} |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \\
&\quad K \in \mathcal{T}_h, \ v \in H^1(K), \ h \in (0, h_0).
\end{aligned} \tag{27}
$$

Inverse inequality:

There exists a constant $C_I > 0$ independent of v, h, and K such that

$$
|v|_{H^1(K)} \leq C_I h_K^{-1} \|v\|_{L^2(K)}, \qquad v \in P^p(K), \ K \in \mathcal{T}_h, \ h \in (0, h_0). \tag{28}
$$

S_h-interpolation:

For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v, \varphi_h) = 0 \qquad \forall \varphi_h \in S_h. \tag{29}$$

Properties of the operator Π_h:

There exists a constant $C_A > 0$ independent of h, K, v such that

$$\begin{aligned} \|\Pi_h v - v\|_{L^2(K)} &\leq C_A h_K^{k+1} |v|_{H^{k+1}(K)}, \\ |\Pi_h v - v|_{H^1(K)} &\leq C_A h_K^{k} |v|_{H^{k+1}(K)}, \\ |\Pi_h v - v|_{H^2(K)} &\leq C_A h_K^{k-1} |v|_{H^{k+1}(K)}, \end{aligned} \tag{30}$$

for all $v \in H^{k+1}(K)$, $K \in \mathcal{T}_h$ and $h \in (0, h_0)$, where $k \in [1, p]$ is an integer.

If u and u_h denote the exact and approximate solutions, then we set $\eta(t) = \Pi_h u(t) - u(t), \xi(t) = u_h(t) - \Pi_h u(t) (\in S_h)$ for a.e. $t \in (0, T)$.

Truncation error in the convection form: **If $\partial\Omega_D = \partial\Omega, \partial\Omega_N = \emptyset$, then**

$$|b_h(u,\xi) - b_h(u_h,\xi)| \qquad (31)$$
$$\leq C \left(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\xi,\xi) \right)^{1/2} \left(h^{p+1} |u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)} \right).$$

If $\partial\Omega_N \neq \emptyset$, then

$$|b_h(u,\xi) - b_h(u_h,\xi)| \qquad (32)$$
$$\leq C \left(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\xi,\xi) \right)^{1/2} \left(h^{p+1/2} |u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)} \right).$$

Coercivity:

An important step in the analysis of error estimates is the *coercivity* **of the form**

$$A_h(u,v) = a_h(u,v) + \varepsilon J_h^\sigma(u,v), \qquad (33)$$

which reads

$$A_h(\varphi_h,\varphi_h) \geq \frac{\varepsilon}{2} \left(|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\varphi_h,\varphi_h) \right), \qquad (34)$$
$$\varphi_h \in S_h, \ \ h \in (0,h_0).$$

We shall discuss the validity of estimate (34) in various situations.

(I) Conforming mesh $\mathcal{T}_h$

Let the mesh $\mathcal{T}_h$ have the standard properties from the finite element method:
if $K, K' \in \mathcal{T}_h$, $K \neq K'$, then $K \cap K' = \emptyset$ or $K \cap K'$ is a common vertex or $K \cap K'$ is a common edge (or $K \cap K'$ is a common face in the case $d = 3$) of K and K'.
In this case we set

$$\sigma|_\Gamma = \frac{C_W}{d(\Gamma)}, \qquad \Gamma \in \mathcal{F}_h. \tag{35}$$

Then the coercivity inequality (34) holds under the following choice of the constant C_W:

$$C_W > 0 \text{ (e. g. } C_W = 1) \text{ for NIPG version,} \tag{36}$$

$$C_W \geq 2C_M(1 + C_I) \quad \text{for SIPG version,} \tag{37}$$

$$C_W \geq C_M(1 + C_I) \quad \text{for IIPG version,} \tag{38}$$

where C_M and C_I are constants from (27) and (28), respectively.

(II) Nonconforming mesh $\mathcal{T}_h$

In this case $\mathcal{T}_h$ is formed by closed triangles with mutually disjoint interiors with hanging nodes in general. Then the coercivity inequality (34) is guaranteed under conditions (36) − (38). However, in this case it is necessary to assume that

$$h_K \leq C_D\, d(\Gamma), \qquad \Gamma \in \mathcal{F}_h, \Gamma \subset \partial K, \tag{39}$$

in order to prove the estimate

$$J_h^\sigma(\eta, \eta) \leq C h^p |u|_{H^{p+1}(\Omega)}. \tag{40}$$

(III) Nonconforming mesh $\mathcal{T}_h$ without assumption (39)

It is obvious that condition (39) is rather restrictive in some cases. In order to avoid it, we change the definition of the weight σ:

$$\begin{aligned} \sigma|_\Gamma &= \frac{2C_W}{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}, \qquad \Gamma \in \mathcal{F}_h^I, \\ \sigma|_\Gamma &= \frac{C_W}{h_{K_\Gamma^{(L)}}}, \qquad \Gamma \in \mathcal{F}_h^D. \end{aligned} \tag{41}$$

Due to theoretical analysis, it is necessary to introduce the assumption of a "local quasiuniformity" of the mesh:

$$h_{K_\Gamma^{(L)}} \leq C_Q\, h_{K_\Gamma^{(R)}}, \qquad \Gamma \in \mathcal{F}_h^I. \tag{42}$$

(Hence, $C_Q \geq 1$.)
Then the coercivity inequality (34) holds under the following choice of C_W:

$$C_W > 0 \text{ **(e. g. $C_W = 1$) for NIPG version**}, \tag{43}$$
$$C_W \geq 2C_M(1 + C_I)\,(1 + C_Q) \quad \text{**for SIPG version**}, \tag{44}$$
$$C_W \geq C_M(1 + C_I)\,(1 + C_Q) \quad \text{**for IIPG version**}. \tag{45}$$

Proof of the coercivity inequality (34) in the case (III) for SIPG version:
Using the definition of the forms a_h and J_h^σ and the Cauchy and Young's inequalities, we find that for any $\delta > 0$ we have

$$a_h(\varphi_h, \varphi_h) \geq \varepsilon |\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} - \varepsilon\omega - \varepsilon \frac{\delta}{C_W} J_h^\sigma(\varphi_h, \varphi_h),$$

where

$$\omega = \frac{1}{\delta} \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2} \, |\langle \nabla \varphi_h \rangle|^2 \, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla \varphi_h|^2 \mathrm{d}S.$$

In view of (42),

$$\omega \leq \frac{1}{\delta} \frac{1 + C_Q}{2} \sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} |\nabla \varphi_h|^2 \mathrm{d}S.$$

Now, the application of (27) and (28) yields the estimate

$$\omega \leq \frac{1}{2\delta} C_M (1 + C_I) \, (1 + C_Q) \, |\varphi_h|^2_{H^1(\Omega, \mathcal{T}_h)}.$$

If we set $\delta := C_M (1 + C_I) \, (1 + C_Q)$ and use assumption (44), we immediately arrive at (34).

In the IIPG case we can proceed similarly.

Error estimates

Assumptions:

- (H),
- regularity of u,
- regularity of the mesh,
- $u_h^0 = \Pi_h u^0$,
- σ, $d(\Gamma), h_K$ and C_W satisfy assumptions from the cases (I) or (II) or (III).

Then the error $e_h = u - u_h$ satisfies the estimate

$$\max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} \tag{46}$$

$$+\frac{\varepsilon}{2}\int_0^t \left(|e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(e_h(\vartheta), e_h(\vartheta))\right) d\vartheta$$

$$\leq C\, h^{2p}, \qquad h \in (0, h_0), \tag{47}$$

with a constant $C > 0$ independent of h.

Sketch of the proof

Let us subtract the relations valid for the exact and approximate solutions, set $\varphi_h = \xi$ and use the coercivity inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}|\xi(t)|^2_{H^1(\Omega,\mathcal{T}_h)} + \frac{\varepsilon}{2}J_h^\sigma(\xi(t),\xi(t)) \quad (48)$$
$$\leq b_h(u(t),\xi(t)) - b_h(u_h(t),\xi(t)) - \left(\frac{\partial\eta(t)}{\partial t},\xi(t)\right)$$
$$-a_h(\eta(t),\xi(t)) - \varepsilon J_h^\sigma(\eta(t),\xi(t)) \quad \textbf{for a.a. } (0,T).$$

Now we estimate individual terms in (48):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \varepsilon|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + \varepsilon J_h^\sigma(\xi,\xi) \quad (49)$$
$$\leq C\Big\{\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)\left(\|\xi\|_{L^2(\Omega)} + h^{p+1/2}|u|_{H^{p+1}(\Omega)}\right)$$
$$+h^{p+1}\,|\partial u/\partial t|_{H^{p+1}(\Omega)}\,\|\xi\|_{L^2(\Omega)}$$
$$+\varepsilon\, h^p|u|_{H^{p+1}(\Omega)}\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)\Big\}$$

Now we apply Young's inequality:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \varepsilon|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + \varepsilon J^\sigma_h(\xi,\xi) \qquad (50)\\
\leq\ & \frac{\varepsilon}{2}\left(J^\sigma_h(\xi,\xi) + |\xi|^2_{H^1(\Omega,\mathcal{T}_h)}\right) + C\left\{\left(1+\frac{1}{\varepsilon}\right)\|\xi\|^2_{L^2(\Omega)}\right.\\
&\left. + \frac{1}{\varepsilon}\left(\left(\varepsilon^2 h^{2p} + h^{2p+1}\right)|u|^2_{H^{p+1}(\Omega)}\right) + h^{2p+2}\,|\partial u/\partial t|^2_{H^{p+1}(\Omega)}\right\}\\
&\qquad \textbf{a. e. in } (0,T).
\end{aligned}
$$

The integration of (50) from 0 **to** $t \in [0,T]$ **and the relation** $\xi(0) = u^0_h - \Pi_h u^0 = 0$ **yield**

$$
\begin{aligned}
&\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^t \left(|\xi(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J^\sigma_h(\xi(\vartheta),\xi(\vartheta))\right)\mathrm{d}\vartheta \qquad (51)\\
\leq\ & C\left\{\left(1+\frac{1}{\varepsilon}\right)\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\,\mathrm{d}\vartheta + \frac{1}{\varepsilon}h^{2p}\int_0^t \left(\left(\varepsilon^2 + h\right)|u(\vartheta)|^2_{H^{p+1}(\Omega)}\right)\mathrm{d}\vartheta\right.\\
&\left. + h^{2p+2}\int_0^t |\partial u(\vartheta)/\partial t|^2_{H^{p+1}(\Omega)}\,\mathrm{d}\vartheta\right\}, \qquad t \in [0,T].
\end{aligned}
$$

Using Gronwall's lemma, we get

$$
\begin{aligned}
&\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^t \left(|\xi(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\xi(\vartheta),\xi(\vartheta))\right) \mathrm{d}\vartheta \qquad (52)\\
\le\ & C\left((\varepsilon + h/\varepsilon)\|u\|^2_{L^2(0,T;H^{p+1}(\Omega))} + h^2\|\partial u/\partial t\|^2_{L^2(0,T;H^{p+1}(\Omega))}\right)\\
&\times h^{2p} \exp\left(\tilde{C}\frac{1+\varepsilon}{\varepsilon}t\right), \quad t \in [0,T],
\end{aligned}
$$

(C and $\tilde{C}$ are constants independent of t, h, ε, u).

Now, since $e_h = \xi + \eta$ and thus,

$$
\begin{aligned}
\|e_h\|^2_{L^2(\Omega)} &\le 2\left(\|\xi\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)}\right), \qquad (53)\\
|e_h|^2_{H^1(\Omega,\mathcal{T}_h)} &\le 2\left(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + |\eta|^2_{H^1(\Omega,\mathcal{T}_h)}\right),\\
J_h^\sigma(e_h,e_h) &\le 2\left(J_h^\sigma(\xi,\xi) + J_h^\sigma(\eta,\eta)\right),
\end{aligned}
$$

we use the above result, estimate the terms with η and obtain the sought error estimate.

Optimal error estimates

The error estimate (46) is optimal in the $L^2(H^1)$-norm, but suboptimal in the $L^\infty(L^2)$-norm.
We carried out the analysis of the $L^\infty(L^2)$-optimal error estimate under the following assumptions.
Assumptions (B):
– the discrete diffusion form a_h is symmetric (i.e. we consider the SIPG version),
– the polygonal domain Ω is convex,
– the exact solution u satisfies the regularity condition,
– conditions (H) are satisfied,
– $u_h^0 = \Pi_h u^0$,
– $\Gamma_D = \partial\Omega$ and $\Gamma_N = \emptyset$.

The application of the Aubin-Nitsche technique based on the use of the elliptic dual problem considered for each $z \in L^2(\Omega)$:

$$-\Delta\psi = z \quad \text{in } \Omega, \qquad \psi|_{\partial\Omega} = 0. \tag{54}$$

Then the weak solution $\psi \in H^2(\Omega)$ and there exists a constant $C > 0$, independent of z, such that

$$\|\psi\|_{H^2(\Omega)} \leq C\|z\|_{L^2(\Omega)}. \tag{55}$$

For each $h \in (0, h_0)$ and $t \in [0, T]$ we define the function $u_h^*(t)$ as the "A_h-projection" of $u(t)$ on S_h, i. e. a function satifying the conditions

$$u_h^*(t) \in S_h, \qquad A_h(u_h^*(t), \varphi_h) = A_h(u(t), \varphi_h) \quad \forall\, \varphi_h \in S_h, \tag{56}$$

and set $\chi = u - u_h^*$.
Using the elliptic dual problem (54), we proved the existence of a constant $C > 0$ such that

$$\|\chi\|_{L^2(\Omega)} \leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}, \tag{57}$$

$$\|\chi_t\|_{L^2(\Omega)} \leq Ch^{p+1}|u_t|_{H^{p+1}(\Omega)}, \;\; h \in (0, h_0). \tag{58}$$

This, the estimate of the truncation error in the form b_h (31), multiple application of Young's inequality and Gronwall's lemma represent important tools for obtaining the $L^\infty(L^2)$-error estimate:

Theorem. Let assumptions (B) be fulfilled. Then the error $e_h = u - u_h$ satisfies the estimate

$$\|e_h\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{p+1}, \tag{59}$$

with a constant $C > 0$ independent of h.

Remark *The constant C in the error estimates is of the order $O(\exp(\tilde{C}T/\varepsilon)$, which blows up for $\varepsilon \to 0+$.*
= a consequence of the application of necessary tools for overcoming the nonlinear convective terms, namely Young's inequality and Gronwall's lemma.

Space-time DG method for nonstationary convection-diffusion problems

Goal: to develop a sufficiently accurate and robust method for the numerical simulation of compressible flow

Promising space discretization: discontinuous Galerkin method with interior and boundary penalty

Time discretization???:

first-order: forward or backward Euler,

higher-order: explicit Runge-Kutta, Crank-Nicolson, BDF,

DG in time

Analysis of the space-time DG for convection-diffusion problems

Nonlinear problem

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon \, \Delta u + g \quad \textbf{in } Q_T = \Omega \times (0,T),$$
$$u|_{\partial\Omega\times(0,T)} = u_D, \qquad (60)$$
$$u(x,0) = u^0(x), \quad x \in \Omega$$

$f_s \in C^1(\mathbb{R}), \quad |f_s'| \le C, \quad \varepsilon > 0, \ d = 2, \ 3$

$\Omega \subset \mathbb{R}^d$ **bounded, polygonal (polyhedral) domain**, $T > 0$

Space semidiscretation by the DGFEM

Mesh: let $d = 2$

$\mathcal{T}_h$ = **partition** of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles K with mutually disjoint interiors such that

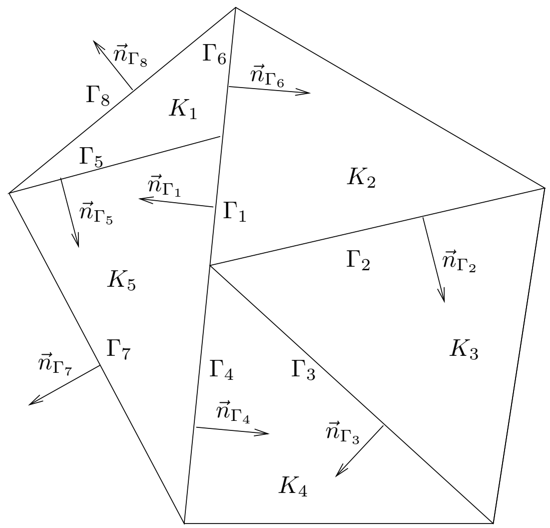
$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K.$$

$\mathcal{F}_h$= the system of all faces of all elements $K \in \mathcal{T}_h$, the set of all interior faces:

$$\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\},$$

the set of all boundary faces:

$$\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}$$

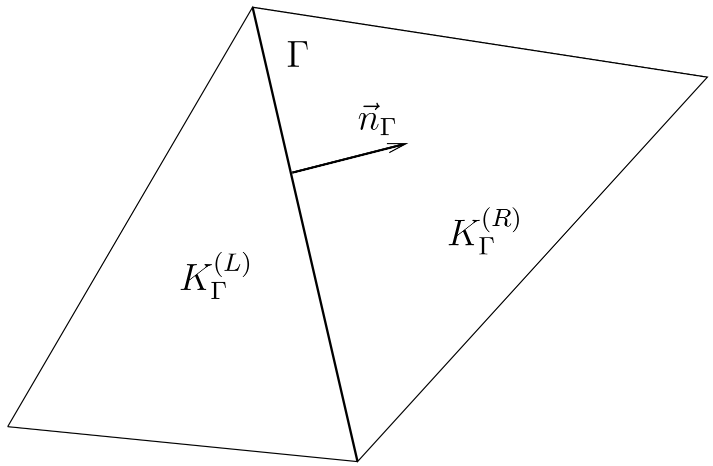


Elements with hanging nodes

$\Gamma \in \mathcal{F}_h \longrightarrow$ **unit normal vector** $\mathbf{n}_\Gamma$.

For $\Gamma \in \mathcal{F}_h^B$ **the normal** $\mathbf{n}_\Gamma$ **has the same orientation as the outer normal to** $\partial\Omega$.

$d(\Gamma) =$ **diameter of** $\Gamma \in \mathcal{F}_h$.



Neighbouring elements

For each $\Gamma \in \mathcal{F}_h^I$ there exist two neighbouring elements $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(R)} \cap \partial K_\Gamma^{(L)}$. We use a convention that $K_\Gamma^{(R)}$ lies in the direction of $\mathbf{n}_\Gamma$ and $K_\Gamma^{(L)}$ lies in the opposite direction to $\mathbf{n}_\Gamma$, see Figure.

h_K - diameter of $K \in \mathcal{T}_h$, ρ_K - radius of the largest ball inscribed in K, $h = \max_{K \in \mathcal{T}_h} h_K$.

Broken Sobolev space

$$H^k(\Omega, \mathcal{T}_h) = \{\varphi; \varphi|_K \in H^k(K) \ \forall K \in \mathcal{T}_h\}$$

with seminorm

$$|\varphi|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} |\varphi|^2_{H^k(K)}\right)^{1/2}$$

$\varphi \in H^1(\Omega, \mathcal{T}_h)$ - in general discontinuous on interfaces $\Gamma \in \mathcal{F}_h^I$

$\varphi_\Gamma^{(L)}$ and $\varphi_\Gamma^{(R)}$ - the values of φ on Γ considered from the interior and the exterior of $K_\Gamma^{(L)}$,

$\langle \varphi \rangle_\Gamma := (\varphi_\Gamma^{(L)} + \varphi_\Gamma^{(R)})/2$, $[\varphi]_\Gamma := \varphi_\Gamma^{(L)} - \varphi_\Gamma^{(R)}$.

Approximate solution sought in the space

$$S_h^p = \{\varphi \in L^2(\Omega); \varphi|_K \in \mathcal{P}^p(K)\ \forall K \in \mathcal{T}_h\}$$

$p \geq 1$ is an integer.

Derivation of a DG space semidiscretization

- multiply the PDE by any $\varphi_h \in S_h^p$,
- integrate over each $K \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all elements
- in convective terms use numerical flux
- add suitable terms - either vanishing or canceling on the basis of the Dirichlet BC.

$\Longrightarrow$

Space semi-discrete problem: Find $u_h \in C^1([0,T]; S_h^p)$ **such that**

$$\left(\frac{\partial u_h}{\partial t}, \varphi_h\right) + A_h(u_h(t), \varphi_h) = \ell_h(\varphi_h)(t) \quad \forall \varphi_h \in S_h^p \quad \forall t \in (0,T), \tag{61}$$

$$(u_h(0), \varphi_h) = (u^0, \varphi_h) \qquad \forall \varphi_h \in S_h^p.$$

$$(u, \varphi) = \int_\Omega u\varphi \, dx,$$

$$A_h(u,\varphi) = a_h(u,\varphi) + b_h(u,\varphi) + \varepsilon J_h(u,\varphi),$$

Definitions of the forms $a_h, b_h, \ldots$

Let $C_W > 0$ be a fixed constant. We introduce the notation

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2C_W} \quad \text{for } \Gamma \in \mathcal{F}_h^I,$$

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}}}{C_W} \quad \text{for } \Gamma \in \mathcal{F}_h^B.$$

$$\begin{aligned} a_h(u,\varphi) &= \varepsilon \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla \varphi \, dx \\ &- \varepsilon \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma (\langle \nabla u \rangle \cdot \mathbf{n}_\Gamma \, [\varphi] + \theta \langle \nabla \varphi \rangle \cdot \mathbf{n}_\Gamma \, [u]) \; dS \\ &- \varepsilon \sum_{\Gamma \in \mathcal{F}_h^B} \int_\Gamma ((\nabla u \cdot \mathbf{n}_\Gamma)\varphi + \theta (\nabla \varphi \cdot \mathbf{n}) u) \; dS, \end{aligned}$$

$$J_h(u,\varphi) = \sum_{\Gamma\in\mathcal{F}_h^I} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi]\; dS$$

$$+ \sum_{\Gamma\in\mathcal{F}_h^B} h(\Gamma)^{-1} \int_\Gamma u\varphi\, dS,$$

$$\ell_h(\varphi)(t) = \int_\Omega g(t)\varphi\, dx$$

$$-\theta\varepsilon \sum_{\Gamma\in\mathcal{F}_h^B} \int_\Gamma h(\Gamma)^{-1} u_D(t)\varphi\, dS$$

$$+\varepsilon \sum_{\Gamma\in\mathcal{F}_h^B} \int_\Gamma u_D(t)(\nabla\varphi\cdot\mathbf{n})\, dS$$

$\theta = -1,\ 0,\ 1$ **for NIPG, IIPG, SIPG**

$$
\begin{aligned}
b_h(u,\varphi) = & -\sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{d} f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \\
& + \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma H\left(u_\Gamma^{(L)}, u_\Gamma^{(R)}, n_\Gamma\right)\,[\varphi]_\Gamma \mathrm{d}S \\
& + \sum_{\Gamma\in\mathcal{F}_h^B}\int_\Gamma H\left(u_\Gamma^{(L)}, u_\Gamma^{(L)}, n_\Gamma\right)\,\varphi_\Gamma^{(L)} \mathrm{d}S
\end{aligned}
$$

H – **numerical flux**: **Lipschitz continuous**,

consistent: $H(v,v,n) = \sum_{s=1}^{d} f_s(v)\,n_s$,

conservative: $H(v,w,n) = -H(w,v,-n)$

Time discretization by the discontinuous Galerkin method:

(C. Johnson, V. Thomée, C. Schwab, D. Schötzau,... for ODE's or for parabolic problems in combination with conforming FEM in space)

Partition $0 = t_0 < t_1 < ... < t_M = T$ **of** $[0, T]$

$I_m = (t_{m-1}, t_m), \tau_m = t_m - t_{m-1}, m = 1, ..., M$

Notation:

$$\begin{aligned} \varphi_m^{\pm} &= \varphi(t_m\pm) = \lim_{t \to t_m\pm} \varphi(t) \\ \{\varphi\}_m &= \varphi_m^+ - \varphi_m^-. \end{aligned}$$

For each time interval $I_m, m = 1, ..., M$, we can consider, in general, a different triangulation $\mathcal{T}_{h,m}$ of the domain Ω. Therefore, for different intervals I_m we have different $S^p_{h,m}, a_{h,m}, b_{h,m}$, $J_{h,m}, \ell_{h,m}, A_{h,m}$, etc.

$\|\cdot\|$ - $L^2(\Omega)$-norm, DG-norm in $H^1(\Omega, \mathcal{T}_{h,m})$:

$$\|\varphi\|_{DG,m} = \left(\sum_{K\in\mathcal{T}_{h,m}} \int_K |\nabla\varphi|^2 \mathrm{d}x + J_{h,m}(\varphi,\varphi) \right)^{1/2}$$

$$h_m = \max_{K\in\mathcal{T}_{h,m}} h_K, \qquad h = \max_{m=1,...,M} h_m, \qquad \tau = \max_{m=1,...,M} \tau_m.$$

Let $p, q \geq 1$ be integers. ($q = 0$ - backward Euler - see M.F., V. Dolejší, J. Hozman, CMAME 2007)

Approximate solution:

$$U(x,t) \in S^{p,q}_{h,\tau} \tag{62}$$
$$= \left\{ \varphi \in L^2(Q_T); \varphi|_{I_m} = \sum_{i=0}^{q} t^i \varphi_i, \quad \text{with } \varphi_i \in S^p_{h,m} \right\},$$

satisfying

$$\int_{I_m} \Big((U', \varphi) + A_{h,m}(U, \varphi) \Big)\, dt \tag{63}$$
$$+ (\{U\}_{m-1}, \varphi^+_{m-1}) = \int_{I_m} \ell_{h,m}(\varphi)\, dt,$$
$$m = 1, \ldots, M, \qquad \forall \varphi \in S^{p,q}_{h,\tau},$$
$$(U_0^-, \varphi) = (u^0, \varphi) \qquad \forall \varphi \in S^p_{h,1}.$$

Space-time interpolation of the exact solution:

$$\begin{aligned} \pi u \in S_{h,\tau}^{p,q}, \\ \int_{I_m} (\pi u - u, \varphi^*)\, dt &= 0 \quad \forall \varphi^* \in S_{h,\tau}^{p,q-1}, \\ \pi u(t_m^-) &= \Pi_m u(t_m^-), \end{aligned} \tag{64}$$

for $m = 1, \ldots, M$

Π_m **is L^2-projection on $S_{h,m}^p$ in space**:
if $v \in L^2(\Omega)$, **then** $\Pi_m v \in S_{h,m}^p$ **and** $(\Pi_m v - v, \varphi) = 0$ **for all** $\varphi \in S_{h,m}^p$.

Main goal: the derivation of the estimation of the error $e = U - u = \xi + \eta$,

$\xi = U - \pi u \in S_{h,\tau}^{p,q}, \quad \eta = \pi u - u$

$$\Longrightarrow$$

$$\begin{aligned}
&\int_{I_m} \Big((\xi', \varphi) + A_{h,m}(\xi, \varphi)\Big)\,\mathrm{d}t + \Big(\{\xi_{m-1}\}, \varphi^+_{m-1}\Big) \\
= &\int_{I_m} \Big(b_{h,m}(u, \varphi) - b_{h,m}(U, \varphi)\Big)\,\mathrm{d}t \\
&\quad - \int_{I_m} \Big((\eta', \varphi) + A_{h,m}(\eta, \varphi)\Big)\,\mathrm{d}t - \Big(\{\eta\}_{m-1}, \varphi^+_{m-1}\Big) \\
&\qquad \forall \varphi \in S^{p,q}_{h,\tau}.
\end{aligned}$$

Derivation of error estimates

Some assumptions

We consider a system of triangulations $\mathcal{T}_{h,m}$, $m = 1, \ldots, M$, $h \in (0, h_0)$ which is shape regular, quasiuniform:

$$\frac{h_K}{\rho_K} \le c_R, \qquad K \in \mathcal{T}_{h,m}, \qquad m = 1, \ldots, M, \qquad h \in (0, h_0),$$
$$h_{K_\Gamma^{(L)}} \le C_Q \, h_{K_\Gamma^{(R)}} \qquad \forall \, \Gamma \in \mathcal{F}_{h,m}^I.$$

Auxiliary results

Multiplicative trace inequality:

$$\|v\|^2_{L^2(\partial K)} \le C_M \left(\|v\|_{L^2(K)} \, |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right),$$
$$v \in H^1(K), \; K \in \mathcal{T}_{h,m}, \; h \in (0, h_0).$$

Inverse inequality:

$$|v|_{H^1(K)} \le C_I h_K^{-1} \|v\|_{L^2(K)}, \qquad v \in P^p(K), \; K \in \mathcal{T}_{h,m}, \; h \in (0, h_0).$$

Approximation properties of Π_m: $\mu = \min(r, p)$

$$\begin{aligned} \|\Pi_m v - v\|_{L^2(K)} &\le C\, h_K^{\mu+1} |v|_{H^{r+1}(K)}, \\ |\Pi_m v - v|_{H^1(K)} &\le C\, h_K^{\mu} |v|_{H^{r+1}(K)}, \\ |\Pi_m v - v|_{H^2(K)} &\le C\, h_K^{\mu-1} |v|_{H^{r+1}(K)}, \\ & \quad v \in H^{r+1}(\Omega), \; K \in \mathcal{T}_{h,m}, \; h \in (0, h_0), \end{aligned}$$

Coercivity of the form $A_{h,m}$:

$$A_{h,m}(\xi,\xi) \geq \frac{\varepsilon}{2}\|\xi\|^2_{DG,m}$$

provided

$$C_W > 0 \text{ for NIPG},$$

$$C_W \geq C_M(1+C_I)(1+C_Q) \text{ for IIPG},$$

$$C_W \geq 2C_M(1+C_I)(1+C_Q) \text{ for SIPG}.$$

Consistency of $b_{h,m}$: For any $\varphi \in S^{p,q}_{h,\mathcal{T}}$ and $k > 0$,

$$\left|b_{h,m}(u,\varphi) - b_{h,m}(U,\varphi)\right|$$
$$\leq \frac{\varepsilon}{k}\|\varphi\|^2_{DG,m} + \frac{C}{\varepsilon}\left(\|\xi\|^2 + \tilde{\sigma}^2_m(\eta)\right),$$

where $\tilde{\sigma}^2_m(\eta) = \sum_{K\in\mathcal{T}_{h,m}} \left(\|\eta\|^2_{L^2(K)} + h^2_K|\eta|^2_{H^1(K)}\right).$

I) Derivation of estimates for ξ

Let us substitute $\varphi := \xi$ and analyze individual terms.

Integration by parts, Young's inequality and above estimates $\Longrightarrow$

$$\|\xi_m^-\|^2 - \|\xi_{m-1}^-\|^2 + \frac{1}{2}\|\{\xi\}_{m-1}\|^2 + \varepsilon\left(1 - \frac{2}{k}\right)\int_{I_m}\|\xi\|_{DG,m}^2\,\mathrm{d}t$$
$$\leq \frac{C}{\varepsilon}\int_{I_m}\|\xi\|^2\,\mathrm{d}t + 2\|\eta_{m-1}^-\|^2 + C\int_{I_m} R_m(\eta)\,\mathrm{d}t,$$

where $k > 0$ and

$$R_m(\eta) = \varepsilon\sigma_m^2(\eta) + \frac{1}{\varepsilon}\tilde{\sigma}_m^2(\eta),$$
$$\sigma_m^2(\eta) = \|\eta\|_{DG,m}^2 + \sum_{K\in\mathcal{T}_{h,m}} h_K^2|\eta|_{H^2(K)}^2$$

II) Estimation of $\int_{I_m} \|\xi\|^2 \mathrm{d}t$

Approach proposed by Ch. Makridakis based on the Radau quadrature formula:

$$0 < \vartheta_1 < \cdots < \vartheta_{q+1} = 1,$$

$w_i > 0, \quad i = 1, \ldots, q+1$ **weights** $\Longrightarrow$

$$\int_0^1 \varphi(t)\, \mathrm{d}t \approx \sum_{i=1}^{q+1} w_i \varphi(\vartheta_i)$$

- **exact for polynomials of degree** $\leq 2q$

Transformed to the interval I_m**:**

$$\int_{I_m} \varphi(t)\, dt = \int_{t_{m-1}}^{t_m} \varphi(t)\, dt \approx \tau_m \sum_{i=1}^{q+1} w_i \varphi(t^{m,i}):$$

$$t^{m,i} = t_{m-1} + \vartheta_i \tau_m, \quad i = 1, \ldots, q+1$$

Set

$\varphi := \tilde{\xi} =$ **interpolation of the function** $\tau_m \xi(t)/(t - t_{m-1})$

at the points $t^{m,i},\ i = 1, \dots, q+1$

Notation:

$$\|v\|_m := \left(\tau_m \sum_{i=1}^{q+1} w_i\, \vartheta_i^{-1} \|v(t^{m,i})\|^2\right)^{1/2}$$

Auxiliary estimates:

a) $$\int_{I_m} (\xi', \tilde{\xi})\, \mathrm{d}t + \left(\xi^+_{m-1}, \tilde{\xi}_{m-1}\right) \geq \frac{1}{2}\left(\|\xi^-_{m-1}\|^2 + \frac{1}{\tau_m}\|\xi\|^2_m\right)$$

b) $$\left\|\tilde{\xi}^+_{m-1}\right\|^2 \leq \frac{c_1}{\tau_m}\|\xi\|^2_m$$

c) $$\int_{I_m} A_{h,m}(\xi, \tilde{\xi})\, \mathrm{d}t \geq \varepsilon \int_{I_m} \|\xi\|^2_{DG,m} \mathrm{d}t$$

d) $$\left|\int_{I_m} A_{h,m}(\eta, \tilde{\xi})\, \mathrm{d}t\right| \leq \frac{\varepsilon}{k}\int_{I_m} \|\xi\|^2_{DG,m}\, \mathrm{d}t + C\,\varepsilon \int_{I_m} \sigma^2_m(\eta)\, \mathrm{d}t.$$

e) $\int_{I_m} (\eta', \tilde{\xi})\,\mathrm{d}t + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right) \le \|\eta^-_{m-1}\|\,\|\tilde{\xi}^+_{m-1}\|$

f) $\left|\int_{I_m} (b_{h,m}(u,\tilde{\xi}) - b_{h,m}(U,\tilde{\xi}))\,\mathrm{d}t\right|$
$\le \frac{\varepsilon}{k}\int_{I_m} \|\xi\|^2_{DG,m}\,\mathrm{d}t + \frac{C}{\varepsilon}\int_{I_m}\|\xi\|^2\,\mathrm{d}t + \frac{C}{\varepsilon}\int_{I_m}\tilde{\sigma}^2_m(\eta)\,\mathrm{d}t.$

$$
\begin{aligned}
R_m(\eta) &= \varepsilon\sigma^2_m(\eta) + \frac{1}{\varepsilon}\tilde{\sigma}^2_m(\eta),\\
\sigma^2_m(\eta) &= \|\eta\|^2_{DG,m} + \sum_{K\in\mathcal{T}_{h,m}} h^2_K|\eta|^2_{H^2(K)}\\
\tilde{\sigma}^2_m(\eta) &= \sum_{K\in\mathcal{T}_{h,m}} \left(\|\eta\|^2_{L^2(K)} + h^2_K|\eta|^2_{H^1(K)}\right)
\end{aligned}
$$

$\Longrightarrow$ **under the assumption that** $\tau_m = O(\varepsilon)$ **we have**

$$\int_{I_m} \|\xi\|^2 \,\mathrm{d}t \le c\,\tau_m(\|\xi_{m-1}^-\|^2 + \|\eta_{m-1}^-\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t).$$

The above estimates imply that

$$\|\xi_m^-\|^2 + \frac{\varepsilon}{2}\int_{I_m} \|\xi\|_{DG,m}^2 \,\mathrm{d}t \le \left(1 + \frac{c}{\varepsilon}\tau_m\right)\|\xi_{m-1}^-\|^2 + C\left(1 + \frac{\tau_m}{\varepsilon}\right)\|\eta_{m-1}^-\|^2 + C\int_{I_m}\left(1 + \frac{\tau_m}{\varepsilon}\right) R_m(\eta)\,\mathrm{d}t, \quad m = 1, \ldots, M,$$

with constants $c,\ C > 0$.

Gronwall's lemma, $\xi_0^- = 0, \; e = \xi + \eta \Longrightarrow$

Abstract error estimate:

$$
\begin{aligned}
&\|e_m^-\|^2 + \frac{\varepsilon}{2} \sum_{j=1}^{m} \int_{I_m} \|e\|_{DG,j}^2 \,\mathrm{d}t \\
&\leq C e^{\frac{c}{\varepsilon} t_m} \left(\sum_{j=1}^{m} \|\eta_j^-\|^2 + \sum_{j=1}^{m} \int_{I_j} \left(1 + \frac{\tau_j}{\varepsilon}\right) R_j(\eta) \,\mathrm{d}t \right) \\
&\quad + 2\|\eta_m^-\|^2 + \varepsilon \sum_{j=1}^{m} \int_{I_j} \|\eta\|_{DG,j}^2 \,\mathrm{d}t, \quad m = 1, \ldots, M.
\end{aligned}
$$

III) Estimates of expressions with η in terms of $h,\ \tau$

Lemma

Let $u \in \mathcal{H} = H^{q+1}(0,T;H^1(\Omega)) \cap C([0,T];H^{p+1}(\Omega))$**,** **assume the regularity of meshes** $\mathcal{T}_{h,m}$ **and let**

$$\hat{C}_S h_K^2 \le \tau_m$$

– not necessary, if the meshes at all time levels are identical.

Then

$$\int_{I_m} |\eta|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,dt \le C h^{2p} |u|^2_{L^2(I_m;H^{p+1}(\Omega))} + C\tau_m^{2q+2} |u|^2_{H^{q+1}(I_m;H^1(\Omega))},$$

$$\int_{I_m} \|\eta\|^2_{L^2(\Omega)}\,dt$$
$$\le Ch^{2p+2}|u|^2_{L^2(I_m;H^{p+1}(\Omega))} + C\tau_m^{2q+2}|u|^2_{H^{q+1}(I_m;L^2(\Omega))},$$

$$h^2\int_{I_m} |\eta|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,dt \le Ch^{2p}|u|^2_{L^2(I_m;H^{p+1}(\Omega))}$$
$$+C\tau_m^{2q+2}|u|^2_{H^{q+1}(I_m;H^1(\Omega))},$$

$$\sum_{m=1}^{M-1} \|\eta_m^-\|^2_{L^2(\Omega)} \le Ch^{2p}\|u\|^2_{C([0,T];H^{p+1}(\Omega))}.$$

$$\int_{I_m} J_{h,m}(\eta,\eta)\,dt \le Ch^{2p}|u|^2_{L^2(I_m;H^{p+1}(\Omega))}$$
$$+C\tau_m^{2q}|u|^2_{H^{q+1}(I_m;L^2(\Omega))}$$

loss of order of accuracy in time!!!

IV) Finally, we finish the proof of the error estimates:

Assumptions:

a) regularity of u:

$u \in \mathcal{H} = H^{q+1}(0,T; H^1(\Omega)) \cap C([0,T]; H^{p+1}(\Omega))$,

b) shape regularity of the space-time meshes

$\Longrightarrow$ **error estimate**

$$\max_{m=1,\dots,M} \|e_m^-\|^2 + \sum_{m=1}^{M} \frac{\varepsilon}{2} \int_{I_m} \|e\|_{DG,m}^2 \mathrm{d}t$$
$$\leq C \left(h^{2p} |u|^2_{C([0,T];H^{p+1}(\Omega))} + \tau^{2q} |u|^2_{H^{q+1}(0,T;H^1(\Omega))} \right).$$

Improvement of error estimates in time

We expect the error in time in the previous estimate of order $O(\tau^{2q+2})$

The loss of accuracy caused by the estimate of the penalty term $J_{h,m}(\eta,\eta)$

It is necessary to estimate

$$\int_{I_m} J_{h,m}(\pi(\Pi_m u) - \Pi_m u,\, \pi(\Pi_m u) - \Pi_m u)\,\mathrm{d}t.$$

Thus, for $\Gamma \in \mathcal{F}^I_{h,m}$ we have to estimate

$$\int_{I_m} (h(\Gamma)^{-1} \int_\Gamma [\pi(\Pi_m u) - \Pi_m u]^2 \,\mathrm{d}S)\,\mathrm{d}t$$

Using the notation $D^{q+1} = \frac{\partial^{q+1}}{\partial t^{q+1}}$**, relations**

$$D^{q+1}[\Pi_m u(x,\cdot)] = [D^{q+1}\Pi_m u(x,\cdot)],$$
$$[D^{q+1}u] = 0, \quad D^{q+1}(\Pi_m u - u) = \Pi_m(D^{q+1}u) - D^{q+1}u$$

and approximation properties of π**, we get**

$$\int_{I_m}(h(\Gamma)^{-1}\int_\Gamma[\pi(\Pi_m u) - \Pi_m u]^2\,\mathrm{d}S)\,\mathrm{d}t$$
$$\leq\ C\,\tau_m^{2q+2}\int_{I_m}(h(\Gamma)^{-1}\int_\Gamma[\Pi_m(D^{q+1}u) - D^{q+1}u]^2\,\mathrm{d}S)\,\mathrm{d}t.$$

Multiplicative trace inequality

$\|v\|_{L^2(\partial K)} \leq C_M(\|v\|_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}\|v\|_{L^2(K)})$ **and approximation properties of** Π_m $\Longrightarrow$

$$\int_{I_m} \Big(\sum_{\Gamma \in \mathcal{F}^I_{h,m}} h(\Gamma)^{-1} \int_\Gamma [\pi(\Pi_m u) - \Pi_m u]^2 \, \mathrm{d}S \Big) \, \mathrm{d}t$$

$$\leq C \, \tau_m^{2q+2} \|u\|^2_{H^{q+1}(I_m, H^1(\Omega))}.$$

If $\Gamma \in \mathcal{F}^B_{h,m}$, the situation is more complicated. Necessary to assume that

$$u_D(x,t) = \sum_{j=0}^{q} \psi_j(x) \, t^j$$

with $\psi_j \in H^{p+1/2}(\partial\Omega)$.

Then a similar process and above results lead to the expected estimate:

$$J_{h,m}(\eta,\eta) \le C\,(h^{2p}|u|^2_{L^2(I_m,H^{p+1}(\Omega))} + \tau^{2q+2}|u|^2_{H^{q+1}(0,T;H^1(\Omega))})$$

If u_D is not polynomial in t of degree $\le q$, then it is necessary to slightly modify the scheme.

Remark: The use of Young's inequality and Gronwall's lemma $\Longrightarrow$ the constant in the error estimate $C \sim \exp(\overline{C}/\varepsilon)$.

Uniform in $\varepsilon \geq 0$ error estimate obtained for a linear convection-diffusion-reaction equation

(in this case Young's inequality, Gronwall's lemma and assumption $\tau_m = O(\varepsilon)$ NOT USED)

Examples

$Q_T = (0,1)^2 \times (0,1)$,

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u - \varepsilon \triangle u + cu = g$$

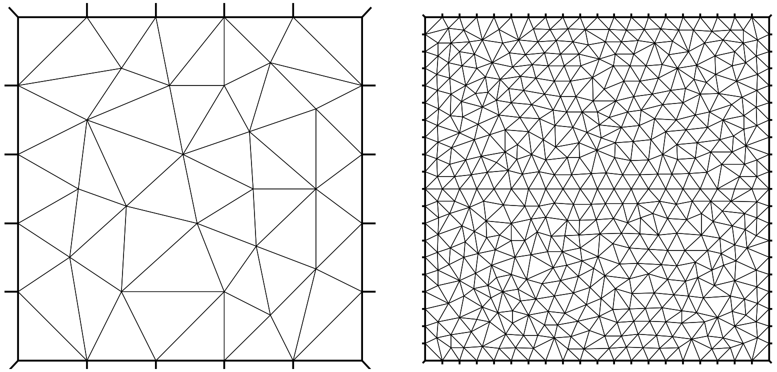
$\mathbf{v} = (v_1, v_2),\ v_1 = v_2 = 1,\ c = 0.5$,

$\varepsilon = 0.005$ **(parabolic case) and** $\varepsilon = 0$ **(hyperbolic case).**

The right-hand side g**, boundary and initial conditions such that they conform to the exact solution**

$$u_{ex}(x_1, x_2, t) = (1 - e^{-t}) \times \left(2x + 2y - xy + 2(1 - e^{v_1(x_1-1)/\nu})(1 - e^{v_2(x_2-1)/\nu})\right),$$

$\nu = 0.05$ determines the steepness of the boundary layer in the exact solution.



Coarse and fine meshes

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\|e_{\tau h}\|_{\sqrt{\varepsilon}L^2(H^1)}$	EOC^0_{space}	EOC^0_{time}	EOC^1_{space}	EOC^1_{time}
0.2838	0.2500	4.5853E-02	1.9970E-01	-	-	-	-
0.2172	0.2000	3.5474E-02	1.5372E-01	0.96	1.15	0.98	1.17
0.1540	0.1667	2.2387E-02	1.2782E-01	1.34	2.52	0.54	1.01
0.1035	0.1000	1.2945E-02	8.9991E-02	1.38	1.07	0.88	0.69
0.0768	0.0769	5.3557E-03	5.7493E-02	2.95	3.36	1.50	1.71
0.0532	0.0526	2.3742E-03	3.7567E-02	2.22	2.14	1.16	1.12
0.0398	0.0400	1.3345E-03	2.6438E-02	1.98	2.10	1.21	1.28
0.0270	0.0270	5.2577E-04	1.6779E-02	2.40	2.38	1.17	1.16
Global order of convergence				2.07	2.11	1.07	1.11

$\varepsilon = 0.005$, $p = 1$, $q = 1$ **(parabolic case)**

h	τ	$e_{\tau h L^2(L^2)}$	$e_{\tau h \sqrt{\varepsilon} L^2(H^1)}$	EOC^0_{space}	EOC^0_{time}	EOC^1_{space}	EOC^1_{time}
0.2838	0.2500	2.0470E-02	8.7193E-02	-	-	-	-
0.2172	0.2000	1.0103E-02	5.5539E-02	2.64	3.16	1.69	2.02
0.1540	0.1667	4.3992E-03	3.4110E-02	2.42	4.56	1.42	2.67
0.1035	0.1000	1.6821E-03	1.7835E-02	2.42	1.88	1.63	1.27
0.0768	0.0769	4.9668E-04	7.7827E-03	4.08	4.65	2.78	3.16
0.0532	0.0526	1.6550E-04	3.3350E-03	3.00	2.90	2.31	2.23
0.0398	0.0400	7.7630E-05	1.8029E-03	2.61	2.76	2.12	2.24
0.0270	0.0270	2.7654E-05	7.0749E-04	2.66	2.63	2.41	2.39
Global order of convergence				2.89	2.78	2.05	2.41

$\varepsilon = 0.005$, $p = 2$, $q = 2$ **(parabolic case)**

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\mathrm{EOC}^0_{\mathrm{space}}$	$\mathrm{EOC}^0_{\mathrm{time}}$
0.2838	0.2500	4.9212E-02	-	-
0.2172	0.2000	3.8843E-02	0.89	1.06
0.1540	0.1667	2.5997E-02	1.17	2.20
0.1035	0.1000	1.5581E-02	1.29	1.00
0.0768	0.0769	6.9089E-03	2.72	3.10
0.0532	0.0526	3.2904E-03	2.02	1.95
0.0398	0.0400	1.8620E-03	1.96	2.07
0.0270	0.0270	7.5458E-04	2.32	2.30
Global order of convergence			1.95	1.99

$\varepsilon = 0$, $p = 1$, $q = 1$ **(hyperbolic case)**

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\mathrm{EOC}^0_{\mathrm{space}}$	$\mathrm{EOC}^0_{\mathrm{time}}$
0.2838	0.2500	2.3451E-02	-	-
0.2172	0.2000	1.2484E-02	2.36	2.83
0.1540	0.1667	6.1746E-03	2.05	3.86
0.1035	0.1000	2.6342E-03	2.14	1.67
0.0768	0.0769	8.0848E-04	3.95	4.50
0.0532	0.0526	2.6400E-04	3.05	2.95
0.0398	0.0400	1.0761E-04	3.09	3.27
0.0270	0.0270	2.7962E-05	3.47	3.44
Global order of convergence			2.87	2.98

$\varepsilon = 0$, $p = 2$, $q = 2$ **(hyperbolic case)**

DGFEM for the solution of compressible flow

Importance of the simulation of compressible flow:

– design of airplanes (investigation of wings and tails vibrations)

– design of steam turbomachines (vibrations of blades)

– car industry (in order to avoid noise)

– civil engineering (interaction of a strong wind with structures - TV towers, cooling towers, bridges etc.)

– medicine (creation of voice)

In all these examples: flow of gases, i.e. compressible flow

for low Mach numbers often incompressible model used

sometimes the compressibility plays an important role

often, the flow in time-dependent domains has to be studied

Continuous problem

Consider compressible flow in a bounded domain $\Omega_t \subset \mathbb{R}^2$ depending on time $t \in [0, T]$. Let the boundary of Ω_t consist of three different parts:

$\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}$

Γ_I **- inlet**

Γ_O **- outlet**

Γ_{W_t} **- impermeable walls that may move in dependence on time.**

Basic equations: *continuity eq., Navier-Stokes eq's, energy equation*

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{N} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = \sum_{s=1}^{N} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s} \tag{65}$$

Notation

$$\begin{aligned} \boldsymbol{w} &= (\rho, \rho v_1, \ldots, \rho v_N, E)^{\mathsf{T}} \in I\!R^m, \qquad m = N+2, \quad (66) \\ \boldsymbol{w} &= \boldsymbol{w}(x,t), \; x \in \Omega_t, \; t \in (0,T), \\ \boldsymbol{f}_i(\boldsymbol{w}) &= (f_{i1}, \ldots, f_{im})^{\mathsf{T}} \\ &= (\rho v_i, \rho v_1 v_i + \delta_{1i}\, p, \ldots, \rho v_N v_i + \delta_{Ni}\, p, (E+p)v_i)^{\mathsf{T}} \\ \boldsymbol{R}_i(\boldsymbol{w}, \nabla \boldsymbol{w}) &= (R_{i1}, \ldots, R_{im})^{\mathsf{T}} \\ &= (0, \tau_{i1}, \ldots, \tau_{iN}, \tau_{i1}\, v_1 + \cdots + \tau_{iN}\, v_N \kappa \partial\theta/\partial x_i)^{\mathsf{T}}, \\ \tau_{ij} &= \lambda\, \mathbf{div} \boldsymbol{v} \delta_{ij} + 2\mu\, d_{ij}(\boldsymbol{v}), \; d_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \end{aligned}$$

ρ – **density,** p – **pressure,** E – **total energy,** $\boldsymbol{v} = (v_1, \dots, v_N)$ – **velocity,** θ **absolute temperature**

Thermodynamical relations:

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2), \qquad \theta = \left(\frac{E}{\rho} - \frac{1}{2}|\boldsymbol{v}|^2 \right) / c_v. \quad (67)$$

$\gamma > 1$ – **Poisson adiabatic constant,** $c_v > 0$ – **specific heat,** $\mu > 0, \lambda = -2\mu/3$ – **viscosity coefficients,** κ – **heat conduction**

Initial condition:

$$\boldsymbol{w}(x, 0) = \boldsymbol{w}^0(x), \qquad x \in \Omega_0, \qquad (68)$$

Boundary conditions

For each $t \in (0, T)$ we prescribe the following conditions:

Inlet:

a) $\rho|_{\Gamma_I} = \rho_D,$

b) $\boldsymbol{v}|_{\Gamma_I} = \boldsymbol{v}_D = (v_{D1}, \ldots, v_{DN})^{\mathsf{T}},$

c) $\theta|_{\Gamma_I} = \theta_D;$

Wall - with a moving part:

a) $\boldsymbol{v}|_{\Gamma_{W_t}} = \boldsymbol{z}_D =$ **velocity of a moving wall**,

b) $\dfrac{\partial \theta}{\partial n}|_{\Gamma_{W_t}} = 0;$

Outlet:

a) $\sum_{i=1}^{N} \tau_{ij} n_i = 0, \quad j = 1, \ldots, N,$ b) $\frac{\partial \theta}{\partial n} = 0$

on Γ_O

Obstacles:

– hyperbolic-parabolic character of the system

– nonlinear singularly perturbed system

– shock waves, contact discontinuities, boundary layers, wakes and their interaction

– moving boundary

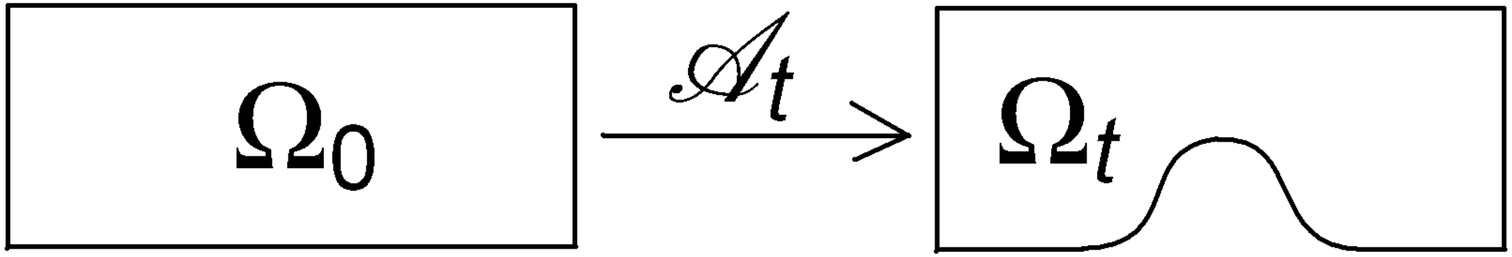
– lack of theoretical results

ALE formulation

Let $N = 2 \Longrightarrow m = 4$.

The dependence of the domain on time is taken into account with the aid of the arbitrary Lagrangian-Eulerian (ALE) method (proposed by T. Hughes et al.), based on a regular one-to-one ALE mapping of the reference domain Ω_0 onto the current configuration Ω_t:

$$\mathcal{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t, \quad \text{i.e. } \mathcal{A}_t : X \in \overline{\Omega}_0 \mapsto x = x(X,t) \in \overline{\Omega}_t.$$



The ALE mapping $\mathcal{A}_t$.

Domain velocity:

$$\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X), t \in [0,T], X \in \Omega_0, \tag{69}$$
$$\boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x),t),\ t \in [0,T],\ x \in \overline{\Omega}_t$$
$$(\boldsymbol{z}|_{\Gamma_{W_t}} = \boldsymbol{z}_D)$$

ALE derivative of a function $f = f(x,t)$ defined for $x \in \Omega_t,\ t \in [0,T]$:

$$\frac{D^A}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t)|_{X=\mathcal{A}_t^{-1}(x)}, \tag{70}$$

where

$$\tilde{f}(X,t) = f(\mathcal{A}_t(X),t),\ X \in \Omega_0.$$

It is possible to show that

$$\frac{D^A f}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \mathrm{grad}\, f = \frac{\partial f}{\partial t} + \mathrm{div}(\boldsymbol{z} f) - f\, \mathrm{div} \boldsymbol{z}. \quad (71)$$

$\Longrightarrow$ **ALE formulation of the Navier-Stokes equations:**

$$\frac{D^A \boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\, \mathrm{div} \boldsymbol{z} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s},$$

$\boldsymbol{g}_s, s = 1, 2,$ - **ALE modified inviscid fluxes:**

$$\boldsymbol{g}_s(\boldsymbol{w}) := \boldsymbol{f}_s(\boldsymbol{w}) - z_s \boldsymbol{w}. \quad (72)$$

Space semidiscretation by the DGFEM

Mesh:

Ω_{ht} = polygonal approximation of Ω_t

$\mathcal{T}_{ht}$ = **partition** of the closure $\overline{\Omega}_{ht}$ of the domain Ω_{ht} into a finite number of closed triangles K with mutually disjoint interiors such that

$$\overline{\Omega}_{ht} = \bigcup_{K \in \mathcal{T}_{ht}} K.$$

$\mathcal{F}_{ht}$= **the system of all faces of all elements** $K \in \mathcal{T}_{ht}$,

the set of all interior faces:

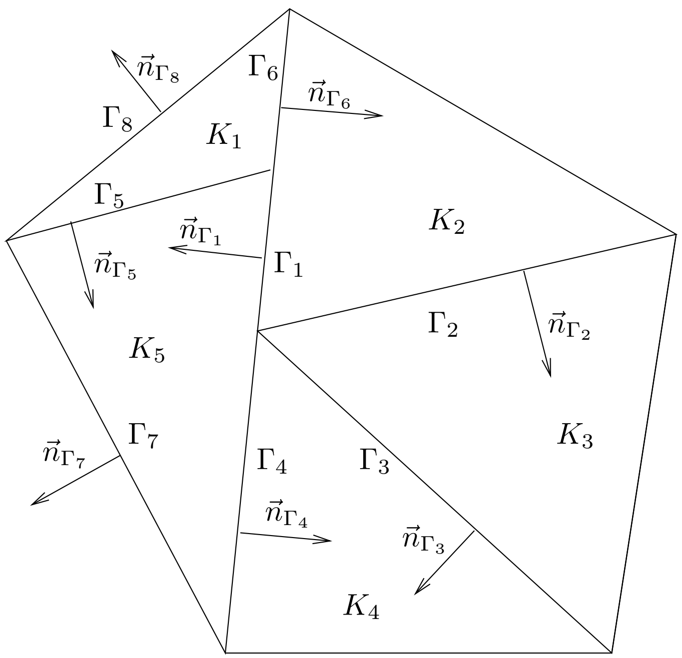
$$\mathcal{F}_{ht}^{I} = \{\Gamma \in \mathcal{F}_{ht};\ \Gamma \subset \Omega\},$$

the set of all boundary faces:

$$\mathcal{F}_{ht}^{B} = \{\Gamma \in \mathcal{F}_{ht};\ \Gamma \subset \partial\Omega_{ht}\},$$

the set of all "Dirichlet" boundary faces:

$$\mathcal{F}_{ht}^{D} = \left\{\Gamma \in \mathcal{F}_{ht}^{B};\ \textbf{a Dirichlet condition on } \Gamma\right\}.$$

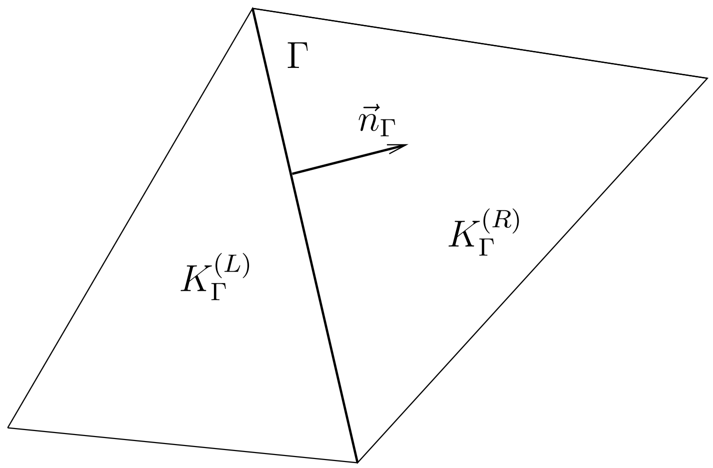


Elements with hanging nodes

$\Gamma \in \mathcal{F}_{ht} \longrightarrow$ **unit normal vector** $\boldsymbol{n}_\Gamma$.

For $\Gamma \in \mathcal{F}_{ht}^B$ **the normal** $\boldsymbol{n}_\Gamma$ **has the same orientation as the outer normal to** $\partial\Omega_{ht}$.

$d(\Gamma) =$ **diameter of** $\Gamma \in \mathcal{F}_{ht}$.



Neighbouring elements

For each $\Gamma \in \mathcal{F}_{ht}^I$ there exist two neighbouring elements $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\Gamma \subset \partial K_\Gamma^{(R)} \cap \partial K_\Gamma^{(L)}$. We use a convention that $K_\Gamma^{(R)}$ lies in the direction of n_Γ and $K_\Gamma^{(L)}$ lies in the opposite direction to n_Γ, see Figure . ($K_\Gamma^{(L)}$, $K_\Gamma^{(R)}$ are called *neighbours*.**)**

Space of approximate solutions

Discontinuous piecewise polynomial functions:

$$\boldsymbol{S}_{ht} = [S_{ht}]^4, \tag{73}$$

$$S_{ht} = \{v; v|_K \in P_r(K)\ \forall K \in \mathcal{T}_{ht}\},$$

$r \geq 0$ – **integer and** $P_r(K)$ **denotes the space of all polynomials on** K **of degree** $\leq r$.

$\varphi \in \boldsymbol{S}_{ht}$ - **in general discontinuous on interfaces** $\Gamma \in \mathcal{F}_{ht}^I$

$\varphi_\Gamma^{(L)}$ **and** $\varphi_\Gamma^{(R)}$ - **the values of** φ **on** Γ **considered from the interior and the exterior of** $K_\Gamma^{(L)}$,

$\langle\varphi\rangle_\Gamma = (\varphi_\Gamma^{(L)} + \varphi_\Gamma^{(R)})/2$, $[\varphi]_\Gamma = \varphi_\Gamma^{(L)} - \varphi_\Gamma^{(R)}$.

Derivation of the discrete problem

- multiply system (71) by a test function $\boldsymbol{\varphi}_h \in \boldsymbol{S}_{ht}$
- integrate over $K \in \mathcal{T}_{ht}$
- use Green's theorem
- sum over all $K \in \mathcal{T}_{ht}$
- introduce the concept of the numerical flux
- introduce suitable terms mutually vanishing for a regular exact solution $\Longrightarrow$

$$\sum_{K\in\mathcal{T}_{ht}} \int_K \frac{D^A\boldsymbol{w}}{Dt} \cdot \boldsymbol{\varphi}_h \, dx$$
$$+b_h(\boldsymbol{w}, \boldsymbol{\varphi}_h) + a_h(\boldsymbol{w}, \boldsymbol{\varphi}_h) + J_h(\boldsymbol{w}, \boldsymbol{\varphi}_h)$$
$$+d_h(\boldsymbol{w}, \boldsymbol{\varphi}_h) = \ell_h(\boldsymbol{w}, \boldsymbol{\varphi}_h)$$

Convection form b_h

$$b_h(\boldsymbol{w}, \boldsymbol{\varphi}_h) = -\sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^{2} \boldsymbol{g}_s(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\, dx$$
$$+ \sum_{\Gamma \in \mathcal{F}_{ht}^I} \int_\Gamma \mathbf{H}_g(\boldsymbol{w}_\Gamma^{(L)}, \boldsymbol{w}_\Gamma^{(R)}, \boldsymbol{n}_\Gamma) \cdot [\boldsymbol{\varphi}_h]_\Gamma\, dS,$$
$$+ \sum_{\Gamma \in \mathcal{F}_{ht}^B} \int_\Gamma \mathbf{H}_g(\boldsymbol{w}_\Gamma^{(L)}, \boldsymbol{w}_\Gamma^{(R)}, \boldsymbol{n}_\Gamma) \cdot \boldsymbol{\varphi}_{h\Gamma}^{(L)}\, dS$$

$\mathbf{H}_g$ - numerical flux

consistent with the fluxes $\boldsymbol{g}_s$:

$\mathbf{H}_g(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{n}) = \sum_{s=1}^2 \boldsymbol{g}_s(w) n_s \quad (\boldsymbol{n} = (n_1, n_2), |\boldsymbol{n}| = 1),$

conservative: $\mathbf{H}_g(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{n}) = -\mathbf{H}_g(\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{n})$

locally Lipschitz-continuous

Viscous form a_h (IIPG)

$$a_h(\boldsymbol{w},\boldsymbol{\varphi}) = \sum_{K\in\mathcal{T}_{ht}} \int_K \sum_{s=1}^{2} \mathbf{R}_s(\boldsymbol{w},\nabla\boldsymbol{w}) \cdot \frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,dx$$

$$- \sum_{\Gamma\in\mathcal{F}_{ht}^{I}} \int_\Gamma \sum_{s=1}^{2} \langle \mathbf{R}_s(\boldsymbol{w},\nabla\boldsymbol{w})\rangle (\boldsymbol{n}_\Gamma)_s \cdot [\boldsymbol{\varphi}]\,dS$$

$$- \sum_{\Gamma\in\mathcal{F}_{ht}^{D}} \int_\Gamma \sum_{s=1}^{2} \mathbf{R}_s(\boldsymbol{w},\nabla\boldsymbol{w}) (\boldsymbol{n}_\Gamma)_s \cdot \boldsymbol{\varphi}\,dS$$

Interior and boundary penalty

$$J_h(\boldsymbol{w},\boldsymbol{\varphi}) = \sum_{\Gamma\in\mathcal{F}_{ht}^{I}} \int_\Gamma \sigma[\boldsymbol{w}] \cdot [\boldsymbol{\varphi}]\,dS$$

$$+ \sum_{\Gamma\in\mathcal{F}_{ht}^{D}} \int_\Gamma \sigma\boldsymbol{w} \cdot \boldsymbol{\varphi}\,dS$$

$$\sigma|_\Gamma = C_W\mu/d(\Gamma)$$

The form d_h

$$d_h(\boldsymbol{w},\boldsymbol{\varphi}) = \sum_{K\in\mathcal{T}_{ht}} \int_K (\boldsymbol{w}\cdot\boldsymbol{\varphi}_h)\,\mathrm{div}\boldsymbol{z}\,dx$$

Right-hand side form ℓ_h

$$\ell_h(\boldsymbol{w},\boldsymbol{\varphi}) = \sum_{\Gamma\in\mathcal{F}_{ht}^D} \int_\Gamma \sum_{s=1}^{2} \sigma\boldsymbol{w}_B\cdot\boldsymbol{\varphi}\,dS.$$

The state $\boldsymbol{w}_B$ defined on the basis of the Dirichlet BC's and extrapolation:

$$\boldsymbol{w}_B = (\rho_D, \rho_D v_{D1}, \rho_D v_{D2}, c_v\rho_D\theta_D + \frac{1}{2}\rho_D|\boldsymbol{v}_D|^2) \quad \textbf{on } \Gamma_I,$$

$$\boldsymbol{w}_B = \boldsymbol{w}_\Gamma^{(L)} \quad \textbf{on } \Gamma_O,$$

$$\boldsymbol{w}_B = (\rho_\Gamma^{(L)}, \rho_\Gamma^{(L)} z_1, \rho_\Gamma^{(L)} z_2, c_v\rho_\Gamma^{(L)}\theta_\Gamma^{(L)} + \frac{1}{2}\rho_\Gamma^{(L)}|\boldsymbol{z}|^2) \quad \textbf{on } \Gamma_{W_t}$$

The approximate solution is defined as $\boldsymbol{w}_h(t) \in \boldsymbol{S}_{ht}$ such that

$$\sum_{K \in \mathcal{T}_{ht}} \int_K \frac{D^A \boldsymbol{w}_h(t)}{Dt} \cdot \boldsymbol{\varphi}_h \, dx$$
$$+ b_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + a_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h)$$
$$+ J_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) + d_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = \ell_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h)$$

holds for all $\boldsymbol{\varphi}_h \in \boldsymbol{S}_{ht}$, all $t \in (0, T)$ and

$\boldsymbol{w}_h(0) = \boldsymbol{w}_h^0$=approximation of the initial state $\boldsymbol{w}^0$

Time discretization

partition $0 = t_0 < t_1 < t_2 \ldots$ **of the time interval** $[0, T]$

$\tau_k = t_{k+1} - t_k$ **- time step**

$\boldsymbol{z}(t_n) \approx \boldsymbol{z}^n$, $\boldsymbol{w}_h(t_n) \approx \boldsymbol{w}_h^n \in \boldsymbol{S}_{ht_n}$ **defined in** Ω_{ht_n} **- causes problems in transition** $t_k \longrightarrow t_{k+1}$

introduce the function

$\widehat{\boldsymbol{w}}_h^k = \boldsymbol{w}_h^k \circ \mathcal{A}_{t_k} \circ \mathcal{A}_{t_{k+1}}^{-1}$ **- defined in the domain** $\Omega_{ht_{k+1}}$

Approximation of the ALE derivative at time t_{k+1} by the finite difference

$$\frac{D^A \boldsymbol{w}_h}{Dt}(x, t_{k+1}) = \frac{\partial \tilde{\boldsymbol{w}}_h}{\partial t}(X, t_{k+1})$$
$$\approx \frac{\tilde{\boldsymbol{w}}_h^{k+1}(X) - \tilde{\boldsymbol{w}}_h^{k}(X)}{\tau_k}$$
$$= \frac{\boldsymbol{w}_h^{k+1}(x) - \widehat{\boldsymbol{w}}_h^{k}(x)}{\tau_k}, \qquad x = \mathcal{A}_{t_{k+1}}(X) \in \Omega_{ht_{k+1}}.$$

Notation:

$(\cdot, \cdot)$ - **scalar product in** $L^2(\Omega_{ht_{k+1}})$

Possible full discretization:

$$\text{(a)} \qquad \boldsymbol{w}_h^{k+1} \in \boldsymbol{S}_{ht_{k+1}},$$

$$\text{(b)} \qquad \left(\frac{\boldsymbol{w}_h^{k+1} - \widehat{\boldsymbol{w}}_h^k}{\tau_k}, \boldsymbol{\varphi}_h \right)$$

$$+ b_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + a_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h)$$

$$+ J_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + d_h\left(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h\right) = \ell_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h)$$

$$\forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_{ht_{k+1}}, \;\; k = 0, 1, \ldots .$$

-**strongly nonlinear algebraic system!!!**

Semi-implicit linearized scheme

Linearization of the form b_h

$$b_h(\boldsymbol{w}, \boldsymbol{\varphi}_h) = - \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^{2} \boldsymbol{g}_s(\boldsymbol{w}) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \, dx$$

$$+ \sum_{\Gamma \in \mathcal{F}_{ht}^I} \int_\Gamma \mathbf{H}_g(\boldsymbol{w}_\Gamma^{(L)}, \boldsymbol{w}_\Gamma^{(R)}, \boldsymbol{n}_\Gamma) \cdot [\boldsymbol{\varphi}_h]_\Gamma \, dS,$$

$$+ \sum_{\Gamma \in \mathcal{F}_{ht}^B} \int_\Gamma \mathbf{H}_g(\boldsymbol{w}_\Gamma^{(L)}, \boldsymbol{w}_\Gamma^{(R)}, \boldsymbol{n}_\Gamma) \cdot \boldsymbol{\varphi}_{h\Gamma}^{(L)} \, dS$$

On the basis of the relation

$$\boldsymbol{g}_s(\boldsymbol{w}_h^{k+1}) = (\mathbb{A}_s(\boldsymbol{w}_h^{k+1}) - z_s^{k+1}\mathbb{I})\boldsymbol{w}_h^{k+1}$$

$$\approx (\mathbb{A}_s(\hat{\boldsymbol{w}}_h^k) - z_s^{k+1}\mathbb{I})\boldsymbol{w}_h^{k+1}$$

we linearize the first term of $b_h(.,.)$:

$$\sum_{K\in\mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^{2} \boldsymbol{g}_s(\boldsymbol{w}_h^{k+1}) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}$$

$$\approx \sum_{K\in\mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^{2} (\mathbb{A}_s(\hat{\boldsymbol{w}}_h^k) - z_s^{k+1}\mathbb{I})\boldsymbol{w}_h^{k+1}) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} dx.$$

The second term of $b_h(.,.)$ is linearized with the aid of the Vijayasundaram numerical flux:

$$\mathbf{H}_g(\boldsymbol{w}_{h\Gamma}^{k+1(L)}, \boldsymbol{w}_{h\Gamma}^{k+1(R)}), \boldsymbol{n}_\Gamma) \approx$$
$$\mathbb{P}_g^+(\langle \hat{\boldsymbol{w}}_h^k \rangle_\Gamma, \boldsymbol{n}_\Gamma)\boldsymbol{w}_{h\Gamma}^{k+1(L)} + \mathbb{P}_g^-(\langle \hat{\boldsymbol{w}}_h^k \rangle_\Gamma, \boldsymbol{n}_\Gamma)\boldsymbol{w}_{h\Gamma}^{k+1(R)},$$

where $\mathbb{P}_g^+$ and $\mathbb{P}_g^-$ are positive and negative parts of the matrix $\mathbb{P}_g(\boldsymbol{w}, \boldsymbol{n}_\Gamma) = \sum_{s=1}^{2} (\mathbb{A}_s(\boldsymbol{w}) - z_s^{k+1}\mathbb{I})(\boldsymbol{n}_\Gamma)_s$

$$\Longrightarrow$$

$$b_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h)$$

$$= -\sum_{K \in \mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^{2} (\mathbb{A}_s(\hat{\boldsymbol{w}}^k(x)) - z_s^{k+1}(x))\mathbb{I})\boldsymbol{w}^{k+1}(x)) \cdot \frac{\partial \boldsymbol{\varphi}_h(x)}{\partial x_s} dx,$$

$$+ \sum_{\Gamma \in \mathcal{F}^I_{ht_{k+1}}} \int_\Gamma \left\{ \mathbb{P}_g^+ \left(\left\langle \hat{\boldsymbol{w}}_h^k \right\rangle_\Gamma, \boldsymbol{n}_\Gamma \right) \boldsymbol{w}_{h\Gamma}^{k+1(L)} \right.$$

$$\left. + \mathbb{P}_g^- \left(\left\langle \hat{\boldsymbol{w}}_h^k \right\rangle_\Gamma, \boldsymbol{n}_\Gamma \right) \boldsymbol{w}_{h\Gamma}^{k+1(R)} \right\} \cdot [\boldsymbol{\varphi}_h]_\Gamma dS$$

$$+ \sum_{\Gamma \in \mathcal{F}^B_{ht_{k+1}}} \int_\Gamma \left\{ \mathbb{P}_g^+ \left(\left\langle \hat{\boldsymbol{w}}_h^k \right\rangle_\Gamma, \boldsymbol{n}_\Gamma \right) \boldsymbol{w}_{h\Gamma}^{k+1(L)} \right.$$

$$\left. + \mathbb{P}_g^- \left(\left\langle \hat{\boldsymbol{w}}_h^k \right\rangle_\Gamma, \boldsymbol{n}_\Gamma \right) \boldsymbol{w}_{h\Gamma}^{k+1(R)} \right\} \cdot \boldsymbol{\varphi}_{h\Gamma} dS$$

Linearization of the form a_h

based on the fact that $\mathrm{R}_s(w, \nabla w)$ is nonlinear in w but linear in ∇w $\Longrightarrow$

$$
\begin{aligned}
& a_h(\boldsymbol{w}^{k+1}, \boldsymbol{\varphi}) \approx \hat{a}_h(\hat{\boldsymbol{w}}^k, \boldsymbol{w}^{k+1}, \boldsymbol{\varphi}) \\
& := \sum_{K \in \mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^{2} \mathbf{R}_s(\hat{\boldsymbol{w}}^k, \nabla \boldsymbol{w}^{k+1}) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s}\, dx \\
& - \sum_{\Gamma \in \mathcal{F}^I_{ht_{k+1}}} \int_\Gamma \sum_{s=1}^{2} \langle \mathbf{R}_s(\hat{\boldsymbol{w}}^k, \nabla \boldsymbol{w}^{k+1}) \rangle (\boldsymbol{n}_\Gamma)_s \cdot [\boldsymbol{\varphi}]\, dS \\
& - \sum_{\Gamma \in \mathcal{F}^D_{ht_{k+1}}} \int_\Gamma \sum_{s=1}^{2} \mathbf{R}_s(\hat{\boldsymbol{w}}^k, \nabla \boldsymbol{w}^{k+1}) (\boldsymbol{n}_\Gamma)_s \cdot \boldsymbol{\varphi}\, dS
\end{aligned}
$$

$\Longrightarrow$ **Semi-implicit discrete problem:**

$$\text{(a)} \qquad \boldsymbol{w}_h^{k+1} \in \boldsymbol{S}_{ht_{k+1}},$$

$$\text{(b)} \qquad \left(\frac{\boldsymbol{w}_h^{k+1} - \widehat{\boldsymbol{w}}_h^k}{\tau_k}, \boldsymbol{\varphi}_h\right) + \widehat{b}_h(\widehat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + \widehat{a}_h(\widehat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + J_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + d_h\left(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h\right) = \ell(\boldsymbol{w}_B^k, \boldsymbol{\varphi})$$

$$\forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_{ht_{k+1}}, \;\; k = 0, 1, \ldots .$$

linear with respect to w_h^{k+1}

Remarks:

– Time discretization is of order 1. Possibility to construct higher order time discretizations using BDF formula and extrapolation in nonlinear terms

– Discrete problem is equivalent on each time level to a linear algebraic system - solved by UMFPACK (direct solver) or GMRES with a block diagonal preconditioning.

– Scheme for the solution of inviscid flow: $\mu = \lambda = \kappa = 0$.

Further important ingredients

- Realization of boundary conditions in the form $\widehat{b}_h$, i.e. the determination of the state $\boldsymbol{w}_{h\Gamma}^{k+1(R)}$ if $\Gamma \subset \partial\Omega_{ht}$ - with the aid of a linearized local initial-boundary value Riemann problem

- Use of isoparametric elements at a curved boundary

- Avoiding the Gibbs phenomenon in high-speed flow = spurious overshoots and undershoots at discontinuities in the numerical solution - with the aid of a discontinuity indicator (M.F., V. Dolejší, C. Schwab, 2003) and local artificial viscosity (M.F., V. Kučera, 2007):

a) Define the discontinuity indicator $g^k(i)$ proposed by M.F., Dolejší and Schwab: Math. Comput. Simul. (2003):

$$g^k(K) = \int_{\partial K} [\hat{\rho}_h^k]^2 \, \mathrm{d}S / (h_K |K|^{3/4}), \qquad K \in \mathcal{T}_{ht_{k+1}}. \quad (74)$$

b) Define the discrete indicator

$$G^k(K) = 0 \text{ if } g^k(K) < 1, \qquad G^k(K) = 1 \text{ if } g^k(K) \geq 1, \quad K \in \mathcal{T}_{ht_{k+1}}. \quad (75)$$

c) To the left-hand side of of the scheme we add the artificial viscosity form

$$\beta_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}) = \nu_1 \sum_{K \in \mathcal{T}_{ht_{k+1}}} h_K G^k(K) \int_K \nabla \boldsymbol{w}_h^{k+1} \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}$$

d) Augment the left-hand side of the scheme by adding the form

$$\tilde{J}_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi})$$
$$= \nu_2 \sum_{\Gamma \in \mathcal{F}^I_{ht_{k+1}}} \frac{1}{2}(G^k(K_\Gamma^{(L)}) + G^k(K_\Gamma^{(R)}) \int_\Gamma [\boldsymbol{w}_h^{k+1}] \cdot [\boldsymbol{\varphi}] \, \mathrm{d}\mathcal{S},$$

Resulting scheme:

$$\text{(a)} \qquad \boldsymbol{w}_h^{k+1} \in \boldsymbol{S}_{ht_{k+1}},$$

$$\text{(b)} \qquad \left(\frac{\boldsymbol{w}_h^{k+1} - \hat{\boldsymbol{w}}_h^k}{\tau_k}, \boldsymbol{\varphi}_h\right)$$

$$+\hat{b}_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + \hat{a}_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h)$$

$$+J_h(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + d_h\left(\boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h\right)$$

$$+\beta_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) + \tilde{J}_h(\hat{\boldsymbol{w}}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = \ell(\boldsymbol{w}_B^k, \boldsymbol{\varphi})$$

$$\forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_{ht_{k+1}}, \quad k = 0, 1, \ldots.$$

This method successfully overcomes problems with the Gibbs phenomenon in the context of the semi-implicit scheme.

Important: $G^k(i)$ vanishes in regions where the solution is regular.

$\Longrightarrow$

The scheme does not produce any nonphysical entropy in these regions.

Example

Burgers equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x_1} + u\,\frac{\partial u}{\partial x_2} = 0 \quad \text{in } \Omega \times (0, T), \tag{76}$$

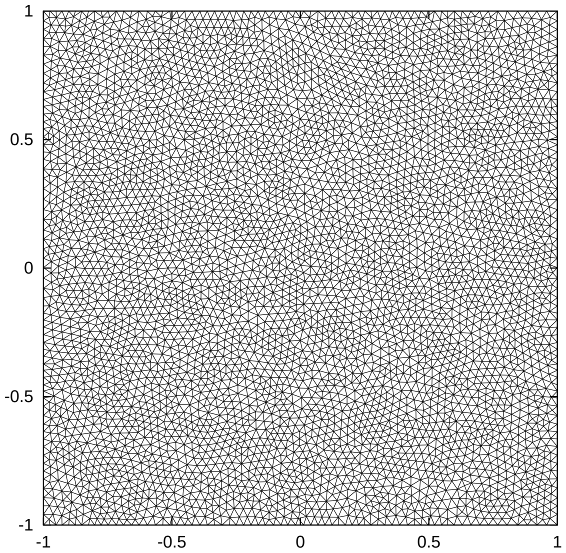
where $\Omega = (-1, 1) \times (-1, 1)$, equipped with initial

condition

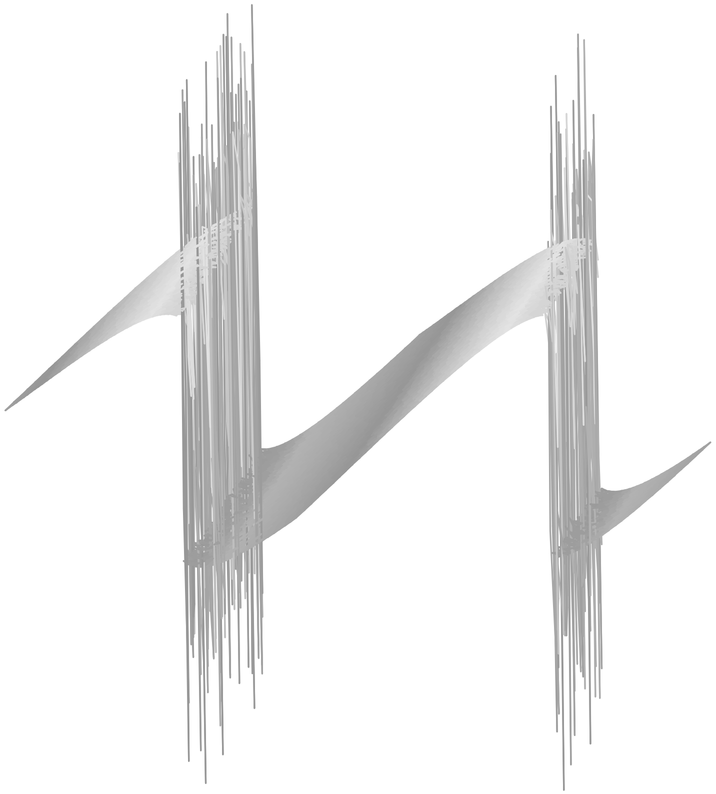
$$u^0(x_1, x_1) = 0.25 + 0.5\,\sin(\pi(x_1 + x_2)), \qquad (x_1, x_2) \in \Omega, \tag{77}$$

and periodic boundary conditions

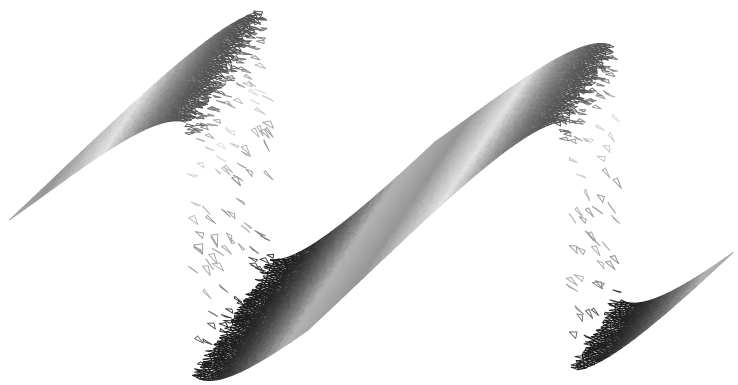
Exact solution at $t = 0.45$



Triangulation used for the numerical solution



Numerical solution computed by DGFEM, at $t = 0.45$



Numerical solution computed by DGFEM with limiting, at $t = 0.45$

Examples: quadratic elements

Flow in fixed domains

1) Inviscid flow

a) Low Mach number flow at incompressible limit

Stationary flow past a Joukowski profile

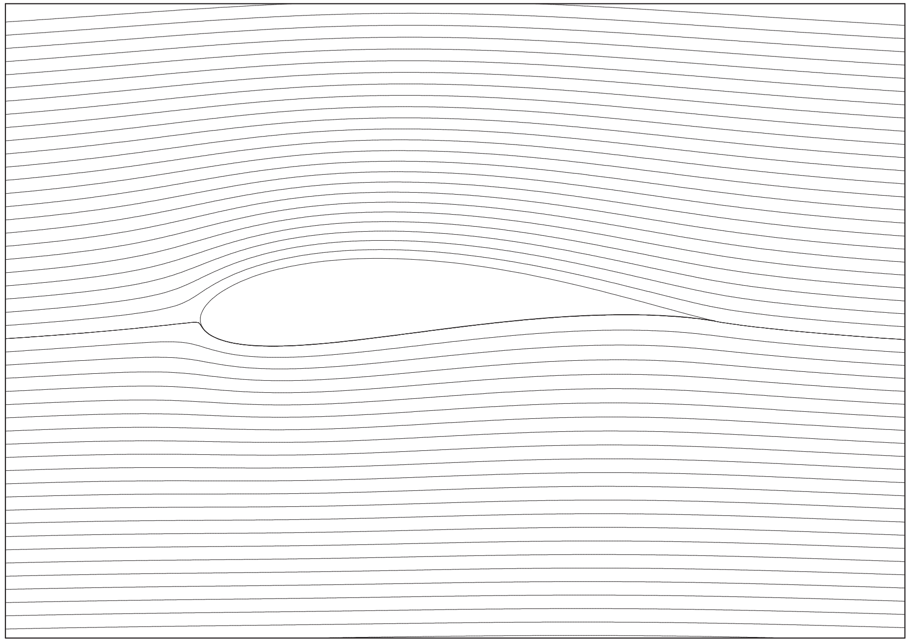
constant far field quantities $\Longrightarrow$ the flow is irrotational and homoentropic

complex function method: exact solution of incompressible inviscid irrotational flow satisfying the Kutta–Joukowski trailing condition, provided

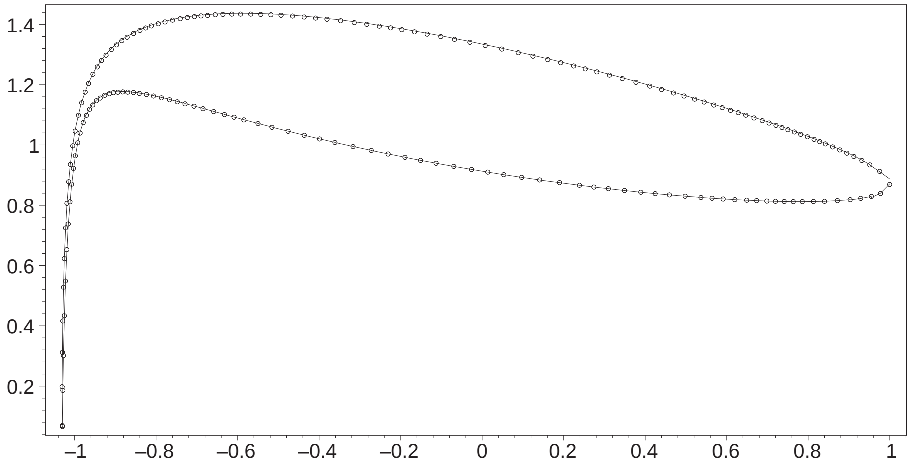
the velocity circulation around the profile, related to the magnitude of the far field velocity,
$\gamma_{\text{ref}} = 0.7158$

Compressible flow: $M_\infty = 10^{-4}$, $\#\mathcal{T}_h = 5418$

The maximum density variation in compressible flow $\rho_{\text{max}} - \rho_{\text{min}} = 1.04 \cdot 10^{-8}$.
Computed velocity circulation related to the magnitude of the far field velocity: $\gamma_{\text{refcomp}} = 0.7205$,
$\Longrightarrow$ **the relative error** 0.66%



Compressible low Mach flow past a Joukowski profile, approximate solution, streamlines

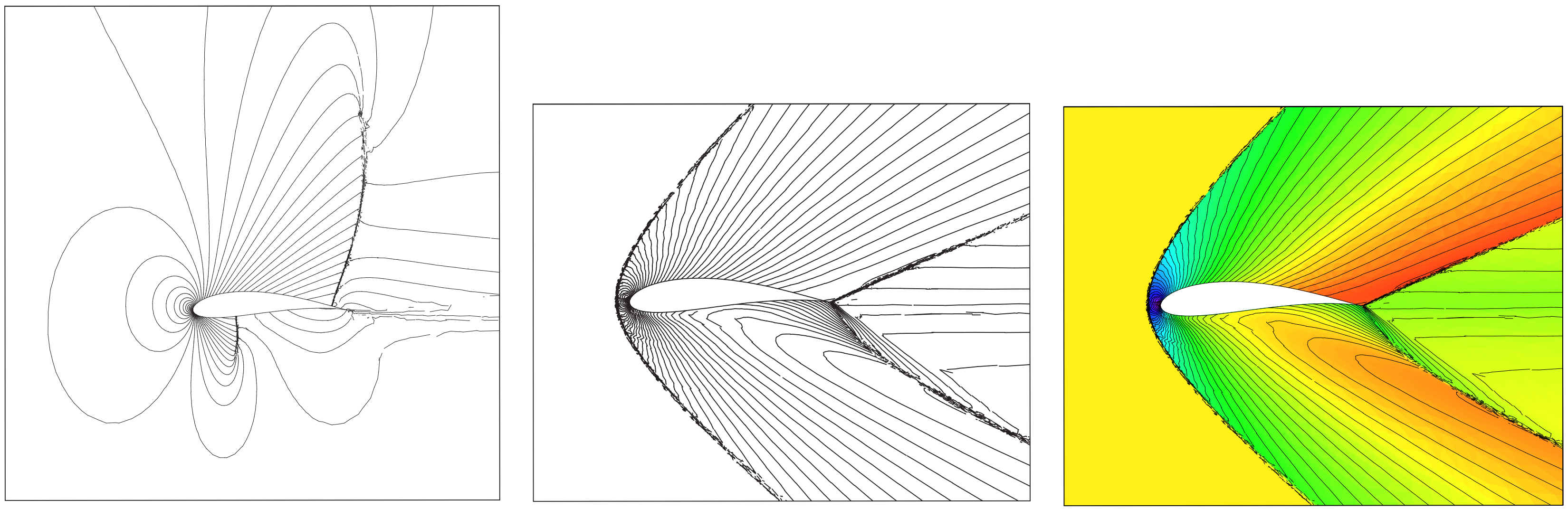


Velocity distribution along the profile: ○○○ – exact solution of incompressible flow, ——— – approximate solution of compressible low Mach flow

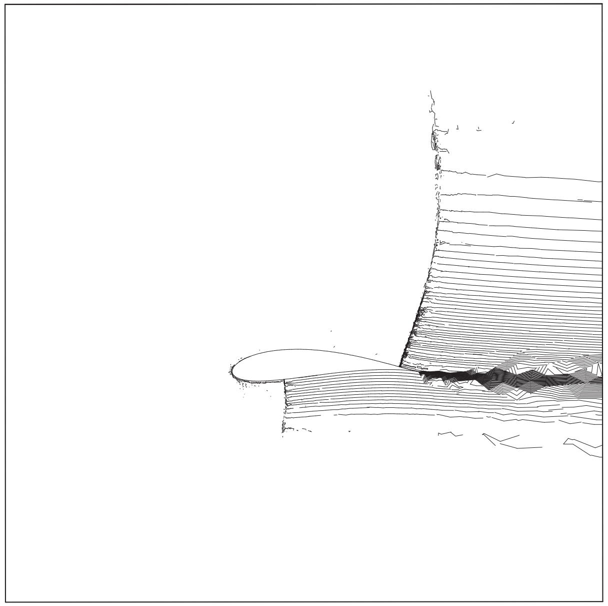
b) **Transonic and hypersonic flow with shock waves past the Joukowski profile**

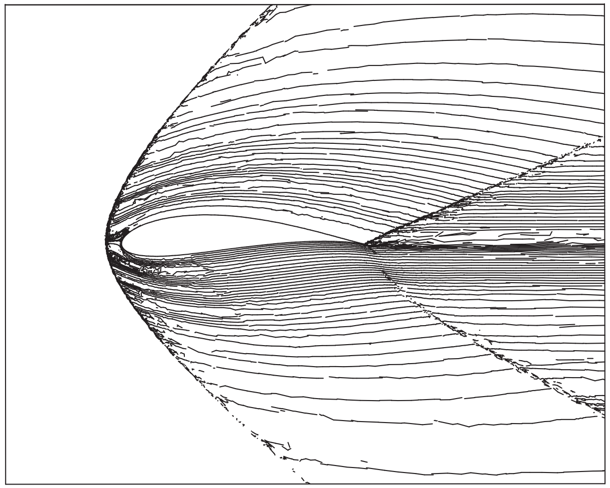
with far field Mach number $M_\infty = 0.8$ **and** $M_\infty = 2.0$**, respectively**

The maximum density variation: $\rho_{max} - \rho_{min} = 0.94$ **for** $M_\infty = 0.8$ **and** $\rho_{max} - \rho_{min} = 2.61$ **for** $M_\infty = 2.0$



Mach number isolines of the flow past a Joukowski profile with $M_\infty = 0.8$ (left) and $M_\infty = 2.0$ (right)





Entropy isolines of the flow past a Joukowski profile with $M_\infty = 0.8$ (left) and $M_\infty = 2.0$ (right)

2) Viscous compressible flow

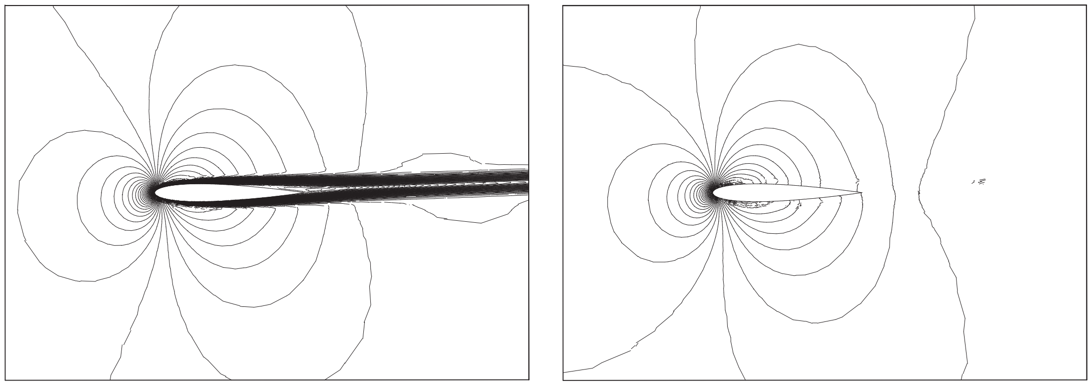
a) Stationary viscous flow past NACA0012 profile

$\theta = 0$ – **IIPG**

far-field Mach number $M = 0.5$

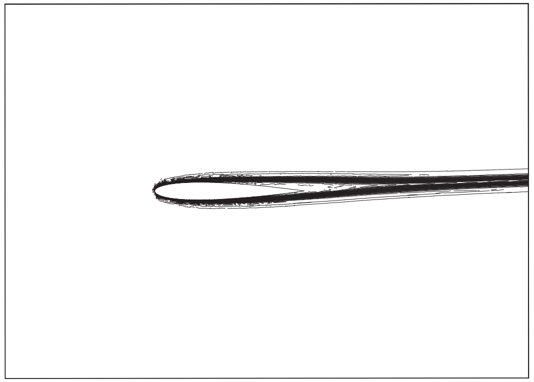
angle of attack $\alpha = 2^\circ$

Reynolds number $Re = 5000$



NACA0012 $\alpha = 2^\circ$ **viscous flow, Mach number**

isolines (left), pressure isolines (right)



entropy isolines.

b) Non-stationary viscous flow past NACA0012 profile

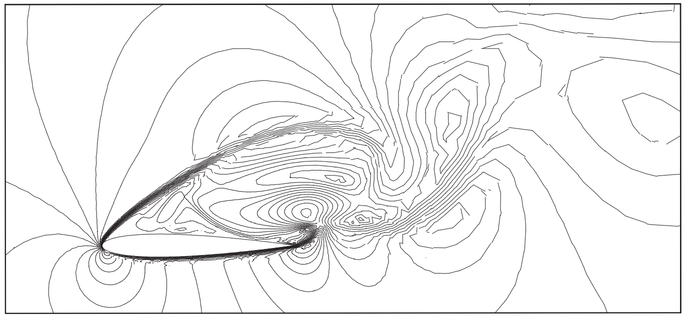
far-field flow has Mach number $M = 0.5$

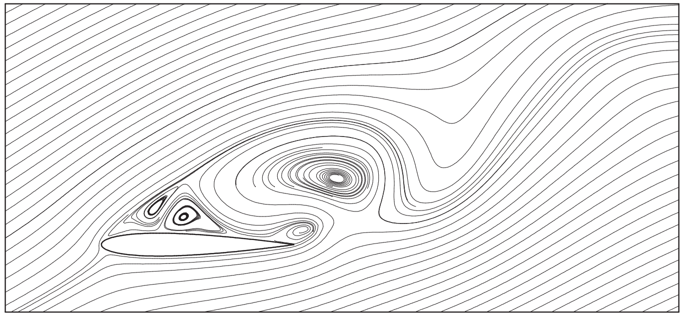
angle of attack $\alpha = 25^\circ$

Reynolds number $Re = 5000$

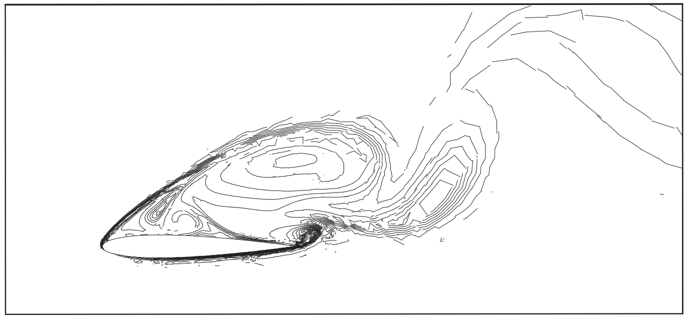
possible to observe an unsteady vortex shedding from the airfoil

figures illustrate the flow situation at time $t = 8.5$





NACA0012 $\alpha = 25^\circ$ viscous flow, Mach number isolines (left), streamlines (right)



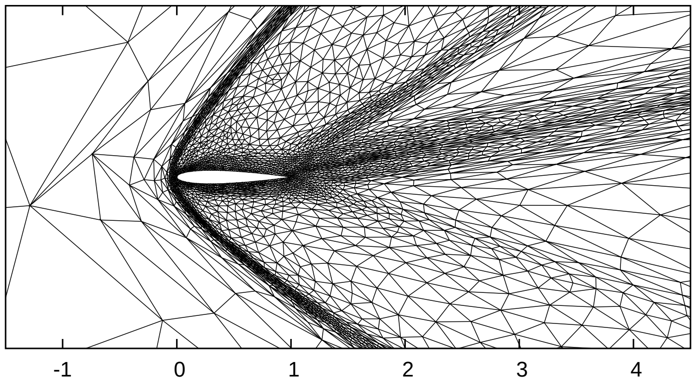
NACA0012 $\alpha = 25^\circ$ viscous flow, entropy isolines

c) Hypersonic viscous flow

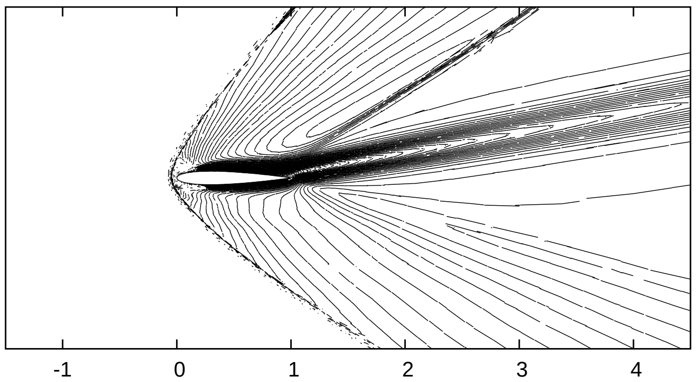
Flow past NACA0012 profile:

Far field Mach number $M_\infty = 2, \alpha = 10^\circ$

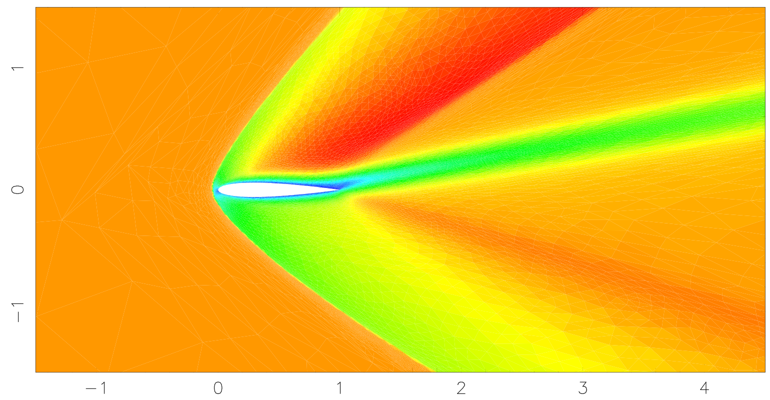
Reynolds number = 1000



Mesh for viscous flow - constructed by ANGENER - V. Dolejší



Mach number isolines for viscous flow



Distribution of the Mach number for viscous flow

Flow in time-dependent domains

1) Compressible inviscid flow in a channel with the initial rectangular shape $\Omega_0 = (-2, 2) \times (0, 1)$, where the lower wall of the channel is moving in the interval $X_1 \in (-1, 1)$:

$$0.45 \sin(0.4t)\,(\cos(\pi X_1) + 1), \;\; X_1 \in (-1, 1). \qquad (78)$$

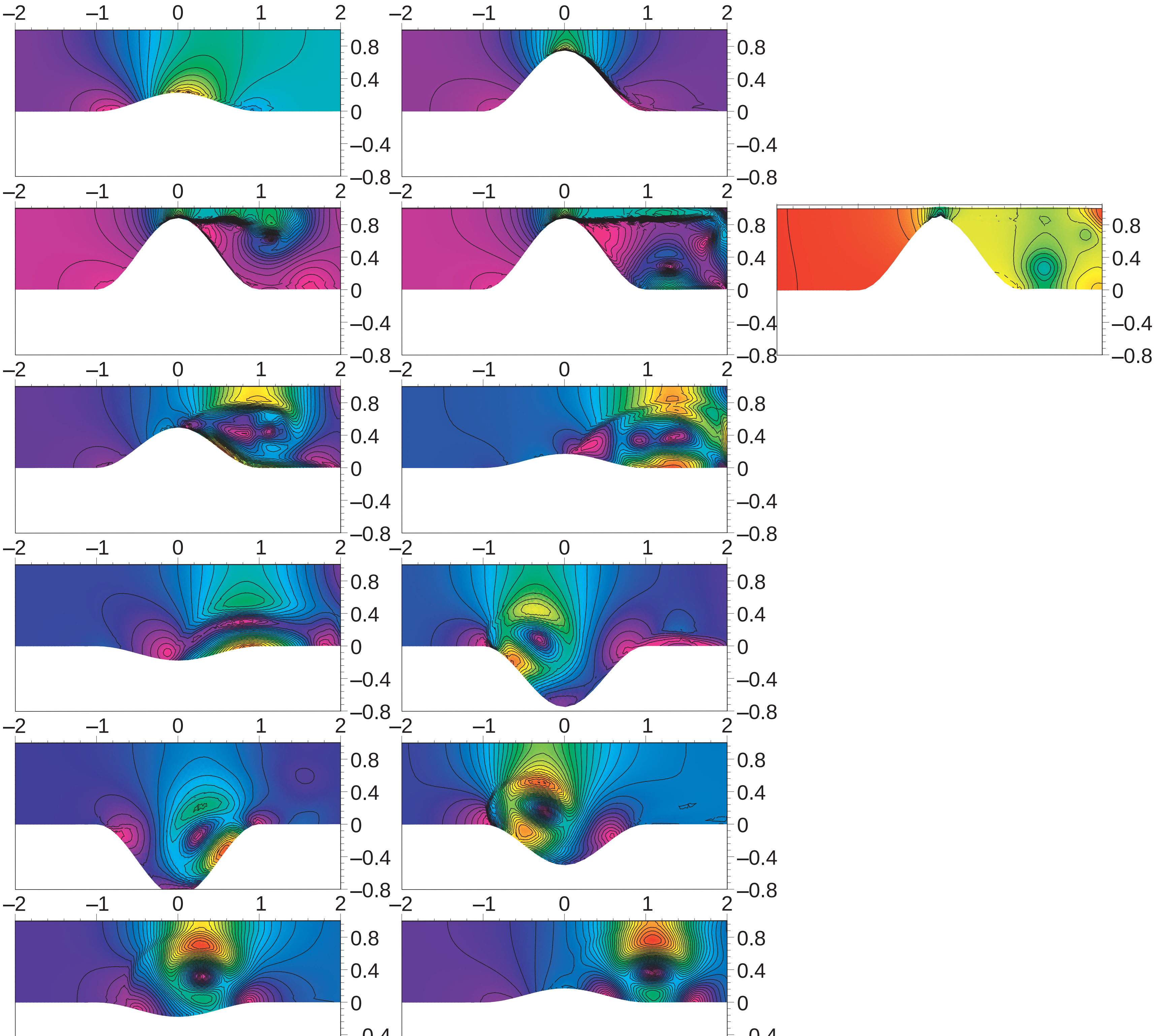
This movement is interpolated to the whole domain resulting in the ALE mapping $\mathcal{A}_t$.

Inlet Mach number $= 0.12$

Figure 1: velocity isolines at different time instants during one period

The solution contains a vortex formation, when the lower wall starts to descend, convected through the domain.

Moreover, we see that a contact discontinuity is developed, when the channel becomes narrow.

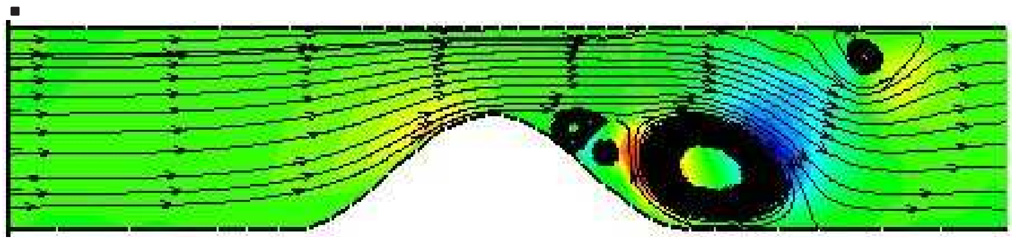


−2 −1 0 1 2
0.8
0.4
0
−0.4
−0.8

2) Compressible viscous flow in a channel with the initial rectangular shape $\Omega_0 = (-5, 5) \times (0, 1)$, where the lower wall of the channel is moving in the interval $X_1 \in (-1, 1)$:

$$0.3 \sin^2(0.04(t - 250.5)) (\cos(\pi X_1) + 1), \;\; X_1 \in (-1, 1).$$

$Re = 6976.74$, **inlet Mach number**$=$ **0.12**



$t = 10.08s$

3) Compressible flow past a moving airfoil

Conclusion

– DGFEM = a promising robust method for the solution of compressible flow
– combination with ALE method allows the solution of flow problems in time dependent domains

Further goals:

– coupling with structural models
– applications to complex FSI problems
– theoretical analysis

Thank you for your attention

Some references:

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