HURWITZ-BRILL-NOETHER THEORY VIA $K3$ SURFACES AND STABILITY CONDITIONS

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ABSTRACT. We develop a novel approach to the Brill–Noether theory of curves endowed with a degree k cover of $\mathbf{P}^1$ via Bridgeland stability conditions on elliptic $K3$ surfaces.

We first develop the Brill–Noether theory on elliptic $K3$ surfaces via the notion of Bridgeland stability type for objects in their derived category. As a main application, we show that curves on elliptic $K3$ surfaces serve as the first known examples of smooth k-gonal curves which are general from the viewpoint of Hurwitz–Brill–Noether theory. In particular, we provide new proofs of the main non-existence and existence results in Hurwitz–Brill–Noether theory. Finally, using degree-k Halphen surfaces, we construct explicit examples of curves defined over number fields which are general from the perspective of Hurwitz–Brill–Noether theory.

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1. INTRODUCTION

The Brill-Noether theorem, asserting that for a general curve C of genus g the dimension of the variety of linear systems

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}$$

equals the Brill-Noether number $\rho(g, r, d) = g-(r+1)(g-d+r)$, is one of the cornerstones of the theory of algebraic curves. After having been formulated as what we would nowadays call, a plausibility statement in the later 19th century, its proof was completed by Griffiths-Harris, Gieseker and Eisenbud-Harris in the 1980s using degeneration and limit linear series, see [GH80], [Gie82], [EH86], [EH83] and [FL81]. A few years later, Lazarsfeld [Laz86] found a completely different approach to this problem by showing that every smooth curve C on a $K3$ surface X with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot C$ satisfies the Brill-Noether theorem. These two approaches, via degeneration and limit linear series respectively via $K3$ surfaces have been then brought together in [ABFS16], where it has been showed that a suitably general curve C on a Halphen surface (that is, a rational surface shown in [ABS17] to be a limit of polarized $K3$ surfaces) satisfies the Brill-Noether theorem.

Hurwitz-Brill-Noether theory is a much more recent development and concerns the loci $W^r_d(C)$, when $[C, A]$ is a general point of the Hurwitz space $\mathcal{H}_{g,k}$ classyfing pairs consisting of a genus g curve C and a pencil $A \in W^1_k(C)$ inducing a degree k cover $C \to \mathbf{P}^1$. Without loss of generality, one may assume that $d \leq g-1$. After initial work of Coppens and Martens [CM99], Pflueger [Pfl17] showed via tropical methods that if

$$\rho_k(g, r, d) := \max_{\ell=0,\dots,r}\{\rho(g, r-\ell, d) - \ell k\} \tag{1}$$

then for such a curve we have $\dim W^r_d(C) \leq \rho_k(g, r, d)$; then Jensen-Ranganathan [JR21], also via tropical geometry, established the existence part and showed that indeed $\dim W^r_d(C) = \rho_k(g, r, d)$. These results have been greatly refined through the notion of *splitting type* of a linear system $L \in W^r_d(C)$, first considered in [Lar21] and [CPJ22]. In particular, the results of H. Larson [Lar21] and E. Larson-H. Larson-Vogt [LLV25], describe via degeneration all the irreducible components of $W^r_d(C)$ for a general point $[C, A] \in \mathcal{H}_{g,k}$ and single out an open subset where these components are smooth. For recent related work on these matters we also refer to [Cop25].

The goal of this paper is to provide a radically different approach to Hurwitz-Brill-Noether theory via *Bridgeland stability conditions.* We consider $K3$ surfaces X with $\mathrm{Pic}(X) \cong \mathbb{Z}\cdot H \oplus \mathbb{Z}\cdot E$, where H is an ample class such that $H^2 = 2g-2$, and $|E|$ is an elliptic pencil with $E \cdot H = k$. In this way, the curves $C \in |H|$ are endowed with a degree k pencil $A := \mathcal{O}_C(E) \in W^1_k(C)$. Such degree k elliptic $K3$ surfaces form an irreducible Noether-Lefschetz divisor in the moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces of genus g.

As a key step toward our final goal in Hurwitz–Brill–Noether theory, we first develop the Brill–Noether theory for elliptic K3 surfaces, as outlined below.

1.1. **Bridgeland stability types.** Let $\mathcal{D}(X)$ denote the bounded derived category of coherent sheaves on X. Stability conditions on $\mathcal{D}(X)$, introduced by Bridgeland in his fundamental papers [Bri07] and [Bri08], generalize to objects in $\mathcal{D}(X)$ the traditional notion of Gieseker stability of sheaves. For $\epsilon \in \mathbb{Q}_{>0}$, we consider the polarization

$$H_\epsilon := E + \epsilon H \in \mathrm{Pic}(X)_{\mathbb{Q}}.$$

on X. With respect to H_ϵ, we study a ray[1] of stability conditions σ_w for $w > 0$. One has a slope function ν_w defined on a fixed abelian subcategory $\mathrm{Coh}^0(X) \subseteq \mathcal{D}(X)$ parametrizing 2-term complexes of sheaves on X, see also Theorem 3.6. Accordingly, one has a concept of σ_w-stability, defined in terms of ν_w-slopes, on the category $\mathrm{Coh}^0(X)$.

For any fixed Mukai vector $v = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2 + \mathrm{ch}_0)$, there are finitely many walls $w_1 < \cdots < w_q$ such that the moduli space $\mathcal{M}_{\sigma_w}(v)$ parametrizing σ_w-stable objects in $\mathrm{Coh}^0(X)$ of class v remains unchanged in the regions between these walls.

[1]To carry out a wall-crossing analysis, we in fact work over a two-dimensional slice of stability conditions. In the Introduction, for simplicity, we focus here on the central ray of this slice, where the walls correspond to certain non-negative real numbers.

In the case of $K3$ surfaces of Picard number 1, as studied in [Bay18, BL17], if $\mathrm{ch}_1(v)$ is minimal, then there is no wall for the class v along the ray σ_w for $w > 0$, that is, the only wall we encounter is the one induced by $\mathcal{O}_X$ (at $w = 0$), via the exact triangle

$$\mathcal{O}_X \otimes \mathrm{Hom}(\mathcal{O}_X, F) \xrightarrow{\mathrm{ev}} F \longrightarrow \mathcal{E},$$

where $\mathcal{E}$ denotes the shifted dual of the *Lazarsfeld–Mukai bundle* [Laz86, Apr13] in the case where F is a line bundle on a curve on X. In our setting however, line bundles of the form $\mathcal{O}_X(eE)$ for $e > 0$ define additional walls, making the wall-crossing analysis significantly more challenging. Motivated by the classical notion of scrollar invariants [CM99] and by H. Larson's recent concept of splitting type [Lar21] for line bundles on curves, we introduce the notion of *Bridgeland stability type* (Definition 5.1) for arbitrary objects in the derived category, which enables us to handle such walls systematically.

Start with a σ_{w_0}-stable object $F \in \mathrm{Coh}^0(X)$ for some $w_0 > 0$. We say that F has *stability type*

$$\overline{e} = \big((e_1, m_1), \dots, (e_p, m_p)\big),$$

where $e_1 > \cdots > e_p \geq 0$ and $m_i > 0$, if the following holds: As we move down from w_0 toward the origin, the object F is first destabilized at a wall $w_1 < w_0$ by an object $\mathcal{O}_X(e_1E) \otimes \mathrm{Hom}(\mathcal{O}_X(e_1E), F) \cong \mathcal{O}_X(e_1E)^{\oplus m_1}$, and the corresponding destabilizing quotient F_1 is σ_{w_1}-stable. We repeat this process with F_1, finding the first wall below w_1 destabilising it, and so on. In total, we obtain a series of destabilising short exact sequences in $\mathrm{Coh}^0(X)$

$$\begin{aligned}
0 \longrightarrow \mathcal{O}_X(e_1E)^{\oplus m_1} &\longrightarrow F \longrightarrow F_1 \longrightarrow 0,\\
0 \longrightarrow \mathcal{O}_X(e_2E)^{\oplus m_2} &\longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0,\\
&\vdots\\
0 \longrightarrow \mathcal{O}_X(e_pE)^{\oplus m_p} &\longrightarrow F_{p-1} \longrightarrow F_p \longrightarrow 0,
\end{aligned}$$

where each F_i is stable along the wall that destabilizes F_{i-1}, and the final object F_p satisfies $\mathrm{Hom}(\mathcal{O}_X, F_p) = 0$.

For the remainder of this subsection, we fix the Mukai vector

$$v := (r_0, H - a_0E, s_0 + r_0),$$

with $r_0 \leq 0$, $a_0 \geq 0$, and $s_0 < 0$. We choose $\epsilon > 0$ sufficiently small, depending on this class v. Our first result states that, for suitable w, any σ_w-stable object of class v has an associated stability type (see Theorem 5.5):

Theorem 1.1. *Fix $w_0 > 0$ such that $\nu_{w_0}(v) < 0$. Every σ_{w_0}-stable object $F \in \mathrm{Coh}^0(X)$ of Mukai vector v admits a stability type $\overline{e} = \big((e_1, m_1), \dots, (e_p, m_p)\big)$ with $p \geq 0$.*

Moreover, the following inequalities hold:

(a) $\displaystyle\sum_{i=1}^{p} m_i \leq \hom(\mathcal{O}_X, F) \leq \sum_{i=1}^{p} m_i(e_i + 1)$,

(b) $m_1(e_1 + 1) \leq \hom(\mathcal{O}_X, F)$.

In other words, flexibility in the choice of the polarization H_ϵ paves the way to performing a systematic study of wall-crossing for v. Observe also that the inequalities in Theorem 1.1 indicate a strong link between the stability type and the Brill-Noether properties of an object in $\mathrm{Coh}^0(X)$.

We expect that the above theorem represents a first step toward a broader framework for studying wall-crossing phenomena on elliptic $K3$ surfaces—a topic of independent interest that we postpone to future work. This perspective has potential applications to Brill–Noether theory on moduli spaces of sheaves (in the spirit of [Mar01, Ley12, CNY23, CNY25] for K-trivial surfaces), as well as to the study of higher rank vector bundles on k-gonal curves (see [BF18, FL21, Fey22] for examples of such study on curves on Picard rank one $K3$ surfaces).

The following theorem describes in geometric terms the moduli spaces of Bridgeland stable objects with a fixed stability type $\overline{e}$ (see Theorem 5.3):

Theorem 1.2. *Let $w > 0$ be generic, and let $\overline{e} = \big((e_1, m_1), \dots, (e_p, m_p)\big)$ be any stability type. Then the subset*

$$\mathcal{M}_{\sigma_w}(v, \overline{e}) := \big\{F \in \mathcal{M}_{\sigma_w}(v) : F \text{ has stability type } \overline{e}\big\}$$

admits a natural scheme structure as an iterated Grassmann bundle inside $\mathcal{M}_{\sigma_w}(v)$. If non-empty, the space $\mathcal{M}_{\sigma_w}(v, \overline{e})$ is smooth, quasi-projective, and irreducible of dimension

$$\Big(v - \sum_{i=1}^{p} m_i(1, e_iE, 1)\Big)^2 + 2 + \sum_{j=1}^{p} m_j\Big(\Big\langle v - \sum_{i=1}^{j} m_i(1, e_iE, 1), (1, e_jE, 1)\Big\rangle - m_j\Big),$$

where $\langle -, - \rangle$ denotes the Mukai pairing.

Theorem 1.2 does not guarantee the non-emptiness of the moduli space $\mathcal{M}_\sigma(v, \overline{e})$. However, it follows from the definition of Bridgeland stability type that the inequalities

$$\Big(v - \sum_{i=1}^{p} m_i(1, e_iE, 1)\Big)^2 + 2 \geq 0, \qquad \nu_w\big(\mathcal{O}_X(e_1E)\big) < \nu_w(v) < 0 \tag{2}$$

are necessary conditions for $\mathcal{M}_\sigma(v, \overline{e})$ to be non-empty.

These conditions also turn out to be sufficient for a distinguished class of stability types that we call *balanced stability types*[2]. These are stability types of the form

$$\overline{e} = \big((e+1, m_1), (e, m_2)\big), \quad \text{with} \quad e, m_1, m_2 \geq 0.$$

Theorem 1.3. *Let $\overline{e}$ be a balanced stability type as above, and assume $m_1 + m_2 \leq k + r_0$. If $w > 0$ satisfies the inequalities (2), then the moduli space $\mathcal{M}_{\sigma_w}(v, \overline{e})$ is non-empty.*

Theorem 1.3, which is Theorem 5.8 in the paper, is our main existence result in the Brill-Noether theory of elliptic $K3$ surfaces.

1.2. **Hurwitz-Brill-Noether theory via Bridgeland stability types.** We now apply our results on the elliptic $K3$ surface X to the study of the Brill–Noether loci $W^r_d(C)$ for pairs $[C, A] \in \mathcal{H}_{g,k}$, where $C \in |H|$ and $A := \mathcal{O}_C(E)$. Our first result in this direction shows that a general curve $C \in |H|$ satisfies the Hurwitz–Brill–Noether theorem. This can be viewed as the Hurwitz-theoretic analogue of Lazarsfeld's classical result [Laz86] for Brill–Noether generality of curves on $K3$ surfaces with Picard number one.

[2]The term "balanced" is used in alignment with H. Larson's terminology in [Lar21].

Theorem 1.4. *For any $d \leq g-1$ and $r \geq 0$, the following hold:*

(a) If $C \in |H|$ is a general curve, then $\dim W^r_d(C) = \rho_k(g,r,d)$.
(b) If $\rho_k(g,r,d) < 0$, then $W^r_d(C) = \emptyset$ for every integral curve $C \in |H|$.

Note that for a *general k-gonal curve* Theorem 1.4 has been established as a combination of results from [Pfl17, JR21, Lar21]. However, not a single example of a smooth k-gonal curve verifying the Hurwitz-Brill-Noether Theorem has been known before and our Theorem 1.4 shows that general curves on an elliptic $K3$ surface enjoy this property.

To prove Theorem 1.4, for an integral curve $C \in |H|$, we regard (the pushforward to X of) any line bundle $L \in \overline{\mathrm{Pic}}^d(C)$ as an H_ϵ-Gieseker stable sheaf on X with Mukai vector $v = (0, H, 1+d-g)$. For $\epsilon > 0$ small enough, Theorem 1.1 asserts that L has an associated Bridgeland stability type. Then one deduces from Theorem 1.2 that in the Gieseker moduli space $\mathcal{M}_{H_\epsilon}(v)$, the subset

$$\left\{F \in \mathcal{M}_{H_\epsilon}(v):\ h^0(X,F) \geq r+1\right\}$$

has dimension at most $g + \rho_k(g,r,d)$, by considering all the stability types compatible with the condition $h^0(X,F) \geq r+1$. Furthermore, under the assumption $\rho_k(g,r,d) < 0$, all the allowed stability types violate inequalities (2). This proves part (b) and the upper bound of part (a) in Theorem 1.4. On the other hand, we prove the existence statement $\dim W^r_d(C) \geq \rho_k(g,r,d)$ by addressing the more general problem of describing irreducible components of $W^r_d(C)$, as outlined below.

Fix a point $[C,A] \in \mathcal{H}_{g,k}$ and any integer $r \geq 0$. For any integer ℓ satisfying the inequalities $\max\{0, r+2-k\} \leq \ell \leq r$, define $e := \left\lfloor \frac{\ell}{r+1-\ell} \right\rfloor$, and write

$$\ell = e(r+1-\ell) + m_1, \quad m_2 := r+1-\ell-m_1, \tag{3}$$

so that $0 \leq m_1 \leq r-\ell$. We then define the following locus (see also (9)):

$$V^r_{d,\ell}(C,A) := \Big\{L \in \mathrm{Pic}^d(C) : h^0(C, L \otimes A^{-e-2}) = 0, \quad h^0(C, L\otimes A^{-e-1}) = m_1, \\ h^0(C, L \otimes A^{-e}) = 2m_1 + m_2, \quad h^0(C,L) = r+1\Big\}\,^{3}.$$

The conditions $h^0\big(C, L\otimes A^{-e-1}\big) = m_1$ and $h^0\big(C, L \otimes A^{-e}\big) = 2m_1 + m_2$ imply, as explained in Proposition 2.6, that $h^0(C,L) \geq r+1$. This shows that $V^r_{d,\ell}(C,A)$ can be realized as a degeneracy locus over (an open subset of) the Brill–Noether variety $W^{m_1-1}_{d-(e+1)k}(C)$, associated to the vector bundle morphism globalizing the multiplication maps

$$H^0(C,A) \otimes H^0\big(C, \omega_C \otimes A^{e+1} \otimes L^\vee\big) \longrightarrow H^0\big(C, \omega_C \otimes A^{e+2} \otimes L^\vee\big).$$

The expected dimension of this degeneracy locus is precisely $\rho(g, r-\ell, d) - \ell k$.

Returning to the setup of elliptic $K3$ surfaces, our next result describes the geometry of the varieties $V^r_{d,\ell}(C,A)$ for curves in elliptic $K3$ surfaces, which in particular completes the proof of Theorem 1.4.

[3] In the language of [Lar21], $V^r_{d,\ell}(C)$ is the locus of degree d line bundles whose splitting type has non-negative part $(\underbrace{e,\ldots,e}_{m_2}, \underbrace{e+1,\ldots,e+1}_{m_1})$.

Theorem 1.5. *Let X be a general $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$ as above. Fix $d \leq g-1$, $r \geq 0$ and $\ell \in \mathbb{Z}$ such that $\max\{0, r+2-k\} \leq \ell \leq r$. Then for a general curve $C \in |H|$, the variety*

$$V^r_{d,\ell}\big(C, \mathcal{O}_C(E)\big)$$

is smooth of the expected dimension $\rho(g, r-\ell, d) - \ell k$. In particular, it is empty if $\rho(g, r-\ell, d) - \ell k < 0$.

In order to prove Theorem 1.5 we apply Theorem 1.3 to the balanced stability type $\overline{e} = \big((e+1, m_1), (e, m_2)\big)$ as defined in (3). It follows that the moduli space $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ of H_ϵ-Gieseker stable sheaves with Mukai vector $v = (0, H, 1+d-g)$ and stability type $\overline{e}$ is non-empty. By Theorem 1.2, this moduli space is smooth, quasi-projective, and irreducible of the expected dimension; notably, this dimension equals $g+\rho(g, r-\ell, d)-\ell k$. In the second part of the proof of Theorem 1.5, we consider the natural support map

$$\mathcal{M}_{H_\epsilon}(v, \overline{e}) \longrightarrow |H|.$$

In Section 7, we prove that this map is dominant by exhibiting reducible curves of the form $C + J$ in the linear system $|H|$, where $C \in |H - E|$ and $J \in |E|$ are chosen generically, such that the fiber over the point $[C + J] \in |H|$ contains a component of dimension $\rho(g, r-\ell, d)-\ell k$. Note that our inductive step alters the parameters according to the rule

$$(g-k,\ d-k,\ r-1,\ \ell-1) \mapsto (g,\ d,\ r,\ \ell),$$

and that under this change the Brill–Noether number in question remains constant, that is, $\rho(g-k, r-\ell, d-k) - (\ell-1)k = \rho(g, r-\ell, d) - \ell k$. Applying Theorems 1.1 and 1.2 to curves in the linear system $|H - E|$, we reduce to the base case $\ell = 0$, which must be treated separately; see Theorem 7.6. Dominance of the support map, when coupled with a close relation between balanced Bridgeland stability types and the non-negative part of the splitting type (see Subsection 6.2), is enough to derive Theorem 1.5.

As a consequence of Theorem 1.5, we have a precise description of the loci $V^r_{d,\ell}(C, A)$ for a general point of the Hurwitz space.

Corollary 1.6. *Fix $d \leq g-1$, $r \geq 0$ and $\ell \in \mathbb{Z}$ such that $\max\{0, r+2-k\} \leq \ell \leq r$. Then for a general point $[C, A] \in \mathcal{H}_{g,k}$, the variety $V^r_{d,\ell}\big(C, A\big)$ is smooth of the expected dimension $\rho(g, r-\ell, d) - \ell k$.*

It is an interesting question to understand in full generality the relation between the Bridgeland stability type and (the non-negative part of) the splitting type of a line bundle on an integral curve $C \in |H|$. There are indications that these two sets invariants might not always coincide. A more conservative guess would be that for any stability type $\overline{e} = \big((e_1, m_1), \ldots, (e_p, m_p)\big)$ and any integral curve $C \in |H|$, the two loci of sheaves $L \in \overline{\mathrm{Pic}}^d(C)$ such that i_*L has stability type $\overline{e}$, respectively those such the non-negative part of the splitting type of L equals $(\underbrace{e_p, \ldots, e_p}_{m_p}, \ldots, \underbrace{e_1, \ldots, e_1}_{m_1})$, to have the same closure in $\overline{\mathrm{Pic}}^d(C)$. We prove this property for balanced stability types in Proposition 6.13.

1.3. **Hurwitz-Brill-Noether theory over number fields.** It is now natural to ask whether one can write down a smooth Hurwitz-Brill-Noether general k-gonal curve of genus g defined over a number field, or even over $\mathbb{Q}$. In Section 8 we explain how the results of this paper can be used to solve this problem.

A Halphen surface of degree $k \geq 2$ is obtained by blowing-up $\mathbf{P}^2$ at points $p_1, \ldots, p_9$ lying on a unique plane cubic J, such that $p_1 + \cdots + p_9 \in J$ is a torsion point of order k (with respect to the group law of J). Set $X := \mathrm{Bl}_{\{p_1,\ldots,p_9\}}(\mathbf{P^2})$ and let h be the hyperplane class on X and $E_1, \ldots, E_9$ the exceptional divisors. Following [ABFS16],

$$\Lambda_g := \left|3gh - gE_1 - \cdots - gE_8 - (g-1)E_9\right|$$

is then a linear system of curves of genus g on X. Note that $h^0\big(C, \mathcal{O}_C(kJ)\big) \geq 2$, that is, every curve $C \in \Lambda_g$ is endowed with a degree k pencil. The following result combines Theorems 8.2 and 8.3:

Theorem 1.7. *A pair $\big[C, \mathcal{O}_C(kJ)\big] \in \mathcal{H}_{g,k}$, where $C \in \Lambda_g$ is a general curve, verifies the Hurwitz-Brill-Noether theorem, that is,*

$$\dim W^r_d(C) = \rho_k(g, r, d).$$

In particular, for every prime k, there exist Hurwitz-Brill-Noether general curves of genus g defined over a number field K with $[K : \mathbb{Q}] \leq k^2 - 1$.

It remains a very interesting open question whether one can write down Hurwitz-Brill-Noether general k-curves of genus g defined over the rationals.

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2. Hurwitz-Brill-Noether loci

In this introductory section, we associate to every smooth curve C endowed with a degree k pencil $(A, V) \in G^1_k(C)$ the Hurwitz-Brill-Noether loci $W^r_{d,\ell}(C, A)$, then we explain a novel derivation of their expected dimension as degeneracy loci over classical Brill-Noether loci associated to C.

We fix integers $g, k \geq 2$ and denote by $\widetilde{\mathcal{H}}_{g,k}$ the Hurwitz stack classifying triples consisting of a stable curve C of genus g, a line bundle $A \in \mathrm{Pic}^k(C)$ and a 2-dimensional space of sections $V \subseteq H^0(C, A)$. When $V = H^0(C, A)$, we refer to points in $\widetilde{\mathcal{H}}_{g,k}$ as pairs $[C, A]$. It is well known that $\widetilde{\mathcal{H}}_{g,k}$ is irreducible of dimension $2g+2k-5 = 3g-3+\rho(g, 1, k)$. Let $\mathcal{H}_{g,k}$ be the open substack of $\widetilde{\mathcal{H}}_{g,k}$ corresponding to triples $[C, A, V]$ as above, where C is a smooth curve. We shall use several times that as long as $g \geq a(k-1)$, we have $\mathrm{Sym}^a H^0(C, A) \cong H^0(C, A^a)$ for a general $[C, A] \in \mathcal{H}_{g,k}$.

We fix a non-negative integer ℓ with $\max\{0, r-k+2\} \leq \ell \leq r$ and such that

$$\rho(g, r-\ell, d) - \ell k \geq 0. \tag{4}$$

We aim to study the loci $W^r_d(C)$ for a triple $[C, A, V] \in \mathcal{H}_{g,k}$. Without loss of generality, by Riemann-Roch, we may assume $d \leq g-1$. Set $e := \left\lfloor \frac{\ell}{r+1-\ell} \right\rfloor$ and write

$$\ell = e(r+1-\ell) + m_1, \tag{5}$$

where $0 \leq m_1 \leq r - \ell$. Setting $m_2 := r+1-\ell-m_1 \geq 1$, we then also have

$$r + 1 = m_1(e+2) + m_2(e+1).$$

Proposition 2.1. *Assume inequality (4) holds. Then for every smooth curve C of genus g, the locus $W^{m_1-1}_{d-(e+1)k}(C)$ is non-empty. Furthermore, $d-(e+1)k \geq 0$.*

Proof. We apply the main existence results of Brill-Noether theory, see e.g. [FL81] and to that end we need to show that the expected dimension $\rho\big(g, m_1-1, d-(e+1)k\big)$ of the determinantal variety $W^{m_1-1}_{d-(e+1)k}(C)$ is non-negative. Indeed, one has

$$\rho\big(g, m_1-1, d-(e+1)k\big) = \rho(g, r-\ell, d) - \ell k + (r-\ell-m_1+1)(g+ek-d+r-\ell+m_1).$$

Note that $r-\ell-m_1 \geq 0$ by (5), whereas

$$g + ek - d + r - \ell + m_1 = g - d + r + ek - e(r+1-\ell) \geq g - d + r - \ell \geq 1,$$

since we assumed that $d \leq g-1$. Since by assumption $\rho(g, r-\ell, d) - \ell k \geq 0$, it follows from [FL81] that $W^{m_1-1}_{d-(e+1)k}(C)$ is non-empty and of dimension at least $\rho\big(g, m_1-1, d-(e+1)k\big)$. Clearly, since $\rho\big(g, m_1-1, d-(e+1)k\big) \geq 0$, it also follows that $d-(e+1)k \geq 0$. □

Proposition 2.2. *Assume $(A, V) \in G^1_k(C)$ is a base point free pencil and let M be a line bundle on C. Then for any $a > 0$ we have the inequality*

$$h^0\big(C, M \otimes A^{a+1}\big) \geq (a+1) \cdot h^0(C, M \otimes A) - a \cdot h^0(C, M).$$

Proof. We prove this via the base point free pencil trick [ACGH85, p. 126]. Since V is a base point free pencil on C, for $m \leq a$ the kernel of the multiplication map $V \otimes H^0(C, M \otimes A^m) \longrightarrow H^0(C, M \otimes A^{m+1})$ can be identified with $H^0(C, M \otimes A^{m-1})$. Therefore

$$h^0(C, M \otimes A^{m+1}) - h^0(C, M \otimes A^m) \geq h^0(C, M \otimes A^m) - h^0(C, M \otimes A^{m-1}),$$

so summing over $m = 1, \ldots, a$ implies that

$$h^0(C, M \otimes A^{a+1}) - h^0(C, M \otimes A) \geq a\big(h^0(C, M \otimes A) - h^0(C, M)\big),$$

as claimed. □

Applying Proposition 2.2 for $a = e+1$, if $M \in W^{m_1-1}_{d-(e+1)k}(C)$ is such that

$$h^0(C, M) = m_1 \quad \text{and} \quad h^0(C, M \otimes A) \geq r - \ell + m_1 + 1 = 2m_1 + m_2, \tag{6}$$

where we recall that m_1 is given by (5), after setting $L := M \otimes A^{e+1} \in \mathrm{Pic}^d(C)$, we have

$$h^0(C, L) \geq (e+1)(r-\ell+m_1+1) - em_1 = e(r-\ell+1) + m_1 + (r-\ell+1) = r+1.$$

Definition 2.3. Let $\widetilde{W}^{m_1-1}_{d-(e+1)k}(C)$ be the union of all components of $W^{m_1-1}_{d-(e+1)k}(C)$ having the expected dimension $\rho\big(g, m_1-1, d-(e+1)k\big)$ and corresponding to a general point M satisfying $H^0(C, M\otimes A^\vee)=0$

Applying [ACGH85, Lemma 3.5], a general point M of each component of $W^{m_1-1}_{d-(e+1)k}(C)$ satisfies $h^0(C,M)=m_1$. Furthermore, from (5) we have $m_1-1\leq r-\ell-1\leq k-3$, hence by applying [CM99, Proposition 2.3.1] (see also Proposition 3.4), we obtain that $\widetilde{W}^{m_1-1}_{d-(e+1)k}(C)\neq\emptyset$ for a general element $[C,A]\in\mathcal{H}_{g,k}$.

Remark 2.4. For a general $(C,A)\in\mathcal{H}_{g,k}$, the equality $W^{m_1-1}_{d-(e+1)k}(C)=\widetilde{W}^{m_1-1}_{d-(e+1)k}(C)$ does not hold whenever $m_1\geq 2$. In fact, using the identity

$$d-(e+m_1)k=\rho\big(g,m_1-1,d-(e+1)k\big)+(m_1-1)(g+ek+m_1-d)$$

we obtain via (5) that $d-(e+m_1)k\geq 0$. In particular line bundles of the form $M=A^{m_1-1}(D)$, with D being an effective divisor on C of degree $d-(e+m_1)k$, satisfy $h^0(C,M)\geq m_1$. Such line bundles depend on $d-(e+m_1)k>\rho\big(g,m_1-1,d-(e+1)k\big)$ parameters and they lie in $W^{m_1-1}_{d-(e+1)k}(C)\setminus\widetilde{W}^{m_1-1}_{d-(e+1)k}(C)$.

The condition (6) defines a determinantal subvariety of (an open subvariety of) the locus $\widetilde{W}^{m_1-1}_{d-(e+1)k}(C)$, as we shall explain.

Definition 2.5. For a smooth curve C, let $V^{m_1-1}_{d-(e+1)k}(C)$ be the subset of $\operatorname{Pic}^{d-(e+1)k}(C)$ parametrizing bundles M such that $h^0(C,M)=m_1$ and $h^0(C,M\otimes A^\vee)=0$.

For $M\in V^{m_1-1}_{d-(e+1)k}(C)$, the Riemann-Roch theorem gives:

$$h^0(C,\omega_C\otimes M^\vee)=g+m_1-1+(e+1)k-d,\quad h^0(C,\omega_C\otimes M^\vee\otimes A)=g+(e+2)k-d-1.$$

We can now take a global version of Definition 2.5 and we denote by

$$\nu\colon \mathcal{V}^{m_1-1}_{d-(e+1)k}\longrightarrow\mathcal{H}_{g,k}\tag{7}$$

the stack of elements $[C,A,V,M]$, where $[C,A,V]\in\mathcal{H}_{g,k}$ and $M\in V^{m_1-1}_{d-(e+1)k}(C)$. Over $\mathcal{V}^{m_1-1}_{d-(e+1)k}$ we have two tautological vector bundles $\mathcal{E}$ and $\mathcal{F}$ with fibres over a point $[C,A,V,M]$ given by

$$\mathcal{E}_{|[C,A,M]}=H^0(C,\omega_C\otimes M^\vee)\quad\text{and}\quad\mathcal{F}_{|[C,A,M]}=H^0(C,\omega_C\otimes M^\vee\otimes A).$$

The local freeness of $\mathcal{E}$ and $\mathcal{F}$ follows from Grauert's theorem. As explained above, $\operatorname{rk}(\mathcal{E})=g+m_1-1+(e+1)k-d$ and $\operatorname{rk}(\mathcal{F})=g+(e+2)k-d-1$. Let $\mathbb{E}$ be the tautological rank 2 vector bundle over $\mathcal{H}_{g,k}$ with fibres $\mathbb{E}_{|[C,A,V]}=V$, for a point $[C,A,V]\in\mathcal{H}_{g,k}$.

There is a morphism of vector bundles

$$\phi\colon\nu^*\mathbb{E}\otimes\mathcal{E}\longrightarrow\mathcal{F}\tag{8}$$

whose fibre over a point $[C,A,V,M]$ is the multiplication map

$$\phi_{C,A,V,M}\colon V\otimes H^0(C,\omega_C\otimes M^\vee)\longrightarrow H^0(C,\omega_C\otimes M^\vee\otimes A).$$

We set $s := g - d + r + e(k - r - 1 + \ell)$ and denote by $\mathfrak{Deg}(\phi)$ the degeneracy locus of the morphism ϕ consisting of those $[C, A, V, M]$ such that dim $\mathrm{Ker}(\phi_{C,A,M}) \geq s$. For a point $[C, A, V] \in \mathcal{H}_{g,k}$, we set $\mathfrak{Deg}_{(C,A,V)}(\phi) := \mathfrak{Deg}(\phi) \cap \nu^{-1}\big([C, A, V]\big) \subseteq V^{m_1-1}_{d-(e+1)k}(C)$.

Proposition 2.6. *For each $[C, A, V] \in \mathcal{H}_{g,k}$ one has an injective map*

$$\mathfrak{Deg}_{(C,A,V)}(\phi) \longrightarrow W^r_d(C) \quad \textit{given by} \quad M \mapsto M \otimes A^{e+1}.$$

Moreover, if $\mathfrak{Deg}_{(C,A,V)}(\phi)$ is non-empty, then all of its components have dimension at least $\rho(g, r - \ell, d) - \ell k$.

Proof. Observe that $M \in V^{m_1-1}_{d-(e+1)k}(C)$ satisfies (6) if and only if $[C, A, V, M] \in \mathfrak{Deg}(\phi)$. Indeed, via the Base Point Free Pencil Trick $\mathrm{Ker}(\phi_{C,A,V,M}) \cong H^0(C, \omega_C \otimes M^\vee \otimes A^\vee)$ and by Riemann-Roch we obtain that

$$h^0(C, A \otimes M) = \dim \mathrm{Ker}(\phi_{C,A,V,M}) + d - ek + 1 - g,$$

from which the conclusion follows. From the general theory of degeneracy loci, see e.g. [ACGH85], every component of $\mathfrak{Deg}_{(C,A,V)}(\phi)$ has dimension at least

$$\dim \widetilde{W}^{m_1-1}_{d-(e+1)k}(C) - s \cdot \big(\mathrm{rk}(\mathcal{F}) - 2 \cdot \mathrm{rk}(\mathcal{E}) + s\big) = g - m_1(g - d + (e+1)k + m_1 - 1) \\ - (g - d + r + e(k - r - 1 + \ell))\,(r - \ell - m_1 + 1) = \rho(g, r - \ell, d) - \ell k.$$

□

Definition 2.7. For a point $[C, A, V] \in \mathcal{H}_{g,k}$, let $W^r_{d,\ell}(C, A)$ be the closure of the image of $\mathfrak{Deg}_{(C,A)}(\phi)$. Inside $W^r_{d,\ell}(C, A)$ we identify the following open subvariety

$$V^r_{d,\ell}(C, A) := \Big\{L \in \mathrm{Pic}^d(C) : h^0(C, L \otimes A^{-e-1}) = m_1, \;\; h^0(C, L \otimes A^{-e-2}) = 0, \\ h^0(C, L \otimes A^{-e}) = 2m_1 + m_2, \;\; h^0(C, L) = r + 1\Big\}. \tag{9}$$

Although each component of $W^r_{d,\ell}(C, A)$ is of dimension at least $\rho(g, r - \ell, d) - \ell k$, Proposition 2.6 does not establish the nonemptiness of $W^r_{d,\ell}(C, A)$ when inequality (4) is satisfied. However, by semicontinuity, we have the following:

Proposition 2.8. *Assume (4) is satisfied. If there exists a point $[C_0, A_0, V_0] \in \widetilde{\mathcal{H}}_{g,k}$ such that $W^r_{d,\ell}(C_0, A_0, V_0)$ has a component of dimension $\rho(g, r-\ell, d)-\ell k$, then $W^r_{d,\ell}(C, A, V) \neq \emptyset$ for a general element $[C, A, V] \in \widetilde{\mathcal{H}}_{g,k}$.*

Remark 2.9. By applying Proposition 2.2, clearly $W^r_{d,\ell}(C, A) \subseteq W^r_d(C)$. However it is not the case that the inclusion $W^{r-1}_{d,\ell}(C, A) \subseteq W^r_{d,\ell}(C, A)$ necessarily holds. This is due to the fact that the description (4) involves the parameter m_1 and it is possible that m_1 decreases as r increases.

Remark 2.10. Some of the loci in Definition 2.7 have a transparent description, others less so. For instance, when $\ell = r$, then $W^r_{d,r}(C, A)$ can be identified with the translate $A^r + W_{d-rk}(C) \subseteq \mathrm{Pic}^d(C)$. Clearly, both $W^r_{d,r}(C, A)$ and $V^r_{d,r}(C, A)$ have the expected dimension $d - rk$ predicted by Proposition 2.6 for a general $[C, A] \in \mathcal{H}_{g,k}$, whenever $rk \leq d \leq g - 1$.

2.1. **Brill-Noether theory via splitting types.** We recall the definition of the splitting type of a linear system following [CPJ22] and [Lar21]. Later we shall establish some connections between this invariant and the stability type of a linear system on a curve on an elliptic $K3$ surface, which is defined in terms of Bridgeland stability. We fix an integral curve C and a finite map

$$\pi\colon C \longrightarrow \mathbf{P}^1$$

of degree k. If $L \in W^r_d(C)$, then π_*L is a rank k vector bundle on $\mathbf{P}^1$ and it splits as a direct sum of line bundles

$$\pi_*L \cong \mathcal{O}_{\mathbf{P}^1}(f_1)^{\oplus n_1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(f_q)^{\oplus n_q} \tag{10}$$

where $f_1 > \cdots > f_q$, $n_i > 0$ and $n_1 + \cdots + n_q = k$. The collection $\overline{f}_L := \big((f_i, n_i)\big)^q_{i=1}$ is called the *splitting type* of L (with respect to π). Note that $f_1n_1+\cdots+n_qf_q = d+1-g-k$, in particular $f_q < 0$. With this notation, if $L \in V^r_{d,\ell}(C)$ then the non-negative part $\big(\pi_*L\big)^{\geq 0}$ of π_*L in the splitting (10) is given by

$$\big(\pi_*L\big)^{\geq 0} \cong \mathcal{O}_{\mathbf{P}^1}(e+1)^{\oplus m_1} \oplus \mathcal{O}_{\mathbf{P}^1}(e)^{\oplus m_2}.$$

In particular, $m_1 + m_2 = r + 1 - \ell$ is the rank of $\big(\pi_*L\big)^{\geq 0}$. Since $\pi_*(L)$ must also have at least one negative summand, we obtain that $r + 1 - \ell \leq \mathrm{rk}(\pi_*L) - 1 = k - 1$, that is, $\max\{0, r + 2 - k\} \leq \ell$, which is precisely our original assumption on ℓ.

3. Stability conditions on elliptic $K3$ surfaces

3.1. **Degree k elliptic $K3$ surfaces.** The first step in our analysis is the following observation, which using the surjectivity of the period map for $K3$ surfaces, produces smooth curves endowed with a pencil of prescribed degree.

Proposition 3.1. *Fix integers $g \geq 3$ and $k \geq 2$. Then there exists a smooth $K3$ surface X with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$ such that*

(a) $H^2 = 2g - 2$, $E \cdot H = k$ and $E^2 = 0$,
(b) E is a smooth, irreducible elliptic curve,
(c) H is ample and base point free.

Proof. This is well known to the experts. By [Knu03, Lemma 2.2] one only needs to show ampleness of H. Since H is nef and lies in the positive cone, it suffices to check $H \cdot R > 0$ for all (-2)-curves R on X. Such curves exist only when $k|g$, and are of class $H - \frac{g}{k}E$ (to discard effectiveness of $\frac{g}{k}E - H$, one uses that any divisor in the linear system $\left|\frac{g}{k}E\right|$ is a union of $\frac{g}{k}$ curves in the elliptic pencil $|E|$, cf. [SD74, Proposition 2.6]). Hence $H \cdot R = g - 2 > 0$ as required. □

In the sequel, we fix integers $g \geq 3$ and $k \geq 2$, and a $K3$ surface X as given in Proposition 3.1. Throughout the paper we refer to such an X as a *degree k elliptic $K3$ surface*. We have already used [SD74, Proposition 2.6] guaranteeing the reducibility of any curve in a linear system $|qE|$ with $q \geq 2$. For later use, we summarize this statement:

Lemma 3.2. *For any $q \geq 1$, the following hold:*

(a) The natural map $\mathrm{Sym}^q H^0(\mathcal{O}_X(E)) \longrightarrow H^0(\mathcal{O}_X(qE))$ is an isomorphism. In particular, $h^0(X, \mathcal{O}_X(qE)) = q + 1$ and $h^1(X, \mathcal{O}_X(qE)) = q - 1$.

(b) Under this isomorphism, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-E)^{\oplus q} \xrightarrow{i} \mathrm{Sym}^q H^0(\mathcal{O}_X(E)) \otimes \mathcal{O}_X \cong \mathcal{O}_X^{\oplus q+1} \xrightarrow{\mathrm{ev}} \mathcal{O}_X(qE) \longrightarrow 0 \tag{11}$$

where ev *is the evaluation of global sections. More precisely, given a basis* (s,t) *of* $H^0(X, \mathcal{O}_X(E))$*, the maps* ev *and* i *are respectively given by the matrices*

$$\begin{pmatrix} s^q & s^{q-1}t & \dots & st^{q-1} & t^q \end{pmatrix}, \quad \begin{pmatrix} t & 0 & 0 & \dots & 0 & 0 \\ -s & t & 0 & \dots & 0 & 0 \\ 0 & -s & t & \dots & 0 & 0 \\ 0 & 0 & -s & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -s & t \\ 0 & 0 & 0 & \dots & 0 & -s \end{pmatrix}.$$

3.2. **Multiples of pencils on curves on $K3$ surfaces.** It is known that for a general element $[C, A] \in \mathcal{H}_{g,k}$, the dimensions $h^0(C, A^a)$ are as small as possible, see [Bal89] or [CM99, Proposition 2.1.1]. Those (deformation-theoretic) proofs cannot be easily extended to cover the case of curves lying on a $K3$ surface. The following statement is of independent interest and will turn out to be of importance in showing the non-emptiness of the variety $W^r_{d,r+2-k}(C, A)$, if C lies on an elliptic $K3$ surface.

Theorem 3.3. *Let X be a degree k elliptic $K3$ surface with* $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$*. Denoting by E_C the restriction $\mathcal{O}_C(E)$, then for a general curve $C \in |H|$ we have that*

$$h^0\big(C, E_C^a\big) = \max\big\{ak + 1 - g, a + 1\big\}.$$

Theorem 3.3 states that $h^0(C, E_C^a) = a + 1$ for $g \geq a(k-1)$, whereas for $g < a(k-1)$ one has $H^1(C, E_C^a) = 0$, therefore by Riemann-Roch $h^0\big(C, E_C^a\big) = ak + 1 - g$.

Proof. Let us write $g = n(k-1) + b$, where $b \leq k - 2$. We prove by induction on n that for a general element $C \in |H|$ one has

$$h^0\big(C, E_C^a\big) = \max\{ak + 1 - g, a + 1\}. \tag{12}$$

Assume first $g \leq k - 1$. We pick any smooth curve $C \in |H|$ and consider the twist of the short exact sequence $0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$ by $\mathcal{O}_X(E)$. Taking cohomology we obtain the exact sequence

$$H^1(\mathcal{O}_X(E)) \longrightarrow H^1(C, E_C) \longrightarrow H^2(\mathcal{O}_X(E - C)).$$

Since $(C-E)^2 = 2g-2-2k \leq -4$, it follows that $0 = h^0(\mathcal{O}_X(C-E)) = h^2(\mathcal{O}_X(E-C))$. Together with the vanishing $h^1(\mathcal{O}_X(E)) = 0$, this yields $h^1(E_C) = 0$ which proves the case $a = 1$. For $a \geq 2$, observe that $h^1(E_C^a) = 0$ since

$$H^1(E_C^a)^\vee \cong H^0(\omega_C \otimes E_C^{-a})) \subset H^0(\omega_C \otimes E_C^{-1}) \cong H^1(E_C)^\vee = 0,$$

which proves (12) for $g \leq k - 1$.

Assume now $g \geq k$, in which case $|H - E| \neq \emptyset$. We aim to establish (12) for a general curve $Y \in |H|$. We pick general curves $C \in |H - E|$ and $J \in |E|$, and write

$C\cdot J = x_1+\cdots+x_k$. We choose a smooth curve $Y_0\in|H|$ and consider the pencil in $|H|$ spanned by Y_0 and $C+J$. Write

$$Y_0\cdot J = y_1+\cdots+y_k \ \text{ and } \ Y_0\cdot C = z_1+\cdots+z_{2g-2-k}.$$

Let $\epsilon\colon \widetilde{X}\longrightarrow X$ be the blow-up of X at the $2g-2$ points $y_1,\ldots,y_k,z_1,\ldots,z_{2g-2-k}$, and denote by $E_{y_1},\ldots,E_{y_k}$ and $E_{z_1},\ldots,E_{z_{2g-2-k}}$ the corresponding exceptional divisors on $\widetilde{X}$. We write C' (resp. J') for the strict transform of C (resp. J). We then have a fibration $f\colon \widetilde{X}\to \mathbf{P}^1$ induced by the linear system

$$\left|\epsilon^*(H)-E_{y_1}-\cdots-E_{y_k}-E_{z_1}-\cdots-E_{z_{2g-2-k}}\right|$$

and having the curve $Y_1:=C'+J'$ as a fibre. To simplify notation, we identify C with C' and J with J' on $\widetilde{X}$.

Assume (12) fails for a given $a\in\mathbb{N}$, for every curve $Y\in|H|$. Then by semicontinuity

$$h^0\big(Y_1,\epsilon^*(E^a)(-J)\big) > \max\{a+1, ak+1-g\}. \tag{13}$$

Note that one has the canonical identification

$$H^0\big(Y_1,\epsilon^*(E^a)(-J)\big)\cong \operatorname{Ker}\Big\{H^0\big(C,E_C^{a-1}\big)\oplus H^0\big(J,\mathcal{O}_J(x_1+\cdots+x_k)\big)\xrightarrow{\text{ev}}\mathbb{C}_{x_1,\ldots,x_k}\Big\}. \tag{14}$$

If $b=0$, thus $g=n(k-1)$, we write $g(C)=g-k=(n-2)(k-1)+k-2$. If $a\geq n+1$, then $a-2\geq n-1$ and by induction we observe that $h^0\big(C,E_C^{a-1}\big)-h^0\big(C,E_C^{a-2}\big)=k$, hence the evaluation $\text{ev}\colon H^0\big(C,E_C^{a-1}\big)\longrightarrow\mathbb{C}_{x_1,\ldots,x_k}$ is surjective. Using (14) we obtain

$$h^0\big(Y_1,\epsilon^*(E^a)(-J)\big)=h^0\big(C,E_C^{a-1}\big)+h^0\big(J,\mathcal{O}_J(x_1+\cdots+x_k)\big)-k=ak+1-g,$$

which contradicts (13). If on the other hand $a\leq n-1$, then $a-1\leq n-2$ and from (14) we obtain $h^0\big(Y_1,\epsilon^*(E^a)(-J)\big)\leq h^0\big(C,E_C^{a-1}\big)+1=a+1$, contradicting (13) again.

If $b\geq 1$, we have $g'=g(C)=g-k=(n-1)(k-1)+b-1$, and by induction (12) holds for C. If $a\leq n$, then $a-1\leq n-1$ and using (14) we obtain

$$h^0\big(Y_1,\epsilon^*(E^a)(-J)\big)\leq h^0\big(C,E_C^{a-1}\big)+1=a+1,$$

which violates (13). If $a\geq n+2$, then by induction $h^0\big(C,E_C^{a-1}\big)=(a-1)k+1-g+k=ak+1-g$ and the map $\text{ev}\colon H^0(C,E_C^{a-1})\to\mathbb{C}_{x_1,\ldots,x_k}$ is surjective, which implies that the map $\text{ev}\colon H^0(C,E_C^{a-1})\to\mathbb{C}_{x_1,\ldots,x_k}$ is surjective as well. It follows from (14) that $h^0\big(Y_1,\epsilon^*(E^a)(-J)\big)=h^0(C,E_C^{a-1})=ak+1-g$, contradiction.

This reasoning leaves the cases (i) $b=0, a=n$, respectively, (ii) $b\geq 1, a=n+1$ uncovered. Since they are handled in an essentially identical way, we describe in details only case (i). Thus $g=n(k-1)$ and by the induction hypotheses $h^0\big(C,E_C^{n-1}\big)=n+1$, while $h^0\big(C,E_C^{n-2}\big)=n-1$. In order to conclude, it suffices to show that the map ev in (14) is surjective.

We specialize further and choose $Y_0\in|H|$ to be a smooth curve general with respect to the property that it passes through the points $x_3,\ldots,x_k$. In this case, keeping the notation above, $y_j=z_j=x_j$ for $j=3,\ldots,k$ and the fibration $f\colon\widetilde{X}\to\mathbf{P}^1$ has the curve $Y_1:=C'+J'+E_{x_3}+\cdots+E_{x_k}$ as one of its fibres. Note $C'\cdot J'=x_1+x_2$ and $E_{x_i}\cdot C'=z_i$

and $E_{x_i} \cdot J' = y_i$ respectively, where $\epsilon(y_i) = \epsilon(z_i) = x_i$ for $i = 3, \ldots, k$. Assuming (12) fails for every curve in $|H|$, we have $h^0\big(Y_1, \epsilon^*(E^n)(-J' - E_{x_3} - \cdots - E_{x_k})\big) \geq n + 2$.

After identifying the curves C' with C and J' with J respectively, the restrictions of the twist $\beta := \mathcal{O}_{\widetilde{X}}\big(-J' - E_{x_3} - \cdots - E_{x_k}\big)$ to the components of Y_1 are given by

$$\beta_{|C} \cong \mathcal{O}_C(-x_1 - \cdots - x_k), \ \beta|_J \cong \mathcal{O}_J(x_1 + x_2) \ \text{ and } \beta_{|E_{x_i}} \cong \mathcal{O}_{E_{x_i}}(1) \ \text{ for } i = 3, \ldots, k.$$

Since each map $\mathrm{ev}_i \colon H^0\big(E_{x_i}, \mathcal{O}_{E_{x_i}}(1)\big) \to \mathbb{C}_{z_i,y_i}$ is an isomorphism, by writing down the Mayer-Vietoris sequence on Y_1 one has the canonical identification

$$H^0\big(Y_1, \epsilon^*(E^n)(-\beta)\big) = \mathrm{Ker}\Big\{H^0\big(C, E_C^{n-1}\big) \oplus H^0\big(J, \mathcal{O}_J(x_1 + x_2)\big) \longrightarrow \mathbb{C}_{x_1,x_2}\Big\}.$$

Now observe that by varying $J \in |E|$ while fixing C, the points $x_1, x_2 \in C$ can be chosen such that the evaluation map $H^0\big(C, E_C^{n-1}\big) \to \mathbb{C}_{x_1,x_2}$ is surjective ($n \geq 2$), for else, we would obtain that $H^0\big(C, E_C^{n-2}\big)$ has codimension 1 in $H^0\big(C, E_C^{n-1}\big)$, which is not the case. Therefore $h^0\big(Y_1, \epsilon^*(E_C^{n-1})(-\beta)\big) = h^0(C, E_C^{n-1}) = n + 1$. This finishes the proof. □

As observed in [CM99] for general k-gonal curves, Theorem 3.3 implies the following:

Proposition 3.4. *Let X be an elliptic $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$ such that $H \cdot E = r + 2$. Then for every $d \leq g - 1$ such that $\rho(g, r, d) \geq 0$, there exists a component Z of $W^r_d(C)$ having dimension $\rho(g, r, d)$, whose general point corresponds to a line bundle L with $H^0(C, L \otimes E_C^\vee) = 0$.*

Proof. We set $a := g - d + r - 1$. Note that $g \geq (a + 1)(r + 1)$, hence by using Theorem 3.3, we have that $h^0(C, A^a) = a + 1$, therefore $h^0(C, \omega_C \otimes A^{-a}) = g - a(r + 1)$. It follows that for a general effective divisor D on C of degree $\rho(g, r, d) = g - (a + 1)(r + 1)$, one has $h^0\big(C, \omega_C \otimes A^{-a}(-D)\big) = r + 1$. Line bundles $L := \omega_C \otimes A^{-a}(-D)$ of this type fill-up a component Z of $W^r_d(C)$ of dimension $\rho(g, r, d)$, see also [CM99, Proposition 2.3.1]. Since Theorem 3.3 also guarantees that $h^0(C, A^{a+1}) = a + 2$, we also have $h^0\big(C, \omega_C \otimes A^{-a-1}\big) = g - (a + 1)(r + 1) = \rho(g, r, d)$, therefore

$$H^0\big(C, L \otimes A^\vee\big) = H^0(C, \omega_C \otimes A^{-a-1}(-D)) = 0,$$

for a general point $L \in Z$, corresponding to a general divisor D of degree $\rho(g, r, d)$. □

3.3. **Derived categories of $K3$ surfaces.** Let $\mathcal{D}(X)$ denote the bounded derived category of coherent sheaves on X. For an object $F \in \mathcal{D}(X)$, we will denote its Chern character, respectively, its Mukai vector by

$$\mathrm{ch}(F) := \big(\mathrm{ch}_0(F), \mathrm{ch}_1(F), \mathrm{ch}_2(F)\big),$$
$$v(F) := \mathrm{ch}(F) \cdot \sqrt{\mathrm{td}(X)} = \big(\mathrm{ch}_0(F), \mathrm{ch}_1(F), \mathrm{ch}_2(F) + \mathrm{ch}_0(F)\big).$$

Under the canonical identifications of $H^0(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ with $\mathbb{Z}$, this rule defines surjective maps $\mathrm{ch}, v \colon K_0(\mathcal{D}(X)) \to \Lambda := \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$. We consider the following symmetric bilinear form on Λ:

$$\begin{aligned} \big\langle (r_1, x_1H + y_1E, s_1), (r_2, x_2H + y_2E, s_2) \big\rangle &:= \\ (x_1H + y_1E) \cdot (x_2H + y_2E) - r_1 s_2 - r_2 s_1. \end{aligned} \tag{15}$$

By Riemann-Roch, for given $F, F' \in \mathcal{D}(X)$ we have

$$-\langle v(F), v(F')\rangle = \chi(F,F') := \sum_i (-1)^i \cdot \operatorname{ext}^i(F,F'),$$

where $\operatorname{ext}^i(F,F') := \dim_{\mathbb{C}} \operatorname{Ext}^i(F,F')$ (we will write $\hom(F,F') := \dim_{\mathbb{C}} \operatorname{Hom}(F,F')$ in the case $i=0$). We also recall that the quantity

$$\Delta(F) := \operatorname{ch}_1(F)^2 - 2\operatorname{ch}_0(F)\operatorname{ch}_2(F) = v(F)^2 + 2\operatorname{ch}_0(F)^2$$

is called the *discriminant* of an object $F \in \mathcal{D}(X)$.

Let us fix $\epsilon \in \mathbb{Q}_{>0}$ and set

$$H_\epsilon := E + \epsilon H \in \operatorname{Pic}(X)_{\mathbb{Q}}. \tag{16}$$

As a result of Proposition 3.1, the class H_ϵ lies in the ample cone. Consider the projection

$$\begin{aligned} \Pi_\epsilon : K_0(\mathcal{D}(X)) \setminus \{F : \operatorname{ch}_0(F) = 0\} &\longrightarrow \mathbb{R}^2 \\ F &\longmapsto \left(\frac{H_\epsilon \cdot \operatorname{ch}_1(F)}{H_\epsilon^2 \cdot \operatorname{ch}_0(F)}, \frac{\operatorname{ch}_2(F)}{H_\epsilon^2 \cdot \operatorname{ch}_0(F)}\right). \end{aligned} \tag{17}$$

The following technical lemma plays a crucial role in the next subsection (as it will guarantee that (21) defines a set of Bridgeland stability conditions). It establishes that there is no sequence of spherical classes whose projections accumulate towards the origin. Recall that a class $F \in K_0(\mathcal{D}(X))$ is called *spherical* if $v(F)^2 = -2$.

Lemma 3.5. *There is no sequence of vectors $v_n = (r_n,\, t_nH + u_nE,\, s_n) \in \Lambda$ such that the following conditions hold:*

(a) $r_n \neq 0$ and $v_n \neq \pm(1,0,1)$,
(b) v_n is the Mukai vector of a spherical class, i.e. $v_n^2 = (t_nH+u_nE)^2 - 2r_ns_n = -2$,
(c) $\frac{H_\epsilon \cdot (t_nH+u_nE)}{r_n} \underset{n}{\longrightarrow} 0$ and $\frac{s_n}{r_n} \underset{n}{\longrightarrow} 1$.

Proof. Write $\epsilon = \frac{a}{b}$ ($a, b \in \mathbb{Z}_{>0}$). If $t_n = 0$, then $s_n = r_n = \pm 1$ by (b) and thus $u_n = 0$ for all $n \gg 0$ by (c). Hence we may assume $t_n \neq 0$ for all n, so that (b) is equivalent to

$$u_nk = \frac{r_ns_n}{t_n} - \frac{1}{t_n} - t_n(g-1). \tag{18}$$

We obtain

$$\begin{aligned} b \cdot \frac{H_\epsilon \cdot (t_nH + u_nE)}{r_n} &= \frac{(aH+bE)(t_nH+u_nE)}{r_n} = \frac{at_n(2g-2) + au_nk + bt_nk}{r_n} = \\ &\overset{(18)}{=} \frac{at_n(g-1) + \frac{ar_ns_n}{t_n} - \frac{a}{t_n} + bt_nk}{r_n} = \frac{t_n}{r_n}\left(a(g-1) + bk\right) + \frac{ar_n}{t_n}\left(\frac{s_n}{r_n} - \frac{1}{r_n^2}\right). \end{aligned} \tag{19}$$

Observe that $\frac{s_n}{r_n} - \frac{1}{r_n^2} > \frac{1}{2}$ for all $n \gg 0$. Otherwise, using condition (c) we would have that $M := \{n \ : \ r_n = s_n = \pm 1\}$ is an infinite set. But then for all $n \in M$

$$0 = (t_nH + u_nE)^2 = t_n(t_n(2g-2) + 2u_nk) \implies u_nk = -(g-1)t_n, \tag{20}$$

and this is a contradiction as it implies

$$0 = \lim_{n\in M} H_\epsilon \cdot (t_nH + u_nE) = \lim_{n\in M}\Big(t_nk + \epsilon(t_n(2g-2) + u_nk)\Big) \overset{(20)}{=} \lim_{n \in M} t_n(k + \epsilon(g-1)).$$

Now since $a(g-1)+bk>0$ and the left-hand side in (19) goes to zero, both terms on the right-hand side must also go to zero as $n\to\infty$. The first term implies that $\frac{t_n}{r_n}\to 0$, but this causes the second term to diverge to infinity which is a contradiction. □

3.4. **A two-dimensional slice of Bridgeland stability conditions.** Given $\epsilon\in\mathbb{Q}_{>0}$, we will work with a two-dimensional slice of stability conditions on $\mathcal{D}(X)$ associated with the polarization H_ϵ, for which we give a brief account. Further details can be found in Bridgeland's original work [Bri07, Bri08].

Given a coherent sheaf F, we define its H_ϵ*-slope* by

$$\mu_{H_\epsilon}(F):=\frac{H_\epsilon\cdot \mathrm{ch}_1(F)}{H_\epsilon^2\cdot \mathrm{ch}_0(F)},$$

with the convention $\mu_{H_\epsilon}(F)=+\infty$ if $\mathrm{ch}_0(F)=0$. This leads to the usual notion of μ_{H_ϵ}-stability on the category $\mathrm{Coh}(X)$ of coherent sheaves. The key idea of Bridgeland was to replace $\mathrm{Coh}(X)$ by other abelian subcategories of $\mathcal{D}(X)$, equipped with a suitable slope.

To that end, for any $b\in\mathbb{R}$ we consider the full subcategories of $\mathrm{Coh}(X)$

$$\mathcal{T}_b:=\Big\{F\in\mathrm{Coh}(X):\mu_{H_\epsilon}(Q)>b \text{ for all quotients } F\twoheadrightarrow Q\Big\} \quad\text{and}$$
$$\mathcal{F}_b:=\Big\{F\in\mathrm{Coh}(X):\mu_{H_\epsilon}(G)\le b \text{ for all subsheaves } G\hookrightarrow F\Big\},$$

which form a torsion pair in $\mathrm{Coh}(X)$, see [Bri08, Definition 3.2] or [Bay18, Proposition 2.2] for further details. Their tilt gives a *bounded t-structure* on $\mathcal{D}(X)$ with heart

$$\mathrm{Coh}^b(X):=\Big\{F\in\mathcal{D}(X):\mathcal{H}^{-1}(F)\in\mathcal{F}_b,\ \mathcal{H}^0(F)\in\mathcal{T}_b,\ \mathcal{H}^i(F)=0 \text{ for } i\neq 0,-1\Big\}.$$

Alternatively, objects of $\mathrm{Coh}^b(X)$ are those isomorphic (in $\mathcal{D}(X)$) to 2-term complexes $G^{-1}\xrightarrow{d}G^0$ with $\ker(d)\in\mathcal{F}_b$ and $\mathrm{coker}(d)\in\mathcal{T}_b$. In particular $\mathrm{Coh}^b(X)$ is an abelian subcategory of $\mathcal{D}(X)$: short exact sequences $0\longrightarrow F'\longrightarrow F\longrightarrow Q\longrightarrow 0$ correspond to those exact triangles $F'\longrightarrow F\longrightarrow Q\longrightarrow F'[1]$ in $\mathcal{D}(X)$ so that $F,F',Q\in\mathrm{Coh}^b(X)$.

For $(b,w)\in\mathbb{R}^2$ we let $Z_{b,w}\colon K_0(\mathcal{D}(X))\to\mathbb{C}$ be the group homomorphism defined as

$$Z_{b,w}(F):=-\mathrm{ch}_2(F)+w\,\mathrm{ch}_0(F)H_\epsilon^2+i\big(\mathrm{ch}_1(F)\cdot H_\epsilon-b\,\mathrm{ch}_0(F)H_\epsilon^2\big).$$

It is clear that $Z_{b,w}$ factors through $\mathrm{ch}\colon K_0(\mathcal{D}(X))\to\Lambda$. Sometimes we will also denote by $Z_{b,w}$ the induced map $\Lambda\to\mathbb{C}$.

Let us first state Bridgeland's result describing stability conditions on $\mathcal{D}(X)$; then we expand upon the statements. By Lemma 3.5, there exists $\delta_\epsilon>0$ (depending on ϵ) such that no projection of a spherical class is at distance less than δ_ϵ of the origin. Accordingly, if we define

$$U_\epsilon:=\Big\{(b,w)\in\mathbb{R}^2\colon 2w>b^2\Big\}\cup\Big\{(b,w)\in\mathbb{R}^2\colon b\neq 0 \text{ and } b^2+w^2<\delta_\epsilon^2\Big\},\tag{21}$$

then Bridgeland's results [Bri08, Lemma 6.2] imply the following:

Theorem 3.6. *For any $(b,w)\in U_\epsilon$, the pair $\sigma_{b,w}:=\big(\mathrm{Coh}^b(X),Z_{b,w}\big)$ is a stability condition on $\mathcal{D}(X)$. Moreover, this assignment defines a continuous map from U_ϵ to the space of stability conditions on $\mathcal{D}(X)$.*

We first explain the notion of $\sigma_{b,w}$-stability and the associated Harder–Narasimhan filtration. Observe that $\operatorname{Im}(Z_{b,w}(F)) \geq 0$ for any nonzero $F \in \operatorname{Coh}^b(X)$, and also $\operatorname{Re}(Z_{b,w}(F)) < 0$ whenever $\operatorname{Im}(Z_{b,w}(F)) = 0$. Intuitively, $\operatorname{Im}(Z_{b,w})$ can be regarded as a notion of "rank" on $\operatorname{Coh}^b(X)$; then the "degree" $-\operatorname{Re}(Z_{b,w})$ is positive for rank zero objects in $\operatorname{Coh}^b(X)$. By considering the *slope function*

$$\nu_{b,w}\colon \operatorname{Coh}^b(X) \to \mathbb{R} \cup \{+\infty\}, \quad \nu_{b,w}(F) := \begin{cases} -\frac{\operatorname{Re}(Z_{b,w}(F))}{\operatorname{Im}(Z_{b,w}(F))} & \text{if } \operatorname{Im}(Z_{b,w}(F)) > 0 \\ +\infty & \text{if } \operatorname{Im}(Z_{b,w}(F)) = 0 \end{cases} \tag{22}$$

we obtain a notion of stability in $\operatorname{Coh}^b(X)$: an object $F \in \operatorname{Coh}^b(X)$ is $\sigma_{b,w}$-*(semi)stable* if and only if for any proper subobject $F' \subset F$ in $\operatorname{Coh}^b(X)$ we have

$$\nu_{b,w}(F') < (\leq)\ \nu_{b,w}(F/F').$$

Every object $F \in \operatorname{Coh}^b(X)$ admits a unique *Harder–Narasimhan* (*HN* for short) *filtration*, namely a finite sequence of objects in $\operatorname{Coh}^b(X)$

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = F$$

whose factors F_i/F_{i-1} are $\sigma_{b,w}$-semistable of decreasing $\nu_{b,w}$-slope.

Furthermore, every $\sigma_{b,w}$-semistable object $F \in \operatorname{Coh}^b(X)$ has a (not necessarily unique) finite *Jordan-Hölder filtration* in $\operatorname{Coh}^b(X)$

$$0 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = F$$

whose factors G_i/G_{i-1} are $\sigma_{b,w}$-stable of the same $\nu_{b,w}$-slope as F. These factors are unique up to relabelling, and are called the *stable factors* of F.

Remark 3.7. We will use the same plane for the image of the projection Π_ϵ defined in (17), and the (b,w)-plane containing U_ϵ. In this way, if $F \in \operatorname{Coh}^b(X)$ and $\operatorname{ch}_0(F) \neq 0$, then $\nu_{b,w}(F)$ is the slope of the line joining the points (b,w) and $\Pi_\epsilon(F)$. If F is moreover $\sigma_{b,w}$-semistable ($2w > b^2$), then $\Pi_\epsilon(F)$ lies outside the region $\left\{(b,w) \in \mathbb{R}^2 \colon 2w > b^2\right\}$.

The fact that stability conditions can be deformed continuously is ensured by a technical requirement called the *support property*. While we refer to [Bay19] for details, here we only consider its main application, namely that stability of objects is governed by a locally finite wall and chamber structure.

Definition 3.8. A *numerical wall* for an object $F \in \mathcal{D}(X)$ is a line segment $\ell \subset U_\epsilon$ determined by an equation of the form $\nu_{b,w}(F) = \nu_{b,w}(F')$, where $F' \in \mathcal{D}(X)$ is such that $(\operatorname{ch}_0(F), H_\epsilon \cdot \operatorname{ch}_1(F), \operatorname{ch}_2(F))$ and $(\operatorname{ch}_0(F'), H_\epsilon \cdot \operatorname{ch}_1(F'), \operatorname{ch}_2(F'))$ are non-proportional. If F is $\sigma_{b,w}$-semistable for $(b,w) \in \ell$ and unstable just above or below ℓ, then ℓ is an *actual wall* for F along which F gets destabilized[4].

Proposition 3.9 (Wall and chamber structure)**.** *Let $v = (v_0, v_1, v_2) \in \Lambda$ be any Mukai vector. Then there exists a locally finite set $\{\mathcal{W}_i^v\}_{i \in I_v}$ of actual walls for objects of Mukai vector v, inducing a chamber decomposition of U_ϵ such that:*

(a) The extension of every actual wall passes through $\Pi_\epsilon(v)$ if $v_0 \neq 0$, or has fixed slope $\frac{v_2}{H_\epsilon \cdot v_1}$ if $v_0 = 0$.

[4] Here, we define numerical (or actual) walls only on our two-dimensional slice U_ϵ, but the intersections of the walls in the full stability manifold with our chosen slice may coincide with the entire slice U_ϵ.

(b) *For any object* $F \in \mathcal{D}(X)$ *with* $v(F) = v$*, the* $\sigma_{b,w}$*-(semi)stability of* F *remains unchanged along a chamber.*

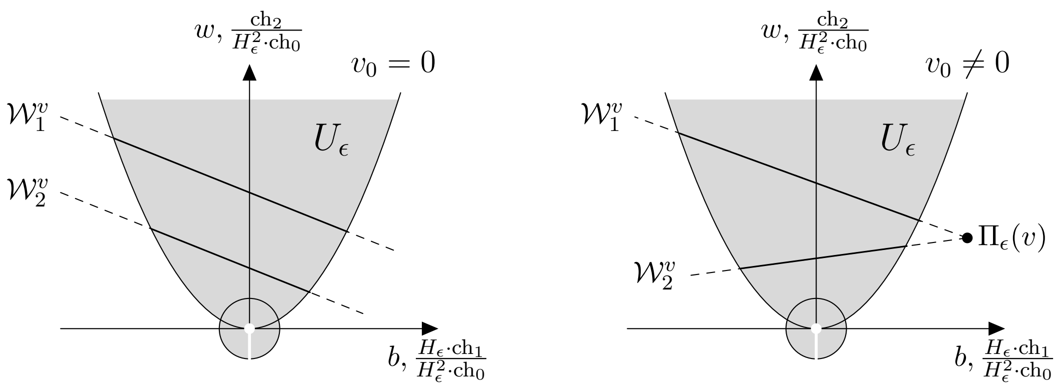

FIGURE 1. Actual walls $\mathcal{W}_i^v$ for v when $v_0 = 0$ and $v_0 \neq 0$

It follows that if σ varies within a chamber, the set of σ-semistable objects with Mukai vector v remains unchanged. Indeed, for any stability condition $\sigma := \sigma_{b,w}$, the moduli stack $\mathfrak{M}_\sigma(v)$ of flat families of σ-semistable objects in $\mathrm{Coh}^b(X)$ of Mukai vector v is an Artin stack of finite type over $\mathbb{C}$ [Tod08, Theorem 1.4 and Section 3]. In cases where a coarse moduli space parameterizing the semistable objects exists, we denote it by $\mathcal{M}_\sigma(v)$. As discussed in detail in [BM14], such a proper coarse moduli space exists, for instance, when v is primitive and σ is generic in the stability manifold with respect to v.

Once a Mukai vector $v \in \Lambda$ is fixed, actual walls for v in U_ϵ are bounded from above (see e.g. [MS17, Section 6.4]); in particular, there is a "largest actual wall". Above it there is the so-called *Gieseker chamber*, where stability agrees with Gieseker stability for sheaves. This gives a fundamental example of Bridgeland stable objects. More precisely:

Theorem 3.10. *Let* $v = (v_0, v_1, v_2) \in \Lambda$*. There exists* $w_0 \in \mathbb{R}$ *such that:*

(a) *If* $v_0 > 0$*, or* $v_0 = 0$ *and* v_1 *is effective, then for all* $w \geq w_0$ *and* $b < \frac{H_\epsilon\cdot v_1}{H_\epsilon^2\cdot v_0}$ *an object* $F \in \mathrm{Coh}^b(X)$ *with* $v(F) = v$ *is* $\sigma_{b,w}$*-(semi)stable if and only if* F *is an* H_ϵ*-Gieseker (semi)stable sheaf.*

(b) *If* $v_0 < 0$*, then for all* $w \geq w_0$ *and* $b > \frac{H_\epsilon\cdot v_1}{H_\epsilon^2\cdot v_0}$ *an object* $F \in \mathrm{Coh}^b(X)$ *with* $v(F) = v$ *is* $\sigma_{b,w}$*-(semi)stable if and only if* $F = R\mathcal{H}om(G, \mathcal{O}_X)[1]$ *for an* H_ϵ*-Gieseker (semi)stable sheaf* G*.*

Let us recall that for any Mukai vector $v \in \Lambda$, there exists a projective variety $\mathcal{M}_{H_\epsilon}(v)$ which is a coarse moduli space parameterizing S-equivalence classes of H_ϵ-Gieseker semistable sheaves, see [HL10, Section 4] for details.

3.5. **Stability of special objects.** This subsection gathers useful lemmas concerning the stability of some distinguished objects along U_ϵ. Noting that the quadratic form $\langle\cdot,\cdot\rangle$ (defined in (15)) has signature $(2,2)$ on $\Lambda_{\mathbb{R}}$, we begin with a remark on the discriminant of semistable objects:

Lemma 3.11. *Let $(b,w)\in U_\epsilon$. Then:*

(a) *If F is a $\sigma_{b,w}$-stable object, then either (i) $\Delta(F)\geq 0$, or (ii) $k|g$, $\mathrm{ch}_0(F)=0$ and $\mathrm{ch}_1(F)=H-\frac{g}{k}E$.*

(b) *Assume $2w>b^2$ and let F be strictly $\sigma_{b,w}$-semistable, with stable factors $\{F_i\}_{i\in I}$. If every factor satisfies $\Delta(F_i)\geq 0$, then $\Delta(F_i)\leq\Delta(F)$ for all i with equality if and only if $\Delta(F_i)=\Delta(F)=0$ and $\mathrm{ch}(F_i)$ is a multiple of $\mathrm{ch}(F)$.*

Proof. Any $\sigma_{b,w}$-stable object F is simple, hence $\hom(F,F)=\mathrm{ext}^2(F,F)=1$ and

$$\Delta(F)-2\,\mathrm{ch}_0(F)^2=-\chi(F,F)\geq -2.$$

If $\mathrm{ch}_0(F)\neq 0$, we get $\Delta(F)\geq 0$. If $\mathrm{ch}_0(F)=0$ and $\Delta(F)<0$ we have $\Delta(F)=-2$, hence $\mathrm{ch}_1(F)^2=-2$ which implies $k|g$ and $\mathrm{ch}_1(F)=\pm(H-\frac{g}{k}E)$. It must be $\mathrm{ch}_1(F)=H-\frac{g}{k}E$ as claimed in part (a), as $F\in\mathrm{Coh}^b(X)$ implies $0\leq \mathrm{Im}Z_{b,w}(F)=H_\epsilon\cdot\mathrm{ch}_1(F)$.

To prove part (b), we first show that the kernel of $Z_{b,w}$ is negative definite with respect to Δ. A class $v\in\ker Z_{b,w}$ can be written as $v=(r,\ xH+yE,\ rwH_\epsilon^2)$ such that

$$yE\cdot H=\frac{br}{\epsilon}H_\epsilon^2-xH^2-\frac{x}{\epsilon}E\cdot H. \tag{23}$$

Since $2w>b^2$, we get

$$\begin{aligned}\Delta(v)&\overset{(23)}{=}H_\epsilon^2\left(\frac{2xbr}{\epsilon}-2r^2w\right)-x^2H^2-\frac{2x^2}{\epsilon}E\cdot H\\ &<H_\epsilon^2\left(\frac{2xr}{\epsilon}b-r^2b^2\right)-x^2H^2-\frac{2x^2}{\epsilon}E\cdot H\leq H_\epsilon^2\frac{x^2}{\epsilon^2}-x^2H^2-\frac{2x^2}{\epsilon}E\cdot H=0\end{aligned}$$

as claimed. Now, since $\nu_{b,w}(F_i)=\nu_{b,w}(F)$ for all $i\in I$, the points $Z_{b,w}(F_i)$ lie in a ray ρ^+ in the upper half plane starting from the origin. Then [BMS16, Lemma 11.7] implies that $(Z_{b,,w})^{-1}(\rho^+)\cap\{\Delta\geq 0\}$ is a convex cone, and one can conclude part (b) via the same argument as in [BMS16, Lemma 3.9]. □

A consequence of Lemma 3.11 is that, for appropriate Mukai vectors of rank 0, the Gieseker chamber contains all stability conditions above the parabola $\{2w=b^2\}\subseteq\mathbb{R}^2$:

Corollary 3.12. *For $q>0$ and $s\in\mathbb{Z}$, the Mukai vector $v=(0,qE,s)\in\Lambda$ has no actual wall intersecting the region $\{2w>b^2\}$ providing that $k\nmid g$, or $k|g$ and $\epsilon<\frac{k}{g+1}$.*

Proof. Assume such an actual wall exists, and let F be an object of Mukai vector v that gets destabilized along this wall. Write $\{F_i\}_{i\in I}$ for the stable factors of F.

Since $\Delta(F)=0$, it suffices to check that $\Delta(F_i)\geq 0$ for all $i\in I$; indeed, in that case it follows from Lemma 3.11.(b) that $\mathrm{ch}(F_i)$ and $\mathrm{ch}(F)$ are proportional for every $i\in I$. This implies $\nu_{b,w}(F_i)=\nu_{b,w}(F)$ for all $(b,w)\in U_\epsilon$, which is a contradiction.

If $\Delta(F_i)<0$ for some i, then by Lemma 3.11.(a) we have $k|g$ and the sum of the Chern characters of stable factors with negative discriminant is of the form $\left(0,a'(H-\frac{g}{k}E),s'\right)$, for some $a'\in\mathbb{Z}_{>0}$ such that

$$a'(H-\tfrac{g}{k}E)\cdot H_\epsilon<qE\cdot H_\epsilon\quad\text{(i.e., } a'(k+\epsilon(g-2))<\epsilon qk). \tag{24}$$

Hence the difference $\mathrm{ch}(F) - \big(0, a'(H - \frac{g}{k}E), s'\big) = \big(0, -a'H + (q - a'\frac{g}{k})E, s - s'\big)$ has non-negative discriminant, which is a contradiction since

$$\big(-a'H + (q - a'\tfrac{g}{k})E\big)^2 = 2a'\left(a'(2g-1) - qk\right) \overset{(24)}{<} 2a'\left(a'(g+1) - \tfrac{a'k}{\epsilon}\right) < 0,$$

(the last inequality being derived from the assumption $\epsilon < \frac{k}{g+1}$). □

For $2w > b^2$, the following lemma characterizes $\sigma_{b,w}$-stable objects with $\mathrm{ch}_2 = 0$ and $\mathrm{ch}_0 \neq 0$ as (shifts of) multiples of the elliptic pencil on X:

Lemma 3.13. *Let $F \in \mathrm{Coh}^b(X)$ satisfy $v(F) = (r, qE, r)$, where $r \neq 0$ and $q \geq 0$. If F is $\sigma_{b,w}$-stable ($2w > b^2$), then $|r| = 1$ and (up to a shift if $r = -1$) $F \cong \mathcal{O}_X(\frac{q}{r}E)$. Conversely, $\mathcal{O}_X(qE)$ (resp. $\mathcal{O}_X(-qE)[1]$) is $\sigma_{b,w}$-stable for all $b < \frac{H_\epsilon \cdot qE}{H_\epsilon^2}$ (resp. for all $b > -\frac{H_\epsilon \cdot qE}{H_\epsilon^2}$) and $2w > b^2$.*

Proof. Note that $\sigma_{b,w}$-stability of F implies $\hom(F,F) = \mathrm{ext}^2(F,F) = 1$, hence

$$2r^2 = \chi(F,F) = 2 - \mathrm{ext}^1(F,F) \leq 2$$

which gives $|r| = 1$. In view of Theorem 3.10, it suffices to check that there is no actual wall for the Mukai vector (r, qE, r) along the region $\{2w > b^2, H_\epsilon \cdot qE - brH_\epsilon^2 > 0\}$.

Any object F with Mukai vector (r, qE, r) has $\Delta(F) = 0$. Thus if F gets destabilized along such a wall, there are stable factors of negative discriminant; otherwise, by Lemma 3.11 all stable factors have Chern character multiple of $\mathrm{ch}(F)$, hence cannot form a wall.

We know that summation of Chern characters of all factors of negative discriminant is of the form $(0, a'(H - \frac{g}{k}E), s)$ for some $a' \in \mathbb{Z}_{>0}$. Moreover, if σ_{b_0,w_0} lies in the actual wall, then this summation computes the same ν_{b_0,w_0}-slope as F:

$$\frac{s}{a'(H - \frac{g}{k}E) \cdot H_\epsilon} = \frac{-rw_0H_\epsilon^2}{qE \cdot H_\epsilon - b_0rH_\epsilon^2}.$$

Thus $rs \leq 0$, which implies

$$\Delta\big(\mathrm{ch}(F) - \big(0, a'(H - \tfrac{g}{k}E), s\big)\big) = -2a'^2 - 2a'qE \cdot H + 2rs < 0,$$

contradicting Lemma 3.11. Therefore there is no such wall, which finishes the proof. □

Along the horizontal line $w = 0$, we have a different behavior (recall the role of δ_ϵ in the definition (21) of U_ϵ):

Lemma 3.14. *Assume $\epsilon < 1$, and consider $\sigma := \sigma_{b,0}$ for any $b \in (-\delta_\epsilon, 0)$. Then:*

(a) $\mathcal{O}_X$ and $\mathcal{O}_X(-E)[1]$ are σ-stable objects.
(b) $\mathcal{O}_X(qE)$ and $\mathcal{O}_X(-(q+1)E)[1]$ are strictly σ-semistable for all $q \geq 1$.

Proof. Since $\mathcal{O}_X$ and $\mathcal{O}_X(-E)[1]$ are σ-semistable by Lemma 3.13, to prove (a) we only need to show that they are not strictly σ-semistable. In the case of $\mathcal{O}_X$, we can get arbitrary close to $\Pi_\epsilon(\mathcal{O}_X) = (0,0)$ as $b \to 0^-$; hence any exact triangle $F_2 \to \mathcal{O}_X \to F_1$ of σ-semistable objects with the same σ-slope satisfies $\Pi_\epsilon(F_1) = \Pi_\epsilon(\mathcal{O}_X) = \Pi_\epsilon(F_2)$. But this implies that $\mathcal{O}_X$ is strictly $\sigma_{b,w}$-semistable for $w > \frac{b^2}{2}$, contradicting Lemma 3.13.

Assume $F_2 \to \mathcal{O}_X(-E)[1] \to F_1$ is an exact triangle with $\mathrm{ch}(F_1) = (r_0,\ tH + aE,\ 0)$, such that F_1 is σ-stable of the same $\nu_{b,0}$-slope as $\mathcal{O}_X(-E)[1]$. Then

$$-2 \leq v(F_1)^2 = t^2H^2 + 2atE \cdot H - 2r_0^2 \tag{25}$$

and in particular

$$-2 - 2taE \cdot H \leq t^2H^2. \tag{26}$$

Also, the inequalities $0 \leq \mathrm{Im} Z_{0,b}(F_1) \leq \mathrm{Im} Z_{0,b}(\mathcal{O}_X(-E)[1])$ for all $b \in (-\delta_\epsilon, 0)$ give

$$0 \leq tE \cdot H + t\epsilon H^2 + a\epsilon E \cdot H \leq \epsilon E \cdot H. \tag{27}$$

Hence, if $t < 0$ we obtain

$$\frac{\epsilon}{t} + \frac{t}{2}\epsilon H^2 \overset{(26)}{\leq} -a\epsilon E \cdot H \overset{(27)}{\leq} tE \cdot H + t\epsilon H^2$$

which is not possible. If $t \geq 1$, we get

$$E \cdot H \leq tE \cdot H \overset{(27)}{\leq} -t\epsilon H^2 + \epsilon E \cdot H - a\epsilon E \cdot H \overset{(26)}{\leq} \epsilon\left(-\frac{t}{2}H^2 + E \cdot H + \frac{1}{t}\right),$$

hence $\frac{H^2}{2} \leq 1$ (by the assumption $\epsilon < 1$) which implies $g = \frac{H^2}{2} + 1 \leq 2$, contradiction.

Thus $t = 0$, and so (25) gives that either $r_0 = 0$ or $r_0 = \pm 1$. The quotient F_1 must have a bigger slope than $\mathcal{O}_X(-E)[1]$ above the horizontal line $w = 0$, as $\mathcal{O}_X(-E)[1]$ is $\sigma_{b,w}$-stable for $w > \frac{b^2}{2}$ by Lemma 3.13; therefore, $r_0 = -1$. But this implies that F has the same Chern character as $\mathcal{O}_X(-E)[1]$, which is not possible. This proves that $\mathcal{O}_X(-E)[1]$ is σ-stable and completes part (a).

For part (b), note that $\mathcal{O}_X(qE)$ and $\mathcal{O}_X(-(q+1)E)[1]$ are σ-semistable if $q \geq 1$, again by Lemma 3.13. They are strictly σ-semistable, as we have nonzero maps $\mathcal{O}_X \to \mathcal{O}_X(qE)$ and $\mathcal{O}_X(-(q+1)E)[1] \to \mathcal{O}_X(-E)[1]$ between objects of the same $\nu_{b,0}$-slope. □

We end with an observation (see [Bay18, Lemma 6.5] for a proof), used in Section 5:

Lemma 3.15. *Let $F_1, F_2 \in \mathrm{Coh}^b(X)$ be σ_{b,w_0}-stable objects with $\nu_{b,w_0}(F_1) = \nu_{b,w_0}(F_2)$ and $\nu_{b,w}(F_1) < \nu_{b,w}(F_2)$ for $w > w_0$. Then for any extension*

$$V^\vee \otimes F_1 = F_1^{\oplus n} \longrightarrow F \longrightarrow F_2$$

induced by an n-dimensional subspace $V \subseteq \mathrm{Ext}^1(F_2, F_1)$, the object F is $\sigma_{b,w}$-stable for all sufficiently small $w > w_0$.

4. Wall-crossing for special Mukai vectors

In this section we show how, by choosing a suitable polarization H_ϵ, wall-crossing for Mukai vectors of the form $(r_0, H - a_0E, s_0)$ ($a_0 \in \mathbb{Z}_{\geq 0}$) becomes significantly simpler. We show in Proposition 4.2 that actual walls for v intersecting the vertical axis $b = 0$ admit an explicit description and can be classified into two types, according to the shape of destabilizing short exact sequences. The following key lemma will be applied repeatedly:

Lemma 4.1. *Fix $m \geq 0$. Then there is $\epsilon_m > 0$ so that, if $\epsilon < \epsilon_m$, then any Chern character $(r, tH + qE, s) \in \Lambda$ with*

$$-rs \leq m\ , \qquad 0 \leq (tH + qE) \cdot H_\epsilon \leq H \cdot H_\epsilon \qquad \text{and} \qquad \Delta(r, tH + qE, s) \geq -2$$

satisfies $t = 0$ or $t = 1$.

Proof. We know

$$-2 \leq \Delta(r, tH + qE, s) = (tH + qE)^2 - 2rs \leq t^2H^2 + 2tqE \cdot H + 2m, \tag{28}$$

and our second condition gives

$$0 \leq tE \cdot H + t\epsilon H^2 + q\epsilon E \cdot H \leq E \cdot H + \epsilon H^2. \tag{29}$$

If $t \geq 2$, then for $\epsilon < \epsilon_m := \frac{E \cdot H}{H^2+m+1}$,

$$(t-1)E \cdot H \overset{(29)}{<} -q\epsilon E \cdot H \overset{(28)}{\leq} \epsilon\Big(\frac{t}{2}H^2 + \frac{m+1}{t}\Big) < (t-1)E \cdot H,$$

which is a contradiction. If $t \leq -1$ the first inequality in (29) gives $q \geq 0$, and so

$$-tE \cdot H \overset{(29)}{<} q\epsilon E \cdot H \overset{(28)}{\leq} \epsilon\Big(-\frac{t}{2}H^2 - \frac{m+1}{t}\Big) < -tE \cdot H$$

for $\epsilon < \epsilon_m$, which is again a contradiction. □

Via Lemma 4.1 we can control all actual walls for the Mukai vectors of interest in later sections:

Proposition 4.2. *Fix a Mukai vector $v = (r_0, H - a_0E, s_0 + r_0)$ for some $a_0 \in \mathbb{Z}_{\geq 0}$. There exists $\epsilon_v > 0$ such that, if $\epsilon < \epsilon_v$ and $F \in \mathrm{Coh}^0(X)$ is σ_{0,w_0}-strictly semistable of Mukai vector v for some $w_0 \geq 0$, then F sits in an exact triangle*

$$Q_1 \longrightarrow F \longrightarrow Q_2$$

where $\nu_{0,w_0}(F) = \nu_{0,w_0}(Q_i)$, and either $\mathrm{ch}_1(Q_1)$ or $\mathrm{ch}_1(Q_2)$ is a multiple of E.
If moreover $(s_0, w_0) \neq (0, 0)$, then up to relabeling the factors:

- *(a) Q_1 is σ_{0,w_0}-stable with $\mathrm{ch}_1(Q_1) = H - aE$ for some $a \geq a_0$, and*
- *(b) Q_2 is σ_{0,w_0}-semistable and either:*
  - *(b_1) it is isomorphic, up to a shift, to $\mathcal{O}_X(mE)^{\oplus \frac{a-a_0}{m}}$ for some $m \in \mathbb{Z}$, or*
  - *(b_2) every σ_{0,w_0}-stable factor of Q_2 has $\mathrm{ch}_0 = 0$ (in particular, $\mathrm{ch}_0(Q_2) = 0$) and ch_1 is a multiple of E.*

Remark 4.3. Let ℓ be the line through the point $(0, w_0)$ that passes through $\Pi_\epsilon(F)$ (if $r_0 \neq 0$) or has slope $\frac{s_0}{(H-a_0E)\cdot H_\epsilon}$ (if $r_0 = 0$). We are interested in the upper part

$$\ell^+ := \ell \cap \big\{(b, w) \in U_\epsilon : w \geq 0\big\}$$

of the line ℓ in U_ϵ. In particular, if $w_0 = 0$, in Proposition 4.2 σ_{0,w_0}-(semi)stability and ν_{0,w_0}-slope refer to $\sigma_{b,w}$-(semi)stability and $\nu_{b,w}$-slope for (b, w) lying on ℓ^+.

Proof of Proposition 4.2. Consider the line segment ℓ^+ as described in Remark 4.3. If F is $\sigma_{b',w'}$-unstable for (b', w') lying just above or just below the line ℓ^+, we consider the Harder-Narasimhan filtration with respect to $\sigma_{b',w'}$

$$Q_0 \subset Q_1 \subset \cdots \subset Q_n = F.$$

If Q_i/Q_{i-1} is strictly $\sigma_{b',w'}$-semistable for some i, then we further refine the filtration to account for the $\sigma_{b',w'}$-stable factors. In case F is $\sigma_{b',w'}$-semistable for (b', w') values just above and below the line ℓ^+, we consider a JH filtration of F with respect to one side.

We denote by $I := \{1, ..., n'\}$ the index set of this refined filtration, for which all quotients $F_i := Q_i/Q_{i-1}$ are $\sigma_{b',w'}$-stable with

$$\nu_{b',w'}(F_1) \geq \nu_{b',w'}(F_2) \geq \cdots \geq \nu_{b',w'}(F_{n'}). \tag{30}$$

Write $\alpha_i := \mathrm{ch}(F_i) = (r_i, t_iH + q_iE, s_i)$, with $i = 1, \ldots, n$. We classify these stable factors into several different types.

Consider $I' := \{i \in I : \ r_i > 0, \ s_i > 0\}$. Take $\epsilon < \epsilon_{m=0}$ (in the notations of Lemma 4.1). Since $r_is_i > 0$ for all $i \in I'$, Lemma 4.1 implies $t_i \in \{0, 1\}$ as $\Delta(F_i) \geq v(F_i)^2 \geq -2$. But if $t_i = 0$, then $v(F_i)^2 = -2r_i(r_i + s_i) < -2$ which is not possible, therefore $t_i = 1$ for all $i \in I'$.

We claim that $|I'| \leq 1$, that is, there is at most one stable factor with $r_i > 0$ and $s_i > 0$. Indeed, consider $\alpha_{I'} := \sum_{i\in I'} \alpha_i$. For any $i \in I'$, we have $\Delta(F_i) = v(F_i)^2 + 2r_i^2 \geq 0$ as $r_i > 0$. Since $\mathrm{Ker}\, Z_{0,w}$ is negative semi-definite for $w \geq 0$ with respect to the quadratic form Δ, [BMS16, Lemma 11.6] implies that $\Delta(\alpha_{I'}) \geq 0$. Therefore, Lemma 4.1 applied to the class $\alpha_{I'}$ yields $\sum_{i\in I'} t_i \in \{0, 1\}$, and so the set I' consists of at most one element. Moreover, there are only finitely many possibilities (depending only on the class v, and not on w_0) for the Chern character $(r_i, H + q_iE, s_i)$ of this factor, as:

- The inequality $H_\epsilon \cdot (H + q_iE) \leq H_\epsilon \cdot (H - a_0E)$ implies $q_i \leq -a_0$.
- Since $r_is_i \neq 0$ and $v(F_i)^2 \geq -2$, we have $(H + q_iE)^2 \geq -2 + 2r_i(r_i + s_i) \geq 2r_is_i$, which implies

$$1 \leq |r_i|, |s_i| \leq \frac{(H - a_0E)^2}{2} \qquad \text{and} \qquad \frac{-H^2}{2E \cdot H} \leq q_i. \tag{31}$$

Similarly, one can consider $I'' := \{i \in I : \ r_i < 0, \ s_i < 0\}$. The same argument yields: $t_i = 1$ for all $i \in I''$, $|I'| \leq 1$, and there are finitely many possibilities for the class of a factor in I'' (satisfying inequalities in (31)).

Step 1. We first assume that the wall has positive slope, that is, $\nu_{0,w_0}(F) > 0$. Then

$$I = I' \cup I'' \cup J' \cup J''$$

where $J' := \{i \in I : \ r_i < 0, \ s_i \geq 0\}$ and $J'' = \{i \in I : r_i = 0, \ s_i > 0\}$. If $i \in J' \cup J''$, then we have

$$0 \geq r_i \geq \sum_{j \notin I'} r_j = r_0 - \sum_{j\in I'} r_j \overset{(31)}{\geq} r_0 - \max\Big\{0, \frac{(H - a_0E)^2}{2}\Big\},$$
$$0 \leq s_i \leq \sum_{j \notin I''} s_j = s_0 - \sum_{j\in I''} s_j \overset{(31)}{\leq} s_0 + \max\Big\{0, \frac{(H - a_0E)^2}{2}\Big\},$$

which implies that there are finitely many possible values of r_i and s_i for all $i \in I$. We can thus find $M > 0$ (depending only on v, and not on the slope of the wall) such that $-r_is_i < M$ for every $i \in I$. It follows from Lemma 4.1 that, if $\epsilon < \epsilon_M$, then we have $t_i \in \{0, 1\}$ for all $i \in I$.

Since $\sum_{i\in I}(t_iH + q_iE) = \mathrm{ch}_1(F) = H - a_0E$, we have $\sum_{i\in I} t_i = 1$, hence there is a unique $i_0 \in I$ with $t_{i_0} = 1$. The corresponding stable factor F_{i_0} satisfies the inequalities

$-2 \leq v(F_{i_0})^2 < (H + q_{i_0}E)^2 + 2M$, therefore

$$q := \frac{-2 - H^2 - 2M}{2E \cdot H} < q_{i_0} \leq -a_0. \tag{32}$$

Any other factor satisfies $t_i = 0$ (in particular $i \in J' \cup J''$) and $q_i \geq 0$. Furthermore,

$$0 \leq \sum_{i \neq i_0} q_i = -a_0 - q_{i_0} \overset{(32)}{<} -a_0 - q \tag{33}$$

which implies $q_i < -a_0 - q$ for all $i \neq i_0$. Then either

(c.1) $r_i = 0$ for all $i \neq i_0$, or
(c.2) $r_{i_1} \neq 0$ for some $i_1 \neq i_0 \in I$.

In case (c.1), all the slopes $\nu_{b',w'}(F_i)$ for $i \neq i_0$ are equal (as $\nu_{b,w}(F_i)$ is independent of b, w). Thus the order in (30) implies that F_{i_0} is either a subobject or quotient of F, as claimed in case (b_2) of the main statement.

In case (c.2) we have $F_{i_1} \in J'$ (that is, $r_{i_1} < 0$, $s_{i_1} \geq 0$), and the slope $\eta(\epsilon)$ of the wall $\mathcal{W}(F, F_{i_1})$, depending on ϵ, satisfies

$$\eta(\epsilon) = \frac{s_{i_1}r_0 - s_0r_{i_1}}{r_0q_{i_1}\epsilon E \cdot H - r_{i_1}H_\epsilon \cdot (H - a_0E)} \underset{\epsilon \to 0^+}{\longrightarrow} \frac{s_{i_1}r_0 - s_0r_{i_1}}{-r_{i_1}E \cdot H}.$$

Since there are finitely many possibilities for $\mathrm{ch}(F_{i_1})$, we may choose ϵ small enough (depending only on v) so that the inequality $\eta(\epsilon) \cdot \epsilon(-a_0 - q)E \cdot H < 1$ holds. On the other hand, we know $\eta(\epsilon) = \nu_{0,w_0}(F_i)$ for all $i \in I$. If $i \neq i_0$ satisfies $i \in J''$, then

$$\nu_{0,w_0}(F_i) = \frac{s_i}{q_iE \cdot H_\epsilon} = \eta(\epsilon) < \frac{1}{\epsilon(-a_0 - q)E \cdot H}$$

and hence

$$s_i < \frac{q_iE \cdot H_\epsilon}{\epsilon(-a_0 - q)E.H} = \frac{q_i}{-a_0 - q} \overset{(33)}{\leq} 1,$$

contradiction. Therefore, in case (c.2) we have $i \in J'$ for all $i \neq i_0$. It follows that $\mathrm{ch}(F_i)$ is proportional to $\mathrm{ch}(F_{i_1}) = (r_{i_1}, q_{i_1}E, 0)$ for all $i \neq i_0$; by Lemma 3.13, we have $r_i = -1$ and $F_i = \mathcal{O}_X(-q_{i_1}E)[1]$ for all $i \neq i_0$. Thus, the main statement in case (b_1) follows from the vanishing

$$\hom\left(\mathcal{O}_X(-q_{i_1}E)[1], \mathcal{O}_X(-q_{i_1}E)[2]\right) = \hom(\mathcal{O}_X, \mathcal{O}_X[1]) = 0.$$

Step 2. Now we consider walls of non-positive slope.

(I) If $\nu_{0,w_0}(F) < 0$, then
$$I = I' \cup I'' \cup \tilde{J}' \cup \tilde{J}''$$
where $\tilde{J}' := \{i \in I : \ r_i > 0, \ s_i \leq 0\}$ and $\tilde{J}'' = \{i \in I : r_i = 0, \ s_i < 0\}$.

(II) If $\nu_{0,w_0}(F) = 0$ and $w_0 > 0$, then
$$I = I' \cup I'' \cup R$$
where $R := \{i \in I : \ r_i = 0, \ s_i = 0\}$.

(III) If $\nu_{0,w_0}(F) = 0$ and $w_0 = 0$, then $s_i = 0$ for all $i \in I$.

Then one can easily apply the same argument as in Step 1 to get the final claim. $\square$

As a result of the proof of Proposition 4.2, we get the well-known fact (see e.g. [MS17, Lemma 6.24]) that there are only finitely many actual walls for the class v intersecting the vertical line $b = 0$ at points with $w > 0$.

The following criterion further restricts the possibilities described in Proposition 4.2:

Lemma 4.4. *In Proposition 4.2, we may choose ϵ_v suitably such that if $0 < \epsilon < \epsilon_v$, then the following hold:*

(1) If $r_0 \leq 0$ and $\nu_{0,w_0}(F) < 0$, or $r_0 \geq 0$ and $\nu_{0,w_0}(F) > 0$, then case (b_2) cannot occur.

(2) If $r_0 \leq 0$, $s_0 < 0$, and $\nu_{0,w_0}(F) \geq 0$, or $r_0 \geq 0$, $s_0 > 0$, and $\nu_{0,w_0}(F) \leq 0$, then case (b_1) cannot occur.

Proof. We first assume $r_0 \leq 0$ and $\nu_{0,w_0}(F) < 0$. Then $\Pi_\epsilon(F)$ lies on the left-hand side of the (b, w)-plane, and the slope of the wall is greater than or equal to the slope of the numerical wall $\mathcal{W}(F, \mathcal{O}_X)$, which tends to $\frac{s_0}{E \cdot H}$ as $\epsilon \to 0^+$. Hence, there is a lower bound for the slope of walls for F with negative slope. On the other hand, in case (b_2) the slope $\nu_{b,w}(Q_2)$ (independent of b, w since $\mathrm{ch}_0(Q_2) = 0$) tends to $-\infty$ as $\epsilon \to 0^+$, and so this case cannot occur. A similar argument shows that case (b_2) cannot occur when $r_0 \geq 0$ and $\nu_{0,w_0}(F) > 0$.

Now suppose $r_0 \leq 0$, $s_0 < 0$, and $\nu_{0,w_0}(F) > 0$. As described in the proof of Proposition 4.2, there are finitely many possible values of q_i for stable factors $F_i = \mathcal{O}_X(-q_i E)[1]$; for each of them, we have

$$\frac{H_\epsilon \cdot \mathrm{ch}_1(F)}{(H_\epsilon)^2 \cdot \mathrm{ch}_0(F)} < \frac{H_\epsilon \cdot \mathrm{ch}_1(F_i)}{(H_\epsilon)^2 \cdot \mathrm{ch}_0(F_i)}$$

as $\epsilon \to 0^+$, which implies that the wall $\mathcal{W}(F, F_i)$ has negative slope, and so the claim follows. Similarly (b_1) cannot occur when $r_0 \geq 0$, $s_0 > 0$, and $\nu_{0,w_0}(F) \leq 0$. □

Let us point out that, via a similar argument as in Proposition 4.2, we can show that Gieseker stability and slope-stability coincide for our particular Mukai vectors.

Lemma 4.5. *Fix $v = (r_0, H - a_0 E, s_0 + r_0)$ for some $a_0 \in \mathbb{Z}_{\geq 0}$ and $r_0 > 0$. We may choose ϵ_v suitably such that if $0 < \epsilon < \epsilon_v$, then a sheaf F of Mukai vector v is H_ϵ-Gieseker stable if and only if it μ_{H_ϵ}-stable.*

Proof. Assume there is a H_ϵ-Gieseker stable sheaf F of class v which is not μ_{H_ϵ}-stable. Arguing as in Proposition 4.2, one finds a lower bound M for the quantity $\mathrm{ch}_0 \cdot \mathrm{ch}_2$ of μ_{H_ϵ}-stable factors of an appropriate filtration of F. This gives a distinguished μ_{H_ϵ}-stable factor Q_1 with $\mathrm{ch}(Q_1) = (r, H - aE, s)$; any other stable factor has ch_1 multiple of E.

We know $0 < r < r_0$ and $a \geq a_0$, and then the condition $v(Q_1)^2 \geq -2$ (together with the inequality $rs \geq M$) provides an upper bound for a depending on the class v (independently of ϵ). As for given a and r the equality

$$\frac{H_\epsilon \cdot (H - aE)}{r} = \frac{H_\epsilon \cdot (H - a_0 E)}{r_0}$$

happens for (at most) one value of ϵ, it follows that for ϵ_v sufficiently small there can be no non-trivial μ_{H_ϵ}-destabilizing factor, and hence F must be μ_{H_ϵ}-stable. □

4.1. **Moduli spaces of stable objects.** We conclude this section with a brief discussion on moduli spaces of $\sigma_{b,w}$-stable objects for our special Mukai vectors v. Keeping the convention of Remark 4.3 (that is, by $\sigma_{0,0}$ we mean a stability condition on the upper part of the numerical wall $\mathcal{W}(\mathcal{O}_X, v)$), we have the following result.

Proposition 4.6. *Fix a Mukai vector* $v = (r_0, H - a_0E, s_0 + r_0)$ *for some* $a_0 \in \mathbb{Z}_{\geq 0}$ *such that* $(r_0, s_0) \neq (0, 0)$*. Pick* $0 < \epsilon < \epsilon_v$ *(such that* ϵ *is generic if* $r_0 s_0 = 0$*) and let* $\sigma := \sigma_{0,w}$ *for* $w \geq 0$ *lie in no actual wall for* v *in* U_ϵ*. Then:*

(a) Any σ*-semistable object of class* v *is* σ*-stable.*

(b) There is a coarse moduli space $\mathcal{M}_\sigma(v)$ *parameterizing* σ*-stable objects of class* v *in* $\mathrm{Coh}^b(X)$*, which is a smooth projective hyperkähler variety of dimension* $v^2 + 2$ *(in particular, it is nonempty if and only if* $v^2 \geq -2$*).*

Proof. We only prove (a). Then (b) follows from a series of fundamental works, explained in detail in [BM14, Section 6]. First, assume $r_0 s_0 \neq 0$. Suppose there exists an object $F \in \mathrm{Coh}^0(X)$ that is strictly σ-semistable inside a chamber. Then F admits a Jordan-Hölder filtration that is fixed within the chamber. However, F has stable factors as described in Proposition 4.2.(b), which do not have the same slope $\nu_{0,w+\delta}$ as F for $\delta > 0$, leading to a contradiction.

Now suppose $r_0 = 0$, $s_0 \neq 0$. If an object $F \in \mathrm{Coh}^0(X)$ of class v is strictly σ-semistable, then by 4.2 there exists a Chern character $(0, (a - a_0)E, s)$ such

$$\frac{s_0}{H_\epsilon \cdot (H - a_0)E} = \frac{s}{(a - a_0)E \cdot H_\epsilon}, \quad 0 \leq (a - a_0)E \cdot H_\epsilon < (H - a_0E) \cdot H_\epsilon.$$

However, one can easily check that this happens only for discrete values of ϵ, so the claim holds for a generic choice of ϵ. A similar argument applies to the case $r_0 \neq 0$, $s_0 = 0$. $\square$

The following provides sufficient conditions to guarantee that, along the walls described in Proposition 4.2, a general stable object does not get destabilized:

Lemma 4.7. *Adopt the notations of Proposition 4.2, and assume* $(s_0, w_0) \neq (0, 0)$*. Let* σ_0^+ *(resp.* σ_0^-*) be a stability condition just above (resp. just below)* ℓ^+*. Then the locus in* $\mathcal{M}_{\sigma_0^+}(v)$ *and* $\mathcal{M}_{\sigma_0^-}(v)$ *of* σ_{0,w_0}*-strictly semistable objects has dimension strictly less than* $v^2 + 2 = \dim \mathcal{M}_{\sigma_0^\pm}(v)$*, providing that we are in one of the following situations:*

(1) Condition (b_1) *holds and:* $\chi(\mathcal{O}_X(mE), v) \leq 0$ *for* $m > 0$ *(or* $m = 0$ *and* ℓ^+ *of negative slope),* $\chi(\mathcal{O}_X(mE), v) \geq 0$ *for* $m < 0$ *(or* $m = 0$ *and* ℓ^+ *of positive slope).*

(2) Condition (b_2) *holds and* $\chi(Q_2, v) + 2(a - a_0) < 0$*.*

Proof. First assume that condition (b_1) holds for $m > 0$, or for $m = 0$ and ℓ^+ of negative slope. When $\nu_{\sigma_0^+}(F) > \nu_{\sigma_0^+}(\mathcal{O}_X(mE))$ (resp. $\nu_{\sigma_0^+}(F) < \nu_{\sigma_0^+}(\mathcal{O}_X(mE))$), the locally closed subset in $\mathcal{M}_{\sigma_0^+}(v)$ of σ_{0,w_0}-semistable objects with $\hom(\mathcal{O}_X(mE), F) = h \geq 1$ (resp. $\hom(F, \mathcal{O}_X(mE)) = h \geq 1$) is either empty or of dimension

$$\begin{aligned}(v - h(1, mE, 1))^2 + 2 + h\left(\langle v - h(1, mE, 1), (1, mE, 1)\rangle - h\right) = \\ = v^2 + 2 - h\langle v, (1, mE, 1)\rangle - h^2 < v^2 + 2,\end{aligned}$$

so the claim follows. The same argument works for $\mathcal{M}_{\sigma_0^-}(v)$. Similarly, considering the shift $\mathcal{O}_X(mE)[1]$, one can cover the remaining cases of the first assertion.

Now suppose condition (b_2) holds, so that $v(Q_2) = (0, (a-a_0)E, s)$. Such objects Q_2 vary in a 2α-dimensional moduli space, where $\alpha = \gcd(a-a_0, s)$, by [BM14, Lemma 7.2]. Since $\hom(Q_1, Q_2) = 0 = \hom(Q_2, Q_1)$ then, if non-empty, the locus of σ_{0,w_0}-strictly semistable objects in $\mathcal{M}_{\sigma_0^\pm}(v)$ has dimension less than or equal to

$$(v - v(Q_2))^2 + 2 + 2\alpha + \langle v - v(Q_2), v(Q_2)\rangle =$$
$$= v^2 + 2 - \langle v, v(Q_2)\rangle + 2\alpha \leq v^2 + 2 - \langle v, v(Q_2)\rangle + 2(a - a_0),$$

which is strictly less than $v^2 + 2$ under our assumption. □

5. Stratification of the moduli of Bridgeland stable objects

Fix an elliptic $K3$ surface X of degree $k > 2$ as in Proposition 3.1, equipped with a two-dimensional slice of Bridgeland stability conditions U_ϵ associated to the polarization $H_\epsilon := E + \epsilon H$ for a fixed $\epsilon \in \mathbb{Q}_{>0}$. In this section, we introduce the notion of *Bridgeland stability type*, which is central in our analysis. It enables us to investigate the Brill-Noether stratification of moduli spaces of Bridgeland stable objects with appropriate Mukai vector, as detailed in Theorem 5.5 and Theorem 5.8. In Section 6, we apply these results to the specific Mukai vector $(0, H, 1 + d - g)$ to study the Brill-Noether theory of line bundles on curves in the linear system $|H|$.

5.1. Bridgeland stability type.

Consider $p \in \mathbb{N}$ and p pairs

$$(e_1, m_1), \ldots, (e_p, m_p) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$$

such that $e_1 > \cdots > e_p \geq 0$. Write $\overline{e} := \big((e_i, m_i)\big)_{i=1}^p$ (so that $\overline{e} = \emptyset$ if $p = 0$).

Definition 5.1. Let $F \in \mathrm{Coh}^0(X)$ be a σ_{0,w_0}-stable object for some $w_0 > 0$, such that $\mathrm{ch}_2(F) \neq 0$. If $p \geq 1$, we say that F is *of (Bridgeland) stability type* $\overline{e}$ if:

(i) When we move down from the point $(0, w_0)$, then F gets destabilized along the wall $\mathcal{W}_1 \subset U_\epsilon$ passing through $(0, w_1)$ for some $w_1 \in [0, w_0)$ via the destabilizing sequence

$$\mathrm{Hom}(\mathcal{O}_X(e_1E), F) \otimes \mathcal{O}_X(e_1E) \xrightarrow{\mathrm{ev}} F \longrightarrow F_1,$$

such that $m_1 = \hom(\mathcal{O}_X(e_1E), F)$ and F_1 is stable along $\mathcal{W}_1$.

(ii) Then we move down the vertical line $b = 0$ and inductively obtain the object F_i along the wall $\mathcal{W}_i \subset U_\epsilon$ where F_{i-1} gets destabilized. The destabilizing sequence is given by

$$\mathrm{Hom}(\mathcal{O}_X(e_iE), F_{i-1}) \otimes \mathcal{O}_X(e_iE) \xrightarrow{\mathrm{ev}} F_{i-1} \longrightarrow F_i,$$

where $m_i = \hom(\mathcal{O}_X(e_iE), F_{i-1})$ and F_i is stable along the wall $\mathcal{W}_i$ which passes through $(0, w_i)$ for some $w_i \in [0, w_{i-1})$.

(iii) The final object F_p is stable along the numerical wall $\mathcal{W}(\mathcal{O}_X, F_p)$ made with the structure sheaf $\mathcal{O}_X$.

If $p = 0$, we say that F is *of (Bridgeland) stability type* $\emptyset$ if it is stable along the numerical wall $\mathcal{W}(\mathcal{O}_X, F)$.

Remark 5.2. Two important points should be noted regarding Definition 5.1:

- If $F \in \mathrm{Coh}^0(X)$ is σ_{0,w_0}-stable of stability type $\overline{e}$, then for any $1 \leq j \leq p$ the quotient F_j is σ_{0,w_j}-stable of stability type $\big((e_i, m_i)\big)_{i=j+1}^{p}$. In particular, the inequality $v(F_p)^2 \geq -2$ holds.
- We know that the object F_i (for $1 \leq i \leq p$) has Mukai vector
$$v(F_i) = v(F) - \sum_{j=1}^{i} m_j v(\mathcal{O}_X(e_j E)),$$
and it is stable along the wall $\mathcal{W}_i$ where it has the same slope as $\mathcal{O}_X(e_i E)$. Since $\mathcal{O}_X(e_i E)$ is also stable along $\mathcal{W}_i$ by Lemma 3.13, we have $\mathrm{Hom}(\mathcal{O}_X(e_i E), F_i) = 0 = \mathrm{Hom}(F_i, \mathcal{O}_X(e_i E))$ which implies
$$\mathrm{ext}^1(F_i, \mathcal{O}_X(e_i E)) = -\chi\left(F_i, \mathcal{O}_X(e_i E)\right) = \langle v(F_i), v(\mathcal{O}_X(e_i E))\rangle.$$

These observations enable us to endow the set of stable objects of a fixed stability type with a natural algebro-geometric structure (given as an open subset of an iterated Grassmannian bundle over a certain Bridgeland moduli space):

Theorem 5.3. *Fix a Mukai vector $v = (r_0, H - a_0 E, s_0 + r_0)$ for some $a_0 \in \mathbb{Z}_{\geq 0}$ and $s_0 \neq 0$ and a stability condition $\sigma_0 := \sigma_{0,w_0}$ with $w_0 > 0$ which does not lie on an actual wall*[5] *for class v. Then for any stability type $\overline{e} = \big((e_i, m_i)\big)_{i=1}^{p}$ where $p \geq 0$, the subset*
$$\mathcal{M}_{\sigma_0}(v, \overline{e}) := \big\{F \in \mathcal{M}_{\sigma_0}(v) : \ F \text{ is of stability type } \overline{e}\big\}$$
admits a natural scheme structure as a locally closed subscheme of $\mathcal{M}_{\sigma_0}(v)$. Moreover, if $\mathcal{M}_{\sigma_0}(v, \overline{e})$ is non-empty, then it is smooth and irreducible of dimension
$$\left(v - \sum_{i=1}^{p} m_i(1, e_i E, 1)\right)^2 + 2 + \sum_{j=1}^{p} m_j \left(\Big\langle v - \sum_{i=1}^{j} m_i(1, e_i E, 1), (1, e_j E, 1)\Big\rangle - m_j\right).$$

Proof. We use induction on the length p of the stability type to endow $\mathcal{M}_{\sigma_0}(v, \overline{e})$ with a scheme structure that satisfies the required properties.

If $p = 0$ (namely $\overline{e} = \emptyset$), then $\mathcal{M}_{\sigma_0}(v, \emptyset) \subseteq \mathcal{M}_{\sigma_0}(v)$ is the (possibly empty) open subset of objects that remain stable along the numerical wall $\mathcal{W}(\mathcal{O}_X, v)$ defined by $\mathcal{O}_X$ and any object of Mukai vector v.

Assume the lemma holds for all Mukai vectors with $\mathrm{ch}_1 = H - aE$ and for all stability types of length $\leq p - 1$. Let σ_1^+ be a stability condition on the vertical line $b = 0$, sufficiently close to $\mathcal{W}_1 = \mathcal{W}(\mathcal{O}_X(e_1 E), v)$ and above $\mathcal{W}_1$. If $\mathcal{M}_{\sigma_0}(v, \overline{e})$ is non-empty, then necessarily $\mathcal{M}_{\sigma_1^+}\big(v - m_1(1, e_1 E, 1),\ \overline{e} \setminus (e_1, m_1)\big)$ must be non-empty. Also by the induction hypothesis, it is a smooth, irreducible, locally closed subscheme of $\mathcal{M}_{\sigma_1^+}(v - m_1(1, e_1 E, 1))$ of dimension
$$\left(v - \sum_{i=1}^{p} m_i(1, e_i E, 1)\right)^2 + 2 + \sum_{j=2}^{p} m_j \left(\Big\langle v - \sum_{i=1}^{j} m_i(1, e_i E, 1), (1, e_j E, 1)\Big\rangle - m_j\right).$$

[5] We always assume $\epsilon > 0$ is generic if $r_0 = 0$.

Let $\mathcal{G}$ denote the Grassmanian bundle over $\mathcal{M}_{\sigma_1^+}\big(v - m_1(1,e_1E,1),\ \overline{e}\setminus(e_1,m_1)\big)$, whose fiber over an object F_1 is given by $\mathrm{Gr}\left(m_1, \mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))\right)$. By Lemma 3.15, we have a natural morphism

$$\varphi\colon \mathcal{G} \longrightarrow \mathcal{M}^{\mathrm{st}}_{\sigma_1^+}(v)$$

which sends a pair $(F_1,V)\in\mathcal{G}$ to the object $F\in\mathcal{M}_{\sigma_1^+}(v)$ sitting in the exact triangle

$$V^\vee\otimes\mathcal{O}_X(e_1E)\xrightarrow{d_1}F\xrightarrow{d_2}F_1\xrightarrow{d_3}V^\vee\otimes\mathcal{O}_X(e_1E)[1]. \tag{34}$$

It follows from Definition 5.1 that φ establishes a (set-theoretic) bijection between $\mathcal{G}$ and the subset of σ_1^+-stable objects of class v with stability type $\overline{e}$. Furthermore, φ is a locally closed immersion. This is a consequence of the following:

Claim 5.4. The differential of φ is injective at every point.

Proof of the claim. The tangent space to $\mathcal{G}$ at (F_1,V) is given by

$$T_{[F_1]}\left(\mathcal{M}_{\sigma_1^+}\left(v-m_1(1,e_1E,1),\ \overline{e}\setminus(e_1,m_1)\right)\right)\times T_{[V]}\,\mathrm{Gr}\left(m_1,\mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))\right),$$

and is contained in

$$T_{[F_1]}\left(\mathcal{M}_{\sigma_1^+}\left(v-m_1(1,e_1E,1)\right)\right)\times T_{[V]}\,\mathrm{Gr}\left(m_1,\mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))\right).$$

On the other hand, if $F=\varphi(F_1,V)$, we have canonical identifications:

- $T_{[F_1]}\big(\mathcal{M}_{\sigma_1^+}\left(v-m_1(1,e_1E,1)\right)\big)=\mathrm{Ext}^1(F_1,F_1)$.
- $T_{[V]}\,\mathrm{Gr}\left(m_1,\mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))\right)=\mathrm{Hom}(V,\mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))/V)$. By applying the long exact sequence of $\mathrm{Ext}^i(-,\mathcal{O}_X(e_1E))$ to (34) we also obtain
$$\mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))/V\cong\mathrm{Ext}^1(F,\mathcal{O}_X(e_1E)),$$
and therefore we canonically have
$$T_{[V]}\,\mathrm{Gr}\left(m_1,\mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E))\right)=V^\vee\otimes\mathrm{Ext}^1(F,\mathcal{O}_X(e_1E)).$$
- $T_{[F]}\big(\mathcal{M}_{\sigma_1^+}(v)\big)=\mathrm{Ext}^1(F,F)$.

Applying $\mathrm{Ext}^i(-,F_1)$ to (34) gives $\mathrm{Hom}(F,F_1)=\mathbb{C}$ and the natural inclusion

$$\lambda\colon\ \mathrm{Ext}^1(F_1,F_1)\hookrightarrow\mathrm{Ext}^1(F,F_1).$$

Moreover, applying $\mathrm{Ext}^i(F,-)$ to (34) results in the long exact sequence

$$0\to V^\vee\otimes\mathrm{Ext}^1(F,\mathcal{O}_X(e_1E))\to\mathrm{Ext}^1(F,F)\xrightarrow{\gamma}\mathrm{Ext}^1(F,F_1)\xrightarrow{\gamma'}V^\vee\otimes\mathrm{Ext}^2(F,\mathcal{O}_X(e_1E))\to\dots$$

Note that $\mathrm{im}(\lambda)\subset\ker(\gamma')$, as any composition $F\xrightarrow{d_2}F_1\to F_1[1]\xrightarrow{d_3[1]}V^\vee\otimes\mathcal{O}_X(e_1E)[2]$ vanishes since $\hom(F_1,\mathcal{O}_X(e_1E)[2])=\hom(\mathcal{O}_X(e_1E),F_1)=0$. Thus we get the diagram

$$\begin{array}{ccccc} & & & & \mathrm{Ext}^1(F_1,F_1)\\ & & & & \big\downarrow\\ V^\vee\otimes\mathrm{Ext}^1(F,\mathcal{O}_X(e_1E)) & \hookrightarrow & \mathrm{Ext}^1(F,F) & \twoheadrightarrow & \mathrm{im}(\gamma)\end{array}$$

realizing the inclusion $T_{(F_1,V)}(\mathcal{G})\subseteq T_{[F]}\left(\mathcal{M}_{\sigma_1^+}(v)\right)$ defined by $d\varphi$. $\square$

Since φ is a locally closed immersion, we can endow $\mathcal{M}_{\sigma_1^+}(v,\overline{e})$ with the scheme structure provided by $\mathcal{G}$; it becomes a smooth, irreducible, locally closed subscheme of $\mathcal{M}_{\sigma_1^+}(v)$. On the other hand, every $F_1 \in \mathcal{M}_{\sigma_1^+}\big(v - m_1(1,e_1E,1),\ \overline{e}\setminus(e_1,m_1)\big)$ satisfies

$$\dim \mathrm{Gr}(m_1, \mathrm{Ext}^1(F_1,\mathcal{O}_X(e_1E)) = m_1\left(\mathrm{ext}^1(F_1,\mathcal{O}_X(e_1E)) - m_1\right) =$$
$$= m_1\left(-\chi(F_1,\mathcal{O}_X(e_1E)) - m_1\right) = m_1\big(\langle v - m_1(1,e_1E,1),(1,e_1E,1)\rangle - m_1\big).$$

This implies that $\mathcal{G}$ (hence $\mathcal{M}_{\sigma_1^+}(v,\overline{e})$) has dimension

$$\left(v - \sum_{i=1}^{p} m_i(1,e_iE,1)\right)^2 + 2 + \sum_{j=1}^{p} m_j\left(\langle v - \sum_{i=1}^{j} m_i(1,e_iE,1),(1,e_jE,1)\rangle - m_j\right).$$

Finally, to conclude the proof we simply observe that $\mathcal{M}_{\sigma_0}(v,\overline{e})$ is an open subset of $\mathcal{M}_{\sigma_1^+}(v,\overline{e})$, consisting of those objects in $\mathcal{M}_{\sigma_1^+}(v,\overline{e})$ that are σ_0-stable. $\square$

5.2. **Stratification of Brill-Noether loci.** We now establish the main result of this section, which provides a stratification of moduli of stable objects. Fix a primitive Mukai vector $v\in\Lambda$ and $\sigma\in U_\epsilon$ lying on no actual wall for v, therefore there is a proper coarse moduli space $\mathcal{M}_\sigma(v)$ parameterizing σ-semistable object of class v in $\mathrm{Coh}^b(X)$. For each $r\geq -1$ we define the closed subscheme

$$W^r_\sigma(v) := \Big\{F\in\mathcal{M}_\sigma(v)\colon\ \hom(\mathcal{O}_X,F)\geq r+1\Big\}\subseteq \mathcal{M}_\sigma(v). \tag{35}$$

Let $V^r_\sigma(v)$ be the subset of $W^r_\sigma(v)$ corresponding to objects F with $\hom(\mathcal{O}_X,F)=r+1$.

The following result, taking advantage of the wall crossing analysis performed in Section 4, shows that every object in $V^r_\sigma(v)$ has a stability type, which is strongly constrained by r.

Theorem 5.5. *Fix $r\geq -1$ and a Mukai vector $v=(r_0, H-a_0E, s_0+r_0)\in\Lambda$ with $r_0\leq 0$, $a_0\geq 0$ and $s_0<0$. There exists $\epsilon(v,r)>0$ such that if $\epsilon<\epsilon(v,r)$ and $w_0>0$ satisfies $\nu_{0,w_0}(v)<0$, then*

$$V^r_{\sigma_{0,w_0}}(v)\subseteq \bigcup_{\overline{e}\in I}\mathcal{M}_{\sigma_{0,w_0}}(v,\overline{e}),$$

where I is the (finite) set of stability types $\overline{e}=\big((e_i,m_i)\big)_{i=1}^p$ with $p\geq 0$ and

(a) $\sum_{i=1}^p m_i\leq r+1\leq \sum_{i=1}^p m_i(e_i+1)$,
(b) $m_1(e_1+1)\leq r+1$.

Proof. First assume $r\geq 0$. Take $F\in V^r_{\sigma_{0,w_0}}(v)$ with $\nu_{0,w_0}(F)<0$. Note that F cannot be stable along the numerical wall $\mathcal{W}(F,\mathcal{O}_X)$, as $\hom(\mathcal{O}_X,F)=r+1>0$. Hence we encounter an actual wall $\mathcal{W}_1$ for F passing through $(0,w_1)$, for some $w_1\in[0,w_0)$.

We apply Proposition 4.2 for the class v and the wall $\mathcal{W}_1$. The assumption $r_0\leq 0$ gives $\nu_{0,w_1}(F)\leq\nu_{0,w_0}(F)<0$, thus by Lemma 4.4 the wall is of type (b_1), namely F gets destabilized via a short exact sequence

$$\mathcal{O}_X(e_1E)\otimes\mathrm{Hom}(\mathcal{O}_X(e_1E),F)\longrightarrow F\longrightarrow F_1$$

such that $e_1 \geq 0$ and F_1 is σ_{0,w_1}-stable. Writing $m_1 := \hom(\mathcal{O}_X(e_1E), F)$, we get $m_1(e_1+1) \leq r+1$ by applying $\mathrm{Hom}(\mathcal{O}_X, -)$ to the above triangle.

If F_1 is stable along the numerical wall $\mathcal{W}(\mathcal{O}_X, F_1)$, then F has stability type $\big((e_1, m_1)\big)$. In this case $\hom(\mathcal{O}_X, F_1) = 0$ which implies $m_1(e_1+1) = r+1$, hence inequalities (a) and (b) are trivially satisfied. Otherwise, F_1 gets destabilized along an actual wall $\mathcal{W}_2$ passing through $(0, w_2)$, with $w_2 \in [0, w_1)$. Since $\mathrm{ch}_0(F_1) = r_0 - m_1 < 0$ and $\mathrm{ch}_2(F_1) = s_0 < 0$, we can apply Proposition 4.2 and Lemma 4.4 with the class $v(F_1)$. It follows that the destabilizing subobject of F_1 along $\mathcal{W}_2$ is $\mathcal{O}_X(e_2E) \otimes \mathrm{Hom}(\mathcal{O}_X(e_2E), F_1)$, where $0 \leq e_2 < e_1$.

We apply this procedure inductively, constructing F_i via the destabilizing sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{O}_X(e_iE), F_{i-1}) \otimes \mathcal{O}_X(e_iE) \longrightarrow F_{i-1} \longrightarrow F_i \longrightarrow 0,$$

so that F_i is stable along $\mathcal{W}_i := \mathcal{W}(\mathcal{O}_X(e_iE), F_{i-1})$ and $m_i := \hom(\mathcal{O}_X(e_iE), F_{i-1})$. If F_i is stable along the numerical wall $\mathcal{W}(\mathcal{O}_X, F_i)$, then the process ends. Eventually, we obtain that F has some stability type $\overline{e} = \{(e_i, m_i)\}_{i=1}^p$ with $p \geq 1$. It only remains to prove the required inequalities in (a).

To that end, observe that $h^2(X, \mathcal{O}_X(e_iE)) = 0$ for $1 \leq i \leq p-1$ (as $e_i > 0$) and $h^2(X, \mathcal{O}_X(e_pE)) \leq 1$ (as $e_p \geq 0$). Moreover, $\mathrm{ext}^2(\mathcal{O}_X, F) = \hom(F, \mathcal{O}_X) = 0$ by the σ_{0,w_0}-stability of F. It follows that

$$\begin{aligned} r+1-s_0-2r_0 &= \hom(\mathcal{O}_X, F) - \chi(\mathcal{O}_X, F) = \mathrm{ext}^1(\mathcal{O}_X, F) \\ &\geq \mathrm{ext}^1(\mathcal{O}_X, F_1) \geq \cdots \geq \mathrm{ext}^1(\mathcal{O}_X, F_{p-1}) \geq \mathrm{ext}^1(\mathcal{O}_X, F_p) - m_p \\ &\geq -\chi(\mathcal{O}_X, F_p) - m_p = 2(m_1 + \cdots + m_{p-1}) + m_p - s_0 - 2r_0, \end{aligned}$$

which implies $2(m_1 + \cdots + m_{p-1}) + m_p \leq r+1$, and so $\sum_{i=1}^p m_i \leq r+1$.

On the other hand, since $\hom(\mathcal{O}_X, F_p) = 0$ (recall that F_p is stable along the numerical wall $\mathcal{W}(F_p, \mathcal{O}_X)$), we have $\hom(\mathcal{O}_X, F_{p-1}) = m_p(e_p+1)$. Therefore,

$$\hom(\mathcal{O}_X, F_{p-2}) \leq \hom(\mathcal{O}_X, F_{p-1}) + m_{p-1}(e_{p-1}+1) = m_p(e_p+1) + m_{p-1}(e_{p-1}+1).$$

Repeating this process, the inequality $r+1 = \hom(\mathcal{O}_X, F) \leq \sum_{i=1}^p m_i(e_i+1)$ follows.

Finally, for $r = -1$ the assertion is that any $F \in \mathcal{M}_{\sigma_{0,w_0}}(v)$ with $\hom(\mathcal{O}_X, F) = 0$ has empty stability type, i.e. it is stable along the numerical wall $\mathcal{W}(\mathcal{O}_X, F)$. This is an immediate application of Proposition 4.2 and Lemma 4.4. □

Remark 5.6. Given $v = (r_0, H - a_0E, s_0 + r_0) \in \Lambda$ as in Theorem 5.5 and a stability type $\overline{e} = \big((e_i, m_i)\big)_{i=1}^p$ with $e_p > 0$, for $w_0 > 0$ consider the union

$$\widetilde{\mathcal{M}}_{\sigma_{0,w_0}}(v, \overline{e}) := \bigcup_{\overline{e'} \in I_{\overline{e}}} \mathcal{M}_{\sigma_{0,w_0}}\big(v, \overline{e} \cup \overline{e'}\big) \tag{36}$$

where $I_{\overline{e}}$ is the set of types $\overline{e'} = \big((e'_i, m'_i)\big)_{i'=1}^{p'}$ ($p' \geq 0$) with $e'_1 < e_p$. Let us write $v_p := v - \sum_{i=1}^p m_i(1, e_iE, 1)$. Note that $\widetilde{\mathcal{M}}_{\sigma_{0,w_0}}(v, \overline{e})$ is defined by a finite union, indexed by the finite set of $\overline{e'} \in I_{\overline{e}}$ such that $\left(v_p - \sum_{i=1}^{p'} m'_i(1, e'_iE, 1)\right)^2 \geq -2$.

As a consequence of Theorem 5.5, there is $\epsilon(v, \overline{e}) > 0$ such that, if $\epsilon < \epsilon(v, \overline{e})$ and $w_0 > 0$ satisfies $\nu_{0,w_0}(v) < 0$ and $\mathcal{M}_{\sigma_{0,w_0}}(v, \overline{e}) \neq \emptyset$, then $\widetilde{\mathcal{M}}_{\sigma_{0,w_0}}(v, \overline{e})$ is smooth and irreducible with $\mathcal{M}_{\sigma_{0,w_0}}(v,\overline{e})$ as an open dense subset. Indeed, as proven in Theorem 5.3, $\mathcal{M}_{\sigma_{0,w_0}}(v,\overline{e})$ is (an open subset of) an iterated Grassmann bundle over an open subset $\mathcal{U} \subseteq \mathcal{M}^{\mathrm{st}}_{\sigma_p}(v_p)$; here σ_p lies on the line $b=0$ and the numerical wall $\mathcal{W}(\mathcal{O}_X(e_pE), v_p)$, and $\mathcal{M}^{\mathrm{st}}_{\sigma_p}(v_p) \subseteq \mathcal{M}_{\sigma_p}(v_p)$ is the open subset of stable objects. Precisely, $\mathcal{U}$ parametrizes the objects that remain stable along $\mathcal{W}(\mathcal{O}_X, v_p)$. By Theorem 5.5, for $\epsilon > 0$ small enough, any object in $\mathcal{M}^{\mathrm{st}}_{\sigma_p}(v_p)$ has an associated stability type. Then $\widetilde{\mathcal{M}}_{\sigma_{0,w_0}}(v,\overline{e})$ can be constructed as (an open subset of) an iterated Grassmann bundle over $\mathcal{M}^{\mathrm{st}}_{\sigma_p}(v_p)$.

5.3. **Balanced stability types.** To conclude this section we focus on a particular class of stability types that, in view of H. Larson's nomenclature [Lar21], we will call *balanced*.

Definition 5.7. A stability type $\overline{e}$ is *balanced* if it is of the form $\big((e+1,m_1),(e,m_2)\big)$ for some $e, m_1 \geq 0$ and $m_2 > 0$.

A balanced stability type $\overline{e} = \{(e+1,m_1),(e,m_2)\}$ determines the number of global sections. Indeed, since $\mathrm{Ext}^1\big(\mathcal{O}_X(eE), \mathcal{O}_X((e+1)E)\big) = 0$, one checks that

$$\mathcal{M}_\sigma(v,\overline{e}) \subseteq V^r_\sigma(v)$$

for $r := m_1(e+2) + m_2(e+1) - 1$.

The geometric description in Theorem 5.3 is conditional to the nonemptiness of $\mathcal{M}_\sigma(v,\overline{e})$. This issue can be resolved for balanced stability types:

Theorem 5.8. *Fix a Mukai vector $v = (r_0, H - a_0E, s_0 + r_0) \in \Lambda$ with $r_0 \leq 0$, $a_0 \geq 0$, and $s_0 < 0$. Consider the stability type $\overline{e} := \big((e+1,m_1),(e,m_2)\big)$ for $e, m_1 \geq 0$ and $m_2 > 0$ such that*

$$m_1 + m_2 \leq k + r_0, \tag{37}$$

$$\Big(v - m_1 \cdot v(\mathcal{O}_X((e+1)E)) - m_2 \cdot v(\mathcal{O}_X(eE))\Big)^2 \geq -2. \tag{38}$$

There exists $\epsilon(v,\overline{e}) > 0$ (depending on v and $\overline{e}$), such that if $\epsilon < \epsilon(v,\overline{e})$ and $w_0 > 0$ satisfies $\nu_{0,w_0}(\mathcal{O}_X((e+1)E)) < \nu_{0,w_0}(v) < 0$, then $\mathcal{M}_{\sigma_{0,w_0}}(v,\overline{e})$ is non-empty.

Note that, according to Remark 5.2, inequality (38) is a necessary condition for the non-emptiness of $\mathcal{M}_{\sigma_{0,w_0}}(v,\overline{e})$. Theorem 5.8 establishes that, under the assumption (37), (38) is also a sufficient condition.

The proof of Theorem 5.8 occupies the rest of this section. The essential step is provided by the following result:

Proposition 5.9. *Let $q \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{R}$, and let $Q \in \mathrm{Coh}^b(X)$ be an object such that:*

- *(a)* $\mathrm{ch}(Q) = (r_0, H - a_0E, s_0)$ *with $r_0 < 0$, $a_0 \geq 0$ and $s_0 \leq 0$.*
- *(b)* $\mathrm{Hom}(Q, \mathcal{O}_J) = 0$ *for a general elliptic curve $J \in |E|$.*
- *(c)* $\mathrm{Hom}(Q, \mathcal{O}_X(uE)) = 0$ *for all $u \geq q$.*

If $0 < m \leq k - \mathrm{hom}(\mathcal{O}_X((q-1)E), Q)$, then for a general $V \in \mathrm{Gr}\big(m, \mathrm{Ext}^1(Q, \mathcal{O}_X(qE))\big)$ the object Q_1 sitting in an exact triangle

$$V^\vee \otimes \mathcal{O}_X(qE) \longrightarrow Q_1 \longrightarrow Q$$

satisfies $\mathrm{hom}(Q_1, \mathcal{O}_X(uE)) = 0$ *for all $u > q$.*

Proof. The claim amounts to the injectivity of the map

$$V \otimes \operatorname{Hom}(\mathcal{O}_X(qE), \mathcal{O}_X(uE)) \longrightarrow \operatorname{Ext}^1(Q, \mathcal{O}_X(uE)) \tag{39}$$

for all $u > q$, if $V \in \operatorname{Gr}\left(m, \operatorname{Ext}^1(Q, \mathcal{O}_X(qE))\right)$ is a general subspace.

Note that (39) is the restriction of the map

$$\psi\colon \operatorname{Hom}(Q, \mathcal{O}_X(qE)[1]) \otimes \operatorname{Hom}(\mathcal{O}_X(qE), \mathcal{O}_X(uE)) \longrightarrow \operatorname{Hom}(Q, \mathcal{O}_X(uE)[1])$$

given by composition. Moreover, ψ is the result of applying $\operatorname{Ext}^1(Q, -)$ to the sequence

$$0 \longrightarrow \mathcal{O}_X((q-1)E)^{\oplus u-q} \longrightarrow \operatorname{Hom}(\mathcal{O}_X(qE), \mathcal{O}_X(uE)) \otimes \mathcal{O}_X(qE) \longrightarrow \mathcal{O}_X(uE) \longrightarrow 0,$$

given by twist by $\mathcal{O}_X(qE)$ of the sequence (11). Since $\hom(Q, \mathcal{O}_X(uE)) = 0$, one has

$$\ker(\psi) \cong \operatorname{Hom}\big(Q, \mathcal{O}_X((q-1)[1]\big)^{\oplus u-q}. \tag{40}$$

The inclusion of this space in $\operatorname{Hom}(Q, \mathcal{O}_X(qE)[1]) \otimes \operatorname{Hom}(\mathcal{O}_X(qE), \mathcal{O}_X(uE))$ is described by Lemma 3.2.(b): a tuple $(\alpha_1, ..., \alpha_{u-q}) \in \operatorname{Hom}(Q, \mathcal{O}_X((q-1)E)[1])^{\oplus u-q}$ is mapped to

$$t\alpha_1 \otimes s^{u-q} + (-s\alpha_1 + t\alpha_2) \otimes s^{u-q-1}t + ... + (-s\alpha_{u-q-1} + t\alpha_{u-q}) \otimes st^{u-q-1} - s\alpha_{u-q} \otimes t^{u-q}.$$

We claim that the two subspaces

$$V_t := \Big\{t\alpha : \alpha \in \operatorname{Hom}(Q, \mathcal{O}_X((q-1)E)[1])\Big\} \subseteq \operatorname{Hom}\big(Q, \mathcal{O}_X(qE)[1]\big),$$
$$V_s := \Big\{s\alpha : \alpha \in \operatorname{Hom}(Q, \mathcal{O}_X((q-1)E)[1])\Big\} \subseteq \operatorname{Hom}\big(Q, \mathcal{O}_X(qE)[1]\big)$$

are of codimension $\geq k - \hom(\mathcal{O}_X((q-1)E), Q)$ in $\operatorname{Hom}\big(Q, \mathcal{O}_X(qE)[1]\big)$. Indeed, applying the functor $\operatorname{Hom}(Q, -)$ to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((q-1)E) \longrightarrow \mathcal{O}_X(qE) \longrightarrow \mathcal{O}_J \longrightarrow 0,$$

where $J \subseteq X$ is the elliptic curve corresponding to s or t, results in a long exact sequence

$$\cdots \longrightarrow \operatorname{Hom}\big(Q, \mathcal{O}_X((q-1)E)[1]\big) \longrightarrow \operatorname{Hom}\big(Q, \mathcal{O}_X(qE)[1]\big)$$
$$\xrightarrow{d} \operatorname{Hom}(Q, \mathcal{O}_J[1]) \longrightarrow \operatorname{Hom}\big(Q, \mathcal{O}_X((q-1)E)[2]\big) \longrightarrow \cdots$$

and therefore we have

$$\begin{aligned}\dim \operatorname{im}(d) &\geq \hom\big(Q, \mathcal{O}_J[1]\big) - \hom\big(Q, \mathcal{O}_X((q-1)E)[2]\big)\\ &\geq -\chi(Q, \mathcal{O}_J) - \hom\big(Q, \mathcal{O}_X((q-1)E)[2]\big) = k - \hom(\mathcal{O}_X((q-1)E), Q)\end{aligned}$$

(using that $\hom(Q, \mathcal{O}_J[i]) = 0$ for $i \neq 0, 1, 2$ since $Q, \mathcal{O}_J \in \operatorname{Coh}^b(X)$).

Since $m \leq k - \hom(\mathcal{O}_X((q-1)E), Q)$, a general subspace $V \subseteq \operatorname{Hom}(Q, \mathcal{O}_X(qE)[1])$ of dimension m has trivial intersection with both V_s and V_t. For such a V, we show that

$$\Big(V \otimes \operatorname{Hom}(\mathcal{O}_X(qE), \mathcal{O}_X(uE))\Big) \cap \operatorname{Ker}(\psi) = 0$$

(that is, (39) is injective). Let $(\alpha_1, ..., \alpha_{u-q}) \neq 0$ be an element in the intersection, so that

$$t\alpha_1 \in V, \quad -s\alpha_1 + t\alpha_2 \in V, \quad ... \quad , -s\alpha_{u-q-1} + t\alpha_{u-q} \in V, \text{ and } -s\alpha_{u-q} \in V. \tag{41}$$

Note that for an appropriate choice of $t \in H^0(\mathcal{O}_X(E))$, the following map is injective:

$$\gamma_t\colon \operatorname{Hom}(Q, \mathcal{O}_X((q-1)E)[1]) \longrightarrow \operatorname{Hom}(Q, \mathcal{O}_X(qE)[1]), \quad \alpha \mapsto t\alpha.$$

Indeed, it follows from the vanishing $\mathrm{Hom}(Q, \mathcal{O}_X(qE)) = 0$ that $\ker(\gamma_t) = \mathrm{Hom}(Q, \mathcal{O}_{J(t)})$, where $J(t) \in |E|$ is the elliptic curve corresponding to t. Thus injectivity of γ_t for a general $t \in H^0(\mathcal{O}_X(E))$ is just a consequence of (b). The first condition in (41) yields $t\alpha_1 \in V \cap V_t = 0$, hence $\alpha_1 = 0$ from the injectivity of γ_t. Inductively, the i-th condition in (41) for $i \leq u - q$ implies that $\alpha_i = 0$, which is a contradiction. □

Proof of Theorem 5.8. It suffices to prove the statement for σ_{0,w_0} lying in the Gieseker chamber (if $r_0 = 0$) or just below the horizontal line through $\Pi_\epsilon(v)$ (if $r_0 \neq 0$).

We start with the Mukai vector

$$v_2 := v - m_1 \cdot v(\mathcal{O}_X((e+1)E)) - m_2 \cdot v(\mathcal{O}_X(eE))$$

and the stability condition σ_e^+, which lies just above the line segment connecting $\Pi_\epsilon(v_2)$ to $\Pi_\epsilon(\mathcal{O}_X(eE))$. Note that $v_2^2 \geq -2$ is equivalent to our assumption (38), in particular $\mathcal{M}_{\sigma_e^+}(v_2) \neq \emptyset$. We first check that a general $F_2 \in \mathcal{M}_{\sigma_e^+}(v_2)$ satisfies

$$\hom(\mathcal{O}_X(uE), F_2) = 0 = \hom(F_2, \mathcal{O}_X(uE)), \text{ for all } u \geq 0. \tag{42}$$

Indeed, the vanishings $\hom(\mathcal{O}_X(uE), F_2) = 0$ for $u > e$ and $\hom(F_2, \mathcal{O}_X(uE)) = 0$ for $0 \leq u \leq e$ follow from σ_e^+-stability of F_2. Moreover, when moving (along the vertical line $b = 0$) from σ_e^+ towards the horizontal line passing through $\Pi_\epsilon(v_2)$, Lemma 4.4 implies that we only encounter actual walls of type (b_1) created by $\mathcal{O}_X(uE)$ for $u > e$. Similarly, when moving from σ_e^+ towards the wall $\mathcal{W}(F_2, \mathcal{O}_X)$ we only encounter walls of type (b_1) created by $\mathcal{O}_X(uE)$ for $0 \leq u \leq e$. However, we have

$$\langle v_2, \mathcal{O}_X(uE) \rangle = uE \cdot H - 2r_0 - s_0 + 2m_1 + 2m_2 > 0, \qquad \text{for all } u \geq 0. \tag{43}$$

Thus Lemma 4.7 (and the finiteness of the number of walls for the class v_2) implies (42) for a general $F_2 \in \mathcal{M}_{\sigma_e^+}(v_2)$. In particular, F_2 is σ-stable, where σ lies on the numerical wall $\mathcal{W}(F_2, \mathcal{O}_X)$ or just below the horizontal line passing through $\Pi_\epsilon(v_2)$.

Furthermore, the vanishing $\mathrm{Hom}(F_2, \mathcal{O}_J) = 0$ holds for a general curve $J \in |E|$. Indeed, consider the value $w_0 > 0$ for which $\nu_{0,w_0}(F_2) = 0$. Then F_2 is σ_{0,w_0}-semistable and $\mathcal{O}_J$ is σ_{0,w_0}-stable for any $J \in |E|$ (by Corollary 3.12). If $\mathrm{Hom}(F_2, \mathcal{O}_J) \neq 0$, then $\mathcal{O}_J$ is a stable quotient of Q since $\nu_{0,w_0}(F_2) = 0 = \nu_{0,w_0}(\mathcal{O}_J)$. Hence there can be only finitely many such $\mathcal{O}_J$'s.

We claim that a general $F_2 \in \mathcal{M}_{\sigma_e^+}(v_2)$ satisfies

$$0 < m_2 \leq k - \hom(\mathcal{O}_X((e-1)E), F_2). \tag{44}$$

We first prove the final statement assuming this claim and later return to justify it.

Assuming (44) holds, we apply Proposition 5.9 with F_2 in place of Q and e substituted for q. This shows that a general extension

$$0 \longrightarrow \mathcal{O}_X(eE)^{\oplus m_2} \longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0$$

satisfies $\hom(F_1, \mathcal{O}_X(uE)) = 0$ for all $u > e$. Then F_1 is σ_e^+-stable and does not get destabilized along any wall $\mathcal{W}(\mathcal{O}_X(uE), F_1)$ $(u \geq e+1)$ because $\hom(F_1, \mathcal{O}_X(uE)) = 0$. One can argue as before to check the vanishing $\mathrm{Hom}(F_1, \mathcal{O}_J) = 0$ for a general $J \in |E|$. Thus by applying Proposition 5.9 again, we find that a general extension

$$0 \longrightarrow \mathcal{O}_X((e+1)E)^{\oplus m_1} \longrightarrow F \longrightarrow F_1 \longrightarrow 0$$

satisfies $\hom(F, \mathcal{O}_X(uE)) = 0$ for all $u > e+1$, provided $m_1 \leq k - \hom(\mathcal{O}_X(eE), F_1) = k - m_2$. This clearly holds under Assumption (37) as $r_0 \leq 0$.

Clearly F is stable with respect to stability conditions just above $\mathcal{W}(\mathcal{O}_X((e+1)E), F_1)$. By Lemma 4.4, all negative-slope walls for F are of type (b_1). Since $\hom(F, \mathcal{O}_X(uE)) = 0$ for all $u > e+1$, Proposition 5.9 ensures that F is σ-stable, where: σ lies just below the horizontal line passing through $\Pi_\epsilon(v)$ (if $r_0 \neq 0$), or σ lies in the Gieseker chamber (if $r_0 = 0$). Moreover, by construction, it is of stability type $\overline{e}$. Thus, $\mathcal{M}_\sigma(v, \overline{e}) \neq \emptyset$ as required. Hence, it only remains to prove condition (44) for a general $F_2 \in \mathcal{M}_{\sigma_e^+}(v_2)$.

If $e \geq 1$, then (42) implies that $\hom(\mathcal{O}_X((e-1)E), F_2) = 0$ for a general F_2, and so condition (44) clearly holds. Thus, we may assume $e = 0$. In order to prove (44), we want to control the quantity $\hom(\mathcal{O}_X(-E), F_2) = \hom(\mathcal{O}_X, F_2(E))$. To handle this case, we first prove that F_2 is stable in the Gieseker chamber of v_2. Then we twist F_2 by $\mathcal{O}_X(E)$, and move down to the wall that $\mathcal{O}_X$ makes for $F_2(E)$.

We already know that a general $F_2 \in \mathcal{M}_{\sigma_e^+}(v_2)$ is at least semistable along the horizontal line passing through $\Pi_\epsilon(v_2)$. This horizontal line, as well as all the walls for the class v_2 above it up to the Gieseker chamber, are of type (b_2) by Lemma 4.4. Hence if F_2 is not stable in the Gieseker chamber, then it has a destabilizing factor of class $(0, (q-a_0)E, \theta)$ with $q - a_0 > 0$ and $\theta \geq 0$. Since $k > 2$ implies $-E \cdot H + 2 < 0$, we have the inequality

$$-\big\langle v_2, (0, (q-a_0)E, \theta)\big\rangle + 2(q-a_0) = (q-a_0)(-E\cdot H + 2) - \theta(m_1 + m_2 - r_0) < 0.$$

By Lemma 4.7, this shows that a general $F_2 \in \mathcal{M}_{\sigma_e^+}(v_2)$ is stable in the Gieseker chamber.

By applying Theorem 3.10 and Lemma 4.5, the object $F_2(E) \in \mathrm{Coh}^0(X)$ is also stable in the Gieseker chamber of $v_2(E) := v\big(F_2(E)\big)$[6]. We move down again along the vertical line $b = 0$, and investigate walls for the Mukai vector $v_2(E)$. We distinguish two cases:

Case 1. First assume $\mathrm{ch}_2(F_2(E)) \leq 0$. Since $-\big\langle v_2(E),\ (0, (q-a_0)E, \theta)\big\rangle + 2(q-a_0) < 0$ for $q - a_0 > 0$ and $\theta \geq 0$, and also

$$\langle v_2(E), v(\mathcal{O}_X(mE))\rangle = mk + 2(m_1 + m_2 - r_0) - \mathrm{ch}_2\big(F_2(E)\big) > 0$$

for all $m \geq 0$, then the same argument as above shows that $F_2(E)$ is stable along the numerical wall $\mathcal{W}(F_2(E), \mathcal{O}_X)$. In particular $\hom(\mathcal{O}_X, F_2(E)) = 0$, and thus condition (44) is satisfied.

Case 2. Now assume $\mathrm{ch}_2\big(F_2(E)\big) > 0$. Then the walls for the Mukai vector $v_2(E)$ that we encounter when moving down to the origin can be either of type (b_1) for $m < 0$, or of type (b_2). But the latter does not destabilize a general $F_2(E) \in \mathcal{M}_{H_\epsilon}\big(v_2(E)\big)$ as in Case 1; hence we only need to consider walls made by $\mathcal{O}_X(mE)[1]$ for $m < 0$. Again, we need to distinguish two different subcases:

Case 2.1. First assume $\big\langle v_2(E), v(\mathcal{O}_X(-E))\big\rangle \leq 0$. Then $\big\langle v_2(E), v(\mathcal{O}_X(mE))\big\rangle \leq 0$ for all $m \leq -1$, and hence a general $F_2(E) \in \mathcal{M}_{H_\epsilon}(v_2(E))$ is $\sigma_{0,w}$-stable for all $w > 0$ by Lemma 4.7. If $\chi(\mathcal{O}_X, F_2(E)) \leq 0$, then $\hom(\mathcal{O}_X, F_2(E)) = 0$ for a general F_2 and

[6] Note that twist gives an isomorphism of moduli spaces in the Gieseker chamber, so $F_2(E)$ is also a general object in the moduli space $\mathcal{M}_{H_\epsilon}(v_2(E))$.

(44) is clearly satisfied, so we may assume $\chi(\mathcal{O}_X, F_2(E)) > 0$. Let ℓ be the line passing through $\Pi_\epsilon(F_2(E))$ and the origin $(0,0)$, and define

$$\ell_1 := \ell \cap \big\{(b,w) \in U_\epsilon : w > 0\big\}, \quad \ell_2 := \ell \cap \big\{(b,w) \in U_\epsilon : w < 0\big\}.$$

Here, ℓ_1 represents the numerical wall created by $\mathcal{O}_X[1]$ for the class $v_2 \otimes \mathcal{O}_X(E)$, while ℓ_2 corresponds to the numerical wall created by $\mathcal{O}_X$.

Since $\chi(\mathcal{O}_X, F_2(E)) > 0$, Lemma 4.7 implies that a general F_2 remains stable along ℓ_1 with respect to $\mathcal{O}_X[1]$, and thus $\hom(F_2(E), \mathcal{O}_X[1]) = 0$. Consequently, we have

$$\hom(\mathcal{O}_X, F_2(E)) = \chi(\mathcal{O}_X, F_2(E)),$$

where we use the fact that $\mathrm{Hom}(F_2(E), \mathcal{O}_X) = 0$. Finally, we compute

$$\begin{aligned} k - \hom(\mathcal{O}_X, F_2(E)) &= k - \chi(\mathcal{O}_X, F_2(E)) \\ &= -s_0 - 2r_0 + 2(m_1 + m_2) \geq m_2, \end{aligned}$$

which establishes the inequality (44).

Case 2.2. Now assume $\big\langle v_2(E), v(\mathcal{O}_X(-E))\big\rangle > 0$. Recall that $\mathrm{ch}_2\big(F_2(E)\big) > 0$ by our assumption, hence for all $m \leq -2$ we have

$$\begin{aligned} \big\langle v_2(E), v(\mathcal{O}_X(mE))\big\rangle &= mk - 2r_0 + 2m_1 + 2m_2 - \mathrm{ch}_2\big(F_2(E)\big) \\ &< \; -2k - 2r_0 + 2(m_1 + m_2) \overset{(37)}{\leq} 0. \end{aligned} \tag{45}$$

It follows that a general $F_2(E) \in \mathcal{M}_{H_\epsilon}(v_2(E))$ remains stable up to the wall that $\mathcal{O}_X(-E)[1]$ is making, and then gets destabilized via a short exact sequence

$$F' \longrightarrow F_2(E) \longrightarrow \mathcal{O}_X(-E)^{\oplus h}[1] \tag{46}$$

in $\mathrm{Coh}^0(X)$, where $h = \chi\big(F_2(E), \mathcal{O}_X(-E)[1]\big)$ and the object F' is $\sigma_{b',w'}$-stable for (b', w') in the wall $\mathcal{W}\big(F_2(E), \mathcal{O}_X(-E)[1]\big)$.

Conversely, given an arbitrary $\sigma_{b',w'}$-stable object G' of Mukai vector $v(F')$ and an element $V \in \mathrm{Gr}\big(h, \mathrm{Hom}(\mathcal{O}_X(-E), G'\big)$, the object G sitting in an extension

$$G' \longrightarrow G \longrightarrow \mathcal{O}_X(-E)^{\oplus h}[1]$$

is stable just above $\mathcal{W}(F_2(E), \mathcal{O}_X(-E)[1])$ by Lemma 3.15. For this reason, we may assume that the object F' appearing in (46) is a general $\sigma_{b',w'}$-stable object. Since

$$\big\langle v(F'), v(\mathcal{O}_X)\big\rangle = \big\langle v_2(E), v(\mathcal{O}_X(-2E))[1]\big\rangle \overset{(45)}{\geq} 0,$$

we have $\hom(\mathcal{O}_X, F') = 0$ by the generality of F'. Since also $\hom\big(\mathcal{O}_X, \mathcal{O}_X(-E)[1]\big) = 0$, then $\hom(\mathcal{O}_X, F_2(E)) = 0$, hence (44) holds and the proof is complete.

□

6. Application to Brill-Noether loci on curves

Fix an elliptic $K3$ surface X of degree $k \geq 2$ as described in Proposition 3.1. In this section, we apply the results of Section 5 to study the Brill–Noether loci

$$V_d^r(C) := W_d^r(C) \setminus W_d^{r+1}(C) = \left\{L \in \operatorname{Pic}^d(C) : h^0(C, L) = r+1\right\}$$

for smooth curves $C \in |H|$ and $d, r \in \mathbb{N}$ with $d \leq g-1$.

To this end, consider the moduli space $\mathcal{M}_{H_\epsilon}(v)$ with Mukai vector $v = (0, H, 1+d-g)$, where $\epsilon > 0$ is sufficiently small, and define the locus

$$V^r_{H_\epsilon}(v) := \left\{F \in \mathcal{M}_{H_\epsilon}(v) : h^0(X, F) = r+1\right\}.$$

For an integral curve $C \in |H|$, the locus $V_d^r(C)$ is identified with the fiber over $[C] \in |H|$ of the natural support map

$$V^r_{H_\epsilon}(v) \hookrightarrow \mathcal{M}_{H_\epsilon}(v) \longrightarrow |H|.$$

This identification is independent of ϵ. Through our analysis of $V^r_{H_\epsilon}(v)$, we derive strong implications for the Brill–Noether theory of curves in $|H|$.

The following is the main result of this section, building upon Section 5:

Theorem 6.1. *Assume $d \leq g-1$. Then for $v := (0, H, 1+d-g)$ we have:*

(a) There exists $\epsilon(v,r) > 0$ such that for $0 < \epsilon < \epsilon(v,r)$, one has

$$V^r_{H_\epsilon}(v) \subseteq \bigcup_{\overline{e} \in I} \mathcal{M}_{H_\epsilon}(v, \overline{e}),$$

where I is the finite set of stability types $\overline{e} = \big((e_i, m_i)\big)_{i=1}^p$ with $p \geq 1$, such that $\sum_{i=1}^p m_i \leq r+1 \leq \sum_{i=1}^p m_i(e_i+1)$ and $m_1(e_1+1) \leq r+1$.

(b) For each $\overline{e} = \big((e_i, m_i)\big)_{i=1}^p$ in I, define $\ell_{\overline{e}} := r+1-\sum_{i=1}^p m_i$. Then:

(i) If $\mathcal{M}_{H_\epsilon}(v, \overline{e}) \neq \emptyset$, then $\rho(g, r-\ell_{\overline{e}}, d) - \ell_{\overline{e}} k \geq 0$. Moreover, $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is irreducible, smooth, and of dimension at most $g + \rho(g, r-\ell_{\overline{e}}, d) - \ell_{\overline{e}} k$.

(ii) If $\overline{e}$ is balanced, $\rho(g, r-\ell_{\overline{e}}, d) - \ell_{\overline{e}} k \geq 0$ and $r+1-k \leq \ell_{\overline{e}}$, then $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is non-empty of dimension $g + \rho(g, r-\ell_{\overline{e}}, d) - \ell_{\overline{e}} k$.

Proof. We first assume that $d < g-1$; the case $d = g-1$ will be treated separately in Subsection 6.1. In this case, assertion (a) is a consequence of Theorem 5.5, applied to a stability condition σ_{0,w_0} lying in the Gieseker chamber for the Mukai vector v.

We now prove (b). Fix $\overline{e} = \big((e_i, m_i)\big)_{i=1}^p$ in I, such that $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is non-empty. According to Remark 5.2, the Mukai vector

$$\big(-m_1 - \cdots - m_p,\ H - (m_1 e_1 + \cdots + m_p e_p)E,\ 1+d-g-m_1-\cdots-m_p\big)$$

has square ≥ -2, which reads as

$$2g - 2(m_1 e_1 + \cdots + m_p e_p)k - 2(m_1 + \cdots + m_p)(g-d-1+m_1+\cdots+m_p) \geq 0.$$

It follows that

$$\begin{aligned}0 &\le g - (m_1e_1 + \cdots + m_pe_p)k - (m_1 + \cdots + m_p)(g - d - 1 + m_1 + \cdots + m_p) \le \\ &\le g - \big(r + 1 - (m_1 + \cdots + m_p)\big)k - (m_1 + \cdots + m_p)(g - d - 1 + m_1 + \cdots + m_p) = \\ &= g - \ell_{\overline{e}}k - (r + 1 - \ell_{\overline{e}})(g - d + r - \ell_{\overline{e}}) = \rho(g, r - \ell_{\overline{e}}, d) - \ell_{\overline{e}}k,\end{aligned}$$

where the second inequality follows from the condition $r + 1 \le \sum_{i=1}^{p} m_i(e_i + 1)$. Hence the inequality $\rho(g, r - \ell_{\overline{e}}, d) - \ell_{\overline{e}}k \ge 0$ holds. Furthermore, by Theorem 5.3, $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is irreducible, smooth and quasi-projective of dimension

$$\begin{aligned}&\left(v - \sum_{i=1}^{p} m_i(1, e_iE, 1)\right)^2 + 2 + \sum_{j=1}^{p} m_j \left(\langle v - \sum_{i=1}^{j} m_i(1, e_iE, 1), (1, e_jE, 1)\rangle - m_j\right) = \\ &\qquad = 2g - (m_1e_1 + \cdots + m_pe_p)k - (m_1 + ... + m_p)(g - d - 1) - (m_1 + \cdots + m_p)^2.\end{aligned}$$

Now using the inequality $r + 1 \le \sum_{i=1}^{p} m_i(e_i + 1)$, this dimension is at most

$$\begin{aligned}&g + g - (r + 1 - (m_1 + \cdots + m_p))k - (m_1 + \cdots + m_p)(g - d + m_1 + \cdots + m_p - 1) = \\ &\qquad = g + g - (r - \ell_{\overline{e}} + 1)(g - d + r - \ell_{\overline{e}}) - \ell_{\overline{e}}k = g + \rho(g, r - \ell_{\overline{e}}, d) - \ell_{\overline{e}}k,\end{aligned}$$

which proves the first part of (b). Finally, the second part is an immediate application of Theorem 5.8 for σ_{0,w_0} in the Gieseker chamber of v. □

As a direct consequence, we obtain the following result:

Corollary 6.2. *For $d \le g - 1$, set $\rho_k(g, r, d) := \max_{\ell=0,\dots,r}\{\rho(g, r - \ell, d) - \ell k\}$. Then:*

(a) If $\rho_k(g, r, d) < 0$, then $W^r_d(C) = \varnothing$ for every integral curve $C \in |H|$.
(b) If $C \in |H|$ is a general curve, then $\dim W^r_d(C) \le \rho_k(g, r, d)$.

Proof. Note that $\ell_{\overline{e}} \in \{0, ..., r\}$ for every stability type $\overline{e} \in I$, since $\sum_{i=1}^{p} m_i \le r + 1$. If $\rho_k(g, r, d) < 0$, then it follows from Theorem 6.1.(b) that $\mathcal{M}_{H_\epsilon}(v, \overline{e}) = \varnothing$ for all $\overline{e} \in I$. Therefore $V^r_{H_\epsilon}(v) = \varnothing$ by Theorem 6.1.(a), which implies $V^r_d(C) = \varnothing$ for every integral curve $C \in |H|$. Since $\rho_k(g, r, d)$ is decreasing as a function of r, we also have $V^{r'}_d(C) = \varnothing$ for all $r' \ge r$, namely $W^r_d(C) = \emptyset$ which proves (a).

On the other hand, according to Theorem 6.1 we have $\dim V^r_{H_\epsilon}(v) \le g + \rho_k(g, r, d)$, and hence by considering the support map

$$V^r_{H_\epsilon}(C) \hookrightarrow \mathcal{M}_{H_\epsilon}(v) \longrightarrow |H|$$

we deduce $\dim V^r_d(C) \le \rho_k(g, r, d)$ for a general $C \in |H|$. Since $\rho_k(g, r, d)$ is decreasing in r, it follows that $\dim W^r_d(C) \le \rho_k(g, r, d)$ as well, which proves (b). □

Remark 6.3. Take $a \in \mathbb{Z}_{\ge 0}$ such that $g' := g - ak \ge 1$. Thanks to the results of Section 5, Theorem 6.1 can be similarly proven for the Mukai vector $v = (0, H - aE, 1 + d - g')$ if $d \le g' - 1$. As a consequence, Corollary 6.2 is also valid for the genus $g' = g - ak$ curves in the linear system $|H - aE|$. Whereas here we take $a = 0$ to simplify the statements, this more general version will be required in Section 7.

Example 6.4. Set $v := (0, H, 1+d-g)$ for $d \leq g-1$ and $r \geq 0$, and let $\ell \in \mathbb{Z}$ satisfy $\max\{0, r+1-k\} \leq \ell \leq r$. Consider the integers $e, m_1 \geq 0$ and $m_2 > 0$ satisfying

$$r+1 = m_1(e+2) + m_2(e+1), \quad r+1-\ell = m_1 + m_2.$$

If $\rho(g, r-\ell, d) - \ell k \geq 0$, then Theorem 6.1.(b) asserts the non-emptiness of the moduli space $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ for the balanced stability type $\overline{e} := \big((e+1, m_1), (e, m_2)\big)$. We spell out in the geometric language of kernel vector bundles the meaning of a sheaf $L \in \mathcal{M}_{H_\epsilon}(v)$ having stability type $\overline{e}$.

Since $\operatorname{Ext}^1\big(\mathcal{O}_X(eE), \mathcal{O}_X((e+1)E)\big) = 0$, any $L \in \mathcal{M}_{H_\epsilon}(v, \overline{e})$ fits in an exact sequence

$$0 \longrightarrow \mathcal{O}_X((e+1)E)^{\oplus m_1} \oplus \mathcal{O}_X(eE)^{\oplus m_2} \longrightarrow L \longrightarrow F_2 \longrightarrow 0$$

in $\operatorname{Coh}^b(X)$ $(-1 \ll b < 0)$, where F_2 is stable along the numerical wall $\mathcal{W}(F_2, \mathcal{O}_X)$. More precisely, F_2 is a complex with two cohomologies: $\mathcal{H}^{-1}(F_2)$ (resp. $\mathcal{H}^0(F_2)$) arises as the kernel (resp. the cokernel) of the evaluation map

$$H^0\big(L(-(e+1)E)\big) \otimes \mathcal{O}_X((e+1)E) \bigoplus \frac{H^0(L(-eE))}{H^0(\mathcal{O}_X(E)) \otimes H^0(L(-(e+1)E))} \otimes \mathcal{O}_X(eE) \xrightarrow{\text{ev}} L. \quad (47)$$

If $L \in \mathcal{M}_{H_\epsilon}(v, \overline{e})$ is general, then by Theorem 3.10.(b) we can write

$$F_2 = R\mathcal{H}om(G, \mathcal{O}_X)[1],$$

where G is general in the moduli space $\mathcal{M}_{H_\epsilon}\big(r-\ell+1, H-\ell E, g-d+r-\ell\big)$. If moreover $\ell < r$, then such a general G is locally free: this implies $\mathcal{H}^0(F_2) = 0$ (in other words, the evaluation (47) is surjective). When $\ell = 0$ (so that $m_1 = e = 0$ and $m_2 = r+1$) and L is supported on a smooth curve, G is simply the Lazarsfeld-Mukai bundle [Laz86] of L.

6.1. **The case of degree** $g-1$**.** Throughout this subsection we fix $a \in \mathbb{Z}_{\geq 0}$ such that $g' := g - ak \geq 1$. We now introduce a notion of stability type for stable objects of Mukai vector $v = (0, H-aE, 0)$. This case requires an *ad hoc* treatment (it was skipped in Definition 5.1), as all the numerical walls $\mathcal{W}(v, \mathcal{O}_X(eE))$, where $e \geq 0$, coalesce into the horizontal line $w = 0$[7]. Nevertheless, we find analogues of the results in Section 5 for this modified definition; using these analogues, the proof of Theorem 6.1 for $d < g-1$ extends naturally to the case $d = g-1$.

Definition 6.5. A stable object $F_0 \in \mathcal{M}^{\text{st}}_{H_\epsilon}(0, H-aE, 0)$ is said to be *of stability type* $\overline{e} = \big((e_i, m_i)\big)_{i=1}^p \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$, where $p \geq 0$ and $e_1 > e_2 > \cdots > e_p \geq 0$, if for all $b \in (-\delta_\epsilon, 0)$ there exist short exact sequences in $\operatorname{Coh}^b(X)$

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_X(e_1E), F_0) \otimes \mathcal{O}_X(e_1E) \xrightarrow{\text{ev}_1} F_0 \longrightarrow F_1 \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_X(e_2E), F_1) \otimes \mathcal{O}_X(e_2E) \xrightarrow{\text{ev}_2} F_1 \longrightarrow F_2 \longrightarrow 0,$$
$$\vdots$$
$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}_X(e_pE), F_{p-1}) \otimes \mathcal{O}_X(e_pE) \xrightarrow{\text{ev}_p} F_{p-1} \longrightarrow F_p \longrightarrow 0,$$

such that for all $i = 0, \ldots, p$:

[7] This also justifies the need to incorporate a small ball around the origin in the definition (21) of the region U_ϵ of Bridgeland stability conditions.

(a) The object F_i is $\sigma_{b,0}$-semistable.
(b) $\mathrm{Hom}(\mathcal{O}_X(-uE)[1], F_i) = 0$ for all $u \geq 1$, and $m_i := \hom(\mathcal{O}_X(e_iE), F_{i-1})$.
(c) If $i \neq 0$, then $\mathrm{Hom}(\mathcal{O}_X(tE), F_i) = 0$ for every $t \geq e_i$. Furthermore, the vanishing $\mathrm{Hom}(\mathcal{O}_X(tE), F_p) = 0$ holds for every $t \geq 0$.

Note that $\mathcal{M}_{H_\epsilon}(0, H-aE, 0)$ may contain strictly Gieseker-semistable sheaves for any $\epsilon > 0$; however, we restrict our attention to stable sheaves only. We begin by showing the following:

Lemma 6.6. *If $\epsilon > 0$ is small enough (depending on a and the list $\big((e_i, m_i)\big)_{i=1}^p$), then all quotients F_i in Definition 6.5 are $\sigma_{b,w}$-stable for every $b \in (-\delta, 0)$ and $0 < w \ll 1$.*

Proof. For $i = 0$ the statement is clear, as $\sigma_{b,w}$ lies in the Gieseker chamber of v by Proposition 4.2. For $i \geq 1$, assume $Q_1 \to F_i \to Q_2$ is a short exact sequence destabilizing F_i just above the horizontal wall $w = 0$; we may also assume that Q_1 is stable there. Write $\mathrm{ch}(Q_1) = (-r, tH - qE, 0)$. Since $\nu_{b,w}(Q_1) > \nu_{b,w}(F_i) > 0$ for $w \gg 0$, we have $r > 0$. Moreover, if we pick $\epsilon < \epsilon_{v(F_i)}$, then by Proposition 4.2, $t \in \{0, 1\}$.

If $t = 0$, then stability of Q_1 implies $Q_1 \cong \mathcal{O}_X(-uE)[1]$ for some $u \geq 1$, according to Lemma 3.13. This is not possible due to the vanishing $\hom(\mathcal{O}_X(-uE)[1], F_i) = 0$.

If $t = 1$, then $\mathrm{ch}(Q_2) = \mathrm{ch}(F_i) - \mathrm{ch}(Q_1) = \big(r - \sum_{j=1}^i m_j, (q - a - \sum_{j=1}^i m_j e_j)E,\ 0\big)$. Since we have surjections $F_0 \twoheadrightarrow F_i \twoheadrightarrow Q_2$ in $\mathrm{Coh}^b(X)$ and F_0 is $\sigma_{b,w}$-stable, we find $0 = \nu_{b,w}(F_0) < \nu_{b,w}(Q_2)$ for all $w > 0$. This implies $\mathrm{ch}_0(Q_2) < 0$, i.e. $0 < r < \sum_{j=1}^i m_j$. But then if we choose ϵ small enough, we get

$$\frac{-\mathrm{ch}_0(F_i)}{H_\epsilon \cdot \mathrm{ch}_1(F_i)} = \frac{\sum m_j}{(H - (a + \sum m_j e_j)E) \cdot H_\epsilon} < \frac{\sum m_j - r}{(q - a - \sum m_j e_j)E \cdot H_\epsilon} = \frac{-\mathrm{ch}_0(Q_2)}{H_\epsilon \cdot \mathrm{ch}_1(Q_2)}$$

which yields $\nu_{0,w}(F_i) < \nu_{0,w}(Q_2)$ for all $w \gg 0$, a contradiction. □

Lemma 6.6 is the key step in establishing the analogue of Theorem 5.3.

Theorem 6.7. *Fix $v = (0, H - aE, 0)$ and any stability type $\overline{e} = \big(e_i, m_i\big)_{i=1}^p$ where $p \geq 0$. If ϵ is sufficiently small, the subset*

$$\mathcal{M}_{H_\epsilon}(v, \overline{e}) := \big\{F_0 \in \mathcal{M}_{H_\epsilon}(v) :\ F_0 \text{ is of stability type } \overline{e}\big\}$$

admits a natural scheme structure as a locally closed subscheme of $\mathcal{M}_{H_\epsilon}(v)$. Moreover, if $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is non-empty, then it is smooth and irreducible of dimension

$$\left(v - \sum_{i=1}^p m_i(1, e_iE, 1)\right)^2 + 2 + \sum_{j=1}^p m_j \left(\Big\langle v - \sum_{i=1}^j m_i(1, e_iE, 1), (1, e_jE, 1)\Big\rangle - m_j\right).$$

More precisely, set $v_p := v - \sum_{i=1}^p m_i(1, e_iE, 1)$ and $\sigma := \sigma_{b,w}$ for $b \in (-\delta_\epsilon, 0)$, $0 < w \ll 1$. If $U \subseteq \mathcal{M}_\sigma(v_p)$ is the (possibly empty) open subset of objects F_p with $\hom(\mathcal{O}_X(-E)[1], F_p) = 0$ and $\hom(\mathcal{O}_X, F_p) = 0$, then $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is an open subset of an iterated Grassmannian bundle over U.

Our next result is an analogue of Theorem 5.5.

Theorem 6.8. *Let $v = (0, H - aE, 0)$. There exists $\epsilon(a) > 0$ such that, if $0 < \epsilon < \epsilon(a)$, then every $F_0 \in \mathcal{M}_{H_\epsilon}(v)$ has a stability type $\overline{e} = \big((e_i, m_i)\big)_{i=1}^p$. If $h^0(X, F_0) = r + 1 \geq 1$, then moreover $p \geq 1$ and the following inequalities hold:*

$$\sum_{i=1}^{p} m_i \leq r + 1 \leq \sum_{i=1}^{p} m_i(e_i + 1), \quad m_1(e_1 + 1) \leq r + 1.$$

The key step to prove Theorem 6.8 is the following injectivity result:

Proposition 6.9. *Let $b \in \mathbb{R}$ and $n \in \mathbb{N}$. If $Q \in \mathrm{Coh}^b(X)$ is an object satisfying $\mathrm{Hom}(\mathcal{O}_X(pE), Q) = 0$ for all $p > n$, then the map*

$$\begin{aligned} \varphi\colon \mathrm{Hom}(\mathcal{O}_X(nE), Q) \otimes \mathrm{Hom}(\mathcal{O}_X(mE), \mathcal{O}_X(nE)) &\longrightarrow \mathrm{Hom}(\mathcal{O}_X(mE), Q) \\ f \otimes g &\longmapsto f \circ g \end{aligned}$$

is injective for all $m \leq n$.

Proof. We may assume that Q is not of the form $\mathcal{O}_X(qE)$ for some $q \in \mathbb{Z}$; otherwise, $q \leq n$ and the claim is trivial. Consider the identification

$$\mathrm{Hom}(\mathcal{O}_X(mE), \mathcal{O}_X(nE)) \cong H^0(X, \mathcal{O}_X((n-m)E)) \cong \mathrm{Sym}^{n_0} H^0(X, \mathcal{O}_X(E))$$

provided by Lemma 3.2, where $n_0 := n - m$. Let s, t be a basis of $\mathrm{Hom}(\mathcal{O}_X(-E), \mathcal{O}_X)$, and let $J, F \in |E|$ denote the corresponding (disjoint) curves in the elliptic pencil. For any $i > 0$, by considering for $\ell \gg 0$ the exact triangles

$$\mathcal{O}_X(\ell E) \longrightarrow \mathcal{O}_{iJ} \longrightarrow \mathcal{O}_X((\ell - i)E)[1], \qquad \mathcal{O}_X(\ell E) \longrightarrow \mathcal{O}_{iF} \longrightarrow \mathcal{O}_X((\ell - i)E)[1]$$

the vanishings $\mathrm{Hom}\big(\mathcal{O}_X((\ell - i)E)[1], Q\big) = 0 = \mathrm{Hom}\big(\mathcal{O}_X(\ell E), Q\big)$ imply that

$$\mathrm{Hom}(\mathcal{O}_{iJ}, Q) = 0, \qquad \mathrm{Hom}(\mathcal{O}_{iJ}, Q) = 0. \tag{48}$$

Hence the maps

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_X((i-1)E), Q) &\xrightarrow{\circ s^i} \mathrm{Hom}(\mathcal{O}_X(-E), Q), \\ \mathrm{Hom}(\mathcal{O}_X((i-1)E), Q) &\xrightarrow{\circ t^i} \mathrm{Hom}(\mathcal{O}_X(-E), Q) \end{aligned} \tag{49}$$

are injective for any $i > 0$.

The compositions $s^{n_0}, s^{n_0-1}t, \ldots, st^{n_0-1}, t^{n_0}$ form a basis of $\mathrm{Hom}\big(\mathcal{O}_X(mE), \mathcal{O}_X(nE)\big)$. Assume for the sake of a contradiction that $\ker(\varphi) \neq 0$, namely we have a nonzero tensor

$$\delta_0 \otimes t^{n_0} + \delta_1 \otimes t^{n_0-1}s + \cdots + \delta_{n_0} \otimes s^{n_0} \quad (\delta_j \in \mathrm{Hom}(\mathcal{O}_X(nE), Q))$$

in the kernel. If $i \in \{0, \ldots, n_0 - 1\}$ is minimal such that $\delta_i \neq 0$, then we have

$$(\delta_i \circ t^{n_0-i} + \cdots + \delta_{n_0} \circ s^{n_0-i}) \circ s^i = 0$$

and hence $\delta_i \circ t^{n_0-i} + \cdots + \delta_{n_0} \circ s^{n_0-i} = 0$, by the injectivity of the maps in (49). Thus

$$\delta_i \circ t^{n_0-i} = (-\delta_{i+1} \circ t^{n_0-i-1} - \cdots - \delta_{n_0} \circ s^{n_0-i}) \circ s.$$

Note that since $\delta_i \neq 0$, we have $\delta_i \circ t^{n_0-i} \neq 0$ as well.

The goal is to show

$$\delta_i = \delta_i' \circ s \tag{50}$$

for some nonzero $\delta_i' \in \mathrm{Hom}(\mathcal{O}_X((n+1)E), Q)$, which contradicts our assumption. Consider the two identical morphisms

$$\mathcal{O}_X((m+i)E) \xrightarrow{t^{n_0-i}} \mathcal{O}_X(nE) \xrightarrow{\delta_i} Q$$

$$\mathcal{O}_X((m+i)E) \xrightarrow{s} \mathcal{O}_X((m+i+1)E) \xrightarrow{-\delta_{i+1}\circ t^{n_0-i-1}-\cdots-\delta_{n_0}\circ s^{n_0-i}} Q$$

By the octahedral axiom, we have distinguished triangles

$$\mathcal{O}_{(n_0-i)F} \longrightarrow \mathrm{cone}(\delta_i \circ t^{n_0-i}) \longrightarrow \mathrm{cone}(\delta_i)$$

$$\mathcal{O}_J \longrightarrow \mathrm{cone}(\delta_i \circ t^{n_0-i}) \longrightarrow \mathrm{cone}(-\delta_{i+1}\circ t^{n_0-i-1} - \cdots - \delta_{n_0} \circ s^{n_0-i}).$$

Note that none of the above cones is zero, as we initially assumed that $Q \neq \mathcal{O}_X(qE)$ for all $q \in \mathbb{Z}$. Since $\mathrm{Hom}(\mathcal{O}_J, \mathcal{O}_{(n_0-i)F}) = 0$ (they are sheaves with disjoint supports), we obtain a nonzero map $\psi\colon \mathcal{O}_G \longrightarrow \mathrm{cone}(\delta_i \circ t^{n_0-i}) \longrightarrow \mathrm{cone}(\delta_i)$. Finally, consider the following diagram whose rows are distinguished triangles:

$$\begin{array}{ccccc}
\mathcal{O}_J[-1] & \longrightarrow & \mathcal{O}_X(nE) & \xrightarrow{s} & \mathcal{O}_X((n+1)E) \\
\downarrow \psi & & \downarrow = & & \\
\mathrm{cone}(\delta_i)[-1] & \longrightarrow & \mathcal{O}_X(nE) & \xrightarrow{\delta_i} & Q
\end{array}$$

Note that the square on the left-hand side commutes (up to constant). Indeed:

- $\mathrm{Hom}\big(\mathcal{O}_J[-1], \mathcal{O}_X(nE)\big) \cong H^1(\mathcal{O}_J(-nE)) \cong H^1(J, \mathcal{O}_J) \cong \mathbb{C}$
- The map $\mathcal{O}_J[-1] \to \mathcal{O}_X(nE)$ in the first row is nonzero.
- The composition $\mathcal{O}_J[-1] \to \mathrm{cone}(\delta_i)[-1] \to \mathcal{O}_X(nE)$ is nonzero. Otherwise, we would obtain a nonzero map $\mathcal{O}_J \to Q$, which is not possible by (48).

Therefore, by the axiom TR3 of triangulated categories, we have $\delta_i = \delta_i' \circ s$ for some $\delta_i' \in \mathrm{Hom}(\mathcal{O}_X((n+1)E), Q)$ as required in (50). This concludes the proof. □

Proof of Theorem 6.8. We only prove the existence of a stability type for F_0; the second statement (constraints on the stability type when $h^0(F_0) = r+1$) can be argued *mutatis mutandis* as in the proof of Theorem 5.5.

We construct the F_i's inductively. Assuming $\epsilon < \epsilon_v$, Proposition 4.2 guarantees that there is no actual wall for F_0 passing through the positive part of the vertical line $b = 0$. Hence F_0 is $\sigma_{b,0}$-semistable and $\nu_{b,0}(F_0) = \nu_{b,0}(\mathcal{O}_X)$ for $b \in (-\delta, 0)$. Furthermore, we have $\mathrm{Hom}(\mathcal{O}_X(-uE)[1], F_0) = 0$ for all $u \geq 1$, since F_0 is a sheaf.

For the induction step, let $i \geq 0$ and assume we have constructed $\sigma_{b,0}$-semistable objects $F_0, \ldots, F_i$ and integers $e_1 > \cdots > e_i \geq 0$ satisfying properties (b) and (c). If $\mathrm{Hom}(\mathcal{O}_X(tE), F_i) = 0$ for all $t \geq 0$, then we set $p := i$ and finish the process (the stronger vanishing for F_p in (c) will be satisfied). Otherwise, we define

$$e_{i+1} := \max\big\{t \geq 0 : \mathrm{Hom}(\mathcal{O}_X(tE), F_i) \neq 0\big\}.$$

This maximum is indeed attained: $e_{i+1} < e_i$ if $i \geq 1$ thanks to (c), whereas for $i = 0$ we have $e_1 \leq h^0(F_0) - 1$. We claim that the evaluation map

$$\mathrm{Hom}(\mathcal{O}_X(e_{i+1}E), F_i) \otimes \mathcal{O}_X(e_{i+1}E) \xrightarrow{\mathrm{ev}_{i+1}} F_i$$

is injective in $\mathrm{Coh}^b(X)$ for $b \in (-\delta, 0)$. Indeed, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{O}_X(e_{i+1}E), F_i) \otimes \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(e_{i+1}E)) \otimes \mathcal{O}_X & \longrightarrow & \mathrm{Hom}(\mathcal{O}_X, F_i) \otimes \mathcal{O}_X \\ \downarrow \psi & & \downarrow \\ \mathrm{Hom}(\mathcal{O}_X(e_{i+1}E), F_i) \otimes \mathcal{O}_X(e_{i+1}E) & \xrightarrow{\mathrm{ev}_{i+1}} & F_i \end{array}$$

where the top horizontal arrow is injective in $\mathrm{Coh}^b(X)$ (by Proposition 6.9), and the two vertical arrows are injective as well (since $\mathcal{O}_X$ is $\sigma_{b,0}$-stable by Lemma 3.14). Then

$$\ker(\mathrm{ev}_{i+1}) \subseteq \mathrm{coker}(\psi) = \mathrm{Hom}\big(\mathcal{O}_X(e_{i+1}E), F_i\big) \otimes \mathcal{O}_X(-E)^{\oplus e_{i+1}}[1],$$

and thus $\ker(\mathrm{ev}_{i+1}) = \mathcal{O}_X(-E)^{\oplus t}[1]$ for some $t \geq 0$, as $\mathcal{O}_X(-E)[1]$ is $\sigma_{b,0}$-stable by Lemma 3.14. Since $\hom\big(\mathcal{O}_X(-E)[1], \mathcal{O}_X(e_{i+1}E)\big) = 0$, it follows that $\ker(\mathrm{ev}_{i+1}) = 0$.

We define $F_{i+1} := \mathrm{coker}(\mathrm{ev}_{i+1})$. Clearly F_{i+1} is $\sigma_{b,0}$-semistable, as it is the quotient of two $\sigma_{b,0}$-semistable objects of the same slope. This proves (a). Moreover, since $\hom(\mathcal{O}_X(-uE)[1], F_i) = 0$ for any $u \geq 1$, we know that $\mathrm{Hom}(\mathcal{O}_X(-uE)[1], F_{i+1})$ equals the kernel of the natural map

$$\mathrm{Hom}(\mathcal{O}_X(e_{i+1}E), F_i) \otimes \mathrm{Hom}(\mathcal{O}_X(-uE), \mathcal{O}_X(e_{i+1}E)) \longrightarrow \mathrm{Hom}(\mathcal{O}_X(-uE), F_i).$$

Therefore $\mathrm{Hom}(\mathcal{O}_X(-uE)[1], F_{i+1}) = 0$ by Proposition 6.9, which proves (b). Finally, by construction $\mathrm{Hom}(\mathcal{O}_X(e_{i+1}E), F_{i+1}) = 0$, which in virtue of Lemma 6.10 implies $\mathrm{Hom}(\mathcal{O}_X(tE), F_{i+1}) = 0$ for all $t \geq e_{i+1}$. This proves (c) and concludes the proof. □

Lemma 6.10. *Let $b \in (-\delta, 0)$, and let $Q \in \mathrm{Coh}^b(X)$ satisfy $\hom(\mathcal{O}_X(-E)[1], Q) = 0$ and $\hom(\mathcal{O}_X(tE), Q) = 0$ for some $t \geq 0$. Then $\hom(\mathcal{O}_X((t+1)E), Q) = 0$.*

Proof. Suppose by contradiction that $\hom\big(\mathcal{O}_X((t+1)E), Q\big) \neq 0$. Then the exact triangle

$$\mathcal{O}_X(tE) \longrightarrow \mathcal{O}_X((t+1)E) \longrightarrow \mathcal{O}_J((t+1)E) \cong \mathcal{O}_J$$

for $J \in |E|$ implies that $\hom(\mathcal{O}_J, Q) \neq 0$. Therefore, from the exact triangle

$$\mathcal{O}_X(tE) \longrightarrow \mathcal{O}_J \longrightarrow \mathcal{O}_X((t-1)E)[1]$$

we have $\hom(\mathcal{O}_X((t-1)E)[1], Q) \neq 0$. This contradicts our hypothesis for $t = 0$, whereas for $t \geq 1$ we know that $\mathcal{O}_X((t-1)E) \in \mathrm{Coh}^b(X)$, hence $\hom(\mathcal{O}_X((t-1)E)[1], Q) = 0$ which, again, is a contradiction. □

Finally, we establish the following non-emptiness result, paralleling Theorem 5.8.

Theorem 6.11. *Let $v = (0, H - aE, 0)$ and consider $\overline{e} := \big((e+1, m_1), (e, m_2)\big)$ for $e, m_1 \geq 0$ and $m_2 > 0$ such that*

$$m_1 + m_2 < k, \text{ and}$$

$$\Big(v - m_1 \cdot v\big(\mathcal{O}_X((e+1)E)\big) - m_2 \cdot v\big(\mathcal{O}_X(eE)\big)\Big)^2 \geq -2.$$

Then there exists $\epsilon(a, \overline{e}) > 0$ such that, if $\epsilon < \epsilon(a, \overline{e})$, then $\mathcal{M}_{H_\epsilon}(v, \overline{e})$ is non-empty.

Proof. The proof goes along the same lines of Theorem 5.8; we explain which parts must be adapted. We fix the Mukai vector

$$v_2 := v - m_1 \cdot v\big(\mathcal{O}_X((e+1)E)\big) - m_2 \cdot v\big(\mathcal{O}_X(eE)\big)$$

and a stability condition $\sigma_+ := \sigma_{b,w}$ for fixed $b \in (-\delta_\epsilon, 0)$ and $0 < w \ll 1$. For every $F_2 \in \mathcal{M}_{\sigma_+}(v_2)$, comparison of σ_+-slopes immmediately yields $\mathrm{Hom}(F_2, \mathcal{O}_X(uE)) = 0$ for all $u \geq 0$, $\mathrm{Hom}(F_2, \mathcal{O}_J) = 0$ for every $J \in |E|$, and $\mathrm{Hom}(\mathcal{O}_X(-E)[1], F_2) = 0$.

Furthermore, for a general $F_2 \in \mathcal{M}_{\sigma_+}(v_2)$, we have $\mathrm{Hom}(\mathcal{O}_X, F_2) = 0$ by a dimension count similar to that of Lemma 4.7, together with the $\sigma_{b,0}$-stability of $\mathcal{O}_X$). Therefore, Lemma 6.10 implies that $\mathrm{Hom}(\mathcal{O}_X(uE), F_2) = 0$ for all $u \geq 1$. Also the short exact sequence in $\mathrm{Coh}^b(X)$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(-(u+1)E)^{\oplus u}[1] \longrightarrow \mathcal{O}_X(-uE)^{\oplus u+1}[1] \longrightarrow 0,$$

obtained by shifts and twists of (11), implies $\mathrm{Hom}\big(\mathcal{O}_X(-(u+1)E)[1], F_2\big) = 0$ for all $u \geq 1$.

It follows from Proposition 5.9 that, if

$$0 < m_2 < k - \hom(\mathcal{O}_X((e-1)E), F_2), \tag{51}$$

then for a general $V_2 \in \mathrm{Gr}\left(m_2, \mathrm{Ext}^1(F_2, \mathcal{O}_X(eE))\right)$ the extension

$$0 \longrightarrow V_2^\vee \otimes \mathcal{O}_X(eE) \longrightarrow F_1 \longrightarrow F_2 \longrightarrow 0$$

satisfies $\mathrm{Hom}(F_1, \mathcal{O}_X(uE)) = 0$ for all $u \geq 0$. We have $\mathrm{Hom}(\mathcal{O}_X(-uE)[1], F_1) = 0$ for every $u \geq 1$ as well. The hard part is to prove σ_+-stability of F_1 under the genericity assumption on V_2 (note that Lemma 3.15 cannot be applied):

Claim 6.12. F_1 is σ_+-stable.

Proof of the claim. Assume F_1 is not σ_+-stable, and let $Q_1 \to F_1 \to Q_2$ be a short exact sequence destabilizing F_1 with respect to σ_+, such that Q_1 is σ_+-stable. Arguing as in the proof of Lemma 6.6, we obtain $\mathrm{ch}(Q_1) = (-r, H - qE, 0)$ with $r > 0$.

Therefore $\mathrm{ch}_1(Q_2) = (q-a)E$. As an application of Lemma 4.1, any σ_+-stable factor of a HN factor of Q_2 has ch_1 equal to a multiple of E and $\mathrm{ch}_2 = 0$. Let Q' be such a stable factor. If $\mathrm{ch}_0(Q') \neq 0$, then by Lemma 3.13 Q' is (up to shift) a line bundle $\mathcal{O}_X(uE)$. This implies $\mathrm{ch}_0(Q') \leq 0$ for every stable factor; otherwise, the last stable factor defines a quotient $F_1 \twoheadrightarrow Q_2 \twoheadrightarrow \mathcal{O}_X(uE)$ in $\mathrm{Coh}^b(X)$ for some $u \geq 0$, contradicting the vanishing $\hom(F_1, \mathcal{O}_X(uE)) = 0$.

Hence, $\mathrm{ch}_0(Q_2) \leq 0$. If $\mathrm{ch}_0(Q_2) < 0$, by arguing as in the proof of Lemma 6.6, we obtain $\nu_{0,w}(F_1) < \nu_{0,w}(Q_2)$ for $w \gg 0$, which is a contradiction. Therefore, every σ_+-stable factor Q' of Q_2 has Chern character $(0, q'E, 0)$ for some $q' > 0$. In particular, by Corollary 3.12, Q_2 is a torsion sheaf. We explain how this leads to a contradiction.

If $H^0(X, Q_2) = 0$, then $\mathrm{Hom}(\mathcal{O}_X(eE), Q_2) = 0$ as $\mathcal{O}_X(eE)$ restricts trivially to the (scheme-theoretic) support of Q_2. Since $\mathrm{Hom}(F_2, Q_2) = 0$ as well (F_2 is σ_+-stable with $\nu_{\sigma_+}(F_2) > 0 = \nu_{\sigma_+}(Q_2)$), it follows that $\mathrm{Hom}(F_1, Q_2) = 0$, a contradiction. Suppose $H^0(X, Q_2) \neq 0$. Then, by Lemma 3.2, all irreducible components of the (set-theoretic) support of Q_2 lie in the linear system $|E|$. Pick one such component $J \in |E|$ with

$H^0(Q_2|_J) \neq 0$. Since $\chi(Q_2|_J) = 0$ (as Q_2 is Gieseker semistable), Serre duality on J implies

$$0 \neq h^1(Q_2|_J) = \mathrm{Hom}_{\mathcal{O}_J}(Q_2|_J, \mathcal{O}_J) = \mathrm{Hom}_{\mathcal{O}_X}(Q_2, \mathcal{O}_J).$$

Therefore we have a surjection $F_1 \twoheadrightarrow Q_2 \twoheadrightarrow \mathcal{O}_J$. Note that, as $\mathrm{Hom}(F_2, \mathcal{O}_J) = 0$, $\mathrm{Hom}(F_1, \mathcal{O}_J)$ equals the kernel of the induced map

$$V_2 \otimes \mathrm{Hom}(\mathcal{O}_X(eE), \mathcal{O}_J) \longrightarrow \mathrm{Ext}^1(F_2, \mathcal{O}_J).$$

Since $\mathrm{Hom}(\mathcal{O}_X(eE), \mathcal{O}_J) \cong \mathbb{C}$, this map is the restriction to V_2 of the natural map

$$\psi_J : \mathrm{Ext}^1(F_2, \mathcal{O}_X(eE)) \longrightarrow \mathrm{Ext}^1(F_2, \mathcal{O}_J)$$

obtained by applying $\mathrm{Ext}^1(F_2, -)$ to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((e-1)E) \longrightarrow \mathcal{O}_X(eE) \longrightarrow \mathcal{O}_J \longrightarrow 0.$$

As $\hom(F_2, \mathcal{O}_J) = 0$ and $\mathrm{ext}^2(F_2, \mathcal{O}_X(eE)) = \hom(\mathcal{O}_X(eE), F_2) = 0$, we get the long exact sequence

$$0 \to \mathrm{Ext}^1(F_2, \mathcal{O}_X((e-1)E)) \to \mathrm{Ext}^1(F_2, \mathcal{O}_X((e-1)E))$$
$$\xrightarrow{\psi_J} \mathrm{Ext}^1(F_2, \mathcal{O}_J) \to \mathrm{Ext}^2(F_2, \mathcal{O}_X((e-1)E)) \to 0.$$

Hence $\ker(\psi_J)$ has codimension $\mathrm{ext}^1(F_2, \mathcal{O}_J) - \hom(\mathcal{O}_X((e-1)E, F_2)$ in $\mathrm{Ext}^1(F_2, \mathcal{O}_X(eE))$. A lower bound for this codimension (using again $\hom(F_2, \mathcal{O}_J) = 0$) is

$$-\chi(F_2, \mathcal{O}_J) - \hom(\mathcal{O}_X((e-1)E, F_2) = k - \hom(\mathcal{O}_X((e-1)E, F_2).$$

It follows from (51) that a general choice of V_2 satisfies $V_2 \cap \ker(\psi_J) = 0$ for every $J \in |E|$; note that we require strict inequality in (51), as J varies in a 1-dimensional family. Therefore $\mathrm{Hom}(F_1, \mathcal{O}_J) = 0$ for every $J \in |E|$, which is a contradiction and proves the Claim 6.12. □

Coming back to the proof of Theorem 6.11, assume (51) holds. Applying Proposition 5.9 again, we find that for a general $V_1 \in \mathrm{Gr}\left(m_1, \mathrm{Ext}^1(F_1, \mathcal{O}_X(eE))\right)$ the extension

$$0 \longrightarrow V_1^\vee \otimes \mathcal{O}_X((e+1)E) \longrightarrow F_0 \longrightarrow F_1 \longrightarrow 0$$

satisfies $\hom(F, \mathcal{O}_X(uE)) = 0 = \hom(\mathcal{O}_X(-uE)[1], F_0)$ for every $u \geq 0$. Note that F_0 is $\sigma_{b,0}$-semistable (as an extension of $\sigma_{b,0}$-semistable objects with equal $\nu_{b,0}$-slope). It suffices to check that F_0 is σ_+-stable: since σ_+ lies in the Gieseker chamber for v, then we obtain $F_0 \in \mathcal{M}_{H_\epsilon}(v, \overline{e})$ as required.

We first show that F_0 is σ_+-semistable. Indeed, if $Q_1 \to F_0 \to Q_2$ is a σ_+-destabilizing sequence with Q_1 σ_+-stable, arguing as in the proof of Lemma 6.6 again gives $\mathrm{ch}(Q_1) = (-r, H - qE, 0)$ with $r > 0$. But then $\mathrm{ch}_0(Q_2) = r > 0$, which contradicts the vanishing $\hom(F_0, \mathcal{O}_X(uE)) = 0$ for some $u \geq 0$ as in the proof of Claim 6.12. Since F_0 is σ_+-semistable, we know that it is a Gieseker semistable sheaf of Chern character $(0, H - aE, 0)$. If not σ_+-stable, F_0 must be supported on a reducible curve, and admits a semistable sheaf Q_2 of Chern character $(0, q'E, 0)$ $(q' > 0)$ as a quotient. But arguing as in the proof of Claim 6.12 we find $\mathrm{Hom}(F_0, Q_2) = 0$ for the general choice of V_1, which shows σ_+-stability of F_0.

Therefore, the proof of Theorem 6.11 is complete as long as the inequality (51) holds. If $e \geq 1$, this is trivial since $\hom\left(\mathcal{O}_X\left((e-1)E\right), F_2\right) = 0$. For $e = 0$, this can be checked exactly as in the proof of Theorem 5.8, where one is led to Case 2.1 (since $\mathrm{ch}_2(F_2(E)) = k > 0$ and $\langle v_2(E), v(\mathcal{O}_X(-E))\rangle = 2(m_1 + m_2 - k) \leq 0$). $\square$

Proof of Theorem 6.1 when $d = g-1$. Combining Theorem 6.7, Theorem 6.8, and Theorem 6.11 via the same argument as in the case $d < g-1$ implies the claim. $\square$

6.2. **Stability type versus splitting type.** The main feature of Theorem 6.1 is the stratification of $V^r_{H_\epsilon}(v)$ in terms of Bridgeland stability types. It is interesting to compare this invariant with the *splitting type* recalled in (10) introduced by H. Larson [Lar21].

Take $a \geq 0$ and $g' := g - ak \geq 1$. Recall that for any integral curve $C \in |H - aE|$, if $\pi\colon C \xrightarrow{i} X \to \mathbf{P}^1$ is the induced degree k cover, then the collection $\overline{f}_L = \big((f_i, n_i)\big)_{i=1}^q$ is via (10) the splitting type of $L \in W^r_d(C)$. Note that f_1 is the largest integer with $\mathrm{Hom}\big(\mathcal{O}_{\mathbf{P}^1}(f_1), \pi_*L\big) = \mathrm{Hom}\big(\mathcal{O}_X(f_1E), i_*L\big) \neq 0$, and $n_1 = \hom(\mathcal{O}_X(f_1E), i_*L)$.

A first expectation could be that the stability type of i_*L coincides with the non-negative part $(\overline{f}_L)^{\geq 0} \subseteq \overline{f}_L$ of the splitting type of L. This might be too coarse to hold in full generality, but it is almost true for balanced stability types. More precisely, consider the stability type $\overline{e} = \big((e+1, m_1), (e, m_2)\big)$ with $m_2 > 0$. As usual, write $r := m_1(e+2) + m_2(e+1) - 1$ and $\ell := r + 1 - m_1 - m_2$.

Proposition 6.13. *Let $C \in |H - aE|$ be integral and $L \in \overline{\mathrm{Pic}}^d(C)$ for $d \leq g' - 1$. Then:*

- *(a) If i_*L has stability type $\overline{e}$, then $(\overline{f}_L)^{\geq 0} = \overline{e}$.*
- *(b) If $(\overline{f}_L)^{\geq 0} = \overline{e}$, then $i_*L \in \widetilde{\mathcal{M}}_{H_\epsilon}(v, \overline{e})$, as defined in (36). Equivalently, i_*L has stability type $\overline{e} \cup \overline{e'}$ for some $\overline{e'} \subset \mathbb{Z}_{\leq e-1} \times \mathbb{Z}_{>0}$.*
- *(c) The loci $\big\{L \in \overline{\mathrm{Pic}}^d(C) : (\overline{f}_L)^{\geq 0} = \overline{e}\big\}$ and $\big\{L \in \overline{\mathrm{Pic}}^d(C) : i_*L$ has stability type $\overline{e}\big\}$ have the same closure in $\overline{\mathrm{Pic}}^d(C)$.*

Proof. Write $v = (0, H - aE, 1 + d - g')$. If $i_*L \in \mathcal{M}_{H_\epsilon}(v, \overline{e})$, then using the vanishing $\mathrm{ext}^1(\mathcal{O}_X(eE), \mathcal{O}_X((e+1)E)) = 0$ we have that i_*L sits in a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(eE)^{\oplus m_2} \oplus \mathcal{O}_X((e+1)E)^{\oplus m_1} \longrightarrow i_*L \longrightarrow Q_2 \longrightarrow 0$$

in $\mathrm{Coh}^b(X)$ (for $-1 \ll b < 0$), where Q_2 is stable along the walls $\mathcal{W}(Q_2, \mathcal{O}_X)$ and $\mathcal{W}(Q_2, \mathcal{O}_X(eE))$. In particular $\hom(\mathcal{O}_X, Q_2) = 0 = \hom(\mathcal{O}_X(eE), Q_2)$ and therefore

$$\hom(\mathcal{O}_X, i_*L) = m_1(e+2) + m_2(e+1), \tag{52}$$

$$\hom(\mathcal{O}_X(eE), i_*L) = m_2 + 2m_1. \tag{53}$$

We already know that the first entry of $(\overline{f}_L)^{\geq 0}$ is $(e+1, m_1)$; combining this with $\mathrm{ext}^1(\mathcal{O}_X(eE), \mathcal{O}_X((e+1)E)) = 0$, (53) results in the inclusion $\overline{e} \subset (\overline{f}_L)^{\geq 0}$. Then it must be $\overline{e} = (\overline{f}_L)^{\geq 0}$ thanks to (52), which proves (a).

If L satisfies $(\overline{f}_L)^{\geq 0} = \overline{e}$, then (53) holds and i_*L sits in an extension

$$0 \longrightarrow \mathcal{O}_X((e+1)E)^{\oplus m_1} \longrightarrow i_*L \longrightarrow Q_1 \longrightarrow 0$$

in $\mathrm{Coh}^0(X)$, with Q_1 stable along $\mathcal{W}\big(\mathcal{O}_X((e+1)E), i_*L\big)$. Now it follows from the vanishing $\mathrm{ext}^1(\mathcal{O}_X(eE), \mathcal{O}_X((e+1)E) = 0$ that

$$\hom(\mathcal{O}_X(eE), Q_1) = \hom(\mathcal{O}_X(eE), i_*L) - 2m_1 \overset{(53)}{=} m_2,$$

and hence Q_1 sits in a sequence $0 \to \mathcal{O}_X(eE)^{\oplus m_2} \to Q_1 \to Q_2 \to 0$ with Q_2 stable along $\mathcal{W}(\mathcal{O}_X(eE), Q_2)$. This proves (b). Finally, assertion (c) follows from the inclusion $\widetilde{\mathcal{M}}_{H_\epsilon}(v,\overline{e}) \subseteq \overline{\mathcal{M}_{H_\epsilon}(v,\overline{e})}$ inside the moduli space $\mathcal{M}_{H_\epsilon}(v)$, provided by Remark 5.6. □

Remark 6.14. Regarding the comparison in Proposition 6.13, we observe the following:
(a) It follows from Proposition 6.13 that, if $e = 0$, that is, $\ell < \frac{r+1}{2}$, then the equality $\left\{L \in \overline{\mathrm{Pic}}^d(C) : \ i_*L \text{ has stability type } \overline{e}\right\} = V^r_{d,\ell}(C)$ holds.
(b) It is shown in Section 7 that if $\max\{0, r+2-k\} \leq \ell \leq r$ and $\rho(g', r-\ell, d) - \ell k \geq 0$, the support morphism $\widetilde{\mathcal{M}}_{H_\epsilon}(v,\overline{e}) \to |H - aE|$ is dominant. By generic smoothness and Proposition 6.13.(b), the locus $V^r_{d,\ell}(C)$ is then smooth of dimension $\rho(g', r-\ell, d) - \ell k$ for a general $C \in |H - aE|$.
(c) In contrast, if $r+1-k \geq 0$ and $\ell := r+1-k$ satisfies $\rho(g', r-\ell, d) - \ell k \geq 0$, then the image of the support morphism must be contained in the locus of reducible curves in $|H - aE|$. Otherwise, by Proposition 6.13 we would have a rank 1, torsion-free sheaf L on an integral curve, with $\chi(L) \leq 0$ and splitting type with $r+1-\ell = k$ non-negative entries, which is impossible.

7. Non-emptiness of Hurwitz-Brill-Noether loci via $K3$ surfaces

In this section we provide a new approach, using stability conditions on $K3$ surfaces, to the non-emptiness of the loci $V^r_{d,\ell}(C,A)$ (recall Definition 2.7), for a general element $[C,A] \in \mathcal{H}_{g,k}$. Precisely, for every ℓ with $\max\{r+2-k, 0\} \leq \ell \leq r$ satisfying inequality (4), that is,

$$\rho(g, r-\ell, d) - \ell k \geq 0,$$

we construct a component of $V^r_{d,\ell}(C,A)$ (and by passing to the closure a component of $W^r_{d,\ell}(C,A)$) having precisely this dimension. An immediate consequence is the existence theorem in Hurwitz-Brill-Noether theory.

This is achieved by proving the result for curves on elliptic $K3$ surfaces. As usual, take the integers $e, m_1 \geq 0$, $m_2 > 0$ such that

$$r+1 = m_1(e+2) + m_2(e+1), \quad r+1-\ell = m_1 + m_2.$$

For any $K3$ surface X as in Proposition 3.1 and a general $C \in |H|$, we show that

$$\left\{L \in \mathrm{Pic}^d(C) : i_*L \text{ has stability type } \overline{e} := \{(e+1, m_1), (e, m_2)\}\right\}$$

is smooth of pure dimension $\rho(g, r-\ell, d) - \ell k$. In view of Proposition 6.13 we obtain that, for the pair $(C,A) = (C, \mathcal{O}_C(E)) \in \mathcal{H}_{g,k}$, the locus $V^r_{d,\ell}(C,A)$ has a component of dimension $\rho(g, r-\ell, d) - \ell k$, which as explained in Proposition 2.8, is enough to derive the result for a general $[C,A] \in \mathcal{H}_{g,k}$.

7.1. **Curves on elliptic $K3$ surfaces.** We fix, as throughout the paper, a degree k elliptic $K3$ surface X as in Proposition 3.1. As explained above, we want to prove that the locus

$$\left\{L \in \mathrm{Pic}^d(C) : i_*L \text{ has stability type } \overline{e}\right\}$$

is smooth of pure dimension $\rho(g, r-\ell, d) - \ell k$, for a general curve $C \in |H|$. This amounts to prove that the natural support map

$$\pi_{(g,d,r,\ell)} : \mathcal{M}_{H_\epsilon}\big((0, H, 1 + d - g), \overline{e}\big) \longrightarrow |H|$$

is dominant, since, as proven in Theorem 6.1, the moduli space $\mathcal{M}_{H_\epsilon}\big((0, H, 1+d-g), \overline{e}\big)$ is smooth and irreducible of dimension $g + \rho(g, r - \ell, d) - \ell k$.

Our argument is inductive in nature and relies on a reduction to the case $\ell = 0$. Indeed, it suffices to exhibit a curve $Y \in |H|$ such that the fiber $\pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)$ has an irreducible component of the correct dimension $\rho(g, r - \ell, d) - \ell k$. We will do this by picking Y to be reducible, so that the information is extracted from a curve of lower genus on X.

In order to perform induction on the tuple (g, d, r, ℓ), let us spell out the content of Theorem 6.1 and Remark 6.3 for the linear systems $|H-aE|$, $a \geq 0$. Let $(g', d', r', \ell') \in \mathbb{N}^4$ be a tuple satisfying

$$\begin{aligned} g' := g - ak \geq 1 \text{ for some } a \geq 0, \quad d' \leq g' - 1, \\ \max\{r' + 2 - k, 0\} \leq \ell' \leq r', \quad \rho(g', r' - \ell', d') - \ell' k \geq 0. \end{aligned} \tag{54}$$

Taking the unique $e', m_1' \geq 0$ and $m_2' > 0$ such that $r' + 1 = m_1'(e' + 2) + m_2'(e' + 1)$ and $r' + 1 - \ell' = m_1' + m_2'$, then for $\overline{e'} := \{(e' + 1, m_1), (e', m_2')\}$ the moduli space

$$\mathcal{M}_{H_\epsilon}\left((0, H - aE, 1 + d' - g'), \overline{e'}\right)$$

is smooth and irreducible of dimension $g' + \rho(g', r' - \ell', d') - \ell' k$. We denote by

$$\pi_{(g',d',r',\ell')} \colon \mathcal{M}_{H_\epsilon}\left((0, H - aE, 1 + d' - g'), \overline{e'}\right) \longrightarrow |H - aE|$$

the corresponding support map. Our main reduction is the following:

Proposition 7.1. *If $\ell \geq 1$ and the map $\pi_{(g-k,d-k,r-1,\ell-1)}$ is dominant, then the map $\pi_{(g,d,r,\ell)}$ is dominant.*

Proof. First note that if $\ell \geq 1$, then the tuple $(g', d', r', \ell') := (g - k, d - k, r - 1, \ell - 1)$ satisfies (54). Indeed, by Proposition 2.1 the assumption $\rho(g, r - \ell, d) - \ell k \geq 0$ implies $d \geq k$ (hence $g' \geq 1$ and $d' \geq 0$), and since

$$\rho(g - k, r - \ell, d - k) - (\ell - 1)k = \rho(g, r - \ell, d) - \ell k \geq 0$$

the map $\pi_{(g-k,d-k,r-1,\ell-1)}$ is indeed well defined. Note also that:

- If $m_1 \neq 0$, then $m_1' = m_1 - 1$, $m_2' = m_2 + 1$ and $e' = e$.
- If $m_1 = 0$, then $m_1' = m_2 - 1$, $m_2' = 1$ and $e' = e - 1$.

Assume the map $\pi_{(g-k,d-k,r-1,\ell-1)}$ is dominant. Then for a general curve $C \in |H-E|$, the fiber $\pi^{-1}_{(g-k,d-k,r-1,\ell-1)}(Y) \subseteq \mathrm{Pic}^{d-k}(C)$ is smooth of pure dimension

$$\rho(g - k, r - \ell, d - k) - (\ell - 1)k = \rho(g, r - \ell, d) - \ell k.$$

Having chosen general curves $C \in |H - E|$ and $J \in |E|$, we consider the nodal curve

$$Y := C + J \in |H|$$

and we denote by $x_1, \ldots, x_k$ the points of the (transverse) intersection $C \cdot J$.

Consider a general point $L_C \in \pi^{-1}_{(g-k,d-k,r-1,\ell-1)}\big([C]\big)$. If $m_1 \neq 0$, then L_C fits in a short exact sequence

$$0 \longrightarrow \mathcal{O}_X((e+1)E)^{\oplus m_1-1} \oplus \mathcal{O}_X(eE)^{\oplus m_2+1} \longrightarrow L_C \longrightarrow W \longrightarrow 0$$

in $\mathrm{Coh}^b(X)$ $(-1 \ll b < 0)$, where $W \in \mathcal{M}_{H_\epsilon}\big(-(r-\ell+1), H-\ell E, -(g-d+r-\ell)\big)$ is a general element. More precisely, as pointed out in Example 6.4, we have $W = G[1]$ for a vector bundle G arising as the kernel of a certain evaluation map. A similar statement holds true if $m_1 = 0$, by considering a distinguished triangle

$$0 \longrightarrow \mathcal{O}_X(eE)^{\oplus m_2-1} \oplus \mathcal{O}_X((e-1)E) \longrightarrow L_C \longrightarrow W \longrightarrow 0.$$

We will assume $m_1 \neq 0$ in the rest of the proof; for $m_1 = 0$ one applies, *mutatis mutandis*, the same argument.

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(eE) \longrightarrow \mathcal{O}_X((e+1)E) \longrightarrow \mathcal{O}_J \longrightarrow 0$$

obtained by multiplication by the section defining the curve $J \in |E|$. We can construct the following commutative diagram with exact rows and columns in $\mathrm{Coh}^b(X)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X((e+1)E)^{\oplus m_1-1} \oplus \mathcal{O}_X(eE)^{\oplus m_2+1} & \longrightarrow & L_C & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & \mathcal{O}_X((e+1)E)^{\oplus m_1} \oplus \mathcal{O}_X(eE)^{\oplus m_2} & \longrightarrow & L & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{O}_J & \xrightarrow{=} & \mathcal{O}_J & & & & \end{array} \tag{55}$$

It follows from the middle column that L is a line bundle on the curve $Y = C + J$, such that $L_{|J} \cong \mathcal{O}_J$ and $L_{|C} \cong L_C(x_1 + \cdots + x_k)$. In particular, $\deg(L) = d$. Furthermore, we can read from the second row that the stability type of L equals $\overline{e} = \{(e+1, m_1), (e, m_2)\}$. Therefore $h^0(Y, L) = r+1$ and $L \in \pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)$.

We are going to prove that the irreducible component of $\pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)$ containing L has the right dimension $\rho(g, r-\ell, d) - \ell k$, or equivalently, that this construction fills up (locally) a component of $\pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)$. To this end, observe that since L is locally free, the locally around L the fibre $\pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)$ is isomorphic to the locus, see also (2.7)

$$V^r_{d,\ell}\big(Y, \mathcal{O}_Y(E)\big) := \Big\{N \in \mathrm{Pic}^d(Y) : h^0(Y,N) = r+1,\ h^0(Y, N(-(e+1)E)) = m_1, \\ h^0(Y, N(-eE)) = 2m_1 + m_2\Big\},$$

where $\mathrm{Pic}^d(Y)$ consists of line bundles of bidegree $(d, 0)$ on $Y = C + J$.

One has the following exact sequence

$$0 \longrightarrow (\mathbb{C}^*)^{k-1} \longrightarrow \mathrm{Pic}^d(Y) \longrightarrow \mathrm{Pic}^d(C) \times \mathrm{Pic}^0(J) \longrightarrow 0. \tag{56}$$

Note that if $N \in \mathrm{Pic}^d(Y)$ has a global section $s = (s_C, s_J) \in H^0(C, N_{|Y}) \times H^0(J, N_{|J})$ such that $s(x_i) \neq 0$ for $i = 1, \ldots, k$, then N is uniquely determined by its restrictions

$N_{|C} \in \mathrm{Pic}^d(C)$ and $N_{|J} \in \mathrm{Pic}^0(J)$, because the multipliers corresponding to the fibre of (56) over the point $(N_{|C}, N_{|J})$ are given by the quotients $s_Y(x_i)/s_C(x_i) \in \mathbb{C}^*$.

We now observe that the line bundle L constructed via (55) enjoys this property. Indeed, since $h^0(C, L_C) = r$, $h^0(Y, L) = r + 1$ and $h^0(J, \mathcal{O}_J) = 1$, we have an exact sequence

$$0 \longrightarrow H^0(C, L_C) \longrightarrow H^0(Y, L) \longrightarrow H^0(J, \mathcal{O}_J) \longrightarrow 0.$$

But a section $s_0 \in H^0(Y, L)$ projecting onto the section $1 \in H^0(J, \mathcal{O}_J)$ does not vanish at any of the points $C \cdot J$.

Assume now that $N \in V^r_{d,\ell}\big(Y, \mathcal{O}_Y(E)\big)$, where $\deg(N_{|C}) = d$ and $\deg(N_{|J}) = 0$. Since we are working locally around L, we may assume that

$$\begin{aligned} h^0\big(C, N_{|C}(-x_1 - \cdots - x_k)\big) &\leq r, \\ h^0\big(C, N_{|C}(-x_1 - \cdots - x_k)(-eE)\big) &\leq 2m_1 + m_2 - 1, \\ h^0\big(C, N_{|C}(-x_1 - \cdots - x_k)(-(e+1)E)\big) &\leq m_1 - 1. \end{aligned} \tag{57}$$

On the other hand one has an exact sequence

$$0 \longrightarrow N_{|C}(-x_1 - \cdots - x_k) \longrightarrow N \longrightarrow N_{|J} \longrightarrow 0,$$

from which it follows that $h^0(C, N_{|C}(-x_1 - \cdots - x_k)) \geq h^0(Y, N) - h^0(J, N_{|J}) \geq r$, with equality only if $N_{|J} \cong \mathcal{O}_J$. Therefore, $N_{|J} \cong \mathcal{O}_J$ and N is uniquely determined by its restriction $N_{|C}$. Furthermore,

$$h^0\big(Y, N_{|Y}(-x_1 - \cdots - x_k)(-aE)\big) \geq h^0(C, N(-aE)) - 1$$

for $a = e, e+1$. Therefore, all inequalities in (57) are equalities, which means that $N_{|C}(-x_1 - \cdots - x_k)$ has $\{(e+1, m_1 - 1), (e, m_2 + 1)\}$ as the nonnegative part of its splitting type. Thus, by Proposition 6.13, we find that $N_{|C}(-x_1 - \cdots - x_k)$ lies in $\pi^{-1}_{(g-k,d-k,r-1,\ell-1)}\big([C]\big)$ (since we work locally). Conversely, as already shown, a general line bundle $L_C \in \pi^{-1}_{(g-k,d-k,r-1,\ell-1)}\big([C]\big)$ gives rise via (55) to an element in the fibre $\pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)$. This shows that

$$\dim_L\Big(\pi^{-1}_{(g,d,r,\ell)}\big([Y]\big)\Big) = \dim_{L_C}\Big(\pi^{-1}_{(g-k,d-k,r-1,\ell-1)}\big([C]\big)\Big) = \rho(g, r-\ell, d) - \ell k,$$

which finishes the proof. □

As a consequence of Proposition 7.1, we can establish the dominance of $\pi_{(g,d,r,\ell)}$ for arbitrary ℓ in the following cases:

Corollary 7.2. *Let X be a degree k elliptic $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$. If*

$$\ell \geq \frac{1}{2}\big(g - d + 2r + 1 - k\big),$$

then the map $\pi_{(g,d,r,\ell)}$ is dominant. In particular, $V^r_{d,\ell}\big(C, \mathcal{O}_C(E)\big)$ is non-empty and of dimension $\rho(g, r-\ell, d) - \ell k$ for a general curve $C \in |H|$.

Proof. In view of Proposition 7.1, we may assume $\ell = 0$. Thus we aim to prove that

$$\mathcal{M}_{H_\epsilon}\big((0, H, 1 + d - g), \overline{e}\big) \longrightarrow |H|$$

is a dominant map for $\overline{e} = \{(0, r+1)\}$. This amounts to prove that for a general $C \in |H|$ there exists a line bundle $L \in \mathrm{Pic}^d(C)$ with $h^0(C, L) = r + 1$ and $h^0(C, L(-E)) = 0$.

Since $k \geq g - d + 2r + 1 - k$, we have

$$\rho(g, r - \ell, d) - \ell k < \rho(g, r, d) \quad \text{for every } \ell \geq 1 \tag{58}$$

By Corollary 6.2, (58) implies $\dim W^r_d(C) = \rho(g, r, d)$ for a general curve $C \in |H|$ and also $\dim W^r_d(C) > \dim W^{r+1}_d(C)$, so a general $L \in W^r_d(C)$ has $h^0(C, L) = r + 1$.

On the other hand, any $L \in \mathrm{Pic}^d(C)$ with $h^0(C, L) = r + 1$ and $h^0(C, L(-E)) \neq 0$ has stability type $\overline{e'} = \{(e'_i, m'_i)\}_{i=1}^p$, where $e'_1 \geq 1$ and

$$r + 1 \geq \begin{cases} 2m_1 & \text{if } p = 1 \\ 2(m_1 + \cdots + m_{p-1}) + m_p & \text{if } p \geq 2 \end{cases} \tag{59}$$

(this follows from Theorem 5.5 and its proof). Furthermore, Theorem 6.1 yields

$$\dim \mathcal{M}_{H_\epsilon}\big((0, H, 1 + d - g), \overline{e'}\big) \leq g + \rho(g, r - \ell, d) - \ell k \overset{(58)}{<} g + \rho(g, r, d)$$

where $\ell = r + 1 - (m_1 + \cdots + m_p) \overset{(59)}{\geq} 1$. It follows that, for a general $C \in |H|$, the locus

$$\big\{L \in \mathrm{Pic}^d(C) : h^0(C, L) = r + 1,\ h^0(C, L(-E)) \neq 0\big\}$$

has dimension $< \rho(g, r, d)$, which concludes the proof. □

Corollary 7.3. *For a degree k elliptic $K3$ surface with* $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$*, for a general curve $C \in |H|$, we have that*

$$\dim\ W^r_d(C) = \rho_k(g, r, d).$$

Proof. This follows from Corollary 7.2 recalling that $\ell_{\max} := \frac{1}{2}\big(g - d + 2r + 1 - k\big)$ is the quantity where the maximum of the quadratic function $\rho(g, r - \ell, d) - \ell k$ is attained. Hence, having proven existence for

$$\ell \geq \min\Big\{\frac{1}{2}(g - d + 2r + 1 - k), r\Big\},$$

we have shown the existence of a component of $W^r_d(C)$ of the maximal dimension. □

Next we observe how our results provide a full answer to the existence problem of the loci $V^r_{d,\ell}(C, A)$ in the extremal case when $\ell = r + 2 - k$.

Theorem 7.4. *For a degree k elliptic $K3$ surface with* $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$*, for a general curve $C \in |H|$ we have*

$$\dim\ V^r_{d,r+2-k}\big(C, \mathcal{O}_C(E)\big) = \rho(g, k - 2, d) - (r + 2 - k)k.$$

Proof. Follows immediately by combining Proposition 3.4 with Proposition 7.1. □

7.2. **Halphen surfaces.** To establish an existence result analogous to Theorem 7.4 for all values of ℓ, we need to relax the assumption of working with an arbitrary degree k elliptic $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z}\cdot H \oplus \mathbb{Z}\cdot E$. We recall that such $K3$ surfaces form an irreducible Noether-Lefschetz divisor in the 19-dimensional moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces of genus g and we will use a degeneration of elliptic $K3$ surfaces to *Halphen surfaces* of index k following [CD12, 2.1]. Let $p_1, \ldots, p_9 \in \mathbf{P}^2$ be distinct points and denote by

$$\epsilon\colon X := \mathrm{Bl}_{\{p_1,\ldots,p_9\}}(\mathbf{P}^2) \longrightarrow \mathbf{P}^2$$

the blow-up of $\mathbf{P}^2$ at these points, by $E_1, \ldots, E_9$ the corresponding exceptional divisors on X and set $\mathcal{O}_X(h) := \epsilon^*(\mathcal{O}_{\mathbb{P}^2}(1))$.

Definition 7.5. The surface X is said to be a Halphen surface if there exists an integer $k \geq 2$ such that $\dim \left|-kK_X\right| = 1$ and $\left|-kK_X\right|$ is base point free. The smallest k with these properties is the *index* of X.

From the definition it follows there exists a unique cubic curve $J \in \left|3h - E_1 - \cdots - E_9\right|$, therefore $J \equiv -K_X$ and there exists an irreducible plane curve Z of degree $3k$ having points of multiplicity k at $p_1, \ldots, p_9$. Accordingly, then

$$\mathcal{O}_J\big(3h - E_1 - \cdots - E_9\big) = \mathcal{O}_J(J) \in \mathrm{Pic}^0(J)[k]$$

is a torsion point of order k. Moreover $Z \cdot J = 0$ and there exists an elliptic pencil $f\colon X \to \mathbf{P}^1$ having kJ as a non-reduced fibre.

We fix a Halphen surface X of index k and for $g \geq 1$ define the *du Val* linear system

$$\Lambda_g := \left|3gh - gE_1 - \cdots - gE_8 - (g-1)E_9\right|. \tag{60}$$

It follows from [ABFS16, Lemma 2.4] that the general element $C \in \Lambda_g$ is a smooth curve of genus g. Observe that $C \cdot J = 1$, therefore the point of intersection $p_{10}^{(g)} = C \cdot J$ is a base point of the linear system Λ_g. The point $p_{10}^{(g)} \in J$ is determined by the relation

$$p_{10}^{(g)} = -gp_1 - \cdots - gp_8 - (g-1)p_9 \in J \tag{61}$$

with respect to the group law on J. Blowing up the base point $p_{10}^{(g)}$ and denoting by $|C'|$ the strict transform of the linear system Λ_g, it is shown in [ABS17] that the linear system $|C'|$ is base point free on $X' := \mathrm{Bl}_{p_{10}^{(g)}}(X)$ and maps X' onto a surface $\overline{X} \subseteq \mathbf{P}^g$ that is a limit of polarized $K3$ surfaces of degree $2g-2$. Note that the map $X' \to \overline{X}$ contracts the proper transform of Z to an elliptic singularity q on $\overline{X}$. The main result of [ABFS16] asserts that if the points $p_1, \ldots, p_9 \in \mathbf{P}^2$ are not associated with a Halphen surface of index at most g, then the general curve $C \in \Lambda_g$ satisfies the Petri theorem. This is the case if $p_1, \ldots, p_9$ are chosen generically in $\mathbf{P}^2$.

Assume now that X is a Halphen surface of index k. Then $A := \mathcal{O}_C(kJ) \in W^1_k(C)$ and it is proved in [Arb24, Theorem 3.1] that if $k \leq \frac{g+3}{2}$, then A computes the gonality of C, therefore $\mathrm{gon}(C) = k$. Furthermore, using e.g. [ABS17] it is easy to see that Halphen surfaces appear as limits of degree k elliptic $K3$ surface. Indeed, it follows from [ABS17, Theorem 24] that the map between the deformation functors

$$\mu\colon \mathrm{Def}\big(\overline{X}, \mathcal{O}_{\overline{X}}(1)\big) \to \mathrm{Def}\big(\overline{X}, q\big)$$

is smooth. Here $\mathrm{Def}(\overline{X},q)$ is parametrized by $H^0(\overline{X},T^1_{\overline{X}})$, where $T^1_{\overline{X}}=\mathcal{E}xt^1\big(\Omega^1_{\overline{X}},\mathcal{O}_{\overline{X}}\big)$, whereas $\mathrm{Def}\big(\overline{X},\mathcal{O}_{\overline{X}}(1)\big)$ is parametrized by $\mathrm{Ext}^1\big(Q_{\mathcal{O}_{\overline{X}}(1)},\mathcal{O}_{\overline{X}}\big)$, where $Q_{\mathcal{O}_{\overline{X}}(1)}$ is the corresponding Atiyah class sitting in the exact sequence

$$0\longrightarrow \Omega_{\overline{X}}\longrightarrow Q_{\mathcal{O}_{\overline{X}}(1)}\longrightarrow \mathcal{O}_{\overline{X}}\longrightarrow 0.$$

Since $\dim \mathrm{Def}\big(\overline{X},q\big)>1$, it follows that the codimension one subvariety of $\mathrm{Def}\big(\overline{X},\mathcal{O}_{\overline{X}(1)}\big)$ parametrizing those deformations that also preserve the line bundle $\mathcal{O}_{\overline{X}}(Z)$ cannot be contained in the space of topologically trivial deformations of $(\overline{X},\mathcal{O}_{\overline{X}}(1))$. Since $|\mathcal{O}_{\overline{X}}(1)(-Z)|$ is very ample, we also know that each formal deformation of $\big(\overline{X},\mathcal{O}_{\overline{X}}(1)\big)$ is also effective. Therefore we obtain that the Halphen surface $\overline{X}\subseteq \mathbf{P}^g$ smooths to an elliptic $K3$ surface like in Proposition 3.1.

Before stating our next result, we recall that for a point p on a smooth curve C we introduce the *vanishing sequence* at p

$$a^{\ell}(p)=\big(0\le a_0^{\ell}(p)<\ldots<a_r^{\ell}(p)\le d\big)$$

of a linear system $\ell=(L,V)\in G^r_d(C)$ by ordering the vanishing orders at p of the sections from V. The *ramification sequence* of ℓ at p is the non-decreasing sequence $\alpha^{\ell}(p)=\big(\alpha_0^{\ell}(p)\le\cdots\le\alpha^{\ell}(p)\big)$ obtained by setting $\alpha_i^{\ell}(p):=a_i^{\ell}(p)-i$. Having fixed integers $0<r\le d$, a Schubert index of type (r,d) is a non-decreasing sequence of integers $\bar\alpha=\big(0\le\alpha_0\le\ldots\le\alpha_r\le d-r\big)$. Its *weight* is defined as $|\bar\alpha|:=\alpha_0+\cdots+\alpha_r$.

For a curve Y of compact type, for points $p_1,\ldots,p_s\in Y_{\mathrm{reg}}$ and for Schubert indices $\bar\alpha^1,\ldots,\bar\alpha^s$ of type (r,d), we denote by $\overline{G}^r_d\big(Y,(p_1,\bar\alpha^1),\ldots,(p_s,\bar\alpha^s)\big)$ the variety of limit linear series ℓ of type g^r_d on Y satisfying the inequalities $\alpha^{\ell}(p_i)\ge\bar\alpha^i$, for $i=1,\ldots,s$. It is a determinantal variety of expected dimension

$$\rho\big(g,r,d,\bar\alpha^1,\ldots,\bar\alpha^s\big)=\rho(g,r,d)-\sum_{i=1}^{s}|\bar\alpha^i|. \tag{62}$$

We shall use that if $[J,p_1,p_2]\in\mathcal{M}_{1,2}$ is a 2-pointed elliptic curve such that $\mathcal{O}_J(p_1-p_2)$ is not a torsion bundle, then $G^r_d\big(J,(p_1,\bar\alpha^1),(p_2,\bar\alpha^2)\big)$ has the expected dimension (62) for any choice of the Schubert indices $\bar\alpha^1$ and $\bar\alpha^2$. We refer to [EH86] for basics on the theory of limit linear series.

Theorem 7.6. *Fix a general degree k elliptic $K3$ surface with* $\mathrm{Pic}(X)\cong\mathbb{Z}\cdot H\oplus\mathbb{Z}\cdot E$. *Then for $d\le g-1$ and $k\ge r+2$ such that $\rho(g,r,d)\ge 0$, for a general curve $C\in|H|$, there exists a component Z of $W^r_d(C)$ of dimension $\rho(g,r,d)$, whose general point corresponds to a line bundle L with $H^0(C,L\otimes E_C^{\vee})=0$.*

Proof. We specialize to curves on a degree k Halphen surface. As we shall explain, by semicontinuity via use of limit linear series, it will suffice to establish the conclusion of the theorem for a general curve $C\in\Lambda_g$ on a Halphen surface X of index k as above. Let C degenerate inside Λ_g to the transverse union $J\cup C_{g-1}$, where C_{g-1} is a general curve from the Du Val linear system Λ_{g-1}. Note that C_{g-1} and J meet at the point $p_{10}^{(g-1)}$ which is the base point of Λ_{g-1}, whereas $p_{10}^{(g)}$ lies on J such that $p_{10}^{(g)}-p_{10}^{(g-1)}=p_1+\cdots+p_9\in J$ (with respect to the group law). In particular $\mathcal{O}_J\big(p_{10}^{(g)}-p_{10}^{(g-1)}\big)$ is a k-torsion point. Furthermore, we let C_{g-1} degenerate to the union $J\cup C_{g-2}$, where C_{g-2} is a general member of the Du Val system Λ_{g-2}. Note that $C_{g-2} \cdot J = p_{10}^{(g-2)}$, where the point $p_{10}^{(g-2)} \in J$ is determined by the relation $p_{10}^{(g-1)} - p_{10}^{(g-2)} = p_1 + \cdots + p_9$ with respect to the group law of J. Iterating this procedure, we arrive eventually at a stable curve

$$Y := J_1 \cup \ldots \cup J_g, \tag{63}$$

where each component J_i is a copy of the curve J and $\{p^{(i)}\} = J_i \cap J_{i+1}$, where in the interest of easing the notation we write $p^{(i)} = p_{10}^{(i)}$ for $i = 1, \ldots, g$. The difference $\mathcal{O}_{J_i}\big(p^{(i)} - p^{(i-1)}\big) \in \mathrm{Pic}^0(J_i)$ is torsion of order k for $i = 2, \ldots, g$. We fix furthermore a point $p^{(0)} \in J_1 \setminus \{p^{(1)}\}$ such that $\mathcal{O}_{J_1}\big(p^{(1)} - p^{(0)}\big) \in \mathrm{Pic}^0(J_1)$ has order k. To summarize, $[Y]$ is a limit in $\overline{\mathcal{M}}_g$ of smooth curves $C \in \Lambda_g$ lying on a Halphen surface.

We shall construct a limit linear series $\ell \in \overline{G}^r_d(Y)$ and a limit linear series $\mathfrak{a} \in \overline{G}^1_k(Y)$ such that both ℓ and $\mathfrak{a}$ smooth to linear series $L \in W^r_d(C)$ and $A = \mathcal{O}_C(Z) \in W^1_k(C)$ on nearby smooth curves $C \in \Lambda_g$, such that (i) L belongs to a component of $W^r_d(C)$ of dimension $\rho(g, r, d)$ and (ii) $H^0(C, L \otimes A^\vee) = 0$.

We now specify the aspects $\ell_{J_1}, \ldots, \ell_{J_g}$ of ℓ. We start by setting

$$\ell_{J_1} := \big(\mathcal{O}_{J_1}\big(d \cdot p^{(1)}\big), V_{J_1}\big) \in G^r_d(J_1),$$

where $V_{J_1} = H^0\big(J_1, \mathcal{O}_{J_1}((r+1)p^{(1)})\big) \subseteq H^0\big(J_1, \mathcal{O}_{J_1}(dp^{(1)})\big)$. We observe that

$$\alpha^{\ell_{J_1}}\big(p^{(0)}\big) = (\underbrace{0, \ldots, 0}_{r+1}) \quad \text{and} \quad \alpha^{\ell_{J_1}}\big(p^{(1)}\big) = (\underbrace{d-r-1, \ldots, d-r-1}_{r}, d-r). \tag{64}$$

Note that since $r + 1 \leq k - 1$, certainly $\mathcal{O}_{J_1}\big((r+1)(p^{(1)} - p^{(0)})\big)$ is nontrivial.

Then on J_i with $i = 2, \ldots, r+2$, we choose the linear series

$$\ell_{J_i} = \Big(\mathcal{O}_{J_i}\big((2i-2) \cdot p^{(i-1)} + (d - 2i + 2) \cdot p^{(i)}\big), V_{J_i}\Big) \in G^r_d(J_i),$$

where

$$V_i := H^0\big(\mathcal{O}_{J_i}(i \cdot p^{(i-1)})\big) + H^0\big(\mathcal{O}_{J_i}((r-i+2) \cdot p^{(i)})\big) \subseteq H^0\Big(\mathcal{O}_{J_i}\big((2i-2) \cdot p^{(i-1)} + (d-2i+2) \cdot p^{(i)}\big)\Big).$$

Observe that V_i is $(r+1)$-dimensional since the subspaces $H^0\big(\mathcal{O}_{J_i}(i \cdot p^{(i-1)})\big)$ and $H^0\big(\mathcal{O}_{J_i}((r-i+2) \cdot p^{(i)})\big)$ intersect along the 1-dimensional subspace $\langle \sigma_i \rangle$, where $\sigma_i \in V_i$ is the unique section with $\mathrm{div}(\sigma_i) = (2i-2) \cdot p^{(i-1)} + (d - 2i + 2) \cdot p^{(i)}$. Note that the ramification profile of ℓ_{J_i} equals $\alpha^{\ell_{J_i}}\big(p^{(i-1)}\big) = (\underbrace{i-2, \ldots, i-2}_{i-1}, \underbrace{i-1, \ldots, i-1}_{r-i+2})$ and $\alpha^{\ell_{J_i}}\big(p^{(i)}\big) = (\underbrace{d-r-i, \ldots, d-r-i}_{r-i+1}, \underbrace{d-r-i+1, \ldots, d-r-i+1}_{i})$. In particular, we have $\alpha^{\ell_{J_{r+2}}}\big(p^{(r+1)}\big) = (\underbrace{r, \ldots, r}_{r+1})$.

In the same way, we construct the aspects of ℓ for the next $(g - d + r - 1)$ groups of subchains of $r + 1$ elliptic curves of Y. Precisely, for $a = 1 + b(r+1) + i$, where

$b \leq g-d+r-1$ and $1 \leq i \leq r+1$, we choose the linear series $\ell_{J_a} \in G^r_d(J_a)$ such that

$$\alpha^{\ell_{J_a}}\left(p^{(a-1)}\right) = \big(\underbrace{br+i-1,\dots,br+i-1}_{i},\underbrace{br+i,\dots,br+i}_{r+1-i}\big), \tag{65}$$

where the ramification sequence of ℓ_{J_a} at the point $p^{(a)}$ is determined by the condition that ℓ form a refined limit linear series, that is, $\alpha_j^{\ell_{J_a}}\left(p^{(a)}\right) + \alpha_{r-j}^{\ell_{J_{a+1}}}\left(p^{(a)}\right) = d-r$, for $j = 0,\dots,r$. Note that for $b = g-d+r-1$ and $i = r+1$, that is, for the component labeled by $1+(r+1)(g-d+r)$, we have the ramification

$$\alpha^{\ell_{J_{1+(r+1)(g-d+r)}}}\left(p^{((r+1)(g-d+r))}\right) = \big(\underbrace{(g-d+r)r,\dots,(g-d+r)r}_{r+1}\big).$$

Our choices imply the following equalities

$$\rho\Big(1,r,d,\alpha^{\ell_{J_i}}(p^{(i-1)}),\alpha^{\ell_{J_i}}\left(p^{(i)}\right)\Big) = 0, \quad \text{for } i = 1,\dots,(r+1)(g-d+r). \tag{66}$$

For the last $\rho(g,r,d)$ components of Y, we choose linear series $\ell_{J_a} \in G^r_d(J_a)$ such that

$$\alpha^{\ell_{J_a}}\left(p^{(a-1)}\right) = \big(\underbrace{a-(g-d+r-1),\dots,a-(g-d+r-1)}_{r+1}\big),$$

for $a = 1+(r+1)(g-d+r),\dots,g$. The ramification sequence $\alpha^{\ell_{J_a}}\left(p^{(a)}\right)$ is again determined by the condition that the aspects of ℓ form a refined limit linear series. In particular, observe that

$$\rho\Big(1,r,d,\alpha^{\ell_{J_a}}\left(p^{(a-1)}\right),\alpha^{\ell_{J_a}}\left(p^{(a)}\right)\Big) = 1, \quad \text{for } a = 1+(r+1)(g-d+r),\dots,g, \tag{67}$$

that is, we have an irreducible 1-dimensional family of choices for ℓ_{J_a} in this range.

We claim that the limit linear series $\ell \in \overline{G}^r_d(Y)$ just constructed is smoothable to *every* smooth curve of genus g, in particular to a curve $C \in \Lambda_g$ on a Halphen surface. We consider the stack $\sigma\colon \widetilde{\mathcal{G}}^r_d \to \overline{\mathcal{M}}_g$ of limit linear series over the versal deformation space of Y. We chose a curve

$$Y' := J_1 \cup \ldots \cup J_g, \tag{68}$$

where this time $\{p^i\} = J_i \cap J_{i+1}$ and now we assume that $\mathcal{O}_{J_i}\left(p^i - p^{i-1}\right) \in \operatorname{Pic}^0(J_i)$ is *not* a torsion point for $i = 1,\dots,g$. In the same way, one can construct a limit linear series $\ell' \in \overline{G}^r_d(Y')$ having the *same* ramification profile as $\ell \in \overline{G}^r_d(Y)$ at the points of intersection $J_i \cap J_{i-1}$. Clearly, ℓ and ℓ' belong to the same irreducible component of $\widetilde{\mathcal{G}}^r_d$.

As explained in [EH86, Theorem 3.4], every component of $\widetilde{\mathcal{G}}^r_d$ has dimension at least $3g-3+\rho(g,r,d)$. To conclude that a component of $\widetilde{\mathcal{G}}^r_d$ containing the point $[Y,\ell]$ dominates $\overline{\mathcal{M}}_g$ it thus suffices to show that the local dimension of the fibre $\sigma^{-1}\big([Y']\big)$ at ℓ' has dimension precisely $\rho(g,r,d)$. Assume that $\tilde{\ell}$ is a limit linear series on Y' lying in a component of $\overline{G}^r_d(Y')$ passing through ℓ and having dimension larger than $\rho(g,r,d)$. Then we may assume that $\tilde{\ell}$ is refined and using the additivity of the adjusted Brill-Noether numbers [EH86, Lemma 3.6], we write

$$\rho(g,r,d) = \sum_{a=1}^{g} \rho\Big(1,r,d,\alpha^{\tilde{\ell}_{J_a}}\left(p^{a-1}\right),\alpha^{\tilde{\ell}_{J_a}}\left(p^{a}\right)\Big),$$

and obtain that there exists $a \in \{1, \ldots, g\}$ such that $\rho\Big(1, r, d, \alpha^{\tilde{\ell}_{J_a}}(p^{a-1}), \alpha^{\tilde{\ell}_{J_a}}(p^a)\Big) < 0$. Then there exist integers $0 \le i_1 < i_2 \le r$ such that

$$\alpha_{i_1}^{\tilde{\ell}_{J_a}}(p^{a-1}) + \alpha_{r-i_1}^{\tilde{\ell}_{J_a}}(p^a) = \alpha_{i_2}^{\tilde{\ell}_{J_a}}(p^{a-1}) + \alpha_{r-i_2}^{\tilde{\ell}_{J_a}}(p^a) = d - r,$$

in particular there exist two sections of the J_a-aspect of $\tilde{\ell}$ which vanish only at the points p^{a-1} and p^a, implying that $\mathcal{O}_{J_a}(p^{a-1} - p^a) \in \mathrm{Pic}^0(J_a)$ is torsion, which is a contradiction. Thus ℓ is smoothable to a linear system $L \in W^r_d(C)$ on every smooth curve $C \in \Lambda_g$. Moreover, we may also assume $h^0(C, L) = r + 1$.

We now return to the curve Y and construct a limit linear series $\mathfrak{a} \in \overline{G}^1_k(Y)$ having the ramification profiles

$$\alpha^{\mathfrak{a}_{J_i}}(p^{(i-1)}) = \alpha^{\mathfrak{a}_{J_i}}(p^{(i)}) = (0, k-1), \quad \text{for } i = 1, \ldots, g.$$

Since $Z \cap J = \emptyset$ it follows that $\mathfrak{a}$ smooths to $A = \mathcal{O}_C(Z) \in W^1_k(C)$ for a neighboring smooth curve $C \in \Lambda_g$. Assuming now that $H^0(C, L \otimes A^\vee) \neq 0$, we also obtain by semicontinuity that there exists a section

$$0 \neq \sigma \in V_{J_1} \cap H^0\big(J_1, \mathcal{O}_{J_1}((d-k) \cdot p^{(1)})\big).$$

But using (64) we have $\mathrm{ord}_{p^{(1)}}(\sigma) \ge d - r - 1 > d - k$, since $k \ge r + 2$, which is a contradiction and thus finishes the proof. □

Remark 7.7. It remains an interesting question whether the conclusion of Theorem 7.6 holds for an arbitrary elliptic $K3$ surface like in Proposition 3.1.

8. Hurwitz-Brill-Noether general curves over number fields

The aim of this short section is to explain how curves on Halphen surfaces provide explicit examples of k-gonal curves defined over a number field which are general from the viewpoint of Hurwitz-Brill-Noether theory. The strategy is inspired by the papers [ABFS16] and [FT17], in which explicit examples of smooth curves of genus g defined over the rationals are provided which satisfy the classical Brill-Noether-Petri theorem.

We fix a Halphen surface $X = \mathrm{Bl}_{p_1,\ldots,p_9}(\mathbf{P}^2)$ of index k and consider the Du Val linear system $\Lambda_g = \big|3gh - gE_1 - \cdots - gE_8 - (g-1)E_9\big|$ of curves of genus g defined as in (60). We use all the notation from the previous section.

Definition 8.1. A smooth k-gonal curve C of genus g is said to be Hurwitz-Brill-Noether general if $\dim W^r_d(C) = \rho_k(g, r, d)$ for all integers $r, d > 0$.

Before proving the next result, we recall Pflueger's following definition [Pfl17, Definition 2.5]. For a natural number n we write $[n] := \{1, \ldots, n\}$. Then a *k-uniform displacement tableau* is a function $t\colon [r+1] \times [g-d+r] \to \mathbb{Z}_{>0}$ such that

- $t(x+1, y) > t(x, y)$ and $t(x, y+1) > t(x, y)$, for all x, y, and
- if $t(x, y) = t(x', y')$, then $x - y \equiv x' - y' \mod k$.

In other words, t consists of labeled boxes, where two labels may coincide only if they are k boxes apart.

Theorem 8.2. *Let X be a Halphen surface of index $k \ge 2$. Then a general curve $C \in \Lambda_g$ is Hurwitz-Brill-Noether general.*

Proof. We first show that if $C \in \Lambda_g$ is a general curve, then $\dim W^r_d(C) \leq \rho_k(g,r,d)$. Assume there exists a component Z of $W^r_d(C)$ of dimension $\dim(Z) > \rho_k(g,r,d)$.

As in the proof of Theorem 7.6, we let C degenerate to the transversal union $J \cup C_{g-1}$, where C_{g-1} is a general curve from the Du Val linear system Λ_{g-1} and where C_{g-1} and J meet at the point $p_{10}^{(g-1)}$. Recall that $p_{10}^{(g)}$ lies on J and that $\mathcal{O}_J\big(p_{10}^{(g)} - p_{10}^{(g-1)}\big)$ is a k-torsion point. Under this degeneration, Z specializes to a component Z_{g-1} of the variety of limit linear series $\overline{G}^r_d\big(C_{g-1} \cup J\big)$ having $\dim(Z_{g-1}) > \rho_k(g,r,d)$.

We consider the forgetful map $\pi_g \colon \overline{G}^r_d\big(C_{g-1} \cup J\big) \to G^r_d\big(C_{g-1}\big)$ retaining the C_{g-1}-aspect of each limit linear series. There exists a Schubert index $\bar{\alpha}^{g-1}$ of type (r,d) maximal with respect to the property $\pi_g(Z_{g-1}) \subseteq G^r_d\big(C_{g-1}, (p_{10}^{(g-1)}, \bar{\alpha}^{g-1})\big)$. Therefore for a general point $\ell = (\ell_{C_{g-1}}, \ell_J) \in Z_{g-1}$, we have that $\alpha^{\ell_{C_{g-1}}}\big(p_{10}^{(g-1)}\big) = \bar{\alpha}^{g-1}$.

We let C_{g-1} degenerate to the transversal union $J \cup C_{g-2}$, where $C_{g-2} \in \Lambda_{g-2}$. Then $C_{g-2} \cdot J = p_{10}^{(g-2)}$ and recall that $p_{10}^{(g-2)} \in J$ is such that $p_{10}^{(g-1)} - p_{10}^{(g-2)} = p_1 + \cdots + p_9$. We now consider the forgetful map

$$\pi_{g-1} \colon \overline{G}^r_d\Big(C_{g-2} \cup J, (p_{10}^{(g-1)}, \bar{\alpha}^{g-1})\Big) \longrightarrow G^r_d(C_{g-2})$$

that retains the C_{g-2}-aspect of each limit linear series. The subvariety $\pi_{g-1}(Z_{g-1})$ then specializes to a subvariety $Z_{g-2} \subset \overline{G}^r_d\big(C_{g-2} \cup J, (p_{10}^{(g-1)}, \bar{a}^{g-1})\big)$. We choose the Schubert index $\bar{\alpha}^{g-2}$ maximal with the property that $\pi_{g-1}(Z_{g-2}) \subseteq G^r_d\big(C_{g-2}, (p_{10}^{(g-2)}, \bar{\alpha}^{g-2})\big)$.

Inductively, assume that we have found a Schubert index $\bar{\alpha}^{g-i+1}$ and defined a subvariety $Z_{g-i} \subset \overline{G}^r_d\big(C_{g-i} \cup J, (p_{10}^{(g-i+1)}, \bar{\alpha}^{g-j+1})\big)$. If

$$\pi_{g-i+1} \colon \overline{G}^r_d\Big(C_{g-i} \cup J, (p_{10}^{(g-i+1)}, \bar{\alpha}^{g-i+1})\Big) \longrightarrow G^r_d(C_{g-i})$$

is the map retaining the C_{g-i}-aspect, let $\bar{\alpha}^{g-i}$ be the maximal Schubert index with the property $\pi_{g-i+1}(Z_{g-i}) \subset G^r_d\big(C_{g-i}, (p_{10}^{(g-i)}, \bar{\alpha}^{g-i})\big)$. Then we let C_{g-i} degenerate to the union $C_{g-i-1} \cup J$, where $C_{g-i-1} \in \Lambda_{g-i-1}$ is a general element meeting J at the point $p_{10}^{(g-i-1)}$. Let $Z_{g-i-1} \subset \overline{G}^r_d\big(C_{g-i-1} \cup J, (p_{10}^{(g-i)}, \bar{\alpha}^{g-i})\big)$ be the limiting subvariety of $\pi_{g-i-1}(Z_{g-i})$ under this degeneration.

Just like in the proof of Theorem 7.6, we consider the chain of elliptic curves

$$Y := J_1 \cup \ldots \cup J_g,$$

where $J_i \cong J$ and $\big\{p^{(i)}\big\} = \big\{p_{10}^{(i)}\big\} = J_i \cap J_{i+1}$. We recall that $\mathcal{O}_{J_i}\big(p^{(i)} - p^{(i-1)}\big) \in \mathrm{Pic}^0(J_i)$ is k-torsion. Via the previous argument, after g steps, we find that there exists a family of limit linear series on Y having dimension strictly exceeding $\rho_k(r,d)$ consisting of limit linear series $\ell \in \overline{G}^r_d(Y)$ satisfying the ramification conditions $\alpha^{\ell_{J_i}}\big(p^{(i)}\big) = \bar{\alpha}^i = \big(\alpha_0^i, \alpha_1^i, \ldots, \alpha_r^i\big)$, for $i = 1, \ldots, g$. Set $a_n^i := \alpha_n^i + i$, for $n = 0, \ldots, r$ for the corresponding vanishing sequences.

One has the inequalities $a_n^i + 1 \geq a_n^{i-1} \geq a_n^i$ for all $n = 0, \ldots, r$ and $i = 1, \ldots, g$. Furthermore, if $a_n^{i-1} = a_n^i$, then the J_i-aspect of a general limit linear series of this family has as underlying line bundle $L_i \cong \mathcal{O}_{J_i}\big(a_n^i \cdot p^{(i)} + (d - a_n^i) \cdot p^{(i-1)}\big)$. In particular, since $p^i - p^{i-1}$ is a k-torsion point on J_i, if $a_n^i = a_n^{i-1}$ and $a_{n'}^i = a_{n'}^{i-1}$ for two integers $0 \le n < n' \le r$, then $a_{n'}^i - a_n^i \equiv 0 \bmod k$.

To a limit linear series ℓ we associate a tableau $t\colon [r+1] \times [g-d+r] \to [g]$ by setting

$$t(n,j) := \min\bigl\{i : \alpha_n^i + i = d - r + j\bigr\},$$

that is, $t(n,j)$ is the smallest integer i such that in the non-increasing sequence consisting of $\bar{\alpha}_n^0 := d - r, \bar{\alpha}_n^1, \ldots, \bar{\alpha}_n^i$ there exist $t(n,j)$ equalities. Then, as explained above, t defines a k-uniform displacement tableau. For each label i appearing in $\mathrm{Im}(t)$, the variety $G_d^r\bigl(J_i, (p^{(i-1)}, \alpha^{\ell_{J_i}}(p^{(i-1)})), (p^{(i)}, \bar{\alpha}^i)\bigr)$ appearing as a factor in $\overline{G}_d^r(Y)$ has dimension zero. It follows that $\dim \overline{G}_d^r(Y)$ is bounded from above by the number of labels omitted from $[g]$ in a k-uniform displacement tableau on $[r+1] \times [g-d+r]$. Using Pflueger's result [Pfl17, Section 3], this quantity is bounded from above by $\rho_k(g,r,d)$, thus showing that for a general curve $C \in \Lambda_g$ we have $\dim W_d^r(C) \le \rho_k(g,r,d)$.

To show that $\dim W_d^r(C) \ge \rho_k(g,r,d)$ we use that if X is a Halphen surface of degree k giving rise to the surface $\overline{X} \subseteq \mathbf{P}^g$ of degree $2g-2$, then there exists a family of degree k polarized $K3$ surfaces (X_t, H_t, E_t) such that the corresponding image $X_t \xrightarrow{|H_t|} \mathbf{P}^g$ has as limit the surface $\overline{X}$. Since for a general curve $C_t \in |H_t|$ we have established in Corollary 7.3 that $\dim W_d^r(C) = \rho_k(g,r,d)$, by semicontinuity $\dim W_d^r(C) \ge \rho_k(g,r,d)$. □

As explained in [ABFS16], one can write down rational points $p_1, \ldots, p_9 \in \mathbf{P}^2$ such that the general curve $C \in \Lambda_g$ is defined over $\mathbb{Q}$ and satisfies the Brill-Noether-Petri theorem. In the case of a Halphen surface X of index k, the condition $\mathcal{O}_J(J) \in \mathrm{Pic}^0(J)[k]$ cannot be realized over the rationals. For instance, Mazur [Maz78] showed that if J is an elliptic curve defined over $\mathbb{Q}$, then any prime k dividing $E(\mathbb{Q})_{\mathrm{tors}}$ satisfies $k \le 7$. However we have the following result:

Theorem 8.3. *For every prime k there exists a Hurwitz-Brill-Noether general curve $[C,A] \in \mathcal{H}_{g,k}$ defined over a number field K with $[K:\mathbb{Q}] \le k^2 - 1$.*

Proof. We start with any elliptic curve $J \subseteq \mathbf{P}^2$ defined over $\mathbb{Q}$ and with 8 rational points $p_1, \ldots, p_8$. Then $p_9 \in J$ is determined by the condition that $p_1 + \cdots + p_9$ is a k-torsion point with respect to the group law of J. We apply [LR13, Theorems 2.1 and 5.1], to conclude that the field K of the definition of p_9 satisfies

$$[K:\mathbb{Q}] \le k^2 - 1.$$

Indeed, if $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_k) \cong \mathrm{Aut}\bigl(E[k]\bigr)$ denotes the Galois representation on the k-torsion points of J, then $[K:\mathbb{Q}] = |\mathrm{Im}(\rho)|/|H|$, where H is the subgroup of all matrices of type $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$, where $a \in \mathbb{F}_k$ and $b \in \mathbb{F}_k^*$. Since $|H| = k(k-1)$, it follows that $[K:\mathbb{Q}] \le k^2 - 1$, with equality if and only if ρ is surjective, which in fact happens for all but finitely many primes k.

The general curve $C \in \Lambda_g$ defined by (60) will be then defined over K and by Theorem 8.2 is Hurwitz-Brill-Noether general. □

Remark 8.4. It is an interesting open question whether there exists a smooth k-gonal curve of genus g defined over $\mathbb{Q}$ which is Hurwitz-Brill-Noether general. As explained, curves on Halphen surface of degree k will not provide the answer for all k. Similarly, we do not known whether there exists a degree k elliptic $K3$ surface X like in Proposition 3.1 defined over $\mathbb{Q}$.

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