Minimal supersolutions of BSDEs under volatility uncertainty

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ABSTRACT

We study the existence of minimal supersolutions of BSDEs under a family of mutually singular probability measures. We consider generators that are jointly lower semicontinuous, positive, and either convex in the control variable and monotone in the value variable, or that fulfill a specific normalization property.

KEYWORDS: Minimal Supersolutions of Second Order Backward Stochastic Differential Equations; Model Uncertainty; G-Expectation

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1 Introduction

We study the existence of minimal supersolutions of BSDEs under a general family of mutually singular probability measures. To that end we consider a probability space $(\Omega, \mathcal{F}, P)$ carrying a Brownian motion W. By $(\mathcal{F}_t)$ we denote the Brownian filtration. Given a family Θ of volatility processes θ, we consider the process $\tilde{W} : \tilde{\Omega} \times [0, T] \to \mathbb{S}_d^{>0}$ defined as the stochastic integral

$$\tilde{W}(\theta) = \int \theta^{1/2} dW, \quad \theta \in \Theta,$$

where $\tilde{\Omega} := \Omega \times \Theta$. It generates a raw filtration $\tilde{\mathcal{F}}_t := \sigma(\tilde{W}_s; s \leq t)$, $t \in [0, T]$. The family of measures is now given by $P^\theta[A] := P[A(\theta)]$, $\theta \in \Theta$, for $A \in \tilde{\mathcal{F}}_T$ and in general it is not possible to define a probability measure under which all probability measures P^θ are absolutely continuous.

Following the approach developed in Drapeau, Heyne, and Kupper [10] and Heyne, Kupper, and Mainberger [13] we aim at constructing the candidate value process for the minimal supersolution of a BSDE by taking the essential infimum at each point in time and obtaining the corresponding control process by some compactness arguments. Since the definition of an essential infimum over a set of random variables depends strongly on the underlying probability measure we first provide conditions under which it is possible to define a related notion. More precisely, this is done by only minimizing over random variables with a specific regularity structure and by imposing regularity on the corresponding infimum. Moreover, by assuming that the set of probability measures is relatively compact we also obtain the existence of a sequence approximating the infimum in the capacity sense.

With this at hand, the next step is to adjust the framework of [10] and [13] in order to incorporate measurability with respect to the filtration $(\tilde{\mathcal{F}}_t)$ generated by $\tilde{W}$. Quite often, the analysis in [10, 13] is based on arguments involving supermartingales and their respective right hand limit processes. However, since in general $(\tilde{\mathcal{F}}_t)$ is neither right- nor left-continuous, we cannot resort to these standard procedures while staying adapted. Therefore, we adopt the notion of optional strong supermartingales, which, by a result of Dellacherie and Meyer [7], are làdlàg processes and relieve us of having to take right hand limits. Accordingly, we formulate our BSDE in a stronger sense, that is with respect to stopping times. More precisely, we say a làdlàg process Y and a control process Z constitute a supersolution of a backward stochastic differential equation if

$$\begin{cases} Y_\sigma - \int_\sigma^\tau g_s\left(Y_s, Z_s\right) ds + \int_\sigma^\tau Z_s d\tilde{W}_s \geq Y_\tau, \\ Y_T \geq \xi, \end{cases} \tag{1.1}$$

for all stopping times $0 \leq \sigma \leq \tau \leq T$. Here, Y and Z are adapted and predictable with respect to $(\tilde{\mathcal{F}}_t)$, respectively, and the equation is to be understood in a θ-wise sense, that is for example $\int Z d\tilde{W}$ represents the family of projections $(\int Z(\theta) dW(\theta))_{\theta\in\Theta}$.

Our main result proves that under the same conditions on the generator as in [10] and [13], and under some regularity assumptions on the candidate value process there exists a minimal supersolution to (1.1) in the quasi-sure sense. As mentioned above we construct the candidate value process by taking an appropriate essential infimum. With the candidate value process at hand we obtain the candidate control process by arguing for each θ separately and then aggregating similar to Soner, Touzi, and Zhang [20] and Nutz and Soner [17] by using a result by Karandikar [14].

The super replication problem under model uncertainty introduced by Lyons [16] is relatively recent and has been subject to many studies, see for example Avellaneda, Levy, and Paras [1], Bion-Nadal and Kervarec [2, 3], Denis and Martini [9], Epstein and Ji [11]. Except for the latter, they all take into account a superhedging problem under volatility uncertainty, whereas the latter also takes into account drift uncertainty. It happens that the mathematical techniques underlying the problem of superhedging under volatility uncertainty are related to the theory of capacities introduced by Choquet [5] and to quasi-sure stochastic analysis, see [24], Denis, Hu, and Peng [8], Denis and Martini [9], and the numerous references therein.

The superhedging problem under volatility uncertainty is also closely linked to other mathematical topics. On the one hand, to the so called G-expectations introduced by Peng [18, 19], see also [8] and Soner, Touzi, and Zhang [22] for further studies and references. On the other hand, to fully non-linear parabolic Partial Differential Equations as introduced by Cheridito, Soner, Touzi, and Victoir [4] and second order Backward Stochastic Differential Equations – 2BSDE for short – see [20] for the well posedness, Soner, Touzi, and Zhang [23] for a dual formulation, and Soner and Touzi [21] for the corresponding dynamic programing principle.

The paper is organized as follows. In Section 2 we fix our notations and the setting, and introduce our notion of essential infimum. We define minimal supersolutions and introduce our main conditions in Section 3, which also contains our main result.

2 Setting and Notation

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a d-dimensional Brownian motion W. By $(\mathcal{F}_t)$ we denote the augmented filtration generated by W, which satisfies the usual conditions. Let $L^0(\mathcal{F}_t)$ denote the set of $\mathcal{F}_t$-measurable random variables, where two of them are identified if they agree P-almost surely. For $p > 0$, the space $L^p(\mathcal{F}_t)$ denotes those random variables in $L^0(\mathcal{F}_t)$ with finite p-norm. We fix a finite time horizon $T > 0$ and denote by $\mathcal{T}$ the set of $(\mathcal{F}_t)$-stopping times on Ω with values in $[0, T]$. An $(\mathcal{F}_t)$-optional process $Y : \Omega \times [0, T] \to \mathbb{R}$ is a *strong supermartingale* if $Y_\sigma \in L^1(\mathcal{F}_\sigma)$ and $E[Y_\tau \,|\, \mathcal{F}_\sigma] \leq Y_\sigma$, for all $\sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$.

Let Θ be a family of volatility processes

$$\theta : \Omega \times [0, T] \longrightarrow \mathbb{S}_d^{>0},$$

which are progressively measurable and such that $\int_0^T \|\theta_u^{1/2}\|^2 du < \infty$, P-almost surely. Here, $\mathbb{S}_d^{>0}$ is the set of strictly positive definite $d \times d$-matrices. On the product space $\tilde{\Omega} := \Omega \times \Theta$, we consider the process $\tilde{W} : \tilde{\Omega} \times [0, T] \to \mathbb{R}^d$ defined as the stochastic integral

$$\tilde{W}(\theta) = \int \theta^{1/2} dW, \quad \theta \in \Theta,$$

generating the filtration $(\tilde{\mathcal{F}}_t)$, where $\tilde{\mathcal{F}}_t := \sigma(\tilde{W}_s; s \leq t)$. Since $(\tilde{\mathcal{F}}_t)$ is in general not right-continuous, we also consider $(\tilde{\mathcal{F}}_t^+)$ defined by $\tilde{\mathcal{F}}_t^+ := \bigcap_{s>t} \tilde{\mathcal{F}}_s$, for $t \in [0, T)$, and $\tilde{\mathcal{F}}_T^+ := \tilde{\mathcal{F}}_T$.

On the sigma-algebra $\tilde{\mathcal{F}}_T$, it is in general not possible to define a probability measure under which all probability measures $P^\theta[A] := P[A(\theta)]$, $\theta \in \Theta$, are absolutely continuous. We therefore define the set function $\tilde{P} : \tilde{\mathcal{F}}_T \to [0, 1]$ by

$$\tilde{P}[A] := \sup_{\theta \in \Theta} P^\theta[A], \quad A \in \tilde{\mathcal{F}}_T. \tag{2.1}$$

By $C_0([0, T]; \mathbb{R}^d)$ we denote the space of continuous functions $\varphi : [0, T] \to \mathbb{R}^d$, $\varphi(0) = 0$, equipped with the uniform norm $||\varphi||_\infty := \sup_{0 \leq t \leq T} |\varphi(t)|$.

Remark 2.1. For each $\theta \in \Theta$, let $\mu^\theta[B] := P[\tilde{W}(\theta) \in B]$, where $B \in \mathscr{B}(C_0([0, T]; \mathbb{R}^d))$. By means of [8, Theorem 1], $c(B) := \sup_{\theta \in \Theta} \mu^\theta(B)$ defines a *capacity* on $\mathscr{B}\left(C_0([0, T]; \mathbb{R}^d)\right)$. Since any $A \in \tilde{\mathcal{F}}_T$ is of the form $A = \tilde{W}^{-1}(B)$ for some $B \in \mathscr{B}(C_0([0, T]; \mathbb{R}^d))$, it follows that $\tilde{P}[A] = c(B)$ is a capacity on $\tilde{\mathcal{F}}_T$. In applications, the measures μ^θ, $\theta \in \Theta$, are often mutually singular. ♦

Sometimes we work under the following assumption on the measures μ^θ defined in Remark 2.1.

(RCP) the set $\{\mu^\theta : \theta \in \Theta\}$ is relatively weak*-compact[1].

In the following we summarize some notation of capacity theory, see also [8]. A subset A of $\tilde{\Omega}$ is called a *polar set* if there exists $B \in \tilde{\mathcal{F}}_T$ with $A \subset B$ such that $\tilde{P}[B] = 0$. The set of

[1]That is, the $\sigma(\mathcal{M}_1, C_b)$-topology on the probability measures over the Polish space $C_0([0, T]; \mathbb{R}^d)$.

all polar sets is denoted by $\mathcal{N}$. We say that a property holds *quasi-surely* if this property holds outside a polar set, that is, this property holds P^θ-almost surely for all $\theta \in \Theta$. By $L^0(\tilde{\mathcal{F}}_t)$ we denote the set of $\tilde{\mathcal{F}}_t$-measurable random variables $X : \tilde{\Omega} \to \mathbb{R}$, where two of them are identified if they agree quasi-surely. Equalities and inequalities between $\tilde{\mathcal{F}}_t$-measurable random variables are understood in the quasi-sure sense. For any $X \in L^0(\tilde{\mathcal{F}}_T)$ such that $E[X(\theta)]$ exists for all $\theta \in \Theta$, we define the *upper expectation* of X as

$$\tilde{E}\,[X] := \sup_{\theta\in\Theta} E\,[X(\theta)]\,. \tag{2.2}$$

The set $L^1(\tilde{\mathcal{F}}_T)$ consists of those $X \in L^0(\tilde{\mathcal{F}}_T)$, for which $\tilde{E}[|X|] < +\infty$. In line with [8], let $L^1_b(\tilde{\mathcal{F}}_T)$ be the set of those $X \in L^1(\tilde{\mathcal{F}}_T)$ for which the family $(X(\theta))_{\theta\in\Theta}$ is uniformly integrable in $L^1(\mathcal{F}_T)$.

For any $X \in L^0(\tilde{\mathcal{F}}_t)$, there exists a measurable function $\varphi : C_0([0,T],\mathbb{R}^d) \to \mathbb{R}$ such that $X = \varphi(\tilde{W}^t)$, where $\tilde{W}^t$ is the stopped process $\tilde{W}^t_s := \tilde{W}_{s\wedge t}$. We then define the set of continuous $\tilde{\mathcal{F}}_t$-measurable random variables as

$$C(\tilde{\mathcal{F}}_t) := \left\{\varphi(\tilde{W}^t) : \varphi \in C\left(C_0([0,T];\mathbb{R}^d);\mathbb{R}\right)\right\}.$$

In general it is not possible to define an "essential infimum" for subsets in $L^0(\tilde{\mathcal{F}}_t)$ with respect to the capacity $\tilde{P}$. However, under the assumption that the infimum of a set is continuous, it is an essential infimum in the following sense.

Proposition 2.2. *Let $\mathcal{X}$ be a subset of $C(\tilde{\mathcal{F}}_t)$ such that $X^* = \varphi^*(\tilde{W}^t) \in C(\tilde{\mathcal{F}}_t)$, where φ^* is the infimum over all continuous functions $\varphi : C_0([0,T];\mathbb{R}^d) \to \mathbb{R}$ satisfying $X = \varphi(\tilde{W}^t)$ for some $X \in \mathcal{X}$. Then, for any $\theta \in \Theta$, there exists a sequence (X^n) in $\mathcal{X}$ such that*

$$X^*(\theta) = \left(\inf_{n\in\mathbb{N}} X_n\right)(\theta). \tag{2.3}$$

If in addition (RCP) is fulfilled and $\mathcal{X}$ is directed downward, then there exists a decreasing sequence (X^n) in $\mathcal{X}$ such that for every $\varepsilon > 0$ it holds

$$\lim_{n\to\infty} \tilde{P}\,[(X^n - X^*) \geq \varepsilon] = 0. \tag{2.4}$$

If moreover $\mathcal{X} \subseteq L^1_b(\tilde{\mathcal{F}}_t)$ and $X^* \in L^1_b(\tilde{\mathcal{F}}_t)$, then

$$\lim_{n\to\infty} \tilde{E}\,[X^n - X^*] = 0. \tag{2.5}$$

Proof. Step 1: Fix $\varepsilon > 0$ and $\theta \in \Theta$. There exists a compact set $K \in \mathscr{B}(C_0([0,T];\mathbb{R}^d))$ such that $\mu^\theta(K^c) \leq \varepsilon$. For any $x \in K$ let $\varphi^\varepsilon_x : C_0([0,T];\mathbb{R}) \to \mathbb{R}$ be a continuous function such that $\varphi^\varepsilon_x(\tilde{W}^t) \in \mathcal{X}$ and $|\varphi^*(x) - \varphi^\varepsilon_x(x)| \leq \varepsilon$, and define the open sets

$$O^\varepsilon_x := \left\{y \in C_0([0,T];\mathbb{R}^d) : |\varphi^\varepsilon_x(y) - \varphi^\varepsilon_x(x)| < \varepsilon \text{ and } |\varphi^*(x) - \varphi^*(y)| < \varepsilon\right\}.$$

The family $(O^\varepsilon_x)_{x\in K}$ is an open cover of K, so that by compactness, there exist $x_1,\dots,x_N$ such that $K\subseteq O^\varepsilon_{x_1}\cup\dots\cup O^\varepsilon_{x_N}$. By construction holds $\varphi^\varepsilon_{x_1}\wedge\dots\wedge\varphi^\varepsilon_{x_N}\leq\varphi^*+3\varepsilon$ on the set K. Hence

$$P\left[\varphi^\varepsilon_{x_1}\left(\tilde W^t(\theta)\right)\wedge\dots\wedge\varphi^\varepsilon_{x_N}\left(\tilde W^t(\theta)\right)>\varphi^*\left(\tilde W^t(\theta)\right)+3\varepsilon\right]\leq\varepsilon.$$

This shows that $X^*(\theta)=\operatorname{ess\,inf}\left\{X\in L^0(\mathcal{F}_t):X\in\mathcal{X}(\theta)\right\}$ and by Föllmer and Schied [12, Theorem A.32] there exists a sequence (X^n) in $\mathcal{X}$ such that $X^*(\theta)=(\inf_n X^n)(\theta)$.

Step 2: Fix $\varepsilon>0$. By means of Prohorov's theorem, the relatively weak* compactness of $\{\mu^\theta:\theta\in\Theta\}$ is equivalent to the fact that $\{\mu^\theta:\theta\in\Theta\}$ is tight, see also [8, Theorem 6]. It follows that there exists a compact set $K\in\mathscr{B}(C_0([0,T];\mathbb{R}^d))$ such that $c(K^c)\leq\varepsilon$. We then define

$$X^\varepsilon:=\varphi^\varepsilon_{x_1}\left(\tilde W^t\right)\wedge\dots\wedge\varphi^\varepsilon_{x_N}\left(\tilde W^t\right),$$

where $\varphi^\varepsilon_{x_1},\dots,\varphi^\varepsilon_{x_N}$ are continuous functions which are similarly constructed as in the previous step. Since $\mathcal{X}$ is directed downwards, X^ε is an element of $\mathcal{X}$ and satisfies $\tilde P[(X^\varepsilon-X^*)>3\varepsilon]\leq\varepsilon$. Thus, the sequence $(X^{1/n})_{n\in\mathbb{N}}$ is as desired.

Step 3: Finally, let (X^n-X^*) be a decreasing sequence in $L^1_b(\tilde{\mathcal{F}}_t)$. Since for every $\varepsilon>0$ it holds

$$0\leq\tilde E\left[X^n-X^*\right]\leq\tilde E\left[(X^n-X^*)\,1_{\{(X^n-X^*)>\varepsilon\}}\right]+\varepsilon,$$

and since the first term on the right hand side tends to zero due to the epsilon-delta-criterion for uniformly integrable sets, it follows that $\tilde E\left[X^n-X^*\right]\to 0$. □

3 Minimal Supersolutions of 2BSDEs

Let $M,N:\tilde\Omega\times[0,T]\to\mathbb{R}$ be $(\tilde{\mathcal{F}}_t)$-adapted processes. The process M is called càdlàg, càglàd or làdlàg if the paths of M are càdlàg, càglàd or làdlàg quasi-surely, respectively. Given a làdlàg process, we denote by M^- and M^+ its càglàd and càdlàg version respectively, that is

$$M^-_t:=\lim_{s\nearrow t}M_s,\quad \text{for } t\in]0,T],\quad\text{and}\quad M^-_0:=M_0,$$
$$M^+_t:=\lim_{s\searrow t}M_s,\quad \text{for } t\in[0,T[,\quad\text{and}\quad M^+_T:=M_T,$$

outside the polar set where M is not làdlàg. Two $(\tilde{\mathcal{F}}_t)$-adapted processes $M,N:\tilde\Omega\times[0,T]\to\mathbb{R}$ are modifications of each others, if $M_t=N_t$, for all $t\in[0,T]$. We say that M is a *supermartingale* or a *strong supermartingale*, if $M(\theta)$ is a supermartingale or a strong supermartingale, for all $\theta\in\Theta$, respectively. See [7, Appendix I] for a definition of strong supermartingales.

Let us define the following sets of value and control processes:

- $\mathcal{S}$ is the set of $(\mathcal{F}_t)$-adapted làdlàg processes $Y:\Omega\times[0,T]\to\mathbb{R}$;

- $\tilde{\mathcal{S}}$ is the set of làdlàg processes $Y : \tilde{\Omega} \times [0,T] \to \mathbb{R}$, which have a modification $\hat{Y}$ satisfying $\hat{Y}_t \in C(\tilde{\mathcal{F}}_t) \cap L^1_b(\tilde{\mathcal{F}}_t)$, for all $t \in [0,T]$, and such that $Y(\theta)$ is optional, for all $\theta \in \Theta$.
- For any $\theta \in \Theta$, let $\mathcal{L}(\theta)$ be the set of $(\mathcal{F}_t)$-predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}^d$ such that $P\left[\int_0^T \|Z_u \theta_u^{1/2}\|^2 du < \infty\right] = 1$.
- $\tilde{\mathcal{L}}$ is the set of $(\tilde{\mathcal{F}}_t)$-predictable processes $Z : \tilde{\Omega} \times [0,T] \to \mathbb{R}^d$ such that $Z(\theta) \in \mathcal{L}(\theta)$, for all $\theta \in \Theta$.

A *generator* is a jointly measurable function g from $\tilde{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$ to $\mathbb{R} \cup \{+\infty\}$ such that the mapping $(s,\omega,\theta) \mapsto g_s(\omega,\theta,y,z) : ([0,t] \times \tilde{\Omega}, \mathcal{B}([0,t]) \otimes \tilde{\mathcal{F}}_t) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable, for each t, for all $(y,z) \in \mathbb{R}^{d+1}$. We say that a generator g is

(POS) positive, if $g(\theta,y,z) \geq 0$;

(LSC) if $(y,z) \mapsto g(\theta,y,z)$ is lower semicontinuous;

(MON) increasing, if $g(\theta,y,z) \geq g(\theta,y',z)$, whenever $y \geq y'$;

(MON′) decreasing, if $g(\theta,y,z) \leq g(\theta,y',z)$, whenever $y \geq y'$;

(CON) convex, if $g(\theta,y,\lambda z + (1-\lambda) z') \leq \lambda g(\theta,y,z) + (1-\lambda) g(\theta,y,z')$, for all $\lambda \in (0,1)$;

(NOR) normalized, if $g(\theta,y,0) = 0$;

$P \otimes dt$-almost surely, for all $y, y' \in \mathbb{R}$, all $z, z' \in \mathbb{R}^{1\times d}$ and all $\theta \in \Theta$.

A pair $(Y,Z) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{L}}$ is said to be a *supersolution* of the BSDE with generator g and terminal condition $\xi \in L^0(\tilde{\mathcal{F}}_T)$, if

$$Y_\sigma(\theta) - \int_\sigma^\tau g_u(\theta, Y_u(\theta), Z_u(\theta))du + \int_\sigma^\tau Z_u(\theta) d\tilde{W}_u(\theta) \geq Y_\tau(\theta) \quad \text{and} \quad Y_T(\theta) \geq \xi(\theta), \tag{3.1}$$

for all $\sigma, \tau \in \mathcal{T}$, with $\sigma \leq \tau$, and for all $\theta \in \Theta$. For such a supersolution (Y,Z), we call Y the *value process* and Z its *control process*. In order to exclude doubling strategies we only consider control processes, which are admissible, that is $\int Z(\theta) d\tilde{W}(\theta)$ is a supermartingale, for all $\theta \in \Theta$. We denote the set of such supersolutions by

$$\mathcal{A}(\xi,g) = \{(Y,Z) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{L}} : Z \text{ is admissible and (3.1) holds}\}. \tag{3.2}$$

A pair (Y,Z) is said to be a *minimal supersolution*, if $(Y,Z) \in \mathcal{A}(\xi,g)$, and if for any other supersolution $(Y',Z') \in \mathcal{A}(\xi,g)$, holds $Y_t \leq Y'_t$, for all $t \in [0,T]$. The natural candidate for the value process of a minimal supersolution is the infimum, that is

$$\hat{\mathcal{E}}_t(\xi) := \hat{\mathcal{E}}^g_t(\xi) := \inf\{Y_t : (Y,Z) \in \mathcal{A}(\xi,g)\}, \quad t \in [0,T]. \tag{3.3}$$

The goal is to find a modification $\mathcal{E}(\xi)$ of $\hat{\mathcal{E}}(\xi)$ that belongs to $\tilde{\mathcal{S}}$ and some admissible process Z in $\tilde{\mathcal{L}}$ such that $(\mathcal{E}(\xi), Z)$ fulfills (3.1), that is $(\mathcal{E}(\xi), Z)$ is a minimal supersolution.

We make the following observations.

Lemma 3.1. *Let g be a generator fulfilling (POS), and $\xi \in L^0(\tilde{\mathcal{F}}_T)$ be a terminal condition such that $\xi^-(\theta) \in L^1(\mathcal{F}_T)$, for all $\theta \in \Theta$. Let $(Y, Z) \in \mathcal{A}(\xi, g)$. Then $\xi(\theta) \in L^1(\mathcal{F}_T)$, for all $\theta \in \Theta$, and*

(i) the value process Y is a strong supermartingale such that $Y_\sigma(\theta) \geq -E[\xi^-(\theta) \,|\, \mathcal{F}_\sigma]$, for all $\sigma \in \mathcal{T}$, and all $\theta \in \Theta$.

(ii) it holds $Y^-_\sigma(\theta) \geq Y_\sigma(\theta) \geq Y^+_\sigma(\theta)$, for all $\sigma \in \mathcal{T}$, and all $\theta \in \Theta$. Moreover, we have $Y(\theta) = Y^+(\theta)$, $P \otimes dt$-almost surely, and $(Y^+(\theta), Z(\theta))$ fulfills (3.1).

Proof. As for Item (i), from (3.1) and the positivity of the generator follows

$$Y_0(\theta) + \int_0^\tau Z_u(\theta) d\tilde{W}_u(\theta) \geq Y_\tau(\theta) \geq -\xi^-(\theta) - \int_\tau^T Z_u(\theta) d\tilde{W}_u(\theta), \tag{3.4}$$

for all $\tau \in \mathcal{T}$ and $\theta \in \Theta$. Both sides being integrable by assumption, so is $Y_\tau(\theta) \in L^1(\mathcal{F}_\tau)$. Since $\xi^-(\theta) \leq \xi(\theta) \leq Y_T(\theta)$, we deduce $\xi(\theta) \in L^1(\mathcal{F}_T)$, for all $\theta \in \Theta$. Furthermore, from the admissibility of Z follows $Y_\tau(\theta) \geq -E[\xi^-(\theta) \,|\, \mathcal{F}_\tau]$. Similar to (3.4) we deduce that

$$Y_\sigma(\theta) \geq Y_\tau(\theta) - \int_\sigma^\tau Z_u(\theta) d\tilde{W}_u(\theta), \tag{3.5}$$

for all stopping times $0 \leq \sigma \leq \tau \leq T$.

As for Item (ii), the first statement follows from [7, Appendix 1, Remark 5.c, p. 397], since $Y(\theta)$ is a strong supermartingale and $(\mathcal{F}_t)$ fulfills the usual conditions. To see the second statement, note that by the làdlàg property of Y holds $(Y^+_\sigma(\theta))^- = Y^-_\sigma(\theta)$, for all $\sigma \in \mathcal{T}$. Consequently, for $\sigma \in \mathcal{T}$ such that $(Y^+_\sigma(\theta))^- = Y^+_\sigma(\theta)$, that is $Y^+(\theta)$ does not jump at σ, we have $Y^-_\sigma(\theta) = Y_\sigma(\theta) = Y^+_\sigma(\theta)$, that is $Y(\theta)$ does not jump at σ. Denote with (τ^n) the sequence of stopping times which exausts the jumps of $Y^+(\theta)$, see Dellacherie and Meyer [6, Theorem IV.88B]. Then, the process $\bar{Y}^\theta$ defined by $\bar{Y}^\theta_t := Y^+_t(\theta) + \sum_n 1_{[\tau^n]}(t)(Y_{\tau^n}(\theta) - Y^+_{\tau^n}(\theta))$, for all $t \in [0, T]$, is an optional modification of $Y(\theta)$. Moreover, it holds $\bar{Y}^\theta_\sigma = Y_\sigma(\theta)$, for all $\sigma \in \mathcal{T}$. Hence, by [6, Theorem IV.86] $\bar{Y}^\theta$ is indistinguishable from $Y(\theta)$. Since, by definition $\bar{Y}^\theta = Y^+(\theta)$, $P \otimes dt$-almost surely, we conclude $Y(\theta) = Y^+(\theta)$, $P \otimes dt$-almost surely. Finally, for any $\sigma, \tau \in \mathcal{T}$ let (σ_k) be a sequence of stopping times decreasing to σ. Then,

$$\begin{aligned} &Y^+_\sigma(\theta) - \int_\sigma^\tau g_u(\theta, Y^+_u(\theta), Z_u(\theta)) du + \int_\sigma^\tau Z_u(\theta) d\tilde{W}_u(\theta) \\ &\quad = \lim_k Y_{\sigma_k}(\theta) - \int_{\sigma_k}^\tau g_u(\theta, Y_u(\theta), Z_u(\theta)) du + \int_{\sigma_k}^\tau Z_u(\theta) d\tilde{W}_u(\theta) \geq \lim_k Y_\tau(\theta) \geq Y^+_\tau. \end{aligned}$$ □

Proposition 3.2. *Let g be a generator fulfilling (POS), and $\xi \in L^0(\tilde{\mathcal{F}}_T)$ be a terminal condition such that $\xi^-(\theta) \in L^1$, for all $\theta \in \Theta$. Suppose that $\mathcal{A}(\xi, g) \neq \emptyset$ and that $\hat{\mathcal{E}}_t(\xi) \in C(\tilde{\mathcal{F}}_t)$, for all $t \in [0,T]$. Then $\hat{\mathcal{E}}(\xi)$ is a supermartingale, and the limits*

$$\hat{\mathcal{E}}_t^-(\xi) := \lim_{s\uparrow t, s\in\Pi} \hat{\mathcal{E}}_s(\xi),\ \hat{\mathcal{E}}_0^-(\xi) := \hat{\mathcal{E}}_0(\xi) \quad \text{and} \quad \hat{\mathcal{E}}_t^+(\xi) := \lim_{s\downarrow t, s\in\Pi} \hat{\mathcal{E}}_s(\xi),\ \hat{\mathcal{E}}_T^+(\xi) := \hat{\mathcal{E}}_T(\xi) \tag{3.6}$$

exist, for all $t \in]0,T[$, quasi-surely, where $\Pi := \{kT/n : n \in \mathbb{N}, k = 0, \ldots, n\}$. Moreover, $\hat{\mathcal{E}}^-(\xi)$ is a càglàd supermartingale and $\hat{\mathcal{E}}^+(\xi)$ is a càdlàg supermartingale, which satisfy

$$\hat{\mathcal{E}}^-(\xi) \geq \hat{\mathcal{E}}^+(\xi) \quad \text{and} \quad \hat{\mathcal{E}}_t^-(\xi) \geq \hat{\mathcal{E}}_t(\xi) \geq \hat{\mathcal{E}}_t^+(\xi), \quad \text{for all } t \in [0,T]. \tag{3.7}$$

Proof. Note first that $\hat{\mathcal{E}}(\xi)$ is adapted by definition. Furthermore, given $(Y,Z) \in \mathcal{A}(\xi, g) \neq \emptyset$, Lemma 3.1 implies $\xi(\theta) \in L^1(\mathcal{F}_T)$ and $Y_t(\theta) \geq -E\left[\xi^-(\theta) \,|\, \mathcal{F}_t\right]$, for all $\theta \in \Theta$. Hence $Y_t(\theta) \geq \hat{\mathcal{E}}_t(\xi)(\theta) \geq -E[\xi^-(\theta) \,|\, \mathcal{F}_t]$ and $\hat{\mathcal{E}}_t(\xi)(\theta) \in L^1(\mathcal{F}_T)$, for all $\theta \in \Theta$.
Fix $\theta \in \Theta$. We show that given $t \in [0,T]$ and $\varepsilon > 0$ there exists $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{S} \times \mathcal{L}(\theta)$ fulfilling (3.1), $Y_t^\varepsilon \leq \hat{\mathcal{E}}_t(\theta) + \varepsilon$ and $Y_s^\varepsilon \geq \hat{\mathcal{E}}_s(\theta)$, for all $s \in [0,T]$. By means of Proposition 2.2, there exists a sequence $(Y^n, Z^n) \in \mathcal{A}(\xi, g)$ such that $\hat{\mathcal{E}}_t(\theta) = (\inf_n Y_t^n)(\theta)$ and $\hat{\mathcal{E}}_s \leq Y_s^n$, for all $s \in [0,T]$. From this sequence, we define recursively $(\tilde{Y}^n, \tilde{Z}^n) \in \mathcal{S} \times \mathcal{L}(\theta)$ starting with $\tilde{Y}^0 = Y^0(\theta)$ and $\tilde{Z}^0 = Z^0(\theta)$ and

$$\begin{aligned}
\tilde{Y}^n &= Y^0(\theta)1_{[0,t[} + \tilde{Y}^{n-1}1_{\{\tilde{Y}_t^{n-1} < Y_t^n(\theta)\}}1_{[t,T]} + Y^n(\theta)1_{\{\tilde{Y}_t^{n-1} \geq Y_t^n(\theta)\}}1_{[t,T]},\\
\tilde{Z}^n &= Z^0(\theta)1_{[0,t]} + \tilde{Z}^{n-1}1_{\{\tilde{Y}_t^{n-1} < Y_t^n(\theta)\}}1_{]t,T]} + Z^n(\theta)1_{\{\tilde{Y}_t^{n-1} \geq Y_t^n(\theta)\}}1_{]t,T]},
\end{aligned}$$

for $n \geq 1$. It is clear that $(\tilde{Y}^n, \tilde{Z}^n) \subset \mathcal{S} \times \mathcal{L}(\theta)$ and fulfills (3.1). By construction, $(\tilde{Y}_t^n)$ is decreasing and such that $\hat{\mathcal{E}}_t(\theta) = \inf_n \tilde{Y}_t^n$ and $\tilde{Y}_s^n \geq \hat{\mathcal{E}}_s(\theta)$, for all $s \in [0,T]$. Moreover, [10, Lemma 3.1] shows that $(Y^\varepsilon, Z^\varepsilon)$ defined as

$$\begin{aligned}
Y^\varepsilon &= \tilde{Y}^0 1_{[0,t[} + \sum_n \tilde{Y}^n 1_{[t,T]} 1_{B^n},\\
Z^\varepsilon &= \tilde{Z}^0 1_{[0,t]} + \sum_n \tilde{Z}^n 1_{]t,T]} 1_{B^n},
\end{aligned}$$

where $B^0 = A^0$, $B^n = A^n \setminus A^{n-1}$, and $A^n = \{Y_t^n \leq \hat{\mathcal{E}}_t(\theta) + \varepsilon\}$, for $n \in \mathbb{N}$, is such that $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{S} \times \mathcal{L}(\theta)$, fulfills (3.1) and by construction fulfills $Y_t^\varepsilon \leq \hat{\mathcal{E}}_t(\theta) + \varepsilon$.

For $\varepsilon > 0$, and any $0 \leq s < t \leq T$ we pick $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{S} \times \mathcal{L}(\theta)$ fulfilling (3.1) such that $Y_s^\varepsilon \leq \hat{\mathcal{E}}_s(\theta) + \varepsilon$ and $Y_t^\varepsilon \geq \hat{\mathcal{E}}_t(\theta)$, for all $t \in [0,T]$. Hence

$$\hat{\mathcal{E}}_t(\theta) \leq Y_t^\varepsilon \leq Y_s^\varepsilon - \int_s^t g_u(Y_u^\varepsilon, Z_u^\varepsilon)du + \int_s^t Z_u^\varepsilon d\tilde{W}_u(\theta) \leq \hat{\mathcal{E}}_s(\theta) + \int_s^t Z_u^\varepsilon d\tilde{W}_u(\theta) + \varepsilon. \tag{3.8}$$

Taking conditional expectation on both sides under $\mathcal{F}_s$ followed by sending ε to zero shows the supermartingale property for $\hat{\mathcal{E}}(\theta)$. Hence, $\hat{\mathcal{E}}$ is a supermartingale and the definition of $\tilde{P}$ immediately yields that $\tilde{P}[A] = 0$, where $A \in \tilde{\mathcal{F}}_T$ is the set where the limits in (3.6) do not exist. □

Before we state our main existence result we introduce the following functions. Let $(\varphi_t)_{t\in[0,T]}$ be a family of mappings from $C_0([0,T],\mathbb{R}^d)$ to $\mathbb{R}$. We define the functions

$$\varphi^-,\varphi^+ : C_0([0,T];\mathbb{R}^d) \to \mathbb{R}^{[0,T]}$$

by

$$w \mapsto \left(\limsup_{\Pi\ni q\uparrow t} \varphi_q(w^q)\right)_{t\in[0,T]} \quad \text{and} \quad w \mapsto \left(\limsup_{\Pi\ni q\downarrow t} \varphi_q(w^q)\right)_{t\in[0,T]}, \tag{3.9}$$

where $w^q := w_{\cdot\wedge q}$, respectively.

Our main existence result for minimal supersolutions of BSDE under model uncertainty can now be stated as follows.

Theorem 3.3. *Let g be a generator and $\xi \in L^0(\tilde{\mathcal{F}}_T)$ be a terminal condition. Assume that one of the following assumptions holds:*

- *the generator g fulfills (POS), (LSC), (CON) and either (MON) or (MON′);*
- *the generator g fulfills (POS), (LSC), and (NOR).*

Assume further that (RCP) holds and that there exists $\bar{\theta} \in \Theta$ such that $\mu^{\bar{\theta}}$ is strictly positive. Moreover, suppose that $\xi^- \in L^1_b(\tilde{\mathcal{F}}_T)$, that $\mathcal{A}(\xi,g) \neq \emptyset$, and that $\hat{\mathcal{E}}_t(\xi) \in C(\tilde{\mathcal{F}}_t)$, for all $t \in [0,T]$. Let $(\varphi_t)_{t\in[0,T]}$ denote the family of continuous functions fulfilling $\hat{\mathcal{E}}_t = \varphi_t(\tilde{W}^t)$. Suppose that $(\varphi^-)_t$ and $(\varphi^+)_t$, with φ^-,φ^+ defined in (3.9), are continuous, for all $t \in [0,T]$. Then there exists a unique control process $Z \in \tilde{\mathcal{L}}$ and a modification $\mathcal{E}(\xi) \in \tilde{\mathcal{S}}$ of $\hat{\mathcal{E}}(\xi)$ such that $(\mathcal{E}(\xi), Z)$ is a minimal supersolution.

Proof. Let $\Pi := \{kT/n : n \in \mathbb{N}, k = 0,\dots,n\}$ and $\hat{\mathcal{E}} := \hat{\mathcal{E}}(\xi)$. Further, by Lemma 3.1, for any stopping time $\tau \in \mathcal{T}$ and any supersolution $(Y,Z) \in \mathcal{A}(\xi,g)$ holds $Y_\tau(\theta) \geq -E[\xi^-(\theta) \,|\, \mathcal{F}_\tau]$, for all $\theta \in \Theta$, as well as the family $(E[\xi^-(\theta) \,|\, \mathcal{F}_\tau])_{\theta\in\Theta}$ is uniformly integrable. In particular, $\hat{\mathcal{E}}_t \in L^1_b(\tilde{\mathcal{F}}_t)$, for all $t \in [0,T]$.

Step 1: Let us show that there exists a sequence $((Y^n, Z^n))$ in $\mathcal{A}(\xi,g)$ such that $Y := \inf_{n\in\mathbb{N}} Y^n$ is a làdlàg process satisfying $Y_t \in L^1_b(\tilde{\mathcal{F}}_t)$, for all $t \in [0,T]$, and Y is a modification of $\hat{\mathcal{E}}$. To this end, let $\varphi_t : C_0([0,T];\mathbb{R}^d) \to \mathbb{R}$ denote the continuous functions satisfying $\hat{\mathcal{E}}_t = \varphi_t(\tilde{W}^t)$, for all $t \in [0,T]$ and consider the mappings $\varphi^-,\varphi^+ : C_0([0,T];\mathbb{R}^d) \to \mathbb{R}^{[0,T]}$ given by (3.9). For quasi all $w \in C_0([0,T];\mathbb{R}^d)$ the image $\varphi^-(w)$ is càglàd. Indeed, note first that, by Proposition 3.2, for all $\theta \in \Theta$, P-almost surely,

$$\varphi^-\left(\tilde{W}(\theta)\right) = \left(\limsup_{\Pi\ni q\uparrow t} \varphi_q\left(\tilde{W}^q(\theta)\right)\right)_{t\in[0,T]} = \left(\limsup_{\Pi\ni q\uparrow t} \hat{\mathcal{E}}_q(\theta)\right)_{t\in[0,T]} = \hat{\mathcal{E}}^-(\theta).$$

Now, let $N := \{w \in C_0([0,T];\mathbb{R}^d) : \varphi^-(w) \text{ is not càglàd}\}$. Then, again with Proposition 3.2, for all $\theta \in \Theta$,

$$P\left[\tilde{W}(\theta) \in N\right] = P\left[\varphi^-\left(\tilde{W}(\theta)\right) \text{ is not càglàd}\right] = P\left[\hat{\mathcal{E}}^-(\theta) \text{ is not càglàd}\right] = 0,$$

and hence $c(N) = 0$. By the same arguments we obtain that for quasi all $w \in C_0([0,T];\mathbb{R}^d)$ the image $\varphi^+(w)$ is càdlàg. It follows that for quasi all $w \in C_0([0,T];\mathbb{R}^d)$ the set of jump points

$$\mathcal{J}(w) := \left\{t \in [0,T] : \left(\varphi^-(w)\right)_t > \left(\varphi^+(w)\right)_t\right\}$$

is countable. Indeed, for $N := \{w \in C_0([0,T];\mathbb{R}^d) : \mathcal{J}(\omega) \text{ is uncountable}\}$ we have

$$\begin{aligned} P\left[\tilde{W}(\theta) \in N\right] &= P\left[\mathcal{J}\left(\tilde{W}(\theta)\right) \text{ is uncountable}\right] \\ &= P\left[\hat{\mathcal{E}}_t^-(\theta) > \hat{\mathcal{E}}_t^+(\theta) \text{ for uncountably many } t \in [0,T]\right] = 0, \end{aligned} \tag{3.10}$$

for all $\theta \in \Theta$, which implies $c(N) = 0$. To see the last equality in (3.10), note first that, P-almost surely, $\hat{\mathcal{E}}_t^-(\theta)(\omega) > \hat{\mathcal{E}}_t^+(\theta)(\omega)$ implies that $\hat{\mathcal{E}}_t^+(\theta)(\omega)$ jumps at t. Indeed, suppose that it does not, that is $\hat{\mathcal{E}}_t^+(\theta)(\omega) = \lim_{s\uparrow t}\hat{\mathcal{E}}_s^+(\theta)(\omega)$. Then we can find, for every $\varepsilon > 0$, some $s \in [0,t)$ and a $p \in \mathbb{Q}$ with $s < p < t$, such that

$$|\hat{\mathcal{E}}_t^+(\theta)(\omega) - \hat{\mathcal{E}}_p(\theta)(\omega)| \le |\hat{\mathcal{E}}_t^+(\theta)(\omega) - \hat{\mathcal{E}}_s^+(\theta)(\omega)| + |\hat{\mathcal{E}}_s^+(\theta)(\omega) - \hat{\mathcal{E}}_p(\theta)(\omega)| \le \varepsilon.$$

Hence, for $\varepsilon_n := 1/n$ and the corresponding p_n, with $p_n \le p_{n+1}$, we obtain the contradiction $\hat{\mathcal{E}}_t^+(\theta)(\omega) = \lim_n \hat{\mathcal{E}}_{p_n}(\theta)(\omega) = \hat{\mathcal{E}}_t^-(\theta)(\omega)$. This implies the result since the càdlàg process $\hat{\mathcal{E}}^+(\theta)$ has only countably many jumps.

Recall that $C_0([0,T];\mathbb{R}^d)$ is separable and that by assumption there exists $\bar{\theta}$ such that $\mu^{\bar{\theta}}$ is strictly positive, that is $\mu^{\bar{\theta}}(B) > 0$, for each nonempty open set $B \in \mathcal{B}(C_0([0,T];\mathbb{R}^d))$. This allows us to choose a dense[2] sequence (w_k) in $C_0([0,T];\mathbb{R}^d)$, such that $\mathcal{J}(w_k)$ is countable for all $k \in \mathbb{N}$. Indeed, we start with an arbitrary dense subset $(\bar{w}_k)$ and consider the countable set of balls $(B_{1/m}(\bar{w}_k))_{m,k\in\mathbb{N}}$. Each $B_{1/m}(\bar{w}_k)$ has positive measure under $\mu^{\bar{\theta}}$ and hence contains some $w_{m,k}$ such that $\mathcal{J}(w_{m,k})$ is countable. By construction $(w_{m,k})_{m,k\in\mathbb{N}}$ is a dense subset, which for simplicity is denoted with (w_k). The countable union

$$\mathcal{J} := \bigcup_{k\in\mathbb{N}} \mathcal{J}(w_k)$$

is a countable subset of $[0,T]$. For each $t \in \Pi \cup \mathcal{J}$, there exists by (RCP), and Proposition 2.2, a sequence $(Y^{n,t}, Z^{n,t})_{n\in\mathbb{N}}$ in $\mathcal{A}(\xi, g)$, which satisfies

$$\lim_{n\to\infty} \tilde{E}[(Y_t^{1,t} \wedge \cdots \wedge Y_t^{n,t}) - \hat{\mathcal{E}}_t] = 0.$$

Now, let $((Y^n, Z^n))$ be a sequence running through the countable family $(Y^{n,t}, Z^{n,t})_{n\in\mathbb{N},t\in\Pi\cup\mathcal{J}}$, such that

$$\tilde{E}\left[\left(Y_t^1 \wedge \cdots \wedge Y_t^n\right) - \hat{\mathcal{E}}_t\right] \to 0, \quad \text{for all } t \in \Pi \cup \mathcal{J}. \tag{3.11}$$

Defining $Y := \inf_{n\in\mathbb{N}} Y^n$, it holds $Y_t = \hat{\mathcal{E}}_t$, for all $t \in \Pi \cup \mathcal{J}$, as well as $Y_t \in L_b^1(\tilde{\mathcal{F}}_t)$, for all $t \in [0,T]$ by use of the lower bound $Y_t(\theta) \ge -E[\xi^-(\theta) \,|\, \mathcal{F}_t]$ as argued in the beginning of the proof.

[2]That is, the $\|\cdot\|_\infty$-closure of $\{w_k : k \in \mathbb{N}\}$ is $C_0([0,T];\mathbb{R}^d)$.

We next fix an arbitrary $\theta \in \Theta$ and show that $Y(\theta)$ is a strong supermartingale. Indeed, since $Y^n(\theta)$ is a strong supermartingale, see Lemma 3.1, for each $n \in \mathbb{N}$, it follows

$$E\left[Y_\tau(\theta)\,|\,\mathcal{F}_\sigma\right] \leq \inf_{n\in\mathbb{N}} E\left[Y^n_\tau(\theta)\,|\,\mathcal{F}_\sigma\right] \leq \inf_{n\in\mathbb{N}} Y^n_\sigma(\theta) \leq Y_\sigma(\theta),$$

for all $\sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$. The integrability condition of Y follows from $Y^1_\tau(\theta) \geq Y_\tau(\theta)$ and the fact that $Y^n_\tau(\theta)$ is uniformly bounded from below by $-E[\xi^-(\theta)\,|\,\mathcal{F}_\tau] \in L^1(\mathcal{F}_\tau)$, for all $\tau \in \mathcal{T}$.

The process Y is làdlàg. Indeed, for each $\theta \in \Theta$ the process $Y^n(\theta)$ is $(\mathcal{F}_t)$-optional. Since Y is the countable infimum over the processes Y^n, it follows that $Y(\theta)$ is $(\mathcal{F}_t)$-optional for all $\theta \in \Theta$. Thus, we deduce by means of [7, Appendix 1, Theorem 4, p. 395] that $Y(\theta)$ is làdlàg, for all $\theta \in \Theta$. This shows that quasi all paths of Y are làdlàg. In particular, since $Y_t = \hat{\mathcal{E}}_t$, for all $t \in \Pi$, it follows $Y^- = \hat{\mathcal{E}}^-$ and $Y^+ = \hat{\mathcal{E}}^+$.

It remains to show that $Y_t = \hat{\mathcal{E}}_t$, for all $t \in [0, T]$. Two distinct cases may happen

a) either $\tilde{P}[\hat{\mathcal{E}}^-_t > \hat{\mathcal{E}}^+_t] > 0$. In this case, the set

$$\left\{w \in C_0([0,T];\mathbb{R}^d) : \left(\varphi^-(w)\right)_t > \left(\varphi^+(w)\right)_t\right\}.$$

is open and nonempty. Hence, it contains some w_{k_0} and consequently $t \in \mathcal{J}$, which implies $Y_t = \hat{\mathcal{E}}_t$.

b) or $\tilde{P}[\hat{\mathcal{E}}^-_t > \hat{\mathcal{E}}^+_t] = 0$, that is $\hat{\mathcal{E}}^-_t = \hat{\mathcal{E}}^+_t$. Since Y is a supermartingale and $(\mathcal{F}_t)$ fulfills the usual conditions, it holds $Y^-_t \geq Y_t$, for all $t \in [0, T]$, see Karatzas and Shreve [15, Proposition 1.3.14]. By Proposition 3.2, we get

$$\hat{\mathcal{E}}^-_t = Y^-_t \geq Y_t \geq \hat{\mathcal{E}}_t \geq \hat{\mathcal{E}}^+_t,$$

which in turns implies $Y_t = \hat{\mathcal{E}}_t$.

Step 2: In this step, we construct for each $\theta \in \Theta$ an admissible control process $Z^\theta \in \mathcal{L}(\theta)$, such that $(Y(\theta), Z^\theta)$ fulfills (3.1). We start by considering the sequence $(\hat{Y}^n(\theta)) := ((Y^n)^+(\theta))$ and the limit $\hat{Y} = \inf_n \hat{Y}^n$. Lemma 3.1 implies that $(\hat{Y}^n(\theta), Z^n(\theta))$ fulfills (3.1), for all $n \in \mathbb{N}$. In the following, we argue for a fixed $\theta \in \Theta$, and only indicate dependency on θ if necessary.

Given the first set of assumptions on the generator we want to apply the method introduced in [10] to obtain a process $Z^\theta \in \mathcal{L}(\theta)$ such that $(\hat{Y}^+(\theta), Z^\theta)$ fulfills (3.1). Therefore, we need to construct a sequence $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{S} \times \mathcal{L}(\theta)$, such that $\tilde{Y}^n$ is càdlàg and $(\tilde{Y}^n, \tilde{Z}^n)$ fulfills (3.1), for all $n \in \mathbb{N}$, $(\tilde{Y}^n)$ is monotone decreasing, and $\lim_n \tilde{Y}^n_t = \hat{Y}_t(\theta)$, for all $t \in \Pi$. We proceed as follows and refer to [10, Lemma 3.1] for a justification of the involved pastings. Fix $k \in \mathbb{N}$, $\varepsilon > 0$, and let $\Pi^k := \{iT/2^k : i = 0, \cdots, 2^k - 1\}$. Set $(\tilde{Y}^{1,0}, \tilde{Z}^{1,0}) := (\hat{Y}^1(\theta), Z^1(\theta))$ and, for $n \in \mathbb{N}$, $n \geq 2$,

$$\begin{aligned}\tilde{Y}^{n,0} &:= \tilde{Y}^{n-1,0} 1_{[0,\tau^n_0[} + \hat{Y}^n(\theta) 1_{[\tau^n_0,T]},\\ \tilde{Z}^{n,0} &:= \tilde{Z}^{n-1,0} 1_{[0,\tau^n_0]} + Z^n(\theta) 1_{]\tau^n_0,T]},\end{aligned}$$

where $\tau_0^n := \inf\{t \geq 0 : \tilde{Y}_t^{n-1,0} > \hat{Y}_t^n(\theta)\}$. By construction holds $\lim_n \tilde{Y}_0^{n,0} = \hat{Y}_0(\theta)$ and we may choose $n_0 \in \mathbb{N}$ such that $\tilde{Y}_0^{n_0,0} - \varepsilon \leq \hat{Y}_0(\theta)$. Set $(\tilde{Y}^{\varepsilon,0}, \tilde{Z}^{\varepsilon,0}) := (\tilde{Y}^{n_0,0}, \tilde{Z}^{n_0,0})$. Now, let $(\tilde{Y}^{0,1}, \tilde{Z}^{0,1}) := (\tilde{Y}^{\varepsilon,0}, \tilde{Z}^{\varepsilon,0})$ and set, for $n \in \mathbb{N}$, $n \geq 1$,

$$\tilde{Y}^{n,1} := \tilde{Y}^{n-1,1} 1_{[0,\tau_1^n[} + \hat{Y}^n(\theta) 1_{[\tau_1^n,T]},$$
$$\tilde{Z}^{n,1} := \tilde{Z}^{n-1,1} 1_{[0,\tau_1^n]} + Z^n(\theta) 1_{]\tau_1^n,T]},$$

where $\tau_1^n := \inf\{t \geq 1T/2^k : \tilde{Y}_t^{n-1,1} > \hat{Y}_t^n(\theta)\}$. By construction holds $\lim_n \tilde{Y}_{T/2^k}^{n,1} = \hat{Y}_{T/2^k}(\theta)$ and using the same arguments as in [10, Proposition 3.2.2] we may then construct $(\tilde{Y}^{\varepsilon,T/2^k}, \tilde{Z}^{\varepsilon,T/2^k})$ such that $Y_{iT/2^k}^{\varepsilon,T/2^k} - \varepsilon \leq \hat{Y}_{iT/2^k}(\theta)$, for $i = 0, 1$. The continuation of this procedure yields a pair $(\tilde{Y}^{\varepsilon,\Pi^k}, \tilde{Z}^{\varepsilon,\Pi^k})$ such that $Y_t^{\varepsilon,\Pi^k} - \varepsilon \leq \hat{Y}_t(\theta)$, for all $t \in \Pi^k$. Let now $((\tilde{Y}^n, \tilde{Z}^n) := (\tilde{Y}^{1/n,\Pi^n}, \tilde{Z}^{1/n,\Pi^n}))$. Then, $((\tilde{Y}^n, \tilde{Z}^n))$ fulfills all the requirements, except that it needs not be monotone decreasing. However, this can be achieved by the same pasting arguments as in the last part of Step 2 in the proof of [10, Theorem 4.1]. We denote the resulting sequence again with $((\tilde{Y}^n, \tilde{Z}^n))$ and observe that the method in [10, Theorem 4.1] yields $Z^\theta \in \mathcal{L}(\theta)$ such that $(\tilde{Y}^+, Z^\theta)$ fulfills (3.1), where $\tilde{Y}^+$ is the right hand limit process of the monotone limit $\tilde{Y} = \lim_n \tilde{Y}^n$. Consequently, since $\tilde{Y}$ coincides with $\hat{Y}(\theta)$ on all dyadic rationals, we obtain that $(\hat{Y}^+(\theta), Z^\theta)$ fulfills (3.1).

In the case of the second set of assumptions on the generator we apply the method developed in [13]. More precisely, we apply [13, Theorem 3.6] to the sequence $((\tilde{Y}^n, \tilde{Z}^n))$ constructed in the preceding paragraph and obtain again $Z^\theta \in \mathcal{L}(\theta)$ such that $(\hat{Y}^+(\theta), Z^\theta)$ fulfills (3.1).

Now, we show that $\hat{Y}_t^+(\theta) = Y_t^+(\theta)$, for all $t \in [0,T]$ and $\theta \in \Theta$. On the one hand, from $Y_t^n(\theta) \geq \hat{Y}_t^n(\theta)$, see Lemma 3.1, follows $Y_t(\theta) \geq \hat{Y}_t(\theta)$ and $Y_t^+(\theta) \geq \hat{Y}_t^+(\theta)$. On the other hand, (3.1) implies, for all $s \geq t$, and $\theta \in \Theta$,

$$\hat{Y}_t^n(\theta) \geq -\int_t^s Z_u^n(\theta) d\tilde{W}_u(\theta) + Y_s^n(\theta).$$

By taking conditional expectation we obtain $\hat{Y}_t^n(\theta) \geq E[Y_s^n(\theta) \,|\, \mathcal{F}_t]$. This yields

$$\hat{Y}_t(\theta) \geq \inf_n E[Y_s^n(\theta) \,|\, \mathcal{F}_t] \geq E[Y_s(\theta) \,|\, \mathcal{F}_t].$$

Since $Y(\theta) \geq E[\xi(\theta) \,|\, \mathcal{F}_\cdot]$ we may apply Fatou's lemma and obtain, by sending s to t, that $\hat{Y}_t(\theta) \geq Y_t^+(\theta)$, which in turn implies $\hat{Y}_t^+(\theta) \geq Y_t^+(\theta)$, for all $t \in [0,T]$. Hence $Y_t^+(\theta) = \hat{Y}_t^+(\theta)$, for all $t \in [0,T]$, and we deduce that $(Y^+(\theta), Z^\theta)$ fulfills (3.1).

It remains to show that $(Y(\theta), Z^\theta)$ fulfills (3.1), for all $\theta \in \Theta$. To that end note that, since $Y(\theta)$ is a strong supermartingale and $(\mathcal{F}_t)$ fulfills the usual conditions, by [7, Appendix 1, Remark 5.c, p. 397] it holds $Y_\tau^-(\theta) \geq Y_\tau(\theta) \geq Y_\tau^+(\theta)$, for all stopping times $\tau \in \mathcal{T}$, and by similar arguments as in Lemma 3.1 we have $Y(\theta) = Y^+(\theta)$, $P \otimes dt$-almost surely. For every $\theta \in \Theta$ the jumps of the càdlàg process $Y^+(\theta)$ are exhausted by an increasing sequence of $(\mathcal{F}_t)$-stopping times. Since every $(\mathcal{F}_t)$-stopping time is predictable we may choose, for $0 \leq \sigma \leq \tau \leq T$ with

$\sigma, \tau \in \mathcal{T}$, an increasing sequence (τ_n) of stopping times converging to τ, with $\tau_n < \tau$, for all $n \in \mathbb{N}$. This yields, for all $\theta \in \Theta$,

$$\begin{aligned}
Y_\sigma(\theta) - \int_\sigma^\tau g_u(\theta, Y_u(\theta), Z_u^\theta)du + \int_\sigma^\tau Z_u^\theta d\tilde{W}_u(\theta) \\
\geq \lim_n Y_\sigma^+(\theta) - \int_\sigma^{\tau_n} g_u(\theta, Y_u^+(\theta), Z_u^\theta)du + \int_\sigma^{\tau_n} Z_u^\theta d\tilde{W}_u(\theta) \\
\geq \lim_n Y_{\tau_n}^+(\theta) = (Y_\tau^+)^-(\theta) = Y_\tau^-(\theta) \geq Y_\tau(\theta),
\end{aligned}$$

where the second equality follows from the làdlàg property of Y. Thus, $(Y(\theta), Z^\theta)$ fulfills (3.1), for all $\theta \in \Theta$.

Step 3: In this final step, we provide $Z \in \tilde{\mathcal{L}}$ such that $Z(\theta) = Z^\theta$, for all $\theta \in \Theta$. The argumentation of this aggregation result relies on a result in[14] extended in the present context in [20] and [17]. Since Y^+ is càdlàg and $(Y^+(\theta), Z^\theta)$ fulfills (3.1), we know that $\langle Y^+(\theta), \tilde{W}(\theta)\rangle = \int Z^\theta \theta du$ and that

$$\langle Y^+(\theta), \tilde{W}(\theta)\rangle = Y^+(\theta)\tilde{W}(\theta) - \int Y^-(\theta)d\tilde{W}(\theta) - \int \tilde{W}(\theta)dY^+(\theta), \quad \text{for all } \theta \in \Theta. \tag{3.12}$$

We next argue that the right hand side of the previous expression is $(\tilde{\mathcal{F}}_t^+)$-adapted. Indeed, the process $Y^+\tilde{W}$ is $(\tilde{\mathcal{F}}_t^+)$-adapted and since Y^- and $\tilde{W}$ are càglàd, we know by [14] that there exists an $(\tilde{\mathcal{F}}_t^+)$-adapted process I which coincides with the integral terms θ-wise in the P-almost sure sense. We briefly expose how one constructs such a functional for the first integral term. For each $n \in \mathbb{N}$, we consider the sequence of $(\tilde{\mathcal{F}}_t^+)$-stopping times $\tilde{\tau}_0^n = 0$ and $\tilde{\tau}_{k+1}^n = \inf\{t \geq \tilde{\tau}_k^n : \left|Y_t^+ - Y_{\tilde{\tau}_k^n}^+\right| \geq 2^{-n}\}$. We then define the process I^n through

$$I_t^n := Y_{\tilde{\tau}_k^n}^+ + \sum_{i=0}^{k-1} Y_{\tilde{\tau}_i^n}^+ \left(\tilde{W}_{\tilde{\tau}_{i+1}^n} - \tilde{W}_{\tilde{\tau}_i^n}\right), \quad \text{for } \tilde{\tau}_k^n \leq t < \tilde{\tau}_{k+1}^n, \text{ and } k \geq 0. \tag{3.13}$$

By construction, I^n is an $(\tilde{\mathcal{F}}_t^+)$-adapted process and we define $I = \limsup_n I^n$ which is also $(\tilde{\mathcal{F}}_t^+)$-adapted. By use of the Burkholder-Davis-Gundy inequality[3] holds

$$E\left[\sup_{t\in[0,T]} \left|I_t^n(\theta) - \int_0^t Y_u^-(\theta)d\tilde{W}_u(\theta)\right|\right] \leq C2^{-n}E\left[\left(\int_0^T \theta_u du\right)^{1/2}\right]. \tag{3.14}$$

Since the right hand side of the previous inequality converges to 0 for each $\theta \in \Theta$, it follows that I is an $(\tilde{\mathcal{F}}_t^+)$-adapted process such that $I(\theta) = \int Y^-(\theta)d\tilde{W}(\theta)$, for all $\theta \in \Theta$.

[3]For any $(\mathcal{F}_t)$-adapted process X^θ holds $E\left[\sup_{t\in[0,T]}\left|\int_0^t X_u^\theta d\tilde{W}_u(\theta)\right|\right] \leq CE\left[\left(\int_0^T |X_u^\theta|^2 \theta du\right)^{1/2}\right]$.

Hence, there exists an $(\tilde{\mathcal{F}}_t^+)$-adapted $\mathbb{R}^d$-valued process denoted by $\langle Y^+, \tilde{W}\rangle$, which θ-wise coincides with $\int Z^\theta \theta du$. Since $\langle Y^+, \tilde{W}\rangle$ is θ-wise continuous, we deduce that it is $(\tilde{\mathcal{F}}_t^+)$-predictable, which implies, see [6, IV.61 Remark (c)], that it is $(\tilde{\mathcal{F}}_t)$-predictable. The same argumentation holds for $\langle \tilde{W}, \tilde{W}\rangle$, for which holds $\langle \tilde{W}(\theta), \tilde{W}(\theta)\rangle = \int \theta du$. We define Z by the pathwise left derivatives, which by means of Lebegue's derivative theorem exists dt-almost surely, as follows

$$Z_t := \left(\lim_{h\searrow 0} \frac{\langle Y^+, \tilde{W}\rangle_{t-h} - \langle Y^+, \tilde{W}\rangle_t}{h}\right)\left(\lim_{h\searrow 0} \frac{\langle \tilde{W}, \tilde{W}\rangle_{t-h} - \langle \tilde{W}, \tilde{W}\rangle_t}{h}\right)^{-1}, \quad t \in]0,T], \tag{3.15}$$

and so Z is $(\tilde{\mathcal{F}}_t)$-predictable. Thus, we obtain some $Z \in \tilde{\mathcal{L}}$ such that $Z(\theta) = Z^\theta$ for all $\theta \in \Theta$.

Step 4: From the previous argumentation we know that (Y^+, Z) fulfills (3.1). Hence, uniqueness of Z follows from the Doob-Meyer decomposition under each $\theta \in \Theta$, see [10, Lemma 3.3] for details. □

References

[1] Avellaneda, M., A. Levy, and A. Paras (1995): "Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities," *Applied Mathematical Finance*, 2, 73–88.

[2] Bion-Nadal, J., and M. Kervarec (2010): "Dynamic Risk Measuring under Model Uncertainty: Taking Advantage of the Hidden Probability Measure," *Preprint*.

[3] Bion-Nadal, J., and M. Kervarec (2012): "Risk Measuring under Model Uncertainty," *The Annals of Applied Probability*, 22(1), 213–238.

[4] Cheridito, P., H. M. Soner, N. Touzi, and N. Victoir (2007): "Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs," *Communications on Pure and Applied Mathematics*, 60(7), 1081–1110.

[5] Choquet, G. (1954): "Theory of Capacities," *Annales de l'institut Fourier*, 5, 131–295.

[6] Dellacherie, C., and P.-A. Meyer (1978): *Probabilities and Potential. A*, vol. 29 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam.

[7] ——— (1982): *Probabilities and Potential. B*, vol. 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, Theory of martingales, Translated from the French by J. P. Wilson.

[8] Denis, L., M. Hu, and S. Peng (2011): "Function Spaces and Capacity Related to a Sublinear Expectation: Application to G-Brownian Motion Paths," *Potential Analysis*, 34(2), 139–161.

[9] Denis, L., and C. Martini (2006): "A Theoretical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty," *Annals of Applied Probability*, 16, 827–852.

[10] Drapeau, S., G. Heyne, and M. Kupper (2011): "Minimal Supersolutions of Convex BSDEs," *accepted for publication in Annals of Probability*.

[11] Epstein, L., and S. Ji (2011): "Ambiguous Volatility, Possibility and Utility in Continuous Time," *ArXiv e-prints*.

[12] Föllmer, H., and A. Schied (2004): *Stochastic Finance. An Introduction in Discrete Time*, de Gruyter Studies in Mathematics. Walter de Gruyter, Berlin, New York, 2 edn.

[13] HEYNE, G., M. KUPPER, AND C. MAINBERGER (2012): "Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators," *accepted for publication in Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*.

[14] KARANDIKAR, R. L. (1995): "On Pathwise Stochastic Integration," *Stochastic Processes and their Applications*, 57(1), 11–18.

[15] KARATZAS, I., AND S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edn.

[16] LYONS, T. (1995): "Uncertain Volatility and the Risk-Free Synthesis of Derivatives," *Applied Mathematical Finance*, 2, 117–133.

[17] NUTZ, M., AND H. M. SONER (2012): "Superhedging and Dynamic Risk Measures under Volatility Uncertainty," *SIAM Journal on Control and Optimization*, 50(4), 2065–2089.

[18] PENG, S. (2007): "G-Expectation, G-Brownian Motion and Related Stochatic Calculus of Itô Type," *Stochastic Analysis and Applications*, volume 2 of Abel Symp., 541–567.

[19] ——— (2008): "Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation," *Stochastic Processes and their Applications*, 12, 2223–2253.

[20] SONER, H., N. TOUZI, AND J. ZHANG (2011a): "Wellposedness of Second Order Backward SDEs," *Probability Theory and Related Fields*, pp. 1–42.

[21] SONER, H. M., AND N. TOUZI (2009): "The Dynamic Programming Equation for Second Order Stochastic Target Problems," *SIAM Journal on Control Optimization*, 48(4), 2344–2365.

[22] SONER, H. M., N. TOUZI, AND J. ZHANG (2011b): "Martingale Representation Theorem for the G-Expectation," *Stochastic Processes and their Applications*, 121(2), 265–287.

[23] SONER, H. M., N. TOUZI, AND J. ZHANG (2013): "Dual Formulation of Second Order Target Problems," *Annals of Applied Probability*, 23(1), 308–347.

[24] SONER, M. H., N. TOUZI, AND J. ZHANG (to appear): "Quasi-sure Stochastic Analysis through Aggregation," *Electronic Journal of Probability*.