Minimal Supersolutions of BSDEs with Lower Semicontinuous Generators

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We study minimal supersolutions of backward stochastic differential equations. We show the existence and uniqueness of the minimal supersolution, if the generator is jointly lower semicontinuous, bounded from below by an affine function of the control variable, and satisfies a specific normalization property. Semimartingale convergence is used to establish the main result.[1]

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1 Introduction

On a filtered probability space, the filtration of which is generated by a d-dimensional Brownian motion, we give conditions ensuring that the set $\mathcal{A}(\xi, g)$, consisting of all supersolutions (Y, Z) of a backward stochastic differential equation with terminal condition ξ and generator g, has a minimal element. Recall that a supersolution can be seen as, compare for instance [8, 13, 7], a càdlàg value process Y and a control process Z, such that, for all $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi$$

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is satisfied.

Our ansatz to find the minimal supersolution is partially inspired by the methods and the setting introduced in Drapeau et al. [7]. More precisely, we start by considering the process $\hat{\mathcal{E}}^g(\xi)$, defined by

$$\hat{\mathcal{E}}^g_t(\xi) = \operatorname{ess\,inf}\left\{Y_t \in L^0(\mathcal{F}_t) \,:\, (Y,Z) \in \mathcal{A}(\xi,g)\right\}, \quad t \in [0,T].$$

It was shown in [7] that under a positivity assumption on the generator - this can be relaxed to a linear bound from below - the process $\hat{\mathcal{E}}^g(\xi)$ is in fact a supermartingale. Moreover, under such an assumption, given an adequate space of control processes, it follows that every value process of a supersolution is also a supermartingale. This is one of the key features of the approach in [7], and we will also adhere to the concept of supermartingale supersolutions. It allows us to consider the process $\mathcal{E}^g(\xi) = \lim_{s\downarrow\cdot, s\in\mathbb{Q}} \hat{\mathcal{E}}^g_s(\xi)$ as a candidate for the value process of the minimal supersolution.

Now, given this candidate value process, one needs to find a candidate control process $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. In [7] this was done by constructing a monotone decreasing sequence of supersolutions converging to $\mathcal{E}^g(\xi)$ and by drawing on compactness results for sequences of martingales given in Delbaen and Schachermayer [4]. Owing to this approach, it was possible to characterize the candidate control process as the limit of a sequence of convex combinations of control processes. Therefore, in order to verify that the pair $(\mathcal{E}^g(\xi), \hat{Z})$ is a supersolution, it was crucial that the generator is convex with respect to the control variable. The principal aim of the current paper is to drop this convexity assumption.

In order to obtain the existence of a minimal supersolution without taking convex combinations, we proceed as follows. Our first idea is to use results on semimartingale convergence given in Barlow and Protter [2]. Loosely speaking, given a sequence of special semimartingales that converges uniformly, in some sense to be made precise, to some limit process, their result guarantees that the limit process is also a special semimartingale and that the locale martingale parts converge in $\mathcal{H}^1$ to the local martingale in the decomposition of the limit process. Interpreted in our setting, this implies that, if we can construct a sequence $((Y^n, Z^n))$ of supersolutions such that (Y^n) converges in the $\mathcal{R}^\infty$-norm to $\mathcal{E}^g(\xi)$, then we obtain the existence of a candidate control process $\hat{Z}$ as the limit of the sequence (Z^n).

Now, our second main idea shows how to construct a sequence converging in the sense of [2]. To that end, we prove that, for $\varepsilon > 0$, there exists $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$ such that $\|Y^\varepsilon - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon$. Note that it is not possible to infer the existence of such a supersolution from the approach taken in [7], where the approximating sequence was decreasing, but only uniform on a finite set of rationals. Therefore, we have to develop a new method. The central idea is to define a suitable preorder on the set of supersolutions and to use Zorn's lemma to show the existence of a maximal element. To set up our preorder, we associate with each supersolution (Y, Z) the stopping time τ, at which Y first leaves the ε-neighborhood of $\mathcal{E}^g(\xi)$. With this at hand, we say (Y^1, Z^1) dominates (Y^2, Z^2), if and only if $\tau^1 \geq \tau^2$ and the processes coincide up to τ^2. Given this preorder, we have to show that each totally ordered chain has an upper bound. In order to achieve this, we assume a mild normalization condition on the generator. In its simplest form it states that g equals zero as soon as the control variable is zero. This assumption is well known especially in the context of g-expectations, see for example Peng [12] and [7]. More generally, we ask for a certain very simple SDE to have a solution on some short time interval. Combining this assumption with the supermartingale structure of our setting, in particular with arguments based on supermartingale convergence, yields the existence of an upper bound. Moreover, we can show that the stopping time associated with the maximal element provided by Zorn's lemma equals T.

The previous arguments show that we obtain indeed a pair of candidate processes $(\mathcal{E}^g(\xi), \hat{Z})$. It remains to verify that the candidate pair is an element of $\mathcal{A}(\xi, g)$. However, this is straightforward by assuming that the generator is jointly lower semicontinuous and can be done by similar arguments as in [7].

Let us briefly discuss the existing literature on related problems, a broader discussion of which can be found in [7]. Nonlinear BSDEs were first introduced in Pardoux and Peng [11]. In this seminal work existence and uniqueness results were given for the case of Lipschitz generators and square integrable terminal conditions. Kobylanski [10] studies BSDEs with quadratic generators, whereas Delbaen et al. [5] consider superquadratic BSDEs with positive generators that are convex in z and independent of y. BSDEs with generators that are not locally Lipschitz are studied in Bahlali et al. [1]. Among the first introducing supersolutions of BSDEs were El Karoui et al. [8, Section 2.3]. Further references can also be found in Peng [13], who studies the existence and uniqueness of constrained minimal supersolutions under the assumption of a Lipschitz generator and square integrable terminal conditions. For a link between minimal supersolutions of BSDEs and solutions of reflected BSDEs see Peng and Xu [14]. Most recently, Cheridito and Stadje [3] have analyzed existence and stability of supersolutions of BSDEs. They consider terminal conditions which are functionals of the underlying Brownian motion and generators that are convex in z and Lipschitz in y, and they work with discrete time approximations of BSDEs. Furthermore, the concept of supersolutions is closely related to Peng's g-expectations, see for instance [7, 12], since the mapping $\xi \mapsto \mathcal{E}^g_0(\xi)$ can be seen as a nonlinear expectation.

The remainder of this paper is organized as follows. Setting and notations are specified in Section 2. A precise definition of minimal supersolutions and important structural properties of $\hat{\mathcal{E}}^g(\xi)$, along with the main existence theorem, can then be found in Sections 3.1 and 3.2. Possible relaxations on the assumptions imposed on the generator are discussed in Section 3.3. Finally, we conclude the paper with a generalization of our results to the case of arbitrary continuous local martingales in Section 3.4.

2 Setting and Notations

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where the filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and is assumed to satisfy the usual conditions. For some fixed time horizon $T > 0$ and for all $t \in [0, T]$, the sets of $\mathcal{F}_t$-measurable random variables are denoted by $L^0(\mathcal{F}_t)$, where random variables are identified in the P-almost sure sense. Let furthermore denote $L^p(\mathcal{F}_t)$ the set of random variables in $L^0(\mathcal{F}_t)$ with finite p-norm, for $p \in [1, +\infty]$. Inequalities and strict inequalities between any two random variables or processes X^1, X^2 are understood in the P-almost sure or in the $P \otimes dt$-almost everywhere sense, respectively. We denote by $\mathcal{T}$ the set of stopping times with values in $[0, T]$ and hereby call an increasing sequence of stopping times (τ^n) such that $P[\bigcup_n \{\tau^n = T\}] = 1$, a localizing sequence of stopping times. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of càdlàg progressively measurable processes Y with values in $\mathbb{R}$. For $p \in [1, +\infty[$, we further denote by $\mathcal{H}^p$ the set of càdlàg local martingales M with finite $\mathcal{H}^p$-norm on $[0, T]$, that is $\|M\|_{\mathcal{H}^p} := E[\langle M, M\rangle_T^{p/2}]^{1/p} < \infty$. By $\mathcal{L}^p := \mathcal{L}^p(W)$ we denote the set of $\mathbb{R}^{1\times d}$-valued, progressively measurable processes Z such that $\int ZdW \in \mathcal{H}^p$, that is, $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T |Z_s|^2\, ds)^{p/2}]^{1/p}$ is finite. For $Z \in \mathcal{L}^p$, the stochastic integral $\int ZdW$ is well defined, see Protter [15], and is by means of the Burkholder-Davis-Gundy inequality [15, Theorem 48] a continuous martingale. We further denote by $\mathcal{L} := \mathcal{L}(W)$ the set of $\mathbb{R}^{1\times d}$-valued, progressively measurable processes Z, such that there exists a localizing sequence of stopping times (τ^n) with $Z1_{[0,\tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. For $Z \in \mathcal{L}$, the stochastic integral $\int ZdW$ is well defined and is a continuous local martingale. Furthermore, for a process X, let X^* denote the following expression $X^* := \sup_{t\in[0,T]} |X_t|$, by which we define the norm $\|X\|_{\mathcal{R}^\infty} := \|X^*\|_{L^\infty}$.

We call a càdlàg semimartingale X a special semimartingale, if it can be decomposed into $X = X_0 + M + A$, where M is a local martingale and A a predictable process of finite variation such that $M_0 = A_0 = 0$. Such a decomposition is then unique, compare for instance [15, Chapter III, Theorem 30], and is called the canonical decomposition of X.

3 Minimal Supersolutions of BSDEs

3.1 First Definitions and Structural Properties

Throughout this paper, a generator is a jointly measurable function g from $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$ to $\mathbb{R} \cup \{+\infty\}$ where $\Omega \times [0, T]$ is endowed with the progressive σ-field. Given a generator g and a terminal condition $\xi \in L^0(\mathcal{F}_T)$, a pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}$ is a supersolution of a BSDE, if, for $0 \leq s \leq t \leq T$, holds

$$Y_s - \int_s^t g_u(Y_u, Z_u)du + \int_s^t Z_u dW_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi \ . \tag{3.1}$$

For a supersolution (Y, Z), we call Y the value process and Z its corresponding control process. Note that the formulation in (3.1) is equivalent to the existence of a càdlàg increasing process K, with $K_0 = 0$, such that

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u)du + (K_T - K_t) - \int_t^T Z_u dW_u, \qquad t \in [0, T]. \tag{3.2}$$

Although the notation in (3.2) is standard in the literature concerning supersolutions of BSDEs, see for example El Karoui et al. [8] and [13], we will work with (3.1), since the proof of our main result exploits this structure. A control process Z is said to be admissible, if the continuous local martingale $\int ZdW$ is a supermartingale. Throughout this paper a generator g is said to be

(LSC) if $(y, z) \mapsto g(\omega, t, y, z)$ is lower semicontinuous, for all $(\omega, t) \in \Omega \times [0, T]$.

(POS) positive, if $g(y, z) \geq 0$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$.

(NOR) normalized, if $g_t(y, 0) = 0$, for all $(t, y) \in [0, T] \times \mathbb{R}$.

We are now interested in the set

$$\mathcal{A}(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} \ : \ Z \text{ is admissible and (3.1) holds}\} \tag{3.3}$$

and the process

$$\hat{\mathcal{E}}_t^g(\xi) := \operatorname{ess\,inf} \left\{ Y_t \in L^0(\mathcal{F}_t) \ : \ (Y, Z) \in \mathcal{A}(\xi, g) \right\}, \quad t \in [0, T]. \tag{3.4}$$

A pair (Y, Z) is called minimal supersolution, if $(Y, Z) \in \mathcal{A}(\xi, g)$, and if for any other supersolution $(Y', Z') \in \mathcal{A}(\xi, g)$, holds $Y_t \leq Y'_t$, for all $t \in [0, T]$.

For the proof of our main existence theorem we will need some auxiliary results concerning structural properties of $\hat{\mathcal{E}}^g(\xi)$ and supersolutions (Y, Z) in $\mathcal{A}(\xi, g)$.

Lemma 3.1. *Let g be a generator satisfying* (POS). *Assume further that $\mathcal{A}(\xi, g) \neq \emptyset$ and that for the terminal condition ξ holds $\xi^- \in L^1(\mathcal{F}_T)$. Then $\xi \in L^1(\mathcal{F}_T)$ and, for any $(Y, Z) \in \mathcal{A}(\xi, g)$, the control Z is unique and the value process Y is a supermartingale such that $Y_t \geq E[\xi \,|\, \mathcal{F}_t]$. Moreover, the unique canonical decomposition of Y is given by*

$$Y = Y_0 + M - A\,, \tag{3.5}$$

where $M = \int ZdW$ and A is an increasing, predictable, càdlàg process with $A_0 = 0$.

The proof of Lemma 3.1 can be found in [7, Lemma 3.3].

Proposition 3.2. *Suppose that $\mathcal{A}(\xi, g) \neq \emptyset$ and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If g satisfies* (POS)*, then the process $\hat{\mathcal{E}}^g(\xi)$ is a supermartingale. In particular,*

$$\mathcal{E}_t^g(\xi) := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^g(\xi) , \quad \text{for all } t \in [0,T] \quad \text{and} \quad \mathcal{E}_T^g(\xi) := \xi \tag{3.6}$$

is a càdlàg supermartingale such that

$$\hat{\mathcal{E}}_t^g(\xi) \geq \mathcal{E}_t^g(\xi) , \quad \text{for all } t \in [0,T] .$$

Furthermore, the following two pasting properties hold true.

1. *Let $(Z^n) \subset \mathcal{L}$ be admissible, $\sigma \in \mathcal{T}$, and $(B_n) \subset \mathcal{F}_\sigma$ be a partition of Ω. Then the pasted process $\bar{Z} = Z^1 1_{[0,\sigma]} + \sum_{n \geq 1} Z^n 1_{B_n} 1_{]\sigma,T]}$ is admissible.*
2. *Let $((Y^n, Z^n)) \subset \mathcal{A}(\xi, g)$, $\sigma \in \mathcal{T}$ and $(B_n) \subset \mathcal{F}_\sigma$ be as before. If $Y^1_{\sigma-} 1_{B_n} \geq Y^n_\sigma 1_{B_n}$ holds true for all $n \in \mathbb{N}$, then $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$, where*

$$\bar{Y} = Y^1 1_{[0,\sigma[} + \sum_{n \geq 1} Y^n 1_{B_n} 1_{[\sigma,T]} \quad \text{and} \quad \bar{Z} = Z^1 1_{[0,\sigma]} + \sum_{n \geq 1} Z^n 1_{B_n} 1_{]\sigma,T]} .$$

Proof. The proof of the part concerning the process $\mathcal{E}^g(\xi)$ can be found in [7, Proposition 3.4]. $\bar{Z}$ is admissible by [7, Lemma 3.1.1]. We can approximate σ from below by some foretelling sequence of stopping times (η_m)[2], and then show, analogously to [7, Lemma 3.1.2], that the pair $(\bar{Y}, \bar{Z})$ satisfies Inequality (3.1) and is thus an element of $\mathcal{A}(\xi, g)$. □

Remark 3.3. Whenever a stopping time σ takes values in a countable subset $\mathscr{S}$ of $[0,T]$, the adapted process $\hat{\mathcal{E}}^g(\xi)$ evaluated at σ is defined by

$$\hat{\mathcal{E}}_\sigma^g(\xi) := \sum_{s \in \mathscr{S}} 1_{A_s} \hat{\mathcal{E}}_s^g(\xi) \quad \text{with } A_s := \{\sigma = s\} .$$

It is straigthforward to show that $\hat{\mathcal{E}}_\sigma^g(\xi)$ is $\mathcal{F}_\sigma$-measurable and consistent with (3.4), in the sense that

$$\hat{\mathcal{E}}_\sigma^g(\xi) = \operatorname{ess\,inf} \{ Y_\sigma \, : \, (Y, Z) \in \mathcal{A}(\xi, g) \} .$$

Proposition 3.2 yields that the set $\{ Y_\sigma \, : \, (Y, Z) \in \mathcal{A}(\xi, g) \}$ is directed downwards, see [7, Proposition 3.2.1], and as a consequence we can find, for any $\varepsilon > 0$, some $(Y^\varepsilon, Z^\varepsilon) \in \mathcal{A}(\xi, g)$ such that

$$Y_\sigma^\varepsilon \leq \hat{\mathcal{E}}_\sigma^g(\xi) + \varepsilon . \qquad \blacklozenge$$

[2]Such a sequence satisfying $\eta_m \leq \eta_{m+1} < \sigma$, for all $m \in \mathbb{N}$, and $\lim_m \eta_m = \sigma$, always exists, since in a Brownian filtration every stopping time is predictable, compare Revuz and Yor [16, Corollary V.3.3].

Proposition 3.4. *Let $0 = \tau_0 \le \tau_1 \le \tau_2 \le \cdots$ be a sequence of stopping times converging to the finite stopping time $\tau^* = \lim_{n\to\infty} \tau_n$. Further, let (Y^n) be a sequence of càdlàg supermartingales such that $Y^n_{\tau_n -} \ge Y^{n+1}_{\tau_n}$, and which satisfies $Y^n 1_{[\tau_{n-1},\tau_n[} \ge M 1_{[\tau_{n-1},\tau_n[}$, where M is a uniformly integrable martingale. Then, for any sequence of stopping times $\sigma_n \in [\tau_{n-1}, \tau_n[$, the limit $Y^\infty := \lim_{n\to\infty} Y^n_{\sigma_n}$ exists and the process*

$$\bar{Y} := \sum_{n\ge 1} Y^n 1_{[\tau_{n-1},\tau_n[} + Y^\infty 1_{[\tau^*,\infty[}$$

is a càdlàg supermartingale. Moreover, the limit Y^∞ is independent of the approximating sequence $(Y^n_{\sigma_n})$ and, if all Y^n are continuous and $Y^n_{\tau_n} = Y^{n+1}_{\tau_n}$, for all $n \in \mathbb{N}$, then $\bar{Y}$ is continuous.

Proof. Note that $(Y^n_{\sigma_n})$ is a $(\mathcal{F}_{\sigma_n})$-supermartingale. Indeed, if $(\tilde{\eta}_m) \uparrow \tau_n$ is a foretelling sequence of stopping times, then, with $\eta_m := \tilde{\eta}_m \vee \tau_{n-1}$, the family $((Y^n_{\eta_m})^-)_{m\in\mathbb{N}}$ is uniformly integrable and we obtain

$$\begin{aligned} E[Y^{n+1}_{\sigma_{n+1}} \,|\, \mathcal{F}_{\sigma_n}] = E[E[Y^{n+1}_{\sigma_{n+1}} \,|\, \mathcal{F}_{\tau_n}] \,|\, \mathcal{F}_{\sigma_n}] \le E[Y^{n+1}_{\tau_n} \,|\, \mathcal{F}_{\sigma_n}] \le E[Y^n_{\tau_n -} \,|\, \mathcal{F}_{\sigma_n}] \\ \le \liminf_m E[Y^n_{\eta_m} \,|\, \mathcal{F}_{\sigma_n}] \le \liminf_m Y^n_{\eta_m \wedge \sigma_n} = Y^n_{\sigma_n} \,. \end{aligned}$$

Moreover, $((Y^n_{\sigma_n})^-)$ is uniformly integrable. Hence, the sequence $(Y^n_{\sigma_n})$ converges by the supermartingale convergence theorem, see Dellacherie and Meyer [6, Theorems V.28,29], to some random variable Y^∞, P-almost surely, and thus $\bar{Y}$ is well-defined. Furthermore, the limit Y^∞ is independent of the approximating sequence $(Y^n_{\sigma_n})$. Indeed, for any other sequence $(\tilde{\sigma}_n)$ with $\tilde{\sigma}_n \in [\tau_{n-1}, \tau_n[$, the limit $\lim_n Y^n_{\tilde{\sigma}_n}$ exists by the same argumentation. Now $\lim_n Y^n_{\sigma_n} = \lim_n Y^n_{\tilde{\sigma}_n} = Y^\infty$ holds, since the sequence $(\hat{\sigma}_n)$ defined by

$$\hat{\sigma}_n := \begin{cases} \sigma_{\frac{n}{2}} \vee \tilde{\sigma}_{\frac{n}{2}} & \text{, for } n \text{ even} \\ \sigma_{\frac{n+1}{2}} \wedge \tilde{\sigma}_{\frac{n+1}{2}} & \text{, for } n \text{ odd} \end{cases}$$

satisfies $\hat{\sigma}_n \in [\tau_{n-1}, \tau_n[$ and $\lim_n Y^n_{\hat{\sigma}_n}$ exists. Thus, all limits must coincide. Next, we show that $\bar{Y}^{\sigma_n}$ is a supermartingale, for all $n \in \mathbb{N}$. To this end first observe that, for all $0 \le s \le t$,

$$\begin{aligned} E\left[\bar{Y}^{\sigma_n}_t - \bar{Y}^{\sigma_n}_s \,|\, \mathcal{F}_s\right] = \sum_{k=0}^{n-2} E\big[E\big[\bar{Y}^{\sigma_n}_{(\tau_{k+1}\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k \vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \,|\, \mathcal{F}_s\big] \\ + E\big[E\big[\bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_{n-1}\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_{n-1}\vee s)\wedge t}\big] \,|\, \mathcal{F}_s\big] \\ + E\big[E\big[\bar{Y}^{\sigma_n}_t - \bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} \,|\, \mathcal{F}_{(\sigma_n\vee s)\wedge t}\big] \,|\, \mathcal{F}_s\big] \,. \end{aligned}$$

Note further that, for each $n \in \mathbb{N}$, the process $\bar{Y}^{\sigma_n}$ is càdlàg and can only jump downwards, that is, $\bar{Y}^{\sigma_n}_{t-} \ge \bar{Y}^{\sigma_n}_t$, for all $t \in \mathbb{R}$. Observe to this end that, on the one hand, $\bar{Y}^{\sigma_n}_{\tau_k -} = Y^k_{\tau_k -} \ge Y^{k+1}_{\tau_k} =$

$\bar{Y}^{\sigma_n}_{\tau_k}$, for all $0 \le k \le n-1$, by assumption, where we assumed $\tau_{k-1} < \tau_k$, without loss of generality. On the other hand, Y^k can only jump downwards. Indeed, as càdlàg supermartingales, all Y^k can be decomposed into $Y^k = Y^k_0 + M^k - A^k$, by the Doob-Meyer decomposition theorem [15, Chapter III, Theorem 13], where M^k is a local martingale and A^k a predictable, increasing process with $A^k_0 = 0$. Since in a Brownian filtration every local martingale is continuous, the claim follows.

Thus, for all $0 \le k \le n-2$, and $(\tilde{\eta}_m) \uparrow \tau_{k+1}$ a foretelling sequence of stopping times, it holds with $\eta_m := \tilde{\eta}_m \vee \tau_k$,

$$
\begin{aligned}
&E\big[\bar{Y}^{\sigma_n}_{(\tau_{k+1}\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \\
&\qquad \le E\big[\bar{Y}^{\sigma_n}_{((\tau_{k+1}-)\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \\
&\qquad = E\big[\liminf_m \bar{Y}^{\sigma_n}_{(\eta_m\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_k\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] \\
&\qquad \le E\big[\liminf_m Y^{k+1}_{(\eta_m\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] - Y^{k+1}_{(\tau_k\vee s)\wedge t} \\
&\qquad \le \liminf_m E\big[Y^{k+1}_{(\eta_m\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_k\vee s)\wedge t}\big] - Y^{k+1}_{(\tau_k\vee s)\wedge t} \le 0\,.
\end{aligned}
$$

Moreover, $E[\bar{Y}^{\sigma_n}_t - \bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} \,|\, \mathcal{F}_{(\sigma_n\vee s)\wedge t}] = 0$, as well as

$$
\begin{aligned}
&E\big[\bar{Y}^{\sigma_n}_{(\sigma_n\vee s)\wedge t} - \bar{Y}^{\sigma_n}_{(\tau_{n-1}\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_{n-1}\vee s)\wedge t}\big] \\
&\qquad \le E\big[Y^n_{(\sigma_n\vee s)\wedge t} - Y^n_{(\tau_{n-1}\vee s)\wedge t} \,|\, \mathcal{F}_{(\tau_{n-1}\vee s)\wedge t}\big] \le 0\,.
\end{aligned}
$$

Combining this we obtain that $E\left[\bar{Y}^{\sigma_n}_t \,|\, \mathcal{F}_s\right] \le \bar{Y}^{\sigma_n}_s$. Furthermore, $\lim_n \bar{Y}^{\sigma_n}_t = \bar{Y}_t$, for all $t \in \mathbb{R}$. Indeed, let us write $\lim_n \bar{Y}^{\sigma_n}_t = \lim_n \bar{Y}^{\sigma_n}_t 1_{\{t<\tau^*\}} + \lim_n \bar{Y}^{\sigma_n}_t 1_{\{t\ge\tau^*\}}$. Then, $\lim_n \bar{Y}^{\sigma_n}_t 1_{\{t\ge\tau^*\}} = \lim_n Y^n_{\sigma_n} 1_{\{t\ge\tau^*\}} = Y^\infty 1_{\{t\ge\tau^*\}} = \bar{Y}_t 1_{\{t\ge\tau^*\}}$ and $\lim_n \bar{Y}_{\sigma_n\wedge t} 1_{\{t<\tau^*\}} = \bar{Y}_t 1_{\{t<\tau^*\}}$. Hence, the claim follows. As a consequence of Fatou's lemma it now holds that

$$
E\left[\bar{Y}_t \,|\, \mathcal{F}_s\right] \le \liminf_{n\to\infty} E\left[\bar{Y}^{\sigma_n}_t \,|\, \mathcal{F}_s\right] \le \liminf_{n\to\infty} \bar{Y}^{\sigma_n}_s = \bar{Y}_s\,,
$$

since the family $((\bar{Y}^{\sigma_n}_t)^-)$ is uniformly integrable. Hence, $\bar{Y}$ is a supermartingale, which by construction has right-continuous paths and Karatzas and Shreve [9, Theorem 1.3.8] then yields that $\bar{Y}$ is even càdlàg. Finally, whenever all Y^n are continuous and $Y^n_{\tau_n} = Y^{n+1}_{\tau_n}$ holds, for all $n \in \mathbb{N}$, the process $\bar{Y}$ is continuous per construction. □

3.2 Existence and Uniqueness of Minimal Supersolutions

We are now ready to state our main existence result. Possible relaxations of the assumptions (POS) and (NOR) imposed on the generator are discussed in Section 3.3. Note that it is not our focus to investigate conditions assuring the crucial assumption that $\mathcal{A}(\xi, g) \neq \emptyset$. See [7] and the references therein for further details.

Theorem 3.5. *Let g be a generator satisfying* (LSC), (POS) *and* (NOR) *and $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then $\mathcal{E}^g(\xi)$ is the value process of the unique minimal supersolution, that is, there exists a unique control process $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Observe that Theorem 3.5 and Proposition 3.2 imply that $\mathcal{E}^g(\xi)$ is a modification of $\hat{\mathcal{E}}^g(\xi)$.

Proof. Step 1: Uniform Limit and Verification. Since $\mathcal{A}(\xi, g) \neq \emptyset$, there exist $(Y^b, Z^b) \in \mathcal{A}(\xi, g)$. From now on we restrict our focus to supersolutions $(\bar{Y}, \bar{Z})$ in $\mathcal{A}(\xi, g)$ satisfying $\bar{Y}_0 \leq Y_0^b$. Indeed, since we are only interested in minimal supersolutions, we can paste any value process of $(Y, Z) \in \mathcal{A}(\xi, g)$ at $\tau := \inf\{t > 0 : Y_t^b > Y_t\} \wedge T$ such that $\bar{Y} := Y^b 1_{[0,\tau[} + Y 1_{[\tau,T]}$ satisfies $\bar{Y}_0 \leq Y_0^b$, where the corresponding control $\bar{Z}$ is obtained as in Proposition 3.2.

Assume for the beginning that we can find a sequence $((Y^n, Z^n))$ within $\mathcal{A}(\xi, g)$ such that

$$\lim_{n\to\infty} \|Y^n - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} = 0\,. \tag{3.7}$$

Since all Y^n are càdlàg supermartingales, they are, by the Doob-Meyer decomposition theorem, special semimartingales with canonical decomposition $Y^n = Y_0^n + M^n - A^n$ as in (3.5). The supermartingale property of all $\int Z^n dW$ and $\xi \in L^1(\mathcal{F}_T)$, compare Lemma 3.1, imply that $E[A_T^n] \leq Y_0^b - E[\xi] \in L^1(\mathcal{F}_T)$. Hence, since each A^n is increasing, $\sup_n E[\int_0^T |dA_s^n|] < \infty$. As (3.7) implies in particular that $\lim_{n\to\infty} E[(Y^n - \mathcal{E}^g(\xi))^*] = 0$, it follows from [2, Theorem 1 and Corollary 2] that $\mathcal{E}^g(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^g(\xi) = \mathcal{E}_0^g(\xi) + M - A$ and that

$$\lim_{n\to\infty} \|M^n - M\|_{\mathcal{H}^1} = 0 \quad , \quad \lim_{n\to\infty} E[(A^n - A)^*] = 0\,.$$

The local martingale M is continuous and allows a representation of the form $M = \int \hat{Z} dW$, where $\hat{Z} \in \mathcal{L}$, compare [15, Chapter IV, Theorem 43]. Since

$$E\left[\left(\int_0^T \left(Z_u^n - \hat{Z}_u\right)^2 du\right)^{1/2}\right] \xrightarrow[n\to+\infty]{} 0\,,$$

we have that, up to a subsequence, (Z^n) converges $P \otimes dt$-almost everywhere to $\hat{Z}$ and $\lim_{n\to\infty} \int_0^t Z^n dW = \int_0^t \hat{Z} dW$, for all $t \in [0, T]$, P-almost surely, due to the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n\to\infty} Z^n(\omega) = \hat{Z}(\omega)$, dt-almost everywhere, for almost all $\omega \in \Omega$.

In order to verify that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$, we will use the convergence obtained above. More precisely, for all $0 \leq s \leq t \leq T$, Fatou's lemma together with (3.7) and the lower semicontinuity of the generator yields

$$\mathcal{E}^g_s(\xi) - \int_s^t g_u(\mathcal{E}^g_u(\xi), \hat{Z}_u)du + \int_s^t \hat{Z}_u dW_u$$
$$\geq \limsup_n \left(Y^n_s - \int_s^t g_u(Y^n_u, Z^n_u)du + \int_s^t Z^n_u dW_u \right) \geq \limsup_n Y^n_t = \mathcal{E}^g_t(\xi)\,.$$

The above, the positivity of g and $\mathcal{E}^g(\xi) \geq E[\xi \,|\, \mathcal{F}_\cdot]$ imply that $\int \hat{Z}dW \geq E[\xi \,|\, \mathcal{F}_\cdot] - \mathcal{E}^g_0(\xi)$. Hence, being bounded from below by a martingale, the continuous local martingale $\int \hat{Z}dW$ is a supermartingale. Thus, $\hat{Z}$ is admissible and $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$ and therefore, by Lemma 3.1, $\hat{Z}$ is unique. Since we know by Proposition 3.2 that $\hat{\mathcal{E}}^g_t(\xi) \geq \mathcal{E}^g_t(\xi)$, for all $t \in [0,T]$, we deduce that $\hat{\mathcal{E}}^g_t(\xi) = \mathcal{E}^g_t(\xi)$, for all $t \in [0,T]$, by the definition of $\hat{\mathcal{E}}^g(\xi)$. Hence, $(\mathcal{E}^g(\xi), \hat{Z})$ is the unique minimal supersolution.

Step 2: A preorder on $\mathcal{A}(\xi, g)$. As to the existence of $((Y^n, Z^n))$ satisfying (3.7), it is sufficient to show that, for arbitrary $\varepsilon > 0$, we can find a supersolution $(Y^\varepsilon, Z^\varepsilon)$ satisfying

$$\|Y^\varepsilon - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon\,. \tag{3.8}$$

We define the following preorder[3] on $\mathcal{A}(\xi, g)$

$$(Y^1, Z^1) \preceq (Y^2, Z^2) \;\;\Leftrightarrow\;\; \tau_1 \leq \tau_2 \;\text{ and }\; (Y^1, Z^1)1_{[0,\tau_1[} = (Y^2, Z^2)1_{[0,\tau_1[}\,, \tag{3.9}$$

where, for $i = 1, 2$,

$$\tau_i = \inf\left\{t \geq 0 \;:\; Y^i_t > \mathcal{E}^g_t(\xi) + \varepsilon\right\} \wedge T\,. \tag{3.10}$$

For any totally ordered chain $((Y^i, Z^i))_{i \in I}$ within $\mathcal{A}(\xi, g)$ with corresponding stopping times τ_i, we want to construct an upper bound. If we consider

$$\tau^* = \operatorname*{ess\,sup}_{i \in I} \tau_i\,,$$

we know by the monotonicity of the stopping times that we can find a monotone subsequence (τ_m) of $(\tau_i)_{i \in I}$ such that $\tau^* = \lim_{m\to\infty} \tau_m$. In particular, τ^* is a stopping time. Furthermore, the structure of the preorder (3.9) yields that the value processes of the supersolutions $((Y^m, Z^m))$ corresponding to the stopping times (τ_m) satisfy

$$Y^{m+1}_{\tau_m} \leq Y^{m+1}_{\tau_m-} = Y^m_{\tau_m-} \;\;, \;\text{ for all } m \in \mathbb{N}\,, \tag{3.11}$$

[3]Note that, in order to apply Zorn's lemma, we need a partial order instead of just a preorder. To this end we consider equivalence classes of processes. Two supersolutions $(Y^1, Z^1), (Y^2, Z^2) \in \mathcal{A}(\xi, g)$ are said to be equivalent, if $(Y^1, Z^1) \preceq (Y^2, Z^2)$ and $(Y^2, Z^2) \preceq (Y^1, Z^1)$. This means that they are equal up to their corresponding stopping time $\tau_1 = \tau_2$ as in (3.10). This induces a partial order on the set of equivalence classes and hence the use of Zorn's lemma is justified.

where the inequality follows from the fact that all Y^m are càdlàg supermartingales, see the proof of Proposition 3.4.

Step 3: A candidate upper bound $(\bar{Y}, \bar{Z})$ *for the chain* $((Y^i, Z^i))_{i\in I}$. We construct a candidate upper bound $(\bar{Y}, \bar{Z})$ for $((Y^i, Z^i))_{i\in I}$ satisfying $P[\tau(\bar{Y}) > \tau^* \,|\, \tau^* < T] = 1$, with $\tau(\bar{Y})$ as in (3.10).

To this end, let $(\bar{\sigma}_n)$ be a decreasing sequence of stopping times taking values in the rationals and converging towards τ^* from the right[4]. Then the stopping times $\hat{\sigma}_n := \bar{\sigma}_n \wedge T$ satisfy $\hat{\sigma}_n > \tau^*$ and $\hat{\sigma}_n \in \mathbb{Q}$, on $\{\tau^* < T\}$, for all n big enough. Let us furthermore define the following stopping time

$$\bar{\tau} := \inf\left\{ t > \tau^* \,:\, |\mathcal{E}^g_{\tau^*}(\xi) - \mathcal{E}^g_t(\xi)| > \frac{\varepsilon}{2} \right\} \wedge T\,. \tag{3.12}$$

Due to the right-continuity of $\mathcal{E}^g(\xi)$ in τ^*, it follows that $\bar{\tau} > \tau^*$ on $\{\tau^* < T\}$. We now set

$$\sigma_n := \hat{\sigma}_n \wedge \bar{\tau}\,, \quad \text{for all } n \in \mathbb{N}. \tag{3.13}$$

The above stopping times still satisfy $\lim_{n\to\infty} \sigma_n = \tau^*$ and $\sigma_n > \tau^*$ on $\{\tau^* < T\}$, for all $n \in \mathbb{N}$. We further define the following sets

$$A_n := \left\{ \left| \mathcal{E}^g_{\tau^*}(\xi) - \hat{\mathcal{E}}^g_{\sigma_m}(\xi) \right| < \frac{\varepsilon}{8}\,, \quad \text{for all } m \geq n \right\} \cap \{\sigma_n \in \mathbb{Q} \cup \{T\}\}\,. \tag{3.14}$$

They satisfy $A_n \subset A_{n+1}$ and $\bigcup_n A_n = \Omega$, by definition of the sequence (σ_m)[5]. Note further that $A_n \in \mathcal{F}_{\sigma_n}$, since $\hat{\mathcal{E}}^g_{\sigma_m}(\xi)$ is $\mathcal{F}_{\sigma_m}$-measurable, for all $m \geq n$, see Remark 3.3. Since the range of each σ_n is countable on the set A_n, we deduce by Remark 3.3 that, for each $n \in \mathbb{N}$, there exists $(\tilde{Y}^n, \tilde{Z}^n) \in \mathcal{A}(\xi, g)$ such that

$$\tilde{Y}^n_{\sigma_n} \leq \hat{\mathcal{E}}^g_{\sigma_n}(\xi) + \frac{\varepsilon}{8} \quad \text{on the set } A_n. \tag{3.15}$$

Next we partition Ω into $B_n := A_n \backslash A_{n-1}$, where we set $A_0 := \emptyset$ and $\tau_0 := 0$, and define the candidate upper bound as

$$\bar{Y} = \sum_{m\geq 1} Y^m 1_{[\tau_{m-1},\tau_m[} + 1_{\{\tau^*<T\}} \sum_{n\geq 1} 1_{B_n} \Big(\mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2} \Big) 1_{[\tau^*,\sigma_n[} + 1_{\{\tau^*<T\}} \sum_{n\geq 1} 1_{B_n} \tilde{Y}^n 1_{[\sigma_n,T[}\,, \quad \bar{Y}_T = \xi, \tag{3.16}$$

$$\bar{Z} = \sum_{m\geq 1} Z^m 1_{]\tau_{m-1},\tau_m]} + 1_{\{\tau^*<T\}} \sum_{n\geq 1} \tilde{Z}^n 1_{B_n} 1_{]\sigma_n,T]}\,. \tag{3.17}$$

[4] Compare [9, Problem 2.24].

[5] Since on $\{\tau^* < T\}$, $\bar{\tau} > \tau^*$ and $\lim_n \hat{\sigma}_n = \tau^*$ with $\hat{\sigma}_n \in \mathbb{Q} \cup \{T\}$, it is ensured that there exists some $n_0 \in \mathbb{N}$, depending on ω, such that σ_n takes values in the rationals for all $n \geq n_0$. By definition of $\mathcal{E}^g(\xi)$ as the right-hand side limit of $\hat{\mathcal{E}}^g(\xi)$ on the rationals, the inequality in the definition of A_n is satisfied for all $n \geq n_0$.

Step 4: Verification of $(\bar{Y}, \bar{Z}) \in \mathcal{A}(\xi, g)$. By verifying that the pair $(\bar{Y}, \bar{Z})$ is an element of $\mathcal{A}(\xi, g)$, we identify $(\bar{Y}, \bar{Z})$ as an upper bound for the chain $((Y^i, Z^i))_{i \in I}$. Even more, $P[\tau(\bar{Y}) > \tau^* \mid \tau^* < T] = 1$ holds true, since, on the set B_n, we have $\bar{Y}_t = \mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2} \leq \mathcal{E}^g_t(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n[$, due to the definition of $\bar{\tau}$ in (3.12).

Step 4a: The value process $\bar{Y}$ *is an element of* $\mathcal{S}$. By construction, the only thing to show is that $\bar{Y}_{\tau^*-}$, the left limit at τ^*, exists. This follows from Proposition 3.4, since, by means of $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $\xi \in L^1(\mathcal{F}_T)$, all Y^m are càdlàg supermartingales, see Lemma 3.1, which are bounded from below by a uniformly integrable martingale, more precisely $Y^m \geq E[\xi \mid \mathcal{F}_\cdot]$, for all $m \in \mathbb{N}$, and satisfy (3.11).

Step 4b: The control process $\bar{Z}$ *is an element of* $\mathcal{L}$ *and admissible.* We proceed by defining, for each $n \in \mathbb{N}$, the processes $\bar{Z}^n := \sum_{m=1}^n Z^m 1_{]\tau_{m-1}, \tau_m]} = \bar{Z} 1_{[0,\tau_n]} = Z^n 1_{[0,\tau_n]}$ and $N^n := \int \bar{Z}^n dW = \int Z^n 1_{[0,\tau_n]} dW$, where the equalities follow from (3.9). Observe that $N^{n+1} 1_{[0,\tau_n]} = N^n 1_{[0,\tau_n]}$, for all $n \in \mathbb{N}$, and that (POS), (3.1) and the supermartingale property of $\int Z^n dW$ imply

$$N^n 1_{[\tau_{n-1}, \tau_n[} \geq 1_{[\tau_{n-1}, \tau_n[}(-E[\xi^- \mid \mathcal{F}_\cdot] - Y_0^b) \,. \tag{3.18}$$

By means of (3.18) and since $\xi^- \in L^1(\mathcal{F}_T)$, with $N^\infty := \lim_n N^n_{\tau_{n-1}}$, the process

$$N = \sum_{n \geq 1} N^n 1_{[\tau_{n-1}, \tau_n[} + 1_{[\tau^*, T]} N^\infty$$

is a well-defined continuous supermartingale due to Proposition 3.4. Hence we may define a localizing sequence by setting $\kappa_n := \inf\{t \geq 0 : |N_t| > n\} \wedge T$ and deduce that N is a continuous local martingale, because N^{κ_n} is a uniformly integrable martingale, for all $n \in \mathbb{N}$. Indeed, for each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, the process $(N^m)^{\kappa_n}$, being a bounded stochastic integral, is a martingale. Moreover, the family $(N^m_{\kappa_n \wedge t})_{m \in \mathbb{N}}$ is uniformly integrable and $N_{\kappa_n \wedge t} = \lim_m N^m_{\kappa_n \wedge t}$, for all $t \in [0, T]$. Consequently, $E[N^{\kappa_n}_t \mid \mathcal{F}_s] = \lim_m E[N^m_{\kappa_n \wedge t} \mid \mathcal{F}_s] = \lim_m N^m_{\kappa_n \wedge s} = N^{\kappa_n}_s$, for all $0 \leq s \leq t \leq T$, and the claim follows. Since the quadratic variation of a continuous local martingale is continuous and unique, see [9, page 36], we obtain

$$\int_0^{\tau^*} \bar{Z}^2_u du = \lim_n \int_0^{\kappa_n \wedge \tau^*} \bar{Z}^2_u du = \lim_n \langle N \rangle_{\kappa_n \wedge \tau^*} = \langle N \rangle_{\tau^*} < \infty \,.$$

Observe that $\sigma := \sum_{n \geq 1} 1_{B_n} \sigma_n$ is an element of $\mathcal{T}$. Indeed, $\{\sigma \leq t\} = \bigcup_{n \geq 1}(B_n \cap \{\sigma_n \leq t\}) \in \mathcal{F}_t$, for all $t \in [0, T]$, since $B_n \in \mathcal{F}_{\sigma_n}$. From $\bar{Z} 1_{]\tau^*, \sigma]} = 0$ we get that

$$\int_0^T \bar{Z}^2_u du = \langle N \rangle_{\tau^*} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \int_\sigma^T (\tilde{Z}^n_u)^2 du < \infty \,,$$

since $(\tilde{Z}^n) \subset \mathcal{L}$. Hence we conclude that $\bar{Z} \in \mathcal{L}$. As for the supermartingale property of $\int \bar{Z} dW$, observe that

$$\int_0^{t\wedge\tau^*} \bar{Z}_u dW_u = \lim_{n\to\infty} \int_0^{t\wedge\tau_n} Z_u^n dW_u$$
$$\geq \lim_{n\to\infty} -E\left[\xi^- \mid \mathcal{F}_{t\wedge\tau_n}\right] - Y_0^b = -E\left[\xi^- \mid \mathcal{F}_{t\wedge\tau^*}\right] - Y_0^b\,,$$

where the inequality follows from (3.1) and (POS). Being bounded from below by a martingale, we deduce by Fatou's lemma that $\bar{Z}1_{[0,\tau^*]}$ is admissible. Since $\bar{Z}1_{]\tau^*,\sigma]} = 0$ and all $\tilde{Z}^n$ are admissible, it follows from Proposition 3.2 that $\bar{Z}$ is indeed admissible.

Step 4c: The pair $(\bar{Y}, \bar{Z})$ is a supersolution. Finally, showing that $(\bar{Y}, \bar{Z})$ satisfies (3.1) identifies $(\bar{Y}, \bar{Z})$ as an element of $\mathcal{A}(\xi, g)$. Observe first that, for all $0 \leq s \leq t \leq T$ and all $m \in \mathbb{N}$, the expression $\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u$ can be written as

$$\bar{Y}_s - \int_s^{(\tau_m\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^{(\tau_m\vee s)\wedge t} \bar{Z}_u dW_u$$
$$- \int_{(\tau_m\vee s)\wedge t}^{(\tau^*\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\tau_m\vee s)\wedge t}^{(\tau^*\vee s)\wedge t} \bar{Z}_u dW_u - \int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du$$
$$+ \int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} \bar{Z}_u dW_u - \int_{(\sigma\vee s)\wedge t}^{t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma\vee s)\wedge t}^{t} \bar{Z}_u dW_u\,. \tag{3.19}$$

Now, we have that

$$\bar{Y}_s - \int_s^{(\tau_m\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^{(\tau_m\vee s)\wedge t} \bar{Z}_u dW_u \;\geq\; \bar{Y}_{(\tau_m\vee s)\wedge t}\,, \tag{3.20}$$

by Proposition 3.2, since $((Y^m, Z^m)) \subset \mathcal{A}(\xi, g)$ and $Y^m_{\tau_m-} \geq Y^{m+1}_{\tau_m}$, for all $m \in \mathbb{N}$, due to (3.11). By letting m tend to infinity and noting that

$$\lim_{m\to\infty} \int_{(\tau_m\vee s)\wedge t}^{(\tau^*\vee s)\wedge t} \bar{Z}_u dW_u = 0 \quad \text{and} \quad \lim_{m\to\infty} \int_{(\tau_m\vee s)\wedge t}^{(\tau^*\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du = 0\,,$$

(3.19) and (3.20) yield that

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u$$

$$\geq \bar{Y}_{((\tau^*-)\vee s)\wedge t} - \int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} \bar{Z}_u dW_u$$

$$- \int_{(\sigma\vee s)\wedge t}^{t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma\vee s)\wedge t}^{t} \bar{Z}_u dW_u \,. \quad (3.21)$$

We now use that $\bar{Y}$ can only jump downwards at τ^*. Indeed, since $\bar{Y}$ is càdlàg, in particular $\bar{Y}_{\tau^*-}$, the left limit at τ^*, exists and is unique, P-almost surely. Furthermore, it holds that $\lim_{m\to\infty} \bar{Y}_{\tau_m-} = \bar{Y}_{\tau^*-}$. Indeed, since the left limits $\bar{Y}_{\tau_m-}$ are well-defined, for all $m \in \mathbb{N}$, we can choose a sequence of stopping times (η_m) such that $\eta_m \in [\tau_{m-1}, \tau_m[$ and $|\bar{Y}_{\tau_m-} - \bar{Y}_{\eta_m}| < \frac{1}{m}$. Since $\lim_m \eta_m = \tau^*$ and $\bar{Y}$ is càdlàg , in particular holds $\lim_m \bar{Y}_{\eta_m} = \bar{Y}_{\tau^*-}$ and the claim follows by an application of the triangular inequality. Thus

$$\bar{Y}_{\tau^*-} = \lim_m \bar{Y}_{\tau_m-} = \lim_m Y^m_{\tau_m-} \geq \lim_m Y^m_{\tau_m}$$

$$\geq \lim_m \mathcal{E}^g_{\tau_m}(\xi) + \varepsilon = \mathcal{E}^g_{\tau^*-}(\xi) + \varepsilon \geq \mathcal{E}^g_{\tau^*}(\xi) + \varepsilon > \bar{Y}_{\tau^*} \,.$$

The first and third inequality hold, since càdlàg supermartingale can only jump downwards, see the proof of Proposition 3.4. Hence, (3.21) can be further estimated by

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u$$

$$\geq \bar{Y}_{(\tau^*\vee s)\wedge t} - \int_{(\sigma\vee s)\wedge t}^{t} g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma\vee s)\wedge t}^{t} \bar{Z}_u dW_u \,, \quad (3.22)$$

where we used that

$$\int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} g_u(\bar{Y}_u, \bar{Z}_u)du = \int_{(\tau^*\vee s)\wedge t}^{(\sigma\vee s)\wedge t} \bar{Z}_u dW_u = 0 \,,$$

due to (3.17), the definition of σ, and (NOR). Now observe that $\bar{Y}_{(\tau^*\vee s)\wedge t} \geq \bar{Y}_{(\sigma\vee s)\wedge t}$, since $\bar{Y}1_{[\tau^*,\sigma[} = (\mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2})1_{[\tau^*,\sigma[}$ and $\bar{Y}$ can only jump downwards at σ. Indeed, on the set B_n, by means of (3.16), (3.14), and (3.15) holds

$$\bar{Y}_{\sigma_n-} = \mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2} = \mathcal{E}^g_{\tau^*}(\xi) - \hat{\mathcal{E}}^g_{\sigma_n}(\xi) + \hat{\mathcal{E}}^g_{\sigma_n}(\xi) + \frac{\varepsilon}{2}$$

$$\geq -\frac{\varepsilon}{8} + \hat{\mathcal{E}}^g_{\sigma_n}(\xi) + \frac{\varepsilon}{2} \geq \tilde{Y}^n_{\sigma_n} - \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \tilde{Y}^n_{\sigma_n} = \bar{Y}_{\sigma_n} \,.$$

Consequently,

$$\bar{Y}_s - \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_s^t \bar{Z}_u dW_u$$

$$\geq \bar{Y}_{(\sigma\vee s)\wedge t} - \int_{(\sigma\vee s)\wedge t}^t g_u(\bar{Y}_u, \bar{Z}_u)du + \int_{(\sigma\vee s)\wedge t}^t \bar{Z}_u dW_u \geq \bar{Y}_t \ , \quad (3.23)$$

where the second inequality in (3.23) follows from $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ and Proposition 3.2.

Step 5: The maximal element (Y^M, Z^M)*.* By Zorn's lemma, there exists a maximal element (Y^M, Z^M) in $\mathcal{A}(\xi, g)$ with respect to the preorder (3.9), satisfying, without loss of generality, $Y_T^M = \xi$. Finally, by showing that the corresponding stopping time satisfies $\tau^M = T$, we obtain a supersolution (Y^M, Z^M) satisfying $\|Y^M - \mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty} \leq \varepsilon$, due to the definition of τ^M in analogy to (3.10). Thus, choosing $Y^M = Y^\varepsilon$ in (3.8) finishes our proof.

But on $\{\tau^M < T\}$ we consider the chain consisting only of (Y^M, Z^M) and, analogously to (3.16) and (3.17), construct an upper bound $(\bar{Y}, \bar{Z})$, with corresponding stopping time $\tau(\bar{Y})$ as in (3.10), satisfying $P[\tau(\bar{Y}) > \tau^M \,|\, \tau^M < T] = 1$. This yields $P[\tau^M < T] \leq P[\tau(\bar{Y}) > \tau^M] = 0$, due to the maximality of τ^M. Hence we deduce that $\tau^M = T$. □

The techniques used in the proof of Theorem 3.5 show that $\mathcal{A}(\xi, g)$ exhibits a certain closedness under monotone limits of decreasing supersolutions.

Theorem 3.6. *Let g be a generator satisfying* (LSC)*,* (POS) *and* (NOR) *and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. Let furthermore $((Y^n, Z^n))$ be a decreasing sequence within $\mathcal{A}(\xi, g)$ with pointwise limit $\hat{Y}_t := \lim_n Y_t^n$, for $t \in [0, T]$. Then $\hat{Y}$ is a supermartingale and it holds*

$$\hat{Y}_t \geq Y_t := \lim_{\substack{s\downarrow t \\ s\in\mathbb{Q}}} \hat{Y}_s \,, \quad \textit{for all } t \in [0, T) \,. \quad (3.24)$$

Moreover, with $Y_T := \xi$, there is a sequence $((\tilde{Y}^n, \tilde{Z}^n)) \subset \mathcal{A}(\xi, g)$ such that $\lim_n \|\tilde{Y}^n - Y\|_{\mathcal{R}^\infty} = 0$, and a unique control $Z \in \mathcal{L}$ such that $(Y, Z) \in \mathcal{A}(\xi, g)$.

Proof. The proof is a straightforward adaptation of the proof of Theorem 3.5. □

Now we focus on the question whether it is possible to find a minimal supersolution within $\mathcal{A}(\xi, g)$, the associated control process Z of which belongs to $\mathcal{L}^1$, and $\int Z dW$ therefore constitutes a true martingale instead of only a supermartingale. To this end, we consider the following subset of $\mathcal{A}(\xi, g)$

$$\mathcal{A}^1(\xi, g) := \left\{(Y, Z) \in \mathcal{A}(\xi, g) \,:\, Z \in \mathcal{L}^1\right\} \,. \quad (3.25)$$

By imposing stronger assumptions on the terminal condition ξ, the next theorem yields the existence of a unique minimal supersolution in $\mathcal{A}^1(\xi, g)$.

Theorem 3.7. *Assume that the generator g satisfies* (LSC)*,* (POS) *and* (NOR)*, and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- \,|\, \mathcal{F}_\cdot])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^1(\xi, g) \neq \emptyset$, then there exists a control $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z})$ is the unique minimal supersolution in $\mathcal{A}^1(\xi, g)$.*

Proof. $\mathcal{A}^1(\xi, g) \neq \emptyset$ yields that $\mathcal{A}(\xi, g) \neq \emptyset$, because $\mathcal{A}^1(\xi, g) \subseteq \mathcal{A}(\xi, g)$. Also, from $(E[\xi^- \,|\, \mathcal{F}_\cdot])^*_T \in L^1(\mathcal{F}_T)$ we deduce that $\xi^- \in L^1(\mathcal{F}_T)$. Hence, Theorem 3.5 yields the existence of an unique control $\hat{Z}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$. Verifying that $\hat{Z} \in \mathcal{L}^1$ is done as in [7, Theorem 4.5]. □

3.3 Relaxations of the Conditions (NOR) and (POS)

In this section, we discuss possible relaxations of the conditions (NOR) and (POS) imposed on the generator throughout Sections 3.1 and 3.2.

First, we want to replace (NOR) by the weaker assumption (NOR'). We say that a generator g satisfies

(NOR') if, for all $\tau \in \mathcal{T}$, there exists some stopping time $\delta > \tau$ such that the stochastic differential equation

$$dy_s = -g_s(y_s, 0)ds\,, \quad y_\tau = \mathcal{E}^g_\tau(\xi) + \frac{\varepsilon}{2} \tag{3.26}$$

admits a solution on $[\tau, \delta]$ where we set $g_t(y, 0) = 0$, for all $y \in \mathbb{R}$ and $t > T$.

Remark 3.8. It is possible to relax the condition (NOR') further by requiring that the stochastic differential inequality $dy_s \geq -g_s(y_s, 0)ds$ with initial value $y_\tau = \mathcal{E}^g_\tau(\xi) + \frac{\varepsilon}{2}$ has a càdlàg solution y on $[\tau, \delta)$. ♦

By this we obtain the following extension of Theorem 3.5.

Theorem 3.9. *Let g be a generator satisfying* (LSC)*,* (POS) *and* (NOR') *and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(\mathcal{F}_T)$. If $\mathcal{A}(\xi, g) \neq \emptyset$, then there exists a unique control process $\hat{Z} \in \mathcal{L}$ such that $(\mathcal{E}^g(\xi), \hat{Z}) \in \mathcal{A}(\xi, g)$.*

Proof. The proof is almost the same as the proof of Theorem 3.5. The only difference lies in the definition of $\bar{Y}$ in (3.16). After τ^*, instead of extending by a constant function, we concatenate the value process at τ^* with the solution of the SDE (3.26), started at $y_{\tau^*} = \mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2}$ and denoted by y. We emphasize that the zero control is maintained. We only need to adjust the argumentation in Step 4c. To that end, we introduce the stopping time

$$\kappa := \inf\{t > \tau^* \,:\, \int_{\tau^*}^{t} g_s(y_s, 0)ds > \frac{\varepsilon}{8}\} \wedge \delta\,, \tag{3.27}$$

and use $\bar{\kappa} := \kappa \wedge \bar{\tau}$, with $\bar{\tau}$ as in (3.12), within the definition of the sequence (σ_n) in analogy to (3.13), that is, $\sigma_n = \hat{\sigma}_n \wedge \bar{\kappa}$, for all $n \in \mathbb{N}$. As before, we set $\sigma := \sum_{n \geq 1} 1_{B_n} \sigma_n$. Consequently, $\bar{Y}$ is given by

$$\bar{Y} = \sum_{m \geq 1} Y^m 1_{[\tau_{m-1}, \tau_m[} + 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \Big(\mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2} - \int_{\tau^*}^{\cdot} g_s(y_s, 0) ds \Big) 1_{[\tau^*, \sigma_n[}$$
$$+ 1_{\{\tau^* < T\}} \sum_{n \geq 1} 1_{B_n} \tilde{Y}^n 1_{[\sigma_n, T[} \quad , \quad \bar{Y}_T = \xi.$$

The definition of the stopping time $\bar{\tau}$ implies that, on the set B_n, we have $\bar{Y}_t \leq \mathcal{E}^g_t(\xi) + \varepsilon$, for all $t \in [\tau^*, \sigma_n[$. Indeed, observe that, for $t \in [\tau^*, \sigma_n[$,

$$\bar{Y}_t = \mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2} - \int_{\tau*}^{t} g_s(y_s, 0) ds \leq \mathcal{E}^g_t(\xi) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mathcal{E}^g_t(\xi) + \varepsilon.$$

Furthermore, on the set B_n, by means of (3.14), (3.27), and (3.15),

$$\begin{aligned} \bar{Y}_{\sigma_n -} &= \mathcal{E}^g_{\tau^*}(\xi) + \frac{\varepsilon}{2} - \int_{\tau*}^{\sigma_n} g_s(y_s, 0) ds \\ &= \mathcal{E}^g_{\tau^*}(\xi) - \hat{\mathcal{E}}^g_{\sigma_n}(\xi) + \hat{\mathcal{E}}^g_{\sigma_n}(\xi) + \frac{\varepsilon}{2} - \int_{\tau*}^{\sigma_n} g_s(y_s, 0) ds \\ &\geq \frac{3\varepsilon}{8} + \hat{\mathcal{E}}^g_{\sigma_n}(\xi) - \int_{\tau*}^{\sigma_n} g_s(y_s, 0) ds \geq \frac{2\varepsilon}{8} + \hat{\mathcal{E}}^g_{\sigma_n}(\xi) \geq \bar{Y}_{\sigma_n} . \end{aligned} \tag{3.28}$$

Hence, pasting at the stopping time σ is in accordance with Proposition 3.2. This yields the result. □

As in [7], the positivity assumption (POS) on the generator can be relaxed to a linear lower bound, which, however, has to be consistent with the assumption (NOR'). In the following a generator g is said to be

(LB-NOR') linearly bounded from below under (NOR'), if there exist adapted measurable processes a and b with values in $\mathbb{R}^{1 \times d}$ and $\mathbb{R}$, respectively, such that $g(y, z) \geq a z^T - b$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d}$, and

$$\frac{dP^a}{dP} = \mathcal{E}\left(\int a dW \right)_T \tag{3.29}$$

defines an equivalent probability measure P^a. Furthermore, $\int_0^t b_s ds \in L^1(P^a)$ holds for all $t \in [0, T]$, and a and b are such that the positive generator defined by

$$\bar{g}(y, z) := g\left(y + \int_0^{\cdot} b_s ds, z \right) - a z^T - b \, , \text{ for all } (y, z) \in \mathbb{R} \times \mathbb{R}^{1 \times d} , \tag{3.30}$$

satisfies (NOR').

An (LB-NOR') setting can always be reduced to a setting with generator satisfying (POS) and (NOR'), by using the change of measure (3.29) and $\bar{g}$ defined in (3.30). Hence, Lemma 3.1 and Proposition 3.2, which strongly rely on the property (POS), can be applied. However, we need a slightly different definition of admissibility than before. A control process Z is said to be a-admissible, if $\int ZdW^a$ is a P^a-supermartingale, where $W^a = W - \int a ds$ is a P^a-Brownian motion by Girsanov's theorem.

The set $\mathcal{A}^a(\xi, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} \,:\, Z$ is a-admissible and (3.1) holds$\}$, as well as the process

$$\hat{\mathcal{E}}_t^{g,a}(\xi) = \operatorname{ess\,inf}\{Y_t \in L^0(\mathcal{F}_t) \,:\, (Y, Z) \in \mathcal{A}^a(\xi, g)\}\,, \quad \text{for } t \in [0, T]\,,$$

are defined analogously to (3.3) and (3.4), respectively. We are now ready to state our most general result, which follows from Theorem 3.9 and [7, Theorem 4.16].

Theorem 3.10. *Let g be a generator satisfying* (LSC) *and* (LB-NOR') *and $\xi \in L^0(\mathcal{F}_T)$ a terminal condition such that $\xi^- \in L^1(P^a)$. If in addition $\mathcal{A}^a(\xi, g) \neq \emptyset$, then*

$$\mathcal{E}_t^{g,a}(\xi) := \lim_{s \downarrow t, s \in \mathbb{Q}} \hat{\mathcal{E}}_s^{g,a}(\xi)\,, \quad \textit{for all } t \in [0, T) \quad \textit{and} \quad \mathcal{E}_T^{g,a}(\xi) := \xi$$

is the value process of the unique minimal supersolution, that is, there exists a unique control process $\hat{Z}$ such that $(\mathcal{E}^{g,a}(\xi), \hat{Z}) \in \mathcal{A}^a(\xi, g)$.

3.4 Continuous Local Martingales and Controls in $\mathcal{L}^1$

Under stronger integrability conditions, the techniques used in the proof of Theorem 3.5 can be generalized to the case where the Brownian motion W appearing in the stochastic integral in (3.1) is replaced by a d-dimensional continuous local martingale M. Let us assume that M is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, which satisfies the usual conditions and in which all martingales are continuous and all stopping times are predictable. We consider controls within the set $\mathcal{L}^1 := \mathcal{L}^1(M)$, consisting of all $\mathbb{R}^{1\times d}$-valued, progressively measurable processes Z, such that $\int ZdM \in \mathcal{H}^1$. As before, for $Z \in \mathcal{L}^1$ the stochastic integral $\int ZdM$ is well defined and is by means of the Burkholder-Davis-Gundy inequality a continuous martingale. A pair $(Y, Z) \in \mathcal{S} \times \mathcal{L}^1$ is now called a supersolution of a BSDE, if it satisfies, for $0 \leq s \leq t \leq T$,

$$Y_s - \int_s^t g_u(Y_u, Z_u) d\langle M \rangle_u + \int_s^t Z_u dM_u \geq Y_t \quad \text{and} \quad Y_T \geq \xi\,, \tag{3.31}$$

for a generator g and a terminal condition $\xi \in L^0(\mathcal{F}_T)$. We will focus on the set

$$\mathcal{A}^{M,1}(\xi, g) := \left\{(Y, Z) \in \mathcal{S} \times \mathcal{L}^1 \,:\, (Y, Z) \text{ satisfy } (3.31)\right\}\,.$$

If we assume $\mathcal{A}^{M,1}(\xi,g)$ to be non-empty, Theorem 3.5 combined with compactness results for sequences of $\mathcal{H}^1$-bounded martingales given in [4] yields that

$$\mathcal{E}_t^g(\xi) := \lim_{s\downarrow t, s\in\mathbb{Q}} \hat{\mathcal{E}}_s^g(\xi)\,, \quad \text{for all } t\in[0,T) \quad \text{and} \quad \mathcal{E}_T^g(\xi) := \xi\,,$$

where

$$\hat{\mathcal{E}}_t^g(\xi) := \operatorname{ess\,inf}\left\{Y_t \in L^0(\mathcal{F}_t)\ :\ (Y,Z)\in\mathcal{A}^{M,1}(\xi,g)\right\}\,, \quad t\in[0,T]\,,$$

is the value process of the unique minimal supersolution within $\mathcal{A}^{M,1}(\xi,g)$. Note that Lemma 3.1 and Proposition 3.2 extend to the case where W is substituted by M.

Theorem 3.11. *Assume that the generator g satisfies* (LSC)*,* (POS) *and* (NOR) *and let $\xi \in L^0(\mathcal{F}_T)$ be a terminal condition such that $(E[\xi^- \,|\, \mathcal{F}_\cdot])^* \in L^1(\mathcal{F}_T)$. If $\mathcal{A}^{M,1}(\xi,g)\neq\emptyset$, then there exists a unique control $\hat{Z}$ such that $(\mathcal{E}^g(\xi),\hat{Z})\in\mathcal{A}^{M,1}(\xi,g)$.*

Proof. By assumption, there is some $(Y^b,Z^b)\in\mathcal{A}^{M,1}(\xi,g)$ and we consider, without loss of generality, only those pairs $(Y,Z)\in\mathcal{A}^{M,1}(\xi,g)$ satisfying $Y\leq Y^b$, obtained by suitable pasting as in Proposition 3.2. Using the techniques of the proof of Theorem 3.5, we can find a sequence $((Y^n,Z^n))\subset\mathcal{A}^{M,1}(\xi,g)$ satisfying $\lim_n \|Y^n-\mathcal{E}^g(\xi)\|_{\mathcal{R}^\infty}=0$, in analogy to (3.7). Since $(\int Z^n dM)$ is uniformly bounded in $\mathcal{H}^1$, compare [7, Theorem 4.5], it follows from [2, Theorem 1] that $\mathcal{E}^g(\xi)$ is a special semimartingale with canonical decomposition $\mathcal{E}^g(\xi)=\mathcal{E}_0^g(\xi)+N-A$ and that

$$\lim_{n\to\infty}\left\|\int Z^n dM - N\right\|_{\mathcal{H}^1} = 0\,. \tag{3.32}$$

Moreover, $N\in\mathcal{H}^1$. Now [4, Theorem 1.6] yields the existence of some $\hat{Z}\in\mathcal{L}^1$ such that $N=\int\hat{Z}dM$. By means of (3.32), (Z^n) converges, up to a subsequence, $P\otimes d\langle M\rangle_t$-almost everywhere to $\hat{Z}$ and $\lim_n\int_0^t Z^n dM=\int_0^t\hat{Z}dM$, for all $t\in[0,T]$, P-almost surely, by means of the Burkholder-Davis-Gundy inequality. In particular, $\lim_{n\to\infty}Z^n(\omega)=\hat{Z}(\omega)$, $d\langle M\rangle$-almost everywhere, for almost all $\omega\in\Omega$. Verifying that $(\mathcal{E}^g(\xi),\hat{Z})$ satisfy (3.31) is now done analogously to Step 1 in the proof of Theorem 3.5, and hence we are done. □

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