Expressing Arakelov invariants using hyperbolic heat kernels

Jay Jorgenson and Jürg Kramer

1. Introduction

1.1. In [**1**], S. Arakelov defined an intersection theory of divisors on an arithmetic surface by including a contribution at infinity which is computed using certain Green's functions defined on the corresponding Riemann surface. In his foundational paper [**5**], G. Faltings established fundamental results in the development of Arakelov theory for arithmetic surfaces based on S. Arakelov's original work on this subject. This work was the origin for various developments in arithmetic geometry such as the creation of higher dimensional Arakelov theory by C. Soulé and H. Gillet, or more refined work on arithmetic surfaces by A. Abbes, P. Michel and E. Ullmo, or P. Vojta's work on the Mordell conjecture.

In the setting of Faltings's results [**5**] there are a number of basic analytic quantities attached to compact Riemann surfaces, namely: The canonical volume form μ_{can}, the canonical Green's function g_{can}, the Arakelov volume form μ_{Ar}, and Faltings's delta function δ_{Fal}. For brevity of language, we will use the term *analytic Arakelov invariants* to describe these four analytic quantities arising from the Arakelov theory of algebraic curves. Unquestionably, applications of the general arithmetic results from [**1**] or [**5**] include an understanding of the aforementioned analytic invariants. Various authors have derived numerous identities which express these invariants in terms of the classical Riemann theta function (see, e.g., [**2**], [**3**], [**6**], [**7**], [**9**], [**15**], and [**19**]). In some sense, these identities all utilize the algebraic geometry of the underlying compact Riemann surface since, in many regards, the classical Riemann theta function is a quantity that naturally belongs to the algebraic geometric aspect of Riemann surface theory.

1.2. Summary of the results. In this article, we revisit the problem of understanding the basic analytic invariants from the Arakelov theory of arithmetic surfaces. Specifically, we take the point of view of differential geometry and relate the invariants under consideration to aspects of the hyperbolic geometry of Riemann surface theory, specifically the hyperbolic heat kernel. The main results of this paper are new identities which express each of the above mentioned Arakelov invariants (the canonical volume form, the canonical Green's function, the Arakelov

The first author acknowledges support from a PSC-CUNY grant.

volume form, and Faltings's delta function) explicitly in terms of the hyperbolic heat kernel.

1.3. Outline of the paper. The outline of the paper is as follows. In section 2, we establish the notation we need throughout the article. In section 3, we prove the first of our identities. Theorem 3.5 expresses the canonical volume form in terms of the hyperbolic volume form and the hyperbolic heat kernel. In some sense, Theorem 3.5 is the main result since all subsequent computations use the identity given in Theorem 3.5. Theorem 3.9 relates the canonical Green's function to the hyperbolic Green's function and the hyperbolic heat kernel, from which we immediately obtain Corollary 3.10, which studies the Arakelov volume form in terms of hyperbolic heat kernels. In section 4 we turn our attention to the Faltings's delta function. The main result of section 4, Theorem 4.6, expresses δ_{Fal} in terms of hyperbolic geometry.

1.4. Acknowledgements. Some of the results stated in this article have been used elsewhere in our study of analytic aspects of Arakelov theory (see [**11**], [**12**], and [**13**]). The second named author presented parts of these results in a lecture at the International Conference "Shimura Varieties, Lattices and Symmetric Spaces" held in Ascona in May 2004 and thanks L. Clozel, B. Edixhoven, and E. Ullmo for their insightful comments. We also thank James Cogdell and Andrew McIntyre for their constructive comments concerning preliminary versions of the present article.

2. Background material

2.1. Hyperbolic and canonical metrics. Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H}=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0\}$. We let X be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g_X the genus of X. In a slight abuse of notation, we will throughout this article identify X with a fundamental domain (say, a Ford domain, bounded by geodesic paths) and identify points on X with their pre-images in $\mathbb{H}$. We assume that $g_X>1$ and that Γ has no elliptic and, apart from the identity, no parabolic elements, i.e., X is smooth and compact.

In the sequel, μ denotes a (smooth) metric on X, i.e., μ is a positive $(1,1)$-form on X. We write $\mathrm{vol}_\mu(X)$ for the volume of X with respect to μ. In particular, we let μ_{hyp} denote the hyperbolic metric on X, which is compatible with the complex structure of X, and has constant negative curvature equal to -1. Locally, we have

$$\mu_{\mathrm{hyp}}(z)=\frac{i}{2}\cdot\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{\mathrm{Im}(z)^2}\,.$$

As a shorthand, we write v_X for the hyperbolic volume $\mathrm{vol}_{\mu_{\mathrm{hyp}}}(X)$; we recall that v_X is given by $4\pi(g_X-1)$. The scaled hyperbolic metric μ_{shyp} is simply the rescaled hyperbolic metric μ_{hyp}/v_X, which measures the volume of X to be one.

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f,g\rangle=\frac{i}{2}\int\limits_X f(z)\,\overline{g(z)}\,\mathrm{Im}(z)^k\cdot\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{\mathrm{Im}(z)^2}\qquad(f,g\in S_k(\Gamma))\,.$$

By choosing an orthonormal basis $\{f_1, \ldots, f_{g_X}\}$ of $S_2(\Gamma)$ with respect to the Petersson inner product, the canonical metric μ_{can} of X is given by

$$\mu_{\mathrm{can}}(z) = \frac{1}{g_X} \cdot \frac{i}{2} \sum_{j=1}^{g_X} |f_j(z)|^2 \, \mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

We note that the canonical metric measures the volume of X to be one.

2.2. Green's functions and residual metrics. We denote the Green's function associated to the metric μ by g_μ. It is a function on $X \times X$ characterized by the two properties

$$\mathrm{d}_z \mathrm{d}_z^c g_\mu(z,w) + \delta_w(z) = \frac{\mu(z)}{\mathrm{vol}_\mu(X)}\,,$$

$$\int_X g_\mu(z,w)\mu(z) = 0 \quad (w \in X)\,;$$

recall that $\mathrm{d}^c = (4\pi i)^{-1}(\partial - \bar{\partial})$, and $\mathrm{dd}^c = -(2\pi i)^{-1}\partial\bar{\partial}$. Assuming that z, w are points on X, which are sufficiently close, our convention for the Green's function is such that the sum $g_\mu(z,w) + \log|z-w|^2$ is bounded as w approaches z. Basic and fundamental information concerning the role of Green's functions in Arakelov theory can be found in [**1**], [**5**], [**15**], and [**18**].

Writing $z = x + iy$ and $\mu(z) = e^{2\rho_\mu(z)}(i/2)\mathrm{d}z \wedge \mathrm{d}\bar{z}$, the Laplacian Δ_μ associated to μ is given by

$$\Delta_\mu = -e^{-2\rho_\mu(z)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -4e^{-2\rho_\mu(z)}\frac{\partial^2}{\partial z \partial \bar{z}}\,.$$

We set the convention that an eigenfunction φ with associated eigenvalue λ satisfies the equation $\Delta_\mu \varphi = \lambda\varphi$; hence, in our notation, we have $\lambda \geq 0$. Also, we note that by combining the above formulas, one gets, for any smooth function f on X, the relation

$$-4\pi \mathrm{dd}^c f = (\Delta_\mu f)\mu\,.$$

The Green's function g_μ is the integral kernel which inverts the operator $-\mathrm{dd}^c$ on the space of functions whose integral is zero. More precisely, for any smooth, bounded function f on X, we have the identity

$$\int_X g_\mu(z,\zeta)(-\mathrm{d}_\zeta \mathrm{d}_\zeta^c f(\zeta)) = f(z), \quad \text{provided} \quad \int_X f(\zeta)\mu(\zeta) = 0.$$

If $\mu = \mu_{\mathrm{hyp}}$, $\mu = \mu_{\mathrm{shyp}}$, or $\mu = \mu_{\mathrm{can}}$, we set

$$g_\mu = g_{\mathrm{hyp}}\,, \quad g_\mu = g_{\mathrm{shyp}}\,, \quad g_\mu = g_{\mathrm{can}}\,,$$

respectively. By means of the function $G_\mu = \exp(g_\mu)$, we can now define a metric $\|\cdot\|_{\mu,\mathrm{res}}$ on the canonical line bundle Ω_X^1 of X in the following way. For $z \in X$, we set

$$\|\mathrm{d}z\|_{\mu,\mathrm{res}}^2 = \lim_{w \to z}\left(G_\mu(z,w) \cdot |z-w|^2\right).$$

We call the metric

$$\mu_{\mathrm{res}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{\|\mathrm{d}z\|_{\mu,\mathrm{res}}^2}$$

the residual metric associated to μ. If $\mu = \mu_{\mathrm{hyp}}$, $\mu = \mu_{\mathrm{shyp}}$, or $\mu = \mu_{\mathrm{can}}$, we set

$$\|\cdot\|_{\mu,\mathrm{res}} = \|\cdot\|_{\mathrm{hyp,res}}\,, \quad \|\cdot\|_{\mu,\mathrm{res}} = \|\cdot\|_{\mathrm{shyp,res}}\,, \quad \|\cdot\|_{\mu,\mathrm{res}} = \|\cdot\|_{\mathrm{can,res}}\,,$$

$$\mu_{\mathrm{res}} = \mu_{\mathrm{hyp,res}}\,, \quad \mu_{\mathrm{res}} = \mu_{\mathrm{shyp,res}}\,, \quad \mu_{\mathrm{res}} = \mu_{\mathrm{can,res}}\,,$$

respectively. We recall that the Arakelov metric μ_{Ar} is defined as the residual metric associated to the canonical metric μ_{can}; the corresponding metric on Ω^1_X is denoted by $\|\cdot\|_{\mathrm{Ar}}$. In order to be able to compare the metrics μ_{Ar} and μ_{hyp}, we define the C^∞-function ϕ_{Ar} on X by the equation

$$\mu_{\mathrm{Ar}} = e^{\phi_{\mathrm{Ar}}}\mu_{\mathrm{hyp}}\,. \tag{2.1}$$

2.3. Faltings's delta function and determinants. Recall that the Laplacian on X associated to the metric μ is denoted by Δ_μ, and we write Δ_{hyp} for the hyperbolic Laplacian on X. In general, associated to the Laplacian Δ_μ one has a spectral zeta function $\zeta_\mu(s)$, which gives rise to the regularized determinant $\det^*(\Delta_\mu)$ (see, for example, [**18**] and references therein). We set the notation

$$D_\mu(X) = \log\left(\frac{\det^*(\Delta_\mu)}{\mathrm{vol}_\mu(X)}\right).$$

If $\mu = \mu_{\mathrm{hyp}}$, or $\mu = \mu_{\mathrm{Ar}}$, we set $D_\mu = D_{\mathrm{hyp}}$, or $D_\mu = D_{\mathrm{Ar}}$, respectively. Observing the first Chern form relations

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp}}) = (2g_X - 2)\mu_{\mathrm{shyp}}\,, \quad \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{Ar}}) = (2g_X - 2)\mu_{\mathrm{can}}\,,$$

an immediate application of Polyakov's formula (see [**14**], p. 78) shows the relation

$$D_{\mathrm{Ar}}(X) = D_{\mathrm{hyp}}(X) + \frac{g_X - 1}{6}\int\limits_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z) + \mu_{\mathrm{can}}(z))\,. \tag{2.2}$$

Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ is introduced in [**5**], where also some of its basic properties are given. In [**9**], Faltings's delta function is expressed in terms of Riemann's theta function, and its asymptotic behavior is investigated. As a by-product of the analytic part of the arithmetic Riemann-Roch theorem for arithmetic surfaces, it is shown in [**18**] that

$$\delta_{\mathrm{Fal}}(X) = -6D_{\mathrm{Ar}}(X) + a(g_X)\,, \tag{2.3}$$

where

$$a(g_X) = -2g_X\log(\pi) + 4g_X\log(2) + (g_X - 1)(-24\zeta'_{\mathbb{Q}}(-1) + 1). \tag{2.4}$$

2.4. Heat kernels and heat traces. The heat kernel $K_{\mathbb{H}}(t; z, w)$ on $\mathbb{H}$ ($t \in \mathbb{R}_{>0}$; $z, w \in \mathbb{H}$) is given by the formula

$$K_{\mathbb{H}}(t; z, w) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int\limits_\rho^\infty \frac{re^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(\rho)}}\,\mathrm{d}r\,,$$

where $\rho = d_{\mathbb{H}}(z, w)$ denotes the hyperbolic distance between z and w. The heat kernel $K_{\mathrm{hyp}}(t; z, w)$ associated to X ($t \in \mathbb{R}_{>0}$; $z, w \in X$), resp. the hyperbolic heat kernel $HK_{\mathrm{hyp}}(t;z,w)$ associated to X ($t\in\mathbb{R}_{>0}$; $z,w\in X$) is defined by averaging over the elements of Γ, resp. the elements of Γ different from the identity, namely

$$\begin{aligned} K_{\mathrm{hyp}}(t;z,w) &= \sum_{\gamma\in\Gamma} K_{\mathbb{H}}(t;z,\gamma w)\,, \text{ resp.} \\ HK_{\mathrm{hyp}}(t;z,w) &= \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma w)\,. \end{aligned}$$

If $z=w$, we write $K_{\mathrm{hyp}}(t;z)$ instead of $K_{\mathrm{hyp}}(t;z,z)$, and analogously for the functions $K_{\mathbb{H}}(t;z,z)$, resp. $HK_{\mathrm{hyp}}(t;z,z)$. The hyperbolic heat trace $HTrK_{\mathrm{hyp}}(t)$ ($t\in\mathbb{R}_{>0}$) is now given by

$$HTrK_{\mathrm{hyp}}(t)=\int\limits_X HK_{\mathrm{hyp}}(t;z)\,\mu_{\mathrm{hyp}}(z)\,.$$

We note that the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ ($z,w\in X$; $z\neq w$) relates in the following way to the heat kernel:

$$g_{\mathrm{hyp}}(z,w)=4\pi\int\limits_0^\infty\left(K_{\mathrm{hyp}}(t;z,w)-\frac{1}{v_X}\right)\mathrm{d}t\,. \tag{2.5}$$

To see this, one only has to recall

$$\left(\Delta_{\mathrm{hyp},z}+\frac{\partial}{\partial t}\right)K_{\mathrm{hyp}}(t;z,w)=0,$$

and the known asymptotic behavior of $K_{\mathrm{hyp}}(t;z,w)$ near $t=0$ and $t=\infty$. By analogy, the hyperbolic Green's function on $\mathbb{H}$ is defined through the formula

$$g_{\mathbb{H}}(z,w)=4\pi\int\limits_0^\infty K_{\mathbb{H}}(t;z,w)\,\mathrm{d}t\quad(z,w\in\mathbb{H};z\neq w)\,.$$

Explicit formulas were given evaluating $g_{\mathbb{H}}(z,w)$, namely, from [**8**], p. 31, we have

$$g_{\mathbb{H}}(z,w)=-\log\left(\left|\frac{z-w}{z-\bar{w}}\right|^2\right).$$

2.5. Selberg's zeta function. A thorough study of the Selberg zeta function can be found in [**8**]. For the sake of completeness, we recall here the basic results which we need, referring the reader to [**8**] for further details. Let $H(\Gamma)$ denote a complete set of representatives of inconjugate, primitive, hyperbolic elements in Γ. Denote by ℓ_γ the hyperbolic length of the closed geodesic determined by $\gamma\in H(\Gamma)$ on X; it is well-known that the equality

$$|\mathrm{tr}(\gamma)|=2\cosh(\ell_\gamma/2)$$

holds. For $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$, the Selberg zeta function $Z_X(s)$ associated to X is defined via the Euler product expansion

$$Z_X(s)=\prod_{\gamma\in H(\Gamma)}\prod_{n=0}^{\infty}\left(1-e^{-(s+n)\ell_\gamma}\right).$$

The Selberg zeta function $Z_X(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation. From [**17**], p. 115, we recall the relation

$$D_{\mathrm{hyp}}(X) = \log\left(\frac{Z'_X(1)}{v_X}\right) + b(g_X)\,, \tag{2.6}$$

where

$$b(g_X) = (g_X - 1)(4\zeta'_{\mathbb{Q}}(-1) - 1/2 + \log(2\pi))\,. \tag{2.7}$$

As in [**10**], we define the quantity

$$c_X = \lim_{s\to 1}\left(\frac{Z'_X}{Z_X}(s) - \frac{1}{s-1}\right).$$

From [**10**], Lemma 4.2, we recall the formula

$$c_X = 1 + \int\limits_0^\infty (HTrK_{\mathrm{hyp}}(t) - 1)\,\mathrm{d}t = \int\limits_0^\infty (HTrK_{\mathrm{hyp}}(t) - 1 + e^{-t})\,\mathrm{d}t\,. \tag{2.8}$$

Identity (2.8) is obtained by means of the McKean formula (assuming $\mathrm{Re}(s) > 1$)

$$\frac{Z'_X}{Z_X}(s) = (2s-1)\int\limits_0^\infty HTrK_{\mathrm{hyp}}(t)e^{-s(s-1)t}\,\mathrm{d}t\,,$$

which, observing the relation $\lim\limits_{s\to\infty} Z_X(s) = 1$, has the integrated version

$$\log(Z_X(s)) = -\int\limits_0^\infty HTrK_{\mathrm{hyp}}(t)e^{-s(s-1)t}\,\frac{\mathrm{d}t}{t}\,. \tag{2.9}$$

Observing the identity

$$\log(w) = \int\limits_0^\infty \left(e^{-t} - e^{-wt}\right)\frac{\mathrm{d}t}{t} \tag{2.10}$$

for $w > 0$ and taking $w = s(s-1)$ (with $s \in \mathbb{R}_{>1}$), we can combine (2.10) with the integrated version of the McKean formula (2.9) to get, for $\mathrm{Re}(s) > 1$, the formula

$$-\log\left(\frac{Z_X(s)}{s(s-1)}\right) = \int\limits_0^\infty \Big((HTrK_{\mathrm{hyp}}(t) - 1)e^{-s(s-1)t} + e^{-t}\Big)\frac{\mathrm{d}t}{t}\,.$$

In particular, when letting s approach 1, we get

$$-\log(Z'_X(1)) = \int\limits_0^\infty \left(HTrK_{\mathrm{hyp}}(t) - 1 + e^{-t}\right)\frac{\mathrm{d}t}{t}\,. \tag{2.11}$$

In this way, we have shown how the special values (2.8) and (2.11) are expressed using the hyperbolic heat kernel.

3. Canonical metrics, Arakelov metrics and Green's functions

In this section, we derive an explicit identity which expresses the canonical metric μ_{can} and the hyperbolic metric μ_{hyp} in terms of the hyperbolic heat kernel; see Theorem 3.5. Building on this result, we then obtain a closed-form expression for the canonical Green's function in terms of hyperbolic geometry; see Theorem 3.9. As a consequence of this result, we derive in Corollary 3.10 an identity for the Arakelov volume form in terms of hyperbolic geometry.

LEMMA 3.1. *With the above notations, we have, for all $z, w \in X$, the formula*

$$\begin{aligned} g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = & \int_X g_{\mathrm{hyp}}(z,\zeta)\mu_{\mathrm{can}}(\zeta) + \int_X g_{\mathrm{hyp}}(w,\zeta)\mu_{\mathrm{can}}(\zeta) - \\ & \int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\mu_{\mathrm{can}}(\zeta)\mu_{\mathrm{can}}(\xi)\,. \end{aligned}$$

PROOF. Let $F_L(z,w)$, resp. $F_R(z,w)$, denote the left-, resp. right-hand side of the stated identity. Using the characterizing properties of the Green's functions, one can show directly that we have for fixed $w \in X$

$$\mathrm{d}_z\mathrm{d}_z^c F_L(z,w) = \mathrm{d}_z\mathrm{d}_z^c F_R(z,w) = \mu_{\mathrm{shyp}}(z) - \mu_{\mathrm{can}}(z)\,,$$

and

$$\int_X F_L(z,w)\mu_{\mathrm{can}}(z) = \int_X F_R(z,w)\mu_{\mathrm{can}}(z) = \int_X g_{\mathrm{hyp}}(w,\zeta)\mu_{\mathrm{can}}(\zeta).$$

Consequently, $F_L(z,w) = F_R(z,w)$, again for fixed w. However, it is obvious that F_L and F_R are symmetric in z and w. This completes the proof of the lemma. □

PROPOSITION 3.2. *With the above notations, we have, for all $z \in X$, the formula*

$$g_X\mu_{\mathrm{can}}(z) = \mu_{\mathrm{shyp}}(z) + \frac{1}{2}\mathrm{c}_1(\Omega_X^1, \|\cdot\|_{\mathrm{hyp,res}})(z)\,;$$

here Ω_X^1 denotes the canonical line bundle on X.

PROOF. Let us rewrite the identity in Lemma 3.1 as

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi(z) + \phi(w)\,, \tag{3.1}$$

where

$$\phi(z) = \int_X g_{\mathrm{hyp}}(z,\zeta)\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X\int_X g_{\mathrm{hyp}}(\xi,\zeta)\mu_{\mathrm{can}}(\zeta)\mu_{\mathrm{can}}(\xi)\,.$$

Taking $\mathrm{d}_z\mathrm{d}_z^c$ in relation (3.1), we get the equation

$$\mu_{\mathrm{shyp}}(z) - \mu_{\mathrm{can}}(z) = \mathrm{d}_z\mathrm{d}_z^c\phi(z)\,. \tag{3.2}$$

On the other hand, we have by definition

$$\begin{aligned} \log\|\mathrm{d}z\|^2_{\mathrm{hyp,res}} &= \lim_{w\to z}\left(g_{\mathrm{hyp}}(z,w) + \log|z-w|^2\right), \\ \log\|\mathrm{d}z\|^2_{\mathrm{can,res}} &= \lim_{w\to z}\left(g_{\mathrm{can}}(z,w) + \log|z-w|^2\right). \end{aligned}$$

From this we deduce, again using (3.1),

$$\log\|\mathrm{d}z\|^2_{\mathrm{hyp,res}} - \log\|\mathrm{d}z\|^2_{\mathrm{can,res}} = \lim_{w\to z}\left(g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)\right) = 2\phi(z)\,. \tag{3.3}$$

Now, taking $-\mathrm{d}_z\mathrm{d}_z^c$ of equation (3.3), yields

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}})(z) - \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{can,res}})(z) = -2\mathrm{d}_z\mathrm{d}_z^c\phi(z)\,. \tag{3.4}$$

Combining equations (3.2) and (3.4) leads to

$$2(\mu_{\mathrm{shyp}}(z) - \mu_{\mathrm{can}}(z)) = \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{can,res}})(z) - \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}})(z)\,. \tag{3.5}$$

Recalling

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{can,res}})(z) = (2g_X - 2)\mu_{\mathrm{can}}(z)\,, \tag{3.6}$$

we derive from (3.5)

$$\mu_{\mathrm{shyp}}(z) - \mu_{\mathrm{can}}(z) = \frac{2g_X - 2}{2}\mu_{\mathrm{can}}(z) - \frac{1}{2}\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}})(z)\,,$$

which proves the proposition. $\square$

Remark 3.3. A key step in the proof of Proposition 3.2 is equation (3.6), which was first proved in [**1**] using classical variational theory. An alternate proof of (3.6) is given in [**9**], which first expresses the canonical Green's function in terms of classical Riemann theta functions, from which the differentiation required to prove (3.6) is immediate.

Proposition 3.4. *With the above notations, we have the following formula for the first Chern form of Ω^1_X with respect to $\|\cdot\|_{\mathrm{hyp,res}}$*

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}})(z) = \frac{1}{2\pi}\cdot\mu_{\mathrm{hyp}}(z) + \left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;z)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(z).$$

Proof. By our definitions, we have for $z \in X$

$$\begin{aligned}
\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}})(z) = &\quad -\mathrm{d}_z\mathrm{d}_z^c \log\|\mathrm{d}z\|^2_{\mathrm{hyp,res}} \\
&= -\mathrm{d}_z\mathrm{d}_z^c \lim_{w\to z}\left(g_{\mathrm{hyp}}(z,w) + \log|z-w|^2\right) \\
&= -\mathrm{d}_z\mathrm{d}_z^c \lim_{w\to z}\left(4\pi\int\limits_0^\infty\left(K_{\mathrm{hyp}}(t;z,w) - \frac{1}{v_X}\right)\mathrm{d}t + \log|z-w|^2\right) \\
&= -\mathrm{d}_z\mathrm{d}_z^c \lim_{w\to z}\left(4\pi\int\limits_0^\infty K_{\mathbb{H}}(t;z,w)\,\mathrm{d}t + \log|z-w|^2\right) \\
&\quad -\mathrm{d}_z\mathrm{d}_z^c \lim_{w\to z}\left(4\pi\int\limits_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma w) - \frac{1}{v_X}\right)\mathrm{d}t\right)\,.
\end{aligned}$$

Using the formula for the Green's function $g_{\mathbb{H}}(z,w)$ on $\mathbb{H}$, we obtain for the first summand in the latter sum

$$\begin{aligned} A &= -\mathrm{d}_z\mathrm{d}_z^c \lim_{w\to z}\left(4\pi\int_0^\infty K_{\mathbb{H}}(t;z,w)\,\mathrm{d}t + \log|z-w|^2\right) \\ &= -\mathrm{d}_z\mathrm{d}_z^c \lim_{w\to z}\left(g_{\mathbb{H}}(z,w) + \log|z-w|^2\right) \\ &= -\mathrm{d}_z\mathrm{d}_z^c \log|z-\bar z|^2 = -\frac{2i}{2\pi}\,\partial_z\,\bar\partial_z \log(z-\bar z) \\ &= \frac{i}{\pi}\partial_z\frac{\mathrm{d}\bar z}{z-\bar z} = -\frac{i}{\pi}\cdot\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{(z-\bar z)^2} \\ &= -\frac{i}{\pi}\cdot\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{(2i\mathrm{Im}(z))^2} = \frac{1}{2\pi}\cdot\mu_{\mathrm{hyp}}(z)\,. \end{aligned}$$

For the second summand we obtain

$$B = -4\pi\mathrm{d}_z\mathrm{d}_z^c\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma z) - \frac{1}{v_X}\right)\mathrm{d}t\,.$$

Since this integral converges absolutely, as does the integral of derivatives of the integrand, we can interchange differentiation and integration, yielding

$$\begin{aligned} B &= -4\pi\int_0^\infty \mathrm{d}_z\mathrm{d}_z^c\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z,\gamma z) - \frac{1}{v_X}\right)\mathrm{d}t \\ &= -4\pi\int_0^\infty \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} \mathrm{d}_z\mathrm{d}_z^c\,K_{\mathbb{H}}(t;z,\gamma z)\,\mathrm{d}t\,. \end{aligned}$$

The claimed formula then follows, since $K_{\mathbb{H}}(t;z)$ is independent of z, and recalling, as in section 2, the identity

$$-4\pi\mathrm{d}_z\mathrm{d}_z^c f(z) = (\Delta_{\mathrm{hyp}}f(z))\mu_{\mathrm{hyp}}(z)\,,$$

for any smooth function f on X. □

THEOREM 3.5. *With the above notations, we have, for all $z\in X$, the formula*

$$\mu_{\mathrm{can}}(z) = \mu_{\mathrm{shyp}}(z) + \frac{1}{2g_X}\left(\int_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;z)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(z).$$

PROOF. We simply have to combine Propositions 3.2 and 3.4, and to use that

$$\frac{1}{g_X} + \frac{v_X}{4\pi g_X} = 1\,.$$

□

Lemma 3.6. *For all $z \in X$, let $H(z)$ be defined by*

$$H(z) = \int\limits_0^\infty \left(HK_{\mathrm{hyp}}(t;z) - \frac{1}{v_X} \right) \mathrm{d}t - \frac{c_X - 1}{v_X}\,.$$

Then, $H(z)$ is uniquely characterized by satisfying the integral formula

$$\int\limits_X H(z)\mu_{\mathrm{hyp}}(z) = 0\,,$$

and the differential equation

$$\Delta_{\mathrm{hyp}} H(z) = \int\limits_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)\,\mathrm{d}t\,.$$

Proof. Concerning the integral equation, note that, by interchanging the order of integration, we have

$$\begin{aligned}\int\limits_X H(z)\mu_{\mathrm{hyp}}(z) &= \int\limits_X \left(\int\limits_0^\infty \left(HK_{\mathrm{hyp}}(t;z) - \frac{1}{v_X} \right) \mathrm{d}t - \frac{c_X - 1}{v_X} \right) \mu_{\mathrm{hyp}}(z) \\ &= \int\limits_0^\infty \left(HTrK_{\mathrm{hyp}}(t) - 1 \right) \mathrm{d}t - (c_X - 1) = 0\,,\end{aligned}$$

where the last equality follows from formula (2.8), given in section 2. As for the differential equation, note that for any $z \in X$, we have

$$HK_{\mathrm{hyp}}(t;z) = K_{\mathrm{hyp}}(t;z) - K_{\mathbb{H}}(t;z).$$

Elementary bounds of heat kernels imply that this function and its derivatives are bounded, hence we can interchange differentiation and integration in the definition of $H(z)$. Since both $K_{\mathbb{H}}(t;z)$ and $(c_X - 1)/v_X$ are annihilated by Δ_{hyp}, the result follows. □

Lemma 3.7. *With the above notations, we have, for all $z \in X$, the formula*

$$\int\limits_X g_{\mathrm{hyp}}(z,\zeta)\mu_{\mathrm{can}}(\zeta) = \frac{2\pi}{g_X} H(z)\,.$$

Proof. Using Theorem 3.5, we have

$$\begin{aligned}
&\int\limits_X g_{\mathrm{hyp}}(z,\zeta)\mu_{\mathrm{can}}(\zeta)\\
&\quad= \int\limits_X g_{\mathrm{hyp}}(z,\zeta)\left(\mu_{\mathrm{shyp}}(\zeta)+\frac{1}{2g_X}\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;\zeta)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(\zeta)\right)\\
&\quad= \frac{1}{2g_X}\int\limits_X g_{\mathrm{hyp}}(z,\zeta)\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;\zeta)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(\zeta)\\
&\quad= \frac{1}{2g_X}\int\limits_X g_{\mathrm{hyp}}(z,\zeta)\Delta_{\mathrm{hyp}}H(\zeta)\mu_{\mathrm{hyp}}(\zeta)\,,
\end{aligned}$$

where the last equality follows from Lemma 3.6. Using the integral formula in Lemma 3.6, the assertion is proved by recalling that g_{hyp} inverts the operator $-\mathrm{dd}^c$ on the space of functions whose integral is zero. □

Lemma 3.8. *With the above notations, we have the formula*

$$\int\limits_X\int\limits_X g_{\mathrm{hyp}}(\xi,\zeta)\mu_{\mathrm{can}}(\zeta)\mu_{\mathrm{can}}(\xi) = \frac{\pi}{g_X^2}\int\limits_X H(\xi)\Delta_{\mathrm{hyp}}H(\xi)\mu_{\mathrm{hyp}}(\xi)\,.$$

Proof. From Lemma 3.7, we have

$$\int\limits_X\int\limits_X g_{\mathrm{hyp}}(\xi,\zeta)\mu_{\mathrm{can}}(\zeta)\mu_{\mathrm{can}}(\xi) = \frac{2\pi}{g_X}\int\limits_X H(\xi)\mu_{\mathrm{can}}(\xi)\,.$$

We now employ Theorem 3.5, which gives

$$\begin{aligned}
\int\limits_X H(\xi)\mu_{\mathrm{can}}(\xi) &= \int\limits_X H(\xi)\left(\mu_{\mathrm{shyp}}(\xi)+\frac{1}{2g_X}\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;\xi)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(\xi)\right)\\
&= \frac{1}{2g_X}\int\limits_X H(\xi)\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;\xi)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(\xi)\,,
\end{aligned}$$

where we have used the integral equation from Lemma 3.6 to obtain the last equality. The result follows by using the differential equation from Lemma 3.6. □

Theorem 3.9. *With the above notations, we have the formula*

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi_X(z) + \phi_X(w)\,,$$

where

$$\phi_X(z) = \frac{2\pi}{g_X}H(z) - \frac{\pi}{2g_X^2}\int\limits_X H(\xi)\Delta_{\mathrm{hyp}}H(\xi)\mu_{\mathrm{hyp}}(\xi)\,,$$

and

$$H(z) = \int\limits_0^\infty\left(HK_{\mathrm{hyp}}(t;z) - \frac{1}{v_X}\right)\mathrm{d}t - \frac{c_X-1}{v_X}\,.$$

Proof. The proof is obtained by combining Lemma 3.1, Lemma 3.7 and Lemma 3.8. □

Corollary 3.10. *Let*

$$F(z)=\int\limits_0^\infty\left(HK_{\mathrm{hyp}}(t;z)-\frac{1}{v_X}\right)\mathrm{d}t\,.$$

Then, with the above notations, we have the formula

$$\phi_{\mathrm{Ar}}(z)=-4\pi\left(1-\frac{1}{g_X}\right)F(z)-\frac{\pi}{g_X^2}\int\limits_X F(\xi)\Delta_{\mathrm{hyp}}F(\xi)\mu_{\mathrm{hyp}}(\xi)-\frac{c_X-1}{g_X(g_X-1)}-\log(4)\,.$$

Proof. Using the known formula for $g_{\mathbb{H}}(z,w)$, as stated in section 2, we can write

$$g_{\mathrm{can}}(z,w)-g_{\mathbb{H}}(z,w)=g_{\mathrm{can}}(z,w)+\log|z-w|^2-\log|z-\bar{w}|^2\,.$$

Therefore, when using the definition of the residual metrics as given in section 2, we then have

$$\lim_{w\to z}\left(g_{\mathrm{can}}(z,w)-g_{\mathbb{H}}(z,w)\right)=\log\|\mathrm{d}z\|^2_{\mathrm{can,res}}-\log(2\,\mathrm{Im}(z))^2$$

$$=\log\left(\frac{\|\mathrm{d}z\|^2_{\mathrm{can,res}}}{\mathrm{Im}(z)^2}\right)-\log(4)=\log\left(\frac{\mu_{\mathrm{hyp}}(z)}{\mu_{\mathrm{Ar}}(z)}\right)-\log(4)=-\phi_{\mathrm{Ar}}(z)-\log(4)\,.$$

However, by Theorem 3.9, we have

$$\lim_{w\to z}\left(g_{\mathrm{can}}(z,w)-g_{\mathbb{H}}(z,w)\right)=\lim_{w\to z}\left(g_{\mathrm{hyp}}(z,w)-g_{\mathbb{H}}(z,w)\right)-2\phi_X(z)$$

$$=\lim_{w\to z}\left(4\pi\int\limits_0^\infty\left(K_{\mathrm{hyp}}(t;z,w)-\frac{1}{v_X}\right)\mathrm{d}t-4\pi\int\limits_0^\infty K_{\mathbb{H}}(t;z,w)\,\mathrm{d}t\right)-2\phi_X(z)$$

$$=4\pi\int\limits_0^\infty\left(K_{\mathrm{hyp}}(t;z)-K_{\mathbb{H}}(t;z)-\frac{1}{v_X}\right)\mathrm{d}t-2\phi_X(z)\,.$$

The rest follows from substituting the formula for ϕ_X, which was proved in Theorem 3.9. □

Remark 3.11. As stated in the introduction, prior results have expressed the canonical Green's function in terms of the classical Riemann theta function. Therefore, we now have a complete, closed-form expression for the Riemann theta function in terms of the hyperbolic heat kernel. A potentially fascinating study would be to explore this relation further, either from the point of view of obtaining results in hyperbolic geometry from the algebraic geometry of the theta function, or conversely.

4. Faltings's delta function

Continuing the theme of the previous section, we now derive an expression which evaluates Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ in terms of spectral theoretic information of X coming from hyperbolic geometry. Our method of proof is as follows. First, we use results from [**17**] and [**18**] together with the Polyakov formula (see (2.2)) to express $\delta_{\mathrm{Fal}}(X)$ in terms of hyperbolic information and the conformal factor ϕ_{Ar} (see (2.1)) relating the Arakelov metric μ_{Ar} to the hyperbolic metric

μ_{hyp} on X. Our starting point is the following lemma which collects results stated above.

LEMMA 4.1. *For any X with genus $g_X > 1$, let*

$$c(g_X) = a(g_X) - 6b(g_X) + 6\log(v_X)\,,$$

where $a(g_X)$, resp. $b(g_X)$ are given by formulas (2.4), *resp.* (2.7). *With the above notations, we then have the formula*

$$\delta_{\mathrm{Fal}}(X) = -6\log(Z'_X(1)) - (g_X - 1)\int\limits_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z) + \mu_{\mathrm{can}}(z)) + c(g_X).$$

PROOF. Combining formulas (2.3), (2.2), and (2.6), we obtain

$$\begin{aligned}
\delta_{\mathrm{Fal}}(X) &= -6D_{\mathrm{Ar}}(X) + a(g_X) \\
&= -6D_{\mathrm{hyp}}(X) - (g_X - 1)\int\limits_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z) + \mu_{\mathrm{can}}(z)) + a(g_X) \\
&= -6\log\left(\frac{Z'_X(1)}{v_X}\right) - (g_X - 1)\int\limits_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z) + \mu_{\mathrm{can}}(z)) + \\
&\qquad a(g_X) - 6b(g_X) \\
&= -6\log(Z'_X(1)) - (g_X - 1)\int\limits_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z) + \mu_{\mathrm{can}}(z)) + \\
&\qquad a(g_X) - 6b(g_X) + 6\log(v_X).
\end{aligned}$$

This completes the proof of the lemma. □

REMARK 4.2. For the sake of completeness, let us make explicit the value of $c(g_X)$; a straightforward calculation yields

$$\begin{aligned}
c(g_X) &= a(g_X) - 6b(g_X) + 6\log(v_X) \\
&= 2g_X\left(-24\zeta'_{\mathbb{Q}}(-1) - 4\log(\pi) - \log(2) + 2\right) + 6\log(g_X - 1) + \\
&\qquad 2\left(24\zeta'_{\mathbb{Q}}(-1) + 6\log(\pi) + 9\log(2) - 2\right).
\end{aligned}$$

LEMMA 4.3. *With the above notations, we have*

$$\int\limits_X \phi_{\mathrm{Ar}}(z)\mu_{\mathrm{shyp}}(z) = -\frac{\pi}{g_X^2}\int\limits_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z) - \frac{c_X - 1}{g_X - 1} - \log(4)\,.$$

PROOF. The result follows from Corollary 3.10 together with Lemma 3.6. □

LEMMA 4.4. *With the above notations, we have*

$$\int\limits_X \phi_{\mathrm{Ar}}(z)\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;z)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(z) = -4\pi\left(1 - \frac{1}{g_X}\right)\int\limits_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z)\,.$$

PROOF. The result follows from Corollary 3.10 together with Lemma 3.6. □

PROPOSITION 4.5. *With the above notations, we have*

$$\int_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z)+\mu_{\mathrm{can}}(z)) = -\frac{2\pi}{g_X}\int_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z) - \frac{2(c_X-1)}{g_X-1} - 2\log(4)\,.$$

PROOF. Using Theorem 3.5, we have

$$\int_X \phi_{\mathrm{Ar}}(z)(\mu_{\mathrm{shyp}}(z)+\mu_{\mathrm{can}}(z)) = 2\int_X \phi_{\mathrm{Ar}}(z)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g_X}\int_X \phi_{\mathrm{Ar}}(z)\left(\int_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;z)\,\mathrm{d}t\right)\mu_{\mathrm{hyp}}(z)\,.$$

The result now follows from Lemma 4.3 and Lemma 4.4. □

THEOREM 4.6. *With the above notations, we have*

$$\delta_{\mathrm{Fal}}(X) = 2\pi\left(1-\frac{1}{g_X}\right)\int_X F(z)\Delta_{\mathrm{hyp}}F(z)\mu_{\mathrm{hyp}}(z) - 6\log(Z'_X(1)) + 2c_X + C(g_X)\,,$$

where

$$\begin{aligned} C(g_X) &= a(g_X) - 6b(g_X) + 2(g_X-1)\log(4) + 6\log(v_X) - 2 \\ &= 2g_X\left(-24\zeta'_{\mathbb{Q}}(-1) - 4\log(\pi) + \log(2) + 2\right) + 6\log(g_X-1) + \\ &\quad\; 2\left(24\zeta'_{\mathbb{Q}}(-1) + 6\log(\pi) + 7\log(2) - 3\right)\,. \end{aligned}$$

PROOF. Simply combine Lemma 4.1 with Proposition 4.5. □

REMARK 4.7. Prior results have expressed $\delta_{\mathrm{Fal}}(X)$ in terms of the Riemann theta function; see [**2**], [**3**], [**9**], and [**19**]. As previously stated, Theorem 4.7 allows one to study $\delta_{\mathrm{Fal}}(X)$ using techniques from hyperbolic geometry. This approach was exploited in [**12**], where asymptotic bounds were derived for the behavior of $\delta_{\mathrm{Fal}}(X)$ through covers. Furthermore, it seems possible that one can use Theorem 4.7 to study $\delta_{\mathrm{Fal}}(X)$ through degeneration, thus reproving some of the main results from [**9**] and [**19**]. Finally, we note that the identity proved in Theorem 4.6 is also proved in [**12**] using a completely different analysis of the integral that appears in Lemma 4.1.

References

[1] *Arakelov, S.:* Intersection theory of divisors on an arithmetic surface. Math. USSR Izvestija **8** (1974), 1167–1180.

[2] *Bost, J.-B.:* Fonctions de Green-Arakelov, fonction thêta et courbes de genre 2. C. R. Acad. Sci. Paris Ser. I **305** (1987), 643–646.

[3] *Bost, J.-B.; Mestre, J.-F.; Moret-Bailly, L.:* Sur le calcul des "classes de Chern" des surfaces arithmétiques de genre 2. In: Séminaire sur les pinceaux des courbes elliptiques. Astérisque **183** (1990), 69–105.

[4] *Chavel, I.:* Eigenvalues in Riemannian geometry. Academic Press, Orlando, 1984.

[5] *Faltings, G.:* Calculus on arithmetic surfaces. Ann. of Math. **119** (1984), 387–424.

[6] *Fay, J.:* Kernel functions, analytic torsion, and moduli spaces. Memoirs of AMS **464**. American Mathematical Society, Providence, 1992.

[7] *Guàrdia, J.:* Analytic invariants in Arakelov theory for curves. C. R. Acad. Sci. Paris Ser. I **329** (1999), 41–46.

[8] *Hejhal, D.:* The Selberg trace formula for $\mathrm{PSL}_2(\mathbb{R})$, vol. 2. Lecture Notes in Math. **1001**. Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[9] *Jorgenson, J.:* Asymptotic behavior of Faltings' delta function. Duke Math J. **61** (1990), 221–254.

[10] *Jorgenson, J.; Kramer, J.:* Bounds on special values of Selberg's zeta functions for Riemann surfaces. J. reine angew. Math. **541** (2001), 1–28.

[11] *Jorgenson, J.; Kramer, J.:* Canonical metrics, hyperbolic metrics, and Eisenstein series for $\mathrm{PSL}_2(\mathbb{R})$. In preparation.

[12] *Jorgenson, J.; Kramer, J.:* Bounds on Faltings's delta function through covers. Submitted.

[13] *Jorgenson, J.; Kramer, J.:* Bounds on canonical Green's functions. Submitted.

[14] *Jorgenson, J.; Lundelius, R.:* Continuity of relative hyperbolic spectral theory through metric degeneration. Duke Math. J. **84** (1996), 47–81.

[15] *Lang, S.:* Introduction to Arakelov theory. Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo, 1988.

[16] *Osgood, B.; Phillips, R.; Sarnak, P.:* Extremals of determinants of Laplacians. J. Funct. Analysis **80** (1988), 148–211.

[17] *Sarnak, P.:* Determinants of Laplacians. Commun. Math. Phys. **110** (1987), 113–120.

[18] *Soulé, C.:* Géométrie d'Arakelov des surfaces arithmétiques. In: Séminaire Bourbaki 1988/89. Astérisque **177-178** (1989), 327–343.

[19] *Wentworth, R.:* The asymptotics of the Arakelov-Green's function and Faltings' delta invariant. Commun. Math. Phys. **137** (1991), 427–459.

Department of Mathematics, The City College of New York, 138th and Convent Avenue, New York, NY 10031, USA

E-mail address: jjorgenson@mindspring.com

Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany

E-mail address: kramer@math.hu-berlin.de