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Bounds on Faltings's delta function through covers

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Abstract

Let X be a compact Riemann surface of genus $g_X \geq 1$. In [7], G. Faltings introduced a new invariant $\delta_{\mathrm{Fal}}(X)$ associated to X. In this paper we give explicit bounds for $\delta_{\mathrm{Fal}}(X)$ in terms of fundamental differential geometric invariants arising from X, when $g_X > 1$. As an application, we are able to give bounds for Faltings's delta function for the family of modular curves $X_0(N)$ in terms of the genus only. In combination with work of A. Abbes, P. Michel and E. Ullmo this leads to an asymptotic formula for the Faltings height of the Jacobian $J_0(N)$ associated to $X_0(N)$.

1. Introduction

1.1. In his foundational paper [7], G. Faltings established fundamental results in the development of Arakelov theory for arithmetic surfaces based on S.S. Arakelov's original work on this subject. The article [7] was the origin for various developments in arithmetic geometry such as the creation of higher dimensional Arakelov theory by C. Soulé and H. Gillet, or more refined work on arithmetic surfaces by A. Abbes, P. Michel, and E. Ullmo, or P. Vojta's work on the Mordell conjecture. The ideas from Faltings's original article continue to be used, and further understanding of the ideas developed in [7] often leads to advances in arithmetic algebraic geometry.

Let us now explain the main object of study in this paper, namely Faltings's delta function. To do this, we let X be a compact Riemann surface of positive genus g_X, Ω^1_X the holomorphic cotangent bundle and $\omega_1, \ldots, \omega_{g_X}$ an orthonormal basis of holomorphic 1-forms on X with respect to the Petersson inner product. The canonical metric on X is then defined by means of the $(1,1)$-form

$$\mu_{\mathrm{can}} = \frac{1}{g_X} \cdot \frac{i}{2} \sum_{j=1}^{g_X} \omega_j \wedge \overline{\omega}_j \, .$$

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We note that if $g_X > 1$, the Riemann surface X also carries a hyperbolic metric, which is compatible with the complex structure of X and has negative curvature equal to minus one; we denote the corresponding $(1,1)$-form by μ_{hyp}.

By means of the normalized Green's function $g_{\mathrm{can}}(x,y)$ $(x,y \in X)$ associated to the canonical $(1,1)$-form μ_{can} in the sense of Arakelov one can inductively define a hermitian metric on any line bundle L on X, whose curvature form is proportional to μ_{can}. In particular, if this construction is applied to the line bundle Ω^1_X, the corresponding hermitian metric is such that the isomorphism induced by the residue map from the fiber of $\Omega^1_X(x)$ at x to $\mathbb{C}$ (equipped with the standard hermitian metric) becomes an isometry for all $x \in X$. By means of the hermitian metric thus defined on any line bundle L, G. Faltings constructs in [7] a hermitian metric $\|\cdot\|_1$ on the determinant line bundle $\lambda(L)$ associated to the cohomology of the line bundle L.

Now, there is another way to metrize the determinant line bundle $\lambda(L)$. For this one considers the degree $g_X - 1$ part $\mathrm{Pic}_{g_X-1}(X)$ of the Picard variety of X together with the line bundle $\mathcal{O}(\Theta)$ associated to the theta divisor Θ. By means of Riemann's theta function the line bundle $\mathcal{O}(\Theta)$ can be metrized in a canonical way. By restricting to the case where the degree of L equals $g_X - 1$, and noting that L is of the form $\mathcal{O}_X(E - P_1 - \ldots - P_r)$ with a fixed divisor E on X and suitable points $P_1, \ldots, P_r$ on X, we obtain a natural morphism from X^r to $\mathrm{Pic}_{g_X-1}(X)$ by sending $(P_1, \ldots, P_r)$ to the class of $\mathcal{O}_X(E-P_1-\ldots-P_r)$. By pulling back $\mathcal{O}(\Theta)$ to X^r via this map, extending it to $Y = X^r \times X$ and restricting to the fiber X of the projection from Y to X^r, we obtain a line bundle, which turns out to be isomorphic to $\lambda(L)$. In this way the hermitian metric given by Riemann's theta function on $\mathcal{O}(\Theta)$ induces a second hermitian metric $\|\cdot\|_2$ on $\lambda(L)$. A straightforward calculation shows that the curvature forms of the two metrics thus obtained coincide. Therefore, they agree up to a multiplicative constant, which depends solely on (the isomorphism class of) X. This constant defines Faltings's delta function $\delta_{\mathrm{Fal}}(X)$; for a precise definition, we refer to [7], p. 402.

In [7], p. 403, it is asked to determine the asymptotic behavior of $\delta_{\mathrm{Fal}}(X_t)$ for a family of compact Riemann surfaces X_t, which approach the Deligne-Mumford boundary of the moduli space of stable algebraic curves of a fixed positive genus g_X. This problem was solved in [13] by first expressing Faltings's delta function in terms of Riemann's theta function, thus obtaining asymptotic expansions for all quantities involved in the expression. In the present article, we will address, among other things, the following, related question, namely to estimate $\delta_{\mathrm{Fal}}(X)$ for varying X covering a fixed base Riemann surface X_0 in terms of fundamental geometric invariants of X as well as additional intrinsic quantities coming from X_0.

1.2. In their work A. Abbes, P. Michel, and E. Ullmo investigated the case of the modular curve $X_0(N)$ (N squarefree, $6 \nmid N$) associated to the congruence subgroup $\Gamma_0(N)$ more closely. Using an arithmetic analogue of Noether's formula, which was also obtained in [7], it was shown in [1], [24] that the Faltings height $h_{\mathrm{Fal}}(J_0(N))$ for the Jacobian $J_0(N)$ of $X_0(N)$ has the following asymptotic expression involving Faltings's delta function as the archimedean contribution

$$12 \cdot h_{\mathrm{Fal}}(J_0(N)) = 4g_{X_0(N)} \log(N) + \delta_{\mathrm{Fal}}(X_0(N)) + o(g_{X_0(N)} \log(N))\,; \tag{1}$$

here the genus $g_{X_0(N)}$ of $X_0(N)$ (N squarefree, $6 \nmid N$) is given by the formula (see [29])

$$\begin{aligned} g_{X_0(N)} \quad = \quad & 1 + \frac{1}{12} \cdot N \prod_{p|N} \left(1 + \frac{1}{p}\right) - \frac{1}{2} \cdot d(N) - \\ & \frac{1}{4} \prod_{p|N} \left(1 + \left(\frac{-4}{p}\right)\right) - \frac{1}{3} \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right), \end{aligned}$$

where $d(N)$ denotes the number of divisors of N. In the subsequent work [31], E. Ullmo established another formula for $h_{\mathrm{Fal}}(J_0(N))$ involving a suitable discriminant $\delta_{\mathbb{T}}$ of the Hecke algebra $\mathbb{T}$ of $J_0(N)$, the matrix M_N of all possible Petersson inner products of a certain basis of eigenforms of weight 2 for $\Gamma_0(N)$, and a suitable natural number α, namely

$$h_{\mathrm{Fal}}(J_0(N)) = \frac{1}{2} \log|\delta_{\mathbb{T}}| - \frac{1}{2} \log|\det(M_N)| - \log(\alpha)\,. \tag{2}$$

By estimating congruences for modular forms, as well as estimating $\det(M_N)$ and α, E. Ullmo derives the bounds

$$\begin{aligned} & g_{X_0(N)} \log(N) + o(g_{X_0(N)} \log(N)) \leq \log|\delta_{\mathbb{T}}| \leq \\ & \qquad\qquad 2g_{X_0(N)} \log(N) + o(g_{X_0(N)} \log(N)) \end{aligned} \tag{3}$$

for $\log|\delta_{\mathbb{T}}|$, from which he then derives the following bounds for $h_{\mathrm{Fal}}(J_0(N))$

$$-Bg_{X_0(N)} \leq h_{\mathrm{Fal}}(J_0(N)) \leq \frac{g_{X_0(N)}}{2} \log(N) + o(g_{X_0(N)} \log(N)) \tag{4}$$

with an absolute constant $B > 0$; we note that the lower bound here is due to unpublished work of J.-B. Bost. This estimate in turn allows him to bound $\delta_{\mathrm{Fal}}(X_0(N))$ as follows

$$\begin{aligned} & -4g_{X_0(N)} \log(N) + o(g_{X_0(N)} \log(N)) \leq \delta_{\mathrm{Fal}}(X_0(N)) \leq \\ & \qquad\qquad 2g_{X_0(N)} \log(N) + o(g_{X_0(N)} \log(N))\,. \end{aligned} \tag{5}$$

1.3. The main purpose of this note is to give bounds for $\delta_{\mathrm{Fal}}(X)$ for arbitrary compact Riemann surfaces of genus $g_X > 1$ in terms of fundamental geometric invariants of X. As a first main result, Theorem 4.5 gives a bound for $\delta_{\mathrm{Fal}}(X)$ for any compact Riemann surface of genus $g_X > 1$ in terms of the smallest non-zero eigenvalue, the length of the shortest geodesic, the number of eigenvalues in the interval $[0, 1/4)$, the number of closed, primitive geodesics of length in the interval $(0, 5)$, the supremum over $x \in X$ of the ratio $\mu_{\mathrm{can}}/\mu_{\mathrm{hyp}}$, and the implied constant in the error term of the prime geodesic theorem for X. Applying this result to the situation where X is a finite cover of a fixed Riemann surface X_0 of genus $g_{X_0} > 1$, we obtain as a second main result (see Corollary 4.6) the estimate

$$\delta_{\mathrm{Fal}}(X) = O_{X_0}\left(g_X\left(1 + \frac{1}{\lambda_{X,1}}\right)\right),$$

where $\lambda_{X,1}$ denotes the smallest non-zero eigenvalue on X. We now want to apply our main results to the modular curves $X_0(N)$ with N being such that $g_{X_0(N)} > 1$, and to derive a bound for $\delta_{\mathrm{Fal}}(X_0(N))$ simply in terms of the genus $g_{X_0(N)}$. To do this, we unfortunately cannot apply Corollary 4.6 directly, but rather have to step back to Theorem 4.5, and have to bound all the fundamental geometric quantities in terms of $g_{X_0(N)}$. This can be done by exploiting the arithmetic nature of the situation, e.g., by recalling estimates on the smallest non-zero eigenvalue on $X_0(N)$ given by R. Brooks in [2]. In Theorem 5.6, we end up with the estimate

$$\delta_{\mathrm{Fal}}(X_0(N)) = O(g_{X_0(N)}),$$

thereby improving the bound (5). Plugging this bound into (1), yields

$$h_{\mathrm{Fal}}(J_0(N)) = \frac{g_{X_0(N)}}{3}\log(N) + o(g_{X_0(N)}\log(N)),$$

thereby improving (4). Using (2) in combination with our bound for $h_{\mathrm{Fal}}(J_0(N))$ and E. Ullmo's lower bound for $\log|\det(M_N)|$, we find the lower bound

$$\log|\delta_{\mathbb{T}}| \geq \frac{5}{3}g_{X_0(N)}\log(N) + o(g_{X_0(N)}\log(N)),$$

thereby improving the lower bound in (3).

1.4. The paper is organized as follows. In section 2, we recall and summarize all the notations, definitions and results used later on in this article. In particular, we recall the definitions for the hyperbolic and the canonical metric on a compact Riemann surface X of genus $g_X > 1$, as well as the definitions of the corresponding Green's functions giving rise to the so-called residual metrics on Ω^1_X. Next, we define Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ by means of the regularized determinant associated to the Laplacian with respect to the Arakelov metric on Ω^1_X (which is nothing but the residual metric associated to the canonical metric). This result was obtained in [30] as a by-product of the analytic part of the arithmetic Riemann-Roch theorem for arithmetic surfaces.

By means of Polyakov's formula, we are able to express Faltings's delta function in terms of the regularized determinant associated to the Laplacian with respect to the hyperbolic metric and a local integral involving the conformal factor relating the two metrics under consideration. We end this section by recalling the heat kernel, heat trace, and Selberg's zeta function associated to X, as well as the formula relating the first derivative of Selberg's zeta function to the regularized determinant associated to the hyperbolic Laplacian, which was proved in [28].

In section 3, we weave together the relations collected in section 2. As the main result of section 3, we obtain a representation of $\delta_{\mathrm{Fal}}(X)$ in terms of the genus, the first derivative of Selberg's zeta function for X at $s = 1$, and a triple integral over X involving the hyperbolic heat trace of X.

The formula obtained in section 3, allows us to estimate $\delta_{\mathrm{Fal}}(X)$ in section 4 by suitably extending the techniques developed in [14] to give bounds for the constant term of the logarithmic derivative of Selberg's zeta function at $s = 1$. In this way, we arrive at our main estimate for $\delta_{\mathrm{Fal}}(X)$ given in Theorem 4.5 in terms of the above mentioned fundamental geometric invariants.

In section 5, we then specialize to the case of the modular curves $X_0(N)$. The main focus here is to estimate all the fundamental geometric quantities occuring in Theorem 4.5 in terms of the genus $g_{X_0(N)}$ of $X_0(N)$ only. The problem that one encounters is the following: The family of modular curves $X_0(N)$ which admit hyperbolic metrics do not form a single tower so then the geometric invariants which appear in Theorem 4.5 cannot be readily bounded. Since $X_0(N)$ is an isometric cover of $X_0(N')$ whenever $N'|N$, the hyperbolic modular curves are sufficiently inter-related, in what one could view as a "net" rather than a single "tower", so that one is able to develop uniform bounds for the geometric invariants in Theorem 4.5 in order to bound Faltings's delta function for all modular curves. This leads to the main result stated in Theorem 5.6.

In section 6, we conclude by briefly discussing the arithmetic implications arising from Theorem 5.6 to estimating the Faltings height $h_{\mathrm{Fal}}(J_0(N))$ of the Jacobian $J_0(N)$ of $X_0(N)$ and the discriminant $\delta_{\mathbb{T}}$ of the Hecke algebra $\mathbb{T}$ of $J_0(N)$.

2. Notations and preliminaries

2.1. **Hyperbolic and canonical metrics.** Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. We let X be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g_X the genus of X. Unless otherwise stated, we assume that $g_X > 1$ and that Γ has no elliptic and, apart from the identity, no parabolic elements, i.e., X is smooth and compact. We identify X locally with its universal cover $\mathbb{H}$; we make this identification explicit by denoting the image of $x \in X$ in $\mathbb{H}$ by $z(x)$.

In the sequel μ denotes a (smooth) metric on X, i.e., μ is a positive $(1,1)$-form on X. We write $\mathrm{vol}_\mu(X)$ for the volume of X with respect to μ. In particular, we let $\mu = \mu_{\mathrm{hyp}}$ denote the hyperbolic metric on X, which is compatible with the complex structure of X, and has constant negative curvature equal to minus one. Locally, we have

$$\mu_{\mathrm{hyp}}(x) = \frac{i}{2} \cdot \frac{\mathrm{d}z(x) \wedge \mathrm{d}\bar{z}(x)}{\mathrm{Im}(z(x))^2} \,.$$

We write $\mathrm{vol}_{\mathrm{hyp}}(X)$ for the hyperbolic volume of X; we recall that $\mathrm{vol}_{\mathrm{hyp}}(X)$ is given by $4\pi(g_X - 1)$. The scaled hyperbolic metric $\mu = \mu_{\mathrm{shyp}}$ is simply the rescaled hyperbolic metric $\mu_{\mathrm{hyp}}/\mathrm{vol}_{\mathrm{hyp}}(X)$, which measures the volume of X to be one.

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g \rangle = \frac{i}{2} \int\limits_X f(z(x))\, \overline{g(z(x))}\, \mathrm{Im}(z(x))^k \cdot \frac{\mathrm{d}z(x) \wedge \mathrm{d}\bar{z}(x)}{\mathrm{Im}(z(x))^2} \qquad (f, g \in S_k(\Gamma))\,.$$

By choosing an orthonormal basis $\{f_1, ..., f_{g_X}\}$ of $S_2(\Gamma)$ with respect to the Petersson inner product, the canonical metric $\mu = \mu_{\mathrm{can}}$ of X is given by

$$\mu_{\mathrm{can}}(x) = \frac{1}{g_X} \cdot \frac{i}{2} \sum_{j=1}^{g_X} |f_j(z(x))|^2 \, \mathrm{d}z(x) \wedge \mathrm{d}\bar{z}(x).$$

We note that the canonical metric measures the volume of X to be one. In order to be able to compare the hyperbolic and the canonical metrics, we define

$$d_{\sup,X} := \sup_{x \in X} \left| \frac{\mu_{\mathrm{can}}(x)}{\mu_{\mathrm{shyp}}(x)} \right| .$$

We note that in [17], optimal bounds for $d_{\sup,X}$ through covers were obtained.

2.2. **Green's functions and residual metrics.** We denote the Green's function associated to the metric μ by g_μ. It is a function on $X \times X$ characterized by the two properties

$$\mathrm{d}_x \mathrm{d}_x^c g_\mu(x, y) + \delta_y(x) = \frac{\mu(x)}{\mathrm{vol}_\mu(X)} \,,$$

$$\int\limits_X g_\mu(x, y) \mu(x) = 0 \,.$$

If $\mu = \mu_{\mathrm{hyp}}$, $\mu = \mu_{\mathrm{shyp}}$, or $\mu = \mu_{\mathrm{can}}$, we set

$$g_\mu = g_{\mathrm{hyp}} \,, \quad g_\mu = g_{\mathrm{shyp}} \,, \quad g_\mu = g_{\mathrm{can}} \,,$$

respectively. Note that $g_{\rm hyp}=g_{\rm shyp}$. By means of the function $G_\mu=\exp(g_\mu)$, we can now define a metric $\|\cdot\|_{\mu,\rm res}$ on the canonical line bundle Ω_X^1 of X in the following way. For $x\in X$ and $z(x)$ as above, we set

$$\|{\rm d}z(x)\|^2_{\mu,\rm res}=\lim_{y\to x}\left(G_\mu(x,y)\cdot|z(x)-z(y)|^2\right).$$

We call the metric

$$\mu_{\rm res}(x)=\frac{i}{2}\cdot\frac{{\rm d}z(x)\wedge{\rm d}\bar z(x)}{\|{\rm d}z(x)\|^2_{\mu,\rm res}}$$

the residual metric associated to μ. If $\mu=\mu_{\rm hyp}$, $\mu=\mu_{\rm shyp}$, or $\mu=\mu_{\rm can}$, we set

$$\|\cdot\|_{\mu,\rm res}=\|\cdot\|_{\rm hyp,res}\,,\quad \|\cdot\|_{\mu,\rm res}=\|\cdot\|_{\rm shyp,res}\,,\quad \|\cdot\|_{\mu,\rm res}=\|\cdot\|_{\rm can,res}\,,$$

$$\mu_{\rm res}=\mu_{\rm hyp,res}\,,\quad \mu_{\rm res}=\mu_{\rm shyp,res}\,,\quad \mu_{\rm res}=\mu_{\rm can,res}\,,$$

respectively. Since $g_{\rm hyp}=g_{\rm shyp}$, we have $\mu_{\rm hyp,res}=\mu_{\rm shyp,res}$. We recall that the Arakelov metric $\mu_{\rm Ar}$ is defined as the residual metric associated to the canonical metric $\mu_{\rm can}$; the corresponding metric on Ω^1_X is denoted by $\|\cdot\|_{\rm Ar}$. In order to be able to compare the metrics $\mu_{\rm hyp}$ and $\mu_{\rm Ar}$, we define the C^∞-function $\phi_{\rm Ar}$ on X by the equation

$$\mu_{\rm Ar}=e^{\phi_{\rm Ar}}\mu_{\rm hyp}\,. \tag{6}$$

2.3. **Faltings's delta function and determinants.** We denote the Laplacian on X associated to the metric μ by Δ_μ. We write $\Delta_{\rm hyp}$ for the hyperbolic Laplacian on X; identifying $x\in X$ with $z(x)=\xi+i\eta$ in a fundamental domain for Γ in $\mathbb{H}$, we have

$$\Delta_{\rm hyp}=-\eta^2\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right). \tag{7}$$

We let $\{\phi_{X,n}\}_{n=0}^\infty$ denote an orthonormal basis of eigenfunctions of $\Delta_{\rm hyp}$ on X with eigenvalues

$$0=\lambda_{X,0}<\lambda_{X,1}\le\lambda_{X,2}\le\dots\,,$$

i.e.,

$$\Delta_{\rm hyp}\phi_{X,n}=\lambda_{X,n}\phi_{X,n}\quad(n=0,1,2,\dots).$$

We denote the number of eigenvalues of $\Delta_{\rm hyp}$ lying in the interval $[a,b)$ by $N^{[a,b)}_{{\rm ev},X}$.

To Δ_μ, we have associated the spectral zeta function $\zeta_\mu(s)$, which gives rise to the regularized determinant $\det^*(\Delta_\mu)$. We set the notation

$$D_\mu(X)=\log\left(\frac{\det^*(\Delta_\mu)}{{\rm vol}_\mu(X)}\right).$$

If $\mu=\mu_{\rm hyp}$, or $\mu=\mu_{\rm Ar}$, we set $D_\mu=D_{\rm hyp}$, or $D_\mu=D_{\rm Ar}$, respectively. Observing the first Chern form relations

$${\rm c}_1(\Omega^1_X,\|\cdot\|_{\rm hyp})=(2g_X-2)\mu_{\rm shyp}(x)\,,\quad {\rm c}_1(\Omega^1_X,\|\cdot\|_{\rm Ar})=(2g_X-2)\mu_{\rm can}(x)\,,$$

an immediate application of Polyakov's formula (see [20], p. 78) shows the relation

$$D_{\mathrm{Ar}}(X) = D_{\mathrm{hyp}}(X) + \frac{g_X - 1}{6} \int\limits_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{can}}(x) + \mu_{\mathrm{shyp}}(x)) \,. \tag{8}$$

Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ is introduced in [7], where also some of its basic properties are given. In [13], Faltings's delta function is expressed in terms of Riemann's theta function, and its asymptotic behavior is investigated. As a by-product of the analytic part of the arithmetic Riemann-Roch theorem for arithmetic surfaces, it is shown in [30] that

$$\delta_{\mathrm{Fal}}(X) = -6D_{\mathrm{Ar}}(X) + a(g_X) \,, \tag{9}$$

where

$$a(g_X) = -2g_X \log(\pi) + 4g_X \log(2) + (g_X - 1)(-24\zeta'_{\mathbb{Q}}(-1) + 1). \tag{10}$$

For the sequel, we only have to recall that $a(g_X) = O(g_X)$.

2.4. **Heat kernels and heat traces.** Let $H(\Gamma)$ denote a complete set of representatives of inconjugate, primitive, hyperbolic elements in Γ. Denote by ℓ_γ the hyperbolic length of the closed geodesic determined by $\gamma \in H(\Gamma)$ on X; it is well-known that the equality

$$|\mathrm{tr}(\gamma)| = 2\cosh(\ell_\gamma/2)$$

holds. We denote the number of elements γ in $H(\Gamma)$, whose geodesic representatives have length in the interval $(0,b)$ by $N^{(0,b)}_{\mathrm{geo},X}$.
The heat kernel $K_{\mathbb{H}}(t;z,w)$ on $\mathbb{H}$ ($t \in \mathbb{R}_{>0}$; $z,w \in \mathbb{H}$) is given by the formula

$$K_{\mathbb{H}}(t;z,w) = K_{\mathbb{H}}(t;\rho) := \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int\limits_\rho^\infty \frac{re^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(\rho)}}\,\mathrm{d}r \,,$$

where $\rho := d_{\mathbb{H}}(z,w)$ denotes the hyperbolic distance between z and w. The heat kernel $K_{\mathrm{hyp}}(t;x,y)$ associated to X ($t \in \mathbb{R}_{>0}$; $x,y \in X$), resp. the hyperbolic heat kernel $HK_{\mathrm{hyp}}(t;x,y)$ associated to X ($t \in \mathbb{R}_{>0}$; $x,y \in X$) is defined by averaging over the elements of Γ, resp. the elements of Γ different from the identity, namely

$$\begin{aligned} K_{\mathrm{hyp}}(t;x,y) &= \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t;z(x),\gamma z(y)) \,, \text{ resp.} \\ HK_{\mathrm{hyp}}(t;x,y) &= \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq \mathrm{id}}} K_{\mathbb{H}}(t;z(x),\gamma z(y)) \,. \end{aligned}$$

We note that $K_{\mathrm{hyp}}(t;x,y)$ satisfies the equation

$$\left(\frac{\partial}{\partial t}+\Delta_{\mathrm{hyp},x}\right)K_{\mathrm{hyp}}(t;x,y)=0 \qquad (y\in X),$$

$$\lim_{t\to 0}\int\limits_X K_{\mathrm{hyp}}(t;x,y)\,f(y)\,\mu_{\mathrm{hyp}}(y)=f(x) \qquad (x\in X)$$

for all C^∞-functions f on X. In terms of the eigenfunctions $\{\phi_{X,n}\}_{n=0}^{\infty}$ and eigenvalues $\{\lambda_{X,n}\}_{n=0}^{\infty}$ of Δ_{hyp}, we have

$$K_{\mathrm{hyp}}(t;x,y)=\sum_{n=0}^{\infty}\phi_{X,n}(x)\phi_{X,n}(y)e^{-\lambda_{X,n}t}\,.$$

If $x=y$, we write $HK_{\mathrm{hyp}}(t;x)$ instead of $HK_{\mathrm{hyp}}(t;x,x)$. The hyperbolic heat trace $HK_{\mathrm{hyp}}(t)$ ($t\in\mathbb{R}_{>0}$) is now given by

$$HTrK_{\mathrm{hyp}}(t)=\int\limits_X HK_{\mathrm{hyp}}(t;x)\,\mu_{\mathrm{hyp}}(x)\,.$$

Introducing the function

$$f(u,t)=\frac{e^{-t/4}}{(4\pi t)^{1/2}}\sum_{n=1}^{\infty}\frac{\log(u)}{u^{n/2}-u^{-n/2}}e^{-(n\log(u))^2/4t}, \tag{11}$$

and setting $HTrK_\gamma(t)=f(e^{\ell_\gamma},t)$, we recall the identity

$$HTrK_{\mathrm{hyp}}(t)=\sum_{\gamma\in H(\Gamma)}HTrK_\gamma(t)\,,$$

which is one application of the Selberg trace formula (see [9]). For any $\delta>0$, we now define

$$HTrK_{\mathrm{hyp},\delta}(t)=HTrK_{\mathrm{hyp}}(t)-\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta}}HTrK_\gamma(t)\,. \tag{12}$$

We note that the hyperbolic Green's function $g_{\mathrm{hyp}}(x,y)$ ($x,y\in X;\ x\neq y$) relates in the following way to the heat kernel

$$g_{\mathrm{hyp}}(x,y)=4\pi\int\limits_0^{\infty}\left(K_{\mathrm{hyp}}(t;x,y)-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\,. \tag{13}$$

In particular for the Green's function $g_{\mathbb{H}}(z,w)$ on $\mathbb{H}$ ($z,w\in\mathbb{H};\ z\neq w$), we recall the formulas

$$g_{\mathbb{H}}(z,w)=-\log\left(\left|\frac{z-w}{z-\bar{w}}\right|^2\right)=4\pi\int\limits_0^{\infty}K_{\mathbb{H}}(t;z,w)\,\mathrm{d}t\,.$$

2.5. **Prime geodesic theorem.** Consider the function

$$\pi_X(u) = \sum_{\substack{\gamma \in H(\Gamma) \\ e^{\ell_\gamma} < u}} 1 \,,$$

which is defined for $u \in \mathbb{R}_{>1}$; it is just the number of inconjugate, primitive, hyperbolic elements of Γ such that the corresponding geodesics have length less than $\log(u)$. For any eigenvalue $\lambda_{X,j}$ $(j = 0, 1, 2, \dots)$, $0 \leq \lambda_{X,j} < 1/4$, we put

$$s_{X,j} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_{X,j}} \,,$$

and note that $1/2 < s_{X,j} \leq 1$. Introducing the integral logarithm

$$\mathrm{li}(u^{s_{X,j}}) = \int\limits_{2}^{u^{s_{X,j}}} \frac{\mathrm{d}\xi}{\log(\xi)} \,,$$

the prime geodesic theorem states

$$\left| \pi_X(u) - \sum_{0 \leq \lambda_{X,j} < 1/4} \mathrm{li}(u^{s_{X,j}}) \right| \leq C \cdot u^{3/4} (\log(u))^{-1/2} \tag{14}$$

for $u > 2$ with an implied constant $C > 0$ depending solely on X (see [11], [5], p. 297, or [10], p. 474). Then, we define the Huber constant $C_{\mathrm{Hub},X}$ to be the infimum of all constants $C > 0$ such that (14) holds. With this definition the main result of [15] implies the following: Assume that X is a finite cover of a fixed Riemann surface X_0 of genus $g_{X_0} > 1$, then

$$C_{\mathrm{Hub},X} \leq \deg(X/X_0) \cdot C_{\mathrm{Hub},X_0}, \tag{15}$$

where $\deg(X/X_0)$ denotes the degree of X over X_0. This choice for the error term in the prime geodesic theorem suffices for our purposes, since we are working with general compact Riemann surfaces. Improvements on the error term in certain cases are contained in [4], [12], and [23]. For the purpose of the present article, these results will not be used.

We note that using the function $\pi_X(u)$, the truncated hyperbolic heat trace (12) can be rewritten as

$$HTrK_{\mathrm{hyp},\delta}(t) = \int\limits_{e^{\delta}}^{\infty} f(u,t) \, \mathrm{d}\pi_X(u) \,. \tag{16}$$

2.6. **Selberg's zeta function.** For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, the Selberg zeta function $Z_X(s)$ associated to X is defined via the Euler product expansion

$$Z_X(s) = \prod_{\gamma \in H(\Gamma)} Z_\gamma(s),$$

where the local factors $Z_\gamma(s)$ are given by

$$Z_\gamma(s) = \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)\ell_\gamma}\right) .$$

The Selberg zeta function $Z_X(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation. From [28], p. 115, we recall the relation

$$D_{\mathrm{hyp}}(X) = \log\left(\frac{Z'_X(1)}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) + b(g_X)\,, \tag{17}$$

where

$$b(g_X) = (g_X - 1)(4\zeta'_{\mathbb{Q}}(-1) - 1/2 + \log(2\pi))\,. \tag{18}$$

As in [14], we define the quantity

$$c_X := \lim_{s\to 1}\left(\frac{Z'_X}{Z_X}(s) - \frac{1}{s-1}\right) .$$

From [14], Lemma 4.2, we recall the formula

$$c_X = 1 + \int_0^\infty (HTrK_{\mathrm{hyp}}(t) - 1)\mathrm{d}t = \int_0^\infty (HTrK_{\mathrm{hyp}}(t) - 1 + e^{-t})\mathrm{d}t\,. \tag{19}$$

Identity (19) is obtained by means of the McKean formula

$$\frac{Z'_X}{Z_X}(s) = (2s-1)\int_0^\infty HTrK_{\mathrm{hyp}}(t)e^{-s(s-1)t}\mathrm{d}t\,,$$

which, observing the asymptotic $\lim_{s\to\infty} Z_X(s) = 1$, has the integrated version

$$\log(Z_X(s)) = -\int_0^\infty HTrK_{\mathrm{hyp}}(t)e^{-s(s-1)t}\frac{\mathrm{d}t}{t}\,. \tag{20}$$

Analogously, we find the local versions

$$\frac{Z'_\gamma}{Z_\gamma}(s) = (2s-1)\int_0^\infty HTrK_\gamma(t)e^{-s(s-1)t}\mathrm{d}t\ , \tag{21}$$

$$\log(Z_\gamma(s)) = -\int_0^\infty HTrK_\gamma(t)e^{-s(s-1)t}\frac{\mathrm{d}t}{t}\,.$$

Observing the identity

$$\log(w) = \int_0^\infty \left(e^{-t} - e^{-wt}\right)\frac{\mathrm{d}t}{t} \tag{22}$$

for $w > 0$ and taking $w = s(s-1)$ (with $s \in \mathbb{R}_{>1}$), we can combine (22) with the integrated version of the McKean formula (20) to get

$$-\log(Z'_X(1)) = \int_0^\infty \left(HTrK_{\mathrm{hyp}}(t) - 1 + e^{-t} \right) \frac{\mathrm{d}t}{t}\,. \tag{23}$$

Subtracting (22) form (23), yields the more general formula

$$-\log(Z'_X(1)) - \log(w) = \int_0^\infty \left(HTrK_{\mathrm{hyp}}(t) - 1 + e^{-wt} \right) \frac{\mathrm{d}t}{t}\,, \tag{24}$$

which holds for $w > 0$. Using (12) and the second formula in (21) with $s = 1$, we end up with the formula

$$-\log(Z'_X(1)) - \log(w) + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} \log(Z_\gamma(1)) = \int_0^\infty \left(HTrK_{\mathrm{hyp},\delta}(t) - 1 + e^{-wt} \right) \frac{\mathrm{d}t}{t}. \tag{25}$$

3. Expressing Faltings's delta via hyperbolic geometry

The purpose of this section is to obtain an expression which evaluates Faltings's delta function $\delta_{\mathrm{Fal}}(X)$ in terms of spectral theoretic information of X coming from hyperbolic geometry. Our method of proof is as follows. First, we use results from [28] and [30] together with the Polyakov formula (8) to express $\delta_{\mathrm{Fal}}(X)$ in terms of hyperbolic information and the conformal factor ϕ_{Ar} (see (6)) relating the Arakelov metric μ_{Ar} to the hyperbolic metric μ_{hyp} on X. We then derive and exploit explicit relations between the canonical and hyperbolic Green's functions in order to explicitly evaluate the term involving ϕ_{Ar}. Our starting point is the following lemma which collects results stated above.

3.1. **Lemma.** *For any X with genus $g_X > 1$, let*

$$c(g_X) := a(g_X) - 6b(g_X) + 6\log(\mathrm{vol}_{\mathrm{hyp}}(X))$$

where $a(g_X)$, resp. $b(g_X)$ are given by formulas (10), resp. (18). With the above notations, we then have the formula

$$\delta_{\mathrm{Fal}}(X) = -6\log(Z'_X(1)) - (g_X - 1)\int_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) + c(g_X).$$

Proof. Combining formulas (9), (8), and (17), we obtain

$$\begin{aligned}
&\delta_{\mathrm{Fal}}(X) = -6D_{\mathrm{Ar}}(X) + a(g_X) = \\
&-6D_{\mathrm{hyp}}(X) - (g_X - 1)\int_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) + a(g_X) = \\
&-6\log\left(\frac{Z'_X(1)}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) - (g_X - 1)\int_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) + \\
&\qquad a(g_X) - 6b(g_X) = \\
&-6\log(Z'_X(1)) - (g_X - 1)\int_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) + \\
&\qquad a(g_X) - 6b(g_X) + 6\log(\mathrm{vol}_{\mathrm{hyp}}(X)).
\end{aligned}$$

This completes the proof of the lemma. □

3.2. **Remark.** For the sake of completeness, let us make explicit the value of $c(g_X)$; a straightforward calculation yields

$$\begin{aligned}
&c(g_X) = a(g_X) - 6b(g_X) + 6\log(\mathrm{vol}_{\mathrm{hyp}}(X)) = \\
&2g_X\left(-24\zeta'_{\mathbb{Q}}(-1) - 4\log(\pi) - \log(2) + 2\right) + 6\log(\mathrm{vol}_{\mathrm{hyp}}(X)) + \\
&\qquad \left(48\zeta'_{\mathbb{Q}}(-1) + 6\log(2\pi) - 4\right).
\end{aligned}$$

3.3. **Lemma.** *Let μ_1, resp. μ_2 be any two positive $(1,1)$-forms on X with associated Green's function $g_1(x,y)$, resp. $g_2(x,y)$, and assume that*

$$\int_X \mu_1(x) = \int_X \mu_2(x) = 1.$$

Then, we have the relation

$$\begin{aligned}
&g_1(x,y) - g_2(x,y) = \\
&\int_X g_1(x,\zeta)\mu_2(\zeta) + \int_X g_1(y,\zeta)\mu_2(\zeta) - \int_X\int_X g_1(\xi,\zeta)\mu_2(\zeta)\mu_2(\xi)\,.
\end{aligned} \tag{26}$$

Proof. Let $F_L(x,y)$, resp. $F_R(x,y)$ denote the left-, resp. right-hand side of the stated identity (26). Using the characterizing properties of the Green's functions, one can show directly that we have for fixed $y \in X$

$$\mathrm{d}_x\mathrm{d}_x^c F_L(x,y) = \mathrm{d}_x\mathrm{d}_x^c F_R(x,y) = \mu_1(x) - \mu_2(x)\,,$$

and

$$\int_X F_L(x,y)\mu_2(x) = \int_X F_R(x,y)\mu_2(x) = \int_X g_1(y,\zeta)\mu_2(\zeta).$$

Consequently, $F_L(x,y) = F_R(x,y)$, again for fixed y. However, it is obvious that F_L and F_R are symmetric in x and y. This completes the proof of the lemma. □

3.4. **Remark.** Equation (26) from Lemma 3.3 provides the key identity for the subsequent investigations. Note that a less explicit variant of it can be found in the literature, e.g., [22], Proposition 1.3.

3.5. **Lemma.** *Let μ_1, μ_2 be as in Lemma 3.3. Furthermore, let $\mu_{1,\mathrm{res}}$, resp. $\mu_{2,\mathrm{res}}$ be the residual metrics associated to μ_1, resp. μ_2. Then, we have*

$$\int\limits_X \log\left(\frac{\mu_{2,\mathrm{res}}(x)}{\mu_{1,\mathrm{res}}(x)}\right)(\mu_1(x)+\mu_2(x)) = 0.$$

Proof. Using the definitions of Green's functions and residual metrics given in section 2.2, we get

$$\log\left(\frac{\mu_{2,\mathrm{res}}(x)}{\mu_{1,\mathrm{res}}(x)}\right) = \log\left(\lim_{y\to x}\frac{G_1(x,y)}{G_2(x,y)}\right).$$

Using Lemma 3.3, this implies

$$\log\left(\frac{\mu_{2,\mathrm{res}}(x)}{\mu_{1,\mathrm{res}}(x)}\right) = \lim_{y\to x}\left(g_1(x,y)-g_2(x,y)\right) =$$

$$2\int\limits_X g_1(x,\zeta)\mu_2(\zeta) - \int\limits_X\int\limits_X g_1(\xi,\zeta)\mu_2(\zeta)\mu_2(\xi).$$

Since

$$\int\limits_X\left(2\int\limits_X g_1(x,\zeta)\mu_2(\zeta) - \int\limits_X\int\limits_X g_1(\xi,\zeta)\mu_2(\zeta)\mu_2(\xi)\right)(\mu_1(x)+\mu_2(x)) = 0,$$

the result follows. □

3.6. **Lemma.** *For any X, we have*

$$\log\left(\frac{\mu_{\mathrm{can,res}}(x)}{\mu_{\mathrm{shyp,res}}(x)}\right) = \phi_{\mathrm{Ar}}(x) + 4\pi\int\limits_0^\infty\left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t + \log(4)\,. \tag{27}$$

Proof. The left-hand side of the claimed formula can be expressed as

$$\log\left(\mu_{\mathrm{can,res}}(x)/\mu_{\mathrm{shyp,res}}(x)\right) = \log\left(\mu_{\mathrm{Ar}}(x)/\mu_{\mathrm{hyp,res}}(x)\right) =$$

$$\log\left(e^{\phi_{\mathrm{Ar}}(x)}\mu_{\mathrm{hyp}}(x)/\mu_{\mathrm{hyp,res}}(x)\right) = \phi_{\mathrm{Ar}}(x) + \log\left(\mu_{\mathrm{hyp}}(x)/\mu_{\mathrm{hyp,res}}(x)\right).$$

We now evaluate $\mu_{\mathrm{hyp}}(x)/\mu_{\mathrm{hyp,res}}(x)$ in terms of the heat kernel on X. Working with relation (13), we have

$$g_{\mathrm{hyp}}(x,y) = 4\pi\int\limits_0^\infty \left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z(x),\gamma z(y)) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t -$$
$$\log\left(\left|\frac{z(x)-z(y)}{z(x)-\bar{z}(y)}\right|^2\right) =$$
$$4\pi\int\limits_0^\infty \left(HK_{\mathrm{hyp}}(t;x,y) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t - \log\left(\left|\frac{z(x)-z(y)}{z(x)-\bar{z}(y)}\right|^2\right),$$

from which we derive

$$\lim_{y\to x}\left(g_{\mathrm{hyp}}(x,y) + \log|z(x)-z(y)|^2\right) =$$
$$4\pi\int\limits_0^\infty \left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t + \log(4\,\mathrm{Im}(z(x))^2).$$

This implies

$$\log\left(\mu_{\mathrm{hyp}}(x)/\mu_{\mathrm{hyp,res}}(x)\right) = \log\left(\|\mathrm{d}z(x)\|^2_{\mathrm{hyp,res}}/\mathrm{Im}(z(x))^2\right) =$$
$$\lim_{y\to x}\left(g_{\mathrm{hyp}}(x,y) + \log|z(x)-z(y)|^2\right) - \log(\mathrm{Im}(z(x))^2) =$$
$$4\pi\int\limits_0^\infty \left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t + \log(4).$$

Combining these calculations, we conclude that

$$\log\left(\frac{\mu_{\mathrm{can,res}}(x)}{\mu_{\mathrm{shyp,res}}(x)}\right) = \phi_{\mathrm{Ar}}(x) + 4\pi\int\limits_0^\infty \left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t + \log(4),$$

which proves the lemma. □

3.7. **Proposition.** *For any X with genus $g_X > 1$, let*

$$F(t;x) := HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

Then, we have the formula

$$\int\limits_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) =$$
$$-\frac{2\pi}{g_X}\int\limits_X\int\limits_0^\infty\int\limits_0^\infty F(t_1;x)\Delta_{\mathrm{hyp}}F(t_2;x)\mathrm{d}t_1\mathrm{d}t_2\,\mu_{\mathrm{hyp}}(x) - \frac{2(c_X-1)}{g_X-1} - 2\log(4).$$

Proof. Choosing $\mu_1 = \mu_{\mathrm{shyp}}$ and $\mu_2 = \mu_{\mathrm{can}}$ in Lemma 3.5, shows

$$\int\limits_X \log\left(\frac{\mu_{\mathrm{can,res}}(x)}{\mu_{\mathrm{shyp,res}}(x)}\right)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) = 0\,.$$

Multiplying (27) by $(\mu_{\mathrm{shyp}} + \mu_{\mathrm{can}})$ and integrating over X, we arrive at the relation

$$\int\limits_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) =$$
$$-4\pi\int\limits_X\int\limits_0^\infty\left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\,(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) - 2\log(4)\,.$$

Interchanging the integration and recalling the formula for the hyperbolic volume of X in terms of g_X together with formula (19) gives

$$4\pi\int\limits_X\int\limits_0^\infty\left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\,\mu_{\mathrm{shyp}}(x) =$$
$$\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\int\limits_0^\infty(HTrK_{\mathrm{hyp}}(t) - 1)\mathrm{d}t = \frac{c_X - 1}{g_X - 1}\,,$$

which leads to the relation

$$\int\limits_X \phi_{\mathrm{Ar}}(x)(\mu_{\mathrm{shyp}}(x) + \mu_{\mathrm{can}}(x)) = \tag{28}$$
$$-4\pi\int\limits_X\int\limits_0^\infty\left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\,\mu_{\mathrm{can}}(x) - \frac{c_X - 1}{g_X - 1} - 2\log(4)\,.$$

In order to rewrite the latter integral, we recall the following formula from [18], which gives an explicit relation between the canonical and the scaled hyperbolic metric form, namely

$$\mu_{\mathrm{can}}(x) = \mu_{\mathrm{shyp}}(x) + \frac{1}{2g_X}\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;x)\mathrm{d}t\right)\mu_{\mathrm{hyp}}(x)\,; \tag{29}$$

for the reader's convenience, we add the proof of (29) in Appendix I, section 6. Observing that $\Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;x) = \Delta_{\mathrm{hyp}}HK_{\mathrm{hyp}}(t;x)$, we obtain by means of

(29) and the preceding calculations

$$4\pi \int_X \int_0^\infty \left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) \mathrm{d}t\, \mu_{\mathrm{can}}(x) = \tag{30}$$

$$\frac{c_X - 1}{g_X - 1} + \frac{2\pi}{g_X} \int_X \int_0^\infty \int_0^\infty \left(HK_{\mathrm{hyp}}(t_1;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right)$$

$$\times \Delta_{\mathrm{hyp}} HK_{\mathrm{hyp}}(t_2;x) \mathrm{d}t_1 \mathrm{d}t_2\, \mu_{\mathrm{hyp}}(x)\,.$$

Substituting (30) into (28) and observing that $\Delta_{\mathrm{hyp}} HK_{\mathrm{hyp}}(t_2;x) = \Delta_{\mathrm{hyp}} F(t_2;x)$, completes the proof of the proposition. □

3.8. **Theorem.** *For any X with genus $g_X > 1$, let*

$$F(x) := \int_0^\infty \left(HK_{\mathrm{hyp}}(t;x) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) \mathrm{d}t\,.$$

Then, we have

$$\delta_{\mathrm{Fal}}(X) =$$

$$2\pi \left(1 - \frac{1}{g_X} \right) \int_X F(x) \Delta_{\mathrm{hyp}} F(x) \mu_{\mathrm{hyp}}(x) - 6 \log(Z'_X(1)) + 2c_X + C(g_X)\,,$$

where

$$C(g_X) = a(g_X) - 6b(g_X) + 2(g_X - 1)\log(4) + 6\log(\mathrm{vol}_{\mathrm{hyp}}(X)) - 2 =$$

$$2g_X \left(-24\zeta'_{\mathbb{Q}}(-1) - 4\log(\pi) + \log(2) + 2 \right) + 6\log(\mathrm{vol}_{\mathrm{hyp}}(X)) +$$

$$\left(48\zeta'_{\mathbb{Q}}(-1) + 6\log(2\pi) - 2\log(4) - 6 \right)\,.$$

Proof. Simply combine Lemma 3.1 with Proposition 3.7. □

3.9. **Remark.** From Theorem 3.8, we have a precise expression for $\delta_{\mathrm{Fal}}(X) - C(g_X)$ in terms of hyperbolic data associated to X, all of which can be derived from the trace of the hyperbolic heat kernel. As such, one can extend the hyperbolic expression to general non-compact, finite volume hyperbolic Riemann surfaces, including those which admit elliptic fixed points. Going further, it seems possible to employ the techniques known as Artin formalism, which has been shown to hold for hyperbolic heat kernels, in order to obtain analogous relations for the Faltings delta function as well as the constant $C(g_X)$. Note that since the Arakelov metric does not lift through covers, there is no immediate reason to expect any relations involving $\delta_{\mathrm{Fal}}(X)$ similar to those predicted by the Artin formalism; however, Theorem 3.8 implies that some relations are possible. We leave this problem for further study elsewhere.

4. Analytic bounds

The main result of the section is Theorem 4.5, which states a bound for Faltings's delta function in terms of fundamental invariants from hyperbolic geometry. Propositions 4.1, 4.2, and 4.3 bound the non-trivial quantities in the expression for Faltings's delta function given in Theorem 3.8, and these results, together with Lemma 4.4, are used to prove Theorem 4.5.

4.1. **Proposition.** *For any X with genus $g_X > 1$, let $F(x)$ be as in Theorem 3.8, and set*

$$d_{\mathrm{sup},X} := \sup_{x\in X}\left|\frac{\mu_{\mathrm{can}}(x)}{\mu_{\mathrm{shyp}}(x)}\right|.$$

Then, we have the estimate

$$0 \leq \int\limits_X F(x)\Delta_{\mathrm{hyp}}F(x)\mu_{\mathrm{hyp}}(x) \leq \frac{(d_{\mathrm{sup},X}+1)^2\mathrm{vol}_{\mathrm{hyp}}(X)}{\lambda_{X,1}}\,.$$

Proof. From formula (29), we have the identity

$$g_X\mu_{\mathrm{can}}(x) - g_X\mu_{\mathrm{shyp}}(x) =$$
$$\frac{1}{2}\left(\int\limits_0^\infty \Delta_{\mathrm{hyp}}HK_{\mathrm{hyp}}(t;x)\mathrm{d}t\right)\mu_{\mathrm{hyp}}(x) = \frac{1}{2}\Delta_{\mathrm{hyp}}F(x)\mu_{\mathrm{hyp}}(x)\,,$$

which immediately gives the formula

$$\Delta_{\mathrm{hyp}}F(x) = \frac{2g_X}{4\pi(g_X-1)}\left(\frac{\mu_{\mathrm{can}}(x)}{\mu_{\mathrm{shyp}}(x)} - 1\right)\,,$$

and, hence, leads to the estimate

$$\sup_{x\in X}|\Delta_{\mathrm{hyp}}F(x)| \leq d_{\mathrm{sup},X}+1.$$

Since X is compact, we can expand $F(x)$ in terms of the orthonormal basis of eigenfunctions $\{\phi_{X,n}\}_{n=0}^\infty$ with eigenvalues $\{\lambda_{X,n}\}_{n=0}^\infty$ of Δ_{hyp}, i.e.,

$$F(x) = \sum_{n=0}^\infty a_n\phi_{X,n}(x),$$

from which we derive

$$\Delta_{\mathrm{hyp}}F(x) = \sum_{n=1}^\infty \lambda_{X,n}a_n\phi_{X,n}(x),$$

taking into account that $\lambda_{X,0}=0$. Therefore, we have

$$\int\limits_X F(x)\Delta_{\mathrm{hyp}}F(x)\mu_{\mathrm{hyp}}(x) = \sum_{n=1}^\infty \lambda_{X,n}a_n^2\,.$$

Observing that

$$\int_X (\Delta_{\mathrm{hyp}} F(x))^2 \, \mu_{\mathrm{hyp}}(x) = \sum_{n=1}^{\infty} \lambda_{X,n}^2 a_n^2 \, ,$$

which yields by the above calculations the trivial bound

$$\sum_{n=1}^{\infty} \lambda_{X,n}^2 a_n^2 = \int_X (\Delta_{\mathrm{hyp}} F(x))^2 \, \mu_{\mathrm{hyp}}(x) \leq (d_{\sup,X} + 1)^2 \mathrm{vol}_{\mathrm{hyp}}(X) \, ,$$

and taking into account $\lambda_{X,1} \leq \lambda_{X,n}$ for all $n \geq 1$, we are finally led to the estimate

$$0 \leq \lambda_{X,1} \int_X F(x) \Delta_{\mathrm{hyp}} F(x) \mu_{\mathrm{hyp}}(x) = \lambda_{X,1} \sum_{n=1}^{\infty} \lambda_{X,n} a_n^2 \leq$$

$$\sum_{n=1}^{\infty} \lambda_{X,n}^2 a_n^2 \leq (d_{\sup,X} + 1)^2 \mathrm{vol}_{\mathrm{hyp}}(X) \, .$$

This completes the proof of the proposition. □

4.2. **Proposition.** *For any X with genus $g_X > 1$, we have the lower bound*

$$c_X \geq -4 \log(2g_X - 2).$$

Letting $\alpha = \min\{\lambda_{X,1}, 7/64\}$ and $\varepsilon \in (0, \alpha)$, we have the upper bound

$$c_X \leq 2 + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < 5}} \frac{Z'_\gamma}{Z_\gamma}(1) + \frac{6}{\varepsilon} \left(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)} \right) .$$

Proof. The lower bound is proved in [14], Theorem 3.3. The upper bound comes from the proof of Theorem 4.7 in [14]. Specifically, for any $\delta > 0$, we recall the inequality

$$c_X \leq 1 + \sum_{0 < \lambda_{X,j} < \varepsilon} \frac{1}{\lambda_{X,j}} + \sum_{\substack{\gamma \in H(\Gamma) \\ \ell_\gamma < \delta}} \frac{Z'_\gamma}{Z_\gamma}(1) + C_{X,\varepsilon} e^{-(1-s_\varepsilon)\delta} + 12 N_{\mathrm{ev},X}^{[0,\varepsilon)} e^{-\delta/2}$$

with

$$C_{X,\varepsilon} = \frac{4(4 - 3s_\varepsilon)}{\varepsilon} (C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[\varepsilon,1/4)}), \quad s_\varepsilon = \frac{1}{2} + \sqrt{\frac{1}{4} - \varepsilon} \, .$$

Choosing $\delta = 5$ and ε as stated above, noting that $N_{\mathrm{ev},X}^{[0,\varepsilon)} = 1$, $12e^{-5/2} < 1$, and $7/8 < s_\varepsilon < 1$, i.e., $4(4 - 3s_\varepsilon) < 6$, the claim follows. □

4.3. **Proposition.** *For any X with genus $g_X > 1$, we have the lower bound*

$$-\log(Z'_X(1)) \geq -4 \log(4g_X - 4) - \frac{1}{16} \, .$$

Letting $\alpha = \min\{\lambda_{X,1}, 7/64\}$ *and* $\varepsilon \in (0,\alpha)$, *we have the upper bound*

$$-\log(Z'_X(1)) \leq -\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<5}} \log(Z_\gamma(1)) + 12\left(5+\frac{1}{\varepsilon}\right)\left(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)} + 1\right).$$

Proof. We follow the method of proof used to prove the bounds in Proposition 4.2. Since these calculations are not immediate from the results in [14], it is necessary to give the details. Let $\delta > 0$ to be specified below. Then, using the trivial bounds

$$HTrK_{\mathrm{hyp}}(t) + \mathrm{vol}_{\mathrm{hyp}}(X)K_{\mathbb{H}}(t;0) = \sum_{j=0}^{\infty} e^{-\lambda_{X,j}t} \geq 1$$

for $t \geq \delta$, and

$$HTrK_{\mathrm{hyp}}(t) \geq 0$$

for $0 \leq t \leq \delta$, we get from formula (23) the bound

$$-\log(Z'_X(1)) \geq \int_0^\delta (e^{-t}-1)\frac{\mathrm{d}t}{t} + \int_\delta^\infty \left(e^{-t} - \mathrm{vol}_{\mathrm{hyp}}(X)K_{\mathbb{H}}(t;0)\right)\frac{\mathrm{d}t}{t}.$$

Trivially, one has $e^{-t} - 1 \geq -t$ for $t \geq 0$, so

$$\int_0^\delta (e^{-t}-1)\frac{\mathrm{d}t}{t} \geq -\delta.$$

Using the obvious bound $K_{\mathbb{H}}(t;0) \leq e^{-t/4}/(4\pi t)$, we get

$$\int_\delta^\infty K_{\mathbb{H}}(t;0)\frac{\mathrm{d}t}{t} \leq \frac{e^{-\delta/4}}{\pi\delta^2},$$

which gives

$$\int_\delta^\infty \left(e^{-t} - \mathrm{vol}_{\mathrm{hyp}}(X)K_{\mathbb{H}}(t;0)\right)\frac{\mathrm{d}t}{t} \geq$$
$$-\mathrm{vol}_{\mathrm{hyp}}(X)\int_\delta^\infty K_{\mathbb{H}}(t;0)\frac{\mathrm{d}t}{t} \geq -\mathrm{vol}_{\mathrm{hyp}}(X)\frac{e^{-\delta/4}}{\pi\delta^2},$$

hence

$$-\log(Z'_X(1)) \geq -\delta - \mathrm{vol}_{\mathrm{hyp}}(X)\frac{e^{-\delta/4}}{\pi\delta^2}.$$

Taking $\delta = 4\log(4g_X - 4)$, and using $\log(4g_X-4) \geq \log(4) > 1$, gives the stated lower bound.

For the upper bound, we proceed as in [14], section 4. Letting $s_w = 1/2+\sqrt{1/4-w}$ for $w \in [0,1/4]$, $\delta > 4$, and $f(u,t)$ as in (11), a straightforward calculation yields

$$\int_{e^\delta}^{\infty} f(u,t)\mathrm{dli}(u^{s_w}) = \frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{\delta}^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{(s_w-n/2-nm)\xi}e^{-(n\xi)^2/4t}\mathrm{d}\xi \tag{31}$$

(see also the proof of Lemma 4.3 in [14]). Writing the term with $n=1$ and $m=0$ as

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{\delta}^{\infty} e^{(s_w-1/2)\xi}e^{-\xi^2/4t}\mathrm{d}\xi = e^{-wt} - \frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{-\infty}^{\delta} e^{(s_w-1/2)\xi}e^{-\xi^2/4t}\mathrm{d}\xi\,,$$

we can rewrite (31) as

$$\begin{aligned} e^{-wt} \;=\; & \int_{e^\delta}^{\infty} f(u,t)\mathrm{dli}(u^{s_w}) - \frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{\delta}^{\infty}\sum_{(n,m)\neq(1,0)} e^{(s_w-n/2-nm)\xi}e^{-(n\xi)^2/4t}\mathrm{d}\xi + \\ & \frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{-\infty}^{\delta} e^{(s_w-1/2)\xi}e^{-\xi^2/4t}\mathrm{d}\xi\,, \end{aligned}$$

where the sum is taken over all integer pairs (n,m) with $n \geq 1$, $m \geq 0$, except for the pair $(n,m) = (1,0)$. Using this identity twice, once with $w = 0$, so $s_w = 1$, and again with $w = 1/4$, so $s_w = 1/2$, and recalling formula (16), we obtain the equality

$$HTrK_{\mathrm{hyp},\delta}(t) - 1 + e^{-t/4} = \int_{e^\delta}^{\infty} f(u,t)\,\mathrm{d}\left[\pi_X(u) - \mathrm{li}(u) + \mathrm{li}(u^{1/2})\right] + \tag{32}$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{\delta}^{\infty}\sum_{(n,m)\neq(1,0)} e^{(1-n/2-nm)\xi}e^{-(n\xi)^2/4t}\mathrm{d}\xi - \frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{-\infty}^{\delta} e^{\xi/2}e^{-\xi^2/4t}\mathrm{d}\xi -$$

$$\frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{\delta}^{\infty}\sum_{(n,m)\neq(1,0)} e^{(1/2-n/2-nm)\xi}e^{-(n\xi)^2/4t}\mathrm{d}\xi + \frac{e^{-t/4}}{(4\pi t)^{1/2}}\int_{-\infty}^{\delta} e^{-\xi^2/4t}\mathrm{d}\xi\,.$$

After these preliminary calculations, we turn to bounding $-\log(Z'_X(1))$ from above. For this we recall formula (25) with $w = 1/4$, namely

$$-\log(Z'_X(1)) - \log(1/4) + \sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta}} \log(Z_\gamma(1)) = \tag{33}$$

$$\int_{0}^{\infty}\left(HTrK_{\mathrm{hyp},\delta}(t) - 1 + e^{-t/4}\right)\frac{\mathrm{d}t}{t}\,.$$

As in [14], we substitute expression (32) for the integrand on the right-hand side of (33), interchange the order of integration, and evaluate. First, we do this for the two integrals coming from the term belonging to $(n, m) = (1, 0)$. We follow the convention which defines the K-Bessel function via the integral

$$K_\sigma(a,b) = \int_0^\infty e^{-a^2t-b^2/t}t^\sigma \frac{\mathrm{d}t}{t}$$

for $a, b \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}$; in particular, it can be shown that

$$K_{-1/2}(a,b) = \frac{\sqrt{\pi}}{b}e^{-2ab}.$$

Using this notation, we get

$$\int\limits_0^\infty \left(\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{-\infty}^{\delta} e^{-\xi^2/4t}\mathrm{d}\xi - \frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{-\infty}^{\delta} e^{\xi/2}e^{-\xi^2/4t}\mathrm{d}\xi \right) \frac{\mathrm{d}t}{t} =$$
$$\int\limits_{-\infty}^{0} \left(\frac{1}{\sqrt{4\pi}}K_{-1/2}(1/2,-\xi/2) - \frac{e^{\xi/2}}{\sqrt{4\pi}}K_{-1/2}(1/2,-\xi/2) \right) \mathrm{d}\xi +$$
$$\int\limits_{0}^{\delta} \left(\frac{1}{\sqrt{4\pi}}K_{-1/2}(1/2,\xi/2) - \frac{e^{\xi/2}}{\sqrt{4\pi}}K_{-1/2}(1/2,\xi/2) \right) \mathrm{d}\xi =$$
$$\int\limits_{-\infty}^{0} \frac{1}{\xi}\left(e^{\xi} - e^{\xi/2}\right)\mathrm{d}\xi + \int\limits_{0}^{\delta} \frac{1}{\xi}\left(e^{-\xi/2} - 1\right)\mathrm{d}\xi = \log(2) + \int\limits_{0}^{\delta} \frac{1}{\xi}\left(e^{-\xi/2} - 1\right)\mathrm{d}\xi\,.$$

For the remaining terms, meaning when $(n, m) \neq (1, 0)$, we can integrate term by term to get

$$\sum_{(n,m)\neq(1,0)} \int\limits_0^\infty \Bigg(\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} e^{(1-n/2-nm)\xi}e^{-(n\xi)^2/4t}\mathrm{d}\xi -$$
$$\frac{e^{-t/4}}{(4\pi t)^{1/2}} \int\limits_{\delta}^{\infty} e^{(1/2-n/2-nm)\xi}e^{-(n\xi)^2/4t}\mathrm{d}\xi \Bigg) \frac{\mathrm{d}t}{t} =$$
$$\sum_{(n,m)\neq(1,0)} \int\limits_{\delta}^{\infty} \Bigg(\frac{e^{(1-n/2-nm)\xi}}{\sqrt{4\pi}}K_{-1/2}(1/2,n\xi/2) -$$
$$\frac{e^{(1/2-n/2-nm)\xi}}{\sqrt{4\pi}}K_{-1/2}(1/2,n\xi/2) \Bigg) \mathrm{d}\xi =$$
$$\sum_{(n,m)\neq(1,0)} \int\limits_{\delta}^{\infty} \frac{1}{n\xi}\left(e^{(1-n-nm)\xi} - e^{(1/2-n-nm)\xi}\right)\mathrm{d}\xi\,.$$

Having explicitly evaluated these integrals, we now proceed to estimate the results. For the first case, we observe the trivial inequality

$$\log(2) + \int_0^\delta \frac{1}{\xi}\left(e^{-\xi/2} - 1\right) \mathrm{d}\xi = \log(2) - \int_0^\delta \frac{1}{\xi}\left(1 - e^{-\xi/2}\right) \mathrm{d}\xi \leq \log(2)\,. \tag{34}$$

For the second case, we first notice that for $n \geq 1$, $m \geq 0$, but $(n, m) \neq (1, 0)$, we have $n + nm \geq 2$, which leads to the trivial estimate

$$\left| \sum_{(n,m)\neq(1,0)} \int_\delta^\infty \frac{1}{n\xi}\left(e^{(1-n-nm)\xi} - e^{(1/2-n-nm)\xi}\right) \mathrm{d}\xi \right| \leq$$
$$2 \sum_{(n,m)\neq(1,0)} \int_\delta^\infty \frac{e^{(1-n-nm)\xi}}{n\xi} \mathrm{d}\xi \leq \frac{2e^\delta}{\delta} \sum_{(n,m)\neq(1,0)} \frac{e^{-n(m+1)\delta}}{n(n+nm-1)}\,.$$

In order to further estimate the latter sum, we break it up into three parts, the first one given by $n \geq 2$, $m = 0$, the second one by $n = 1$, $m \geq 1$, and the third one by $n \geq 2$, $m \geq 1$. For the first part, we have the upper bound

$$\frac{2e^\delta}{\delta} \sum_{n=2}^\infty \frac{e^{-n\delta}}{n(n-1)} \leq \frac{2e^{-\delta}}{\delta} \sum_{n=2}^\infty \frac{1}{n(n-1)} = \frac{2e^{-\delta}}{\delta} \leq \frac{2}{\delta}\,. \tag{35}$$

For the second part, we estimate

$$\frac{2e^\delta}{\delta} \sum_{m=1}^\infty \frac{e^{-(m+1)\delta}}{m} \leq \frac{2e^\delta}{\delta} e^{-\delta} \frac{e^{-\delta}}{1-e^{-\delta}} = \frac{2}{\delta} \cdot \frac{1}{e^\delta - 1} \leq \frac{2}{\delta^2}\,. \tag{36}$$

Using the inequality $nm - 1 \geq 1$, we estimate for the third part

$$\frac{2e^\delta}{\delta} \sum_{n=2}^\infty \sum_{m=1}^\infty \frac{e^{-n(m+1)\delta}}{n(n+nm-1)} \leq \frac{2e^\delta}{\delta} \sum_{n=2}^\infty \sum_{m=1}^\infty \frac{e^{-2(m+1)\delta}}{n(n+1)} = \tag{37}$$
$$\frac{2e^\delta}{\delta} \cdot \frac{1}{2} \sum_{m=1}^\infty e^{-2(m+1)\delta} = \frac{e^\delta}{\delta} e^{-2\delta} \frac{e^{-2\delta}}{1-e^{-2\delta}} = \frac{e^{-\delta}}{\delta} \cdot \frac{1}{e^{2\delta}-1} \leq \frac{e^{-\delta}}{2\delta^2} \leq \frac{1}{2\delta^2}\,.$$

Integrating (32) with respect to t from 0 to ∞ and taking into account the estimates (34), (35), (36), (37), we get the upper bound

$$\int_0^\infty \left(HTrK_{\mathrm{hyp}}(t) - 1 + e^{-t/4}\right) \frac{\mathrm{d}t}{t} \leq \tag{38}$$
$$\int_0^\infty \int_{e^\delta}^\infty f(u,t)\, \mathrm{d}\left[\pi_X(u) - \mathrm{li}(u) + \mathrm{li}(u^{1/2})\right] \frac{\mathrm{d}t}{t} + \frac{4\delta + 5}{2\delta^2} + \log(2)\,.$$

In order to further estimate the right-hand side of (38), we proceed as in the first part of the proof of Theorem 4.7 in [14] (see pp. 18–20). For this, we first note that a direct computation establishes the equality

$$F(u) := \int_0^\infty f(u,t)\frac{\mathrm{d}t}{t} = -\log\left(\prod_{n=0}^{\infty}(1-u^{-(n+1)})\right),$$

which shows that the function $F(u)$ is decreasing in u. We now apply Lemma 4.6 of [14] to the right-hand side of (38) with $\varepsilon \in (0,\alpha)$, where $\alpha = \min\{\lambda_{X,1}, 7/64\}$, and $\delta > 4$ to arrive at the upper bound

$$\int_0^\infty \int_{e^\delta}^\infty f(u,t)\,\mathrm{d}\left[\pi_X(u) - \mathrm{li}(u) + \mathrm{li}(u^{1/2})\right]\frac{\mathrm{d}t}{t} \leq \tag{39}$$

$$C'_X \int_{e^\delta}^\infty F(u)\mathrm{dli}(u^{s_\varepsilon}) + 2C'_X F(e^\delta)\mathrm{li}(e^{s_\varepsilon \delta}),$$

where $C'_X = C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)} + 1$ (see also the proof of Theorem 4.7 in [14]). Now, the inequality

$$-\log(1-v^{-1}) \leq \frac{v^{-1}}{1-e^{-\delta}},$$

which is valid for $v \geq e^\delta$, implies the upper bound

$$F(u) \leq \frac{1}{1-e^{-\delta}}\sum_{n=0}^{\infty} u^{-(n+1)} = \frac{1}{1-e^{-\delta}}\cdot\frac{1}{u-1} \leq \frac{2}{\delta(1-e^{-\delta})}\cdot\frac{\log(u)}{u},$$

where the last inequality holds since $\log(u) \geq \delta > 4$. (Note: Although the factor $\log(u)/\delta$ in the above bound can be eliminated by estimating $F(u)$ by other means, the presence of this factor is helpful in the subsequent computations.) Using the elementary inequality

$$\mathrm{li}(u) \leq \frac{2u}{\log(u)}$$

for $u > e^2$, we obtain

$$\frac{\delta}{e^\delta}\mathrm{li}(e^{s_\varepsilon \delta}) \leq \frac{2}{s_\varepsilon}e^{-(1-s_\varepsilon)\delta},$$

where $\varepsilon < 7/64$ and $\delta > 4$. We are now able to estimate the right-hand side of (39) as follows

$$
\begin{aligned}
(40)\qquad & C'_X \int_{e^\delta}^{\infty} F(u)\mathrm{dli}(u^{s_\varepsilon}) + 2C'_X F(e^\delta)\mathrm{li}(e^{s_\varepsilon\delta}) \leq \\
& \frac{2C'_X}{\delta(1-e^{-\delta})}\int_{e^\delta}^{\infty} \frac{\log(u)}{u}\mathrm{dli}(u^{s_\varepsilon}) + \frac{4C'_X}{\delta(1-e^{-\delta})}\frac{\delta}{e^\delta}\mathrm{li}(e^{s_\varepsilon\delta}) = \\
& \frac{2C'_X}{\delta(1-e^{-\delta})}\cdot\frac{e^{-(1-s_\varepsilon)\delta}}{1-s_\varepsilon} + \frac{4C'_X}{e^\delta-1}\mathrm{li}(e^{s_\varepsilon\delta}) \leq \\
& \frac{2C'_X}{\delta^2}\cdot\frac{s_\varepsilon e^{s_\varepsilon\delta}}{\varepsilon} + \frac{4C'_X}{e^\delta-1}\cdot\frac{2e^\delta}{s_\varepsilon\delta}e^{-(1-s_\varepsilon)\delta} \leq \\
& \frac{2C'_X e^{s_\varepsilon\delta}}{\delta^2}\left(\frac{s_\varepsilon}{\varepsilon}+\frac{4}{s_\varepsilon}\right) \leq \frac{2C'_X e^{s_\varepsilon\delta}}{\delta^2}\left(5+\frac{1}{\varepsilon}\right),
\end{aligned}
$$

Combining (33) with the estimates (38), (39), (40), we find the upper bound

$$
-\log(Z'_X(1)) \leq -\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<\delta}} \log(Z_\gamma(1)) + \frac{2C'_X e^{s_\varepsilon\delta}}{\delta^2}\left(5+\frac{1}{\varepsilon}\right) + \frac{4\delta+5}{2\delta^2} - \log(2)\,.
$$

Since we have assumed $\delta > 4$, we can choose $\delta = 5$, for simplicity. Observing $1/2 - \log(2) < 0$, and $2e^5/25 < 12$, we arrive at the claimed upper bound

$$
-\log(Z'_X(1)) \leq -\sum_{\substack{\gamma\in H(\Gamma)\\ \ell_\gamma<5}} \log(Z_\gamma(1)) + 12\left(5+\frac{1}{\varepsilon}\right)\left(C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)} + 1\right).
$$

□

4.4. **Lemma.** *With the above notations, we have the following results:*

(i) *For any $\gamma \in H(\Gamma)$ with $\ell_\gamma \in (0,5)$, we have*

$$
0 \leq -\log(Z_\gamma(1)) \leq \frac{\pi^2}{6\ell_\gamma}\,.
$$

(ii) *For any $\gamma \in H(\Gamma)$ with $\ell_\gamma > 0$, we have*

$$
0 \leq \frac{Z'_\gamma}{Z_\gamma}(1) \leq 3 + \log\left(\frac{1}{\ell_\gamma}\right).
$$

Proof. We start with the following observation. Consider the (up to scaling) unique cusp form of weight 12 with respect to $\mathrm{SL}_2(\mathbb{Z})$

$$
\Delta(z) = e^{2\pi i z}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n z}\right)^{24}
$$

for $z \in \mathbb{H}$; it has functional equation

$$\Delta(z) = (-z)^{-12}\Delta(-1/z).$$

Upon setting $z = -\ell_\gamma/(2\pi i)$, we have

$$Z_\gamma(1)^{24} = e^{\ell_\gamma}\Delta\left(-\frac{\ell_\gamma}{2\pi i}\right).$$

Using the functional equation for $\Delta(z)$, we then obtain the relation

$$Z_\gamma(1)^{24} = e^{\ell_\gamma}\left(\frac{\ell_\gamma}{2\pi i}\right)^{-12}\Delta\left(\frac{2\pi i}{\ell_\gamma}\right) = e^{\ell_\gamma}\left(\frac{\ell_\gamma}{2\pi}\right)^{-12} e^{-(2\pi)^2/\ell_\gamma}\prod_{n=1}^{\infty}\left(1 - e^{-(2\pi)^2 n/\ell_\gamma}\right)^{24}. \tag{41}$$

We now turn to the proof of the lemma.

(i) From the product formula for $Z_\gamma(1)$, it is immediate that $Z_\gamma(1) \leq 1$ for all $\ell_\gamma \geq 0$; hence, we get the lower bound $-\log(Z_\gamma(1)) \geq 0$. Concerning the upper bound, we derive form (41)

$$-\log(Z_\gamma(1)) = -\frac{\ell_\gamma}{24} + \frac{1}{2}\log\left(\frac{\ell_\gamma}{2\pi}\right) + \frac{\pi^2}{6\ell_\gamma} - \sum_{n=1}^{\infty}\log\left(1 - e^{-(2\pi)^2 n/\ell_\gamma}\right).$$

We now use the elementary inequality $-\log(1-x) \leq x/(1-\sigma)$, which holds whenever $x \in [0,\sigma]$, and take $\sigma = e^{-(2\pi)^2/\ell_\gamma}$ to get

$$-\sum_{n=1}^{\infty}\log\left(1 - e^{-(2\pi)^2 n/\ell_\gamma}\right) \leq \frac{1}{1 - e^{-(2\pi)^2/\ell_\gamma}}\sum_{n=1}^{\infty} e^{-(2\pi)^2 n/\ell_\gamma} = \frac{e^{(2\pi)^2/\ell_\gamma}}{(e^{(2\pi)^2/\ell_\gamma} - 1)^2}.$$

Letting $u = (2\pi)^2/\ell_\gamma$, the upper bound becomes

$$\frac{e^u}{(e^u - 1)^2} = \frac{1}{e^u - 1} + \frac{1}{(e^u - 1)^2},$$

which is clearly monotone decreasing in u and, hence, monotone increasing in ℓ_γ. Therefore, for $\ell_\gamma < 5$, we obtain

$$\frac{1}{2}\log\left(\frac{\ell_\gamma}{2\pi}\right) + \frac{e^{(2\pi)^2/\ell_\gamma}}{(e^{(2\pi)^2/\ell_\gamma} - 1)^2} \leq \frac{1}{2}\log\left(\frac{5}{2\pi}\right) + \frac{e^{(2\pi)^2/5}}{(e^{(2\pi)^2/5} - 1)^2} \leq 0,$$

where the last estimate is obtained numerically. With all this, part (i) is proved.

(ii) We begin by writing

$$\frac{Z_\gamma'}{Z_\gamma}(1) = \ell_\gamma\sum_{n=1}^{\infty}\frac{1}{e^{n\ell_\gamma} - 1}.$$

Let $N \geq 1$ be the smallest integer larger than or equal to $1/\ell_\gamma$, i.e., $N - 1 < 1/\ell_\gamma \leq N$. If $n \geq N$, then $n\ell_\gamma \geq 1$, hence, $e^{n\ell_\gamma} \geq 2$. Observing then $e^{n\ell_\gamma} - 1 \geq e^{n\ell_\gamma}/2$, we get

$$\ell_\gamma \sum_{n=N}^{\infty} \frac{1}{e^{n\ell_\gamma} - 1} \leq 2\ell_\gamma \sum_{n=N}^{\infty} e^{-n\ell_\gamma} = 2\ell_\gamma \frac{e^{-(N-1)\ell_\gamma}}{e^{\ell_\gamma} - 1} \leq \frac{2\ell_\gamma}{e^{\ell_\gamma} - 1} \leq 2\,.$$

For $1 \leq n < N$, we use the inequality $e^{n\ell_\gamma} - 1 \geq n\ell_\gamma$, which implies

$$\ell_\gamma \sum_{n=1}^{N-1} \frac{1}{e^{n\ell_\gamma} - 1} \leq \sum_{n=1}^{N-1} \frac{1}{n} \leq 1 + \log(N-1) \leq 1 + \log\left(\frac{1}{\ell_\gamma}\right),$$

from which part (ii) follows. □

4.5. **Theorem.** *For any X with genus $g_X > 1$, put*

$$h(X) = g_X + \frac{1}{\lambda_X}\left(g_X(d_{\mathrm{sup},X} + 1)^2 + C_{\mathrm{Hub},X} + N_{\mathrm{ev},X}^{[0,1/4)}\right) + \frac{1}{\ell_X} N_{\mathrm{geo},X}^{(0,5)},$$

with $\lambda_X = 1/2 \cdot \min\{\lambda_{X,1}, 7/64\}$ and ℓ_X equal to the length of the smallest geodesic on X. Then, we have the bound

$$\delta_{\mathrm{Fal}}(X) = O(h(X))$$

with an implied constant that is universal.

Proof. The result is a summary of the inequalities derived in this section, namely Propositions 4.1, 4.2, 4.3 and Lemma 4.4, when applied to Theorem 3.8, taking, for example, $\varepsilon = \lambda_X$ in Propositions 4.2 and 4.3. □

4.6. **Corollary.** *Let X_1 be a finite degree cover of the compact Riemann surface X_0 of genus $g_{X_0} > 1$. Then, we have the bound*

$$\delta_{\mathrm{Fal}}(X_1) = O_{X_0}\left(g_{X_1}\left(1 + \frac{1}{\lambda_{X_1,1}}\right)\right).$$

In particular, if $\{X_n\}_{n\geq 1}$ is a tower of finite degree covers of X_0 such that there exists a constant $c > 0$ satisfying $\lambda_{X_n,1} \geq c > 0$ for all $n \geq 1$, we have the bound

$$\delta_{\mathrm{Fal}}(X_n) = O_{X_0}(g_{X_n}).$$

Proof. We analyze the bound obtained in Theorem 4.5. The quantity $N_{\mathrm{ev},X_1}^{[0,1/4)}$ is known to have order $O(g_{X_1})$ with an implied constant that is universal (see [3], p. 211, or [33]). The main result in [6] states the bound $d_{\mathrm{sup},X_1} = O_{X_0}(1)$ (see also [16], [17], and [18] with related results). In [15], Theorem 3.4, it is shown that $C_{\mathrm{Hub},X_1} = O_{X_0}(g_{X_1})$. As discussed in the proof of Theorem 4.11

in [14], $N^{(0,5)}_{\mathrm{geo},X_1} = O_{X_0}(g_{X_1})$ (specifically, recall the definition of $r_{\Gamma_0,\Gamma}$ therein). Trivially, one has $\ell_{X_1} \geq \ell_{X_0}$. With all this, we have shown that

$$h(X) = O_{X_0}\left(g_{X_1} + \frac{g_{X_1}}{\lambda_{X_1}}\right).$$

By choosing $\lambda_{X_1} = 1/2 \cdot \min\{\lambda_{X_1,1}, 7/64\}$, the result follows. □

4.7. **Remark.** We view the results stated in Theorem 4.5 and Corollary 4.6 as complementing known theorems answering the asymptotic behavior of Faltings's delta function for a degenerating family of algebraic curves that approach the Deligne-Mumford boundary of the moduli space of stable curves of a fixed positive genus, as first proved in [13]. The expressions derived in [13] were well-suited to answer the question of the asymptotic behavior of $\delta_{\mathrm{Fal}}(X)$ through degeneration, but do not appear to allow one to bound $\delta_{\mathrm{Fal}}(X)$ in terms of more elementary information concerning X, as in Theorem 4.5 or Corollary 4.6. On the other hand, it is possible that the exact expression for $\delta_{\mathrm{Fal}}(X)$ in terms of hyperbolic geometry could be used to understand $\delta_{\mathrm{Fal}}(X)$ through degeneration. Indeed, stated results in [21] study c_X and $\log(Z'_X(1))$ through degeneration, so it would remain to adapt the analysis in [21] to study the integral which we bound in Proposition 4.1.

5. Applications to the modular curves $X_0(N)$

In this section we focus our attention to the sequence of modular curves $X_0(N)$. The purpose of this section is to bound the geometric quantities in Theorem 4.5 in more elementary terms in order to prove an analogue of Corollary 4.6 for the sequence of modular curves $X_0(N)$, which admit hyperbolic metrics. As stated earlier, the issue we encounter is that the set of modular curves $X_0(N)$ which admit hyperbolic metrics does not from a single tower of hyperbolic Riemann surfaces, hence the results cited in the proof of Corollary 4.6 do not apply. However, the family of hyperbolic modular curves forms a different structure, which we refer to as a "net". More specifically, there is a sequence of hyperbolic modular curves, which we parametrize by a set of integers $\mathcal{B}(p_0)$, and every hyperbolic modular curve is a finite degree cover of (possibly several) modular curves corresponding to elements of $\mathcal{B}(p_0)$. In effect, we bound the quantities in Theorem 4.5 by first obtaining uniform bounds for all modular curves that correspond to elements in $\mathcal{B}(p_0)$, after which we utilize bounds through covers by citing the results which prove Corollary 4.6.
In the following definition, $\mathbb{P}$ denotes the set of primes.

5.1. **Definition.** (i) We call $N \in \mathbb{N}$ *base hyperbolic*, if $g_{X_0(N)} > 1$ and if there exists no proper divisor N' of N with $g_{X_0(N')} > 1$.

(ii) For $p_0 \in \mathbb{P}$, set

$$\mathcal{B}_1(p_0) := \{N \text{ base hyperbolic} \,|\, N = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k},\, p_j \leq p_0 \ (j = 1, \ldots, k \in \mathbb{N})\}.$$

(iii) For $p_0 \in \mathbb{P}$ with $g_{X_0(p_0)} > 1$, set

$$\mathcal{B}_2(p_0) := \{p \in \mathbb{P} \,|\, p > p_0\}.$$

(iv) For $p_0 \in \mathbb{P}$ with $g_{X_0(p_0)} > 1$, set

$$\mathcal{B}(p_0) := \mathcal{B}_1(p_0) \cup \mathcal{B}_2(p_0).$$

5.2. **Remark.** (i) For instance, one can choose $p_0 = 23$.
(ii) The set $\mathcal{B}_1(p_0)$ is obviously finite.
(iii) For every $N \in \mathbb{N}$ with $g_{X_0(N)} > 1$, there exists either $N'|N$ with $N' \in \mathcal{B}_1(p_0)$ or $p|N$ with $p \in \mathcal{B}_2(p_0)$. In other words, one can state that for any $N \in \mathbb{N}$ with $g_{X_0(N)} > 1$, there exists $N' \in \mathcal{B}(p_0)$ such that $X_0(N)$ is a finite cover of $X_0(N')$.

5.3. **Proposition.** *Let $N > N_0$ be such that $X_0(N)$ has genus $g_{X_0(N)} > 1$. Then, we have the following:*

(a) *There is a constant $c_1 > 0$, independent of N, such that $\lambda_{X_0(N),1} \geq c_1$.*

(b) *There is a constant $c_2 > 0$, independent of N, such that $N^{[0,1/4)}_{\mathrm{ev},X_0(N)} \leq c_2 \cdot g_{X_0(N)}$.*

(c) *There is a constant $c_3 > 0$, independent of N, such that $\ell_{X_0(N)} \geq c_3$.*

(d) *There is a constant $c_4 > 0$, independent of N, such that $N^{[0,5)}_{\mathrm{geo},X_0(N)} \leq c_4 \cdot g_{X_0(N)}$.*

Proof. (a) In order to prove the first part of the claim, we recall from [2], Theorem 3.1, that

$$\liminf_{N\to\infty} \lambda_{X(N),1} \geq 5/36\,.$$

Hence, there is a constant $c_1 > 0$, independent of N, such that $\lambda_{X(N),1} \geq c_1$ for all $N > N_0$. Since $X(N)$ is a cover of $X_0(N)$, the Raleigh quotient method for estimating eigenvalues, which shows that the smallest eigenvalue decreases through covers, now implies that $\lambda_{X(N),1} \leq \lambda_{X_0(N),1}$. This completes the proof of (a).

(b) This part of the claim follows immediately by quoting the known universal lower bound for the number of small eigenvalues applied to the special case of the modular curves $X_0(N)$; in fact, one can choose $c_2 = 4$ (see [3], or [5], p. 251).

(c) In the subsequent proof, we let $X_0(N) \cong \Delta_0(N)\backslash\mathbb{H}$ with $\Delta_0(N)$ a torsionfree and cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Recall that $\pi_1(X_0(N)) \cong$

$\Delta_0(N)$ and that each homotopy class in $\pi_1(X_0(N))$ can be uniquely represented by a closed geodesic path on $X_0(N)$. Thus, we have a bijection between the elements $\gamma \in \Delta_0(N)$ and closed geodesic paths β on $X_0(N)$ (with a fixed initial point); note that the quantity ℓ_γ introduced in section 2.4 equals the length $\ell_{X_0(N)}(\beta)$ of β.
Let p_0 be as in Definition 5.1, and $p \in \mathcal{B}_2(p_0)$. The hyperbolic Riemann surface $X_0(p_0p)$ is a cover of $X_0(p)$ of degree p_0+1. Let now β be any closed geodesic path on $X_0(p)$ corresponding to $\gamma \in \Delta_0(p)$ of length $\ell_{X_0(p)}(\beta) = \ell_\gamma$. Then, there exists a minimal $d \in \mathbb{N}$, $1 \leq d \leq p_0+1$, such that $\gamma' = \gamma^d \in \Delta_0(p_0p)$. The element $\gamma' \in \Delta_0(p_0p)$ corresponds to a closed geodesic path β' on $X_0(p_0p)$ of length $\ell_{X_0(p_0p)}(\beta') = d \cdot \ell_{X_0(p)}(\beta)$.
On the other hand, $X_0(p_0p)$ is a finite cover of $X_0(p_0)$, hence $\Delta_0(p_0p)$ is a subgroup of $\Delta_0(p_0)$. Viewing $\gamma' \in \Delta_0(p_0p)$ as an element of $\Delta_0(p_0)$, we see that any closed geodesic path β' on $X_0(p_0p)$ descends to a closed geodesic path β'' on $X_0(p_0)$ of the same length. This proves the inequality

$$\ell_{X_0(p_0p)} \geq \ell_{X_0(p_0)}.$$

In particular, we find for any closed geodesic path β on $X_0(p)$ of length $\ell_{X_0(p)}(\beta)$ lifting to the closed geodesic path β' on $X_0(p_0p)$ of length $d \cdot \ell_{X_0(p)}(\beta)$ the estimate

$$\ell_{X_0(p)}(\beta) = \frac{\ell_{X_0(p_0p)}(\beta')}{d} \geq \frac{\ell_{X_0(p_0p)}(\beta')}{p_0+1} \geq \frac{\ell_{X_0(p_0p)}}{p_0+1} \geq \frac{\ell_{X_0(p_0)}}{p_0+1}\,.$$

Therefore, we have for any $p \in \mathcal{B}_2(p_0)$, the bound $\ell_{X_0(p)} \geq \ell_{X_0(p_0)}/(p_0+1)$. We now define

$$c_3 = \min_{N \in \mathcal{B}_1(p_0)} \{\ell_{X_0(N)}, \ell_{X_0(p_0)}/(p_0+1)\} \leq \inf_{N \in \mathcal{B}(p_0)} \{\ell_{X_0(N)}\}\,,$$

which depends solely on p_0. Since $\mathcal{B}_1(p_0)$ is finite, and $\ell_{X_0(N)}$ is positive for any $N \in \mathcal{B}_1(p_0)$, we conclude that c_3 is positive. Now, for any modular curve $X_0(N)$ with $g_{X_0(N)} > 1$, choose $N' \in \mathcal{B}(p_0)$, so that $X_0(N)$ is a finite cover of $X_0(N')$. Using the lower bound $\ell_{X_0(N)} \geq \ell_{X_0(N')}$, together with inequality $\ell_{X_0(N')} \geq c_3$ for $N' \in \mathcal{B}(p_0)$, we find that $\ell_{X_0(N)} \geq c_3$, which completes the proof of part (c).

(d) As in the proof of part (c), we let $X_0(N) \cong \Delta_0(N)\backslash\mathbb{H}$ with $\Delta_0(N)$ a torsionfree and cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Let p_0 be as in Definition 5.1, and $p \in \mathcal{B}_2(p_0)$. Recalling our notations given in section 2.4, we have

$$\begin{aligned}
&N^{[0,5)}_{\mathrm{geo},X_0(p)} = \#\{\gamma \in \Delta_0(p)\,|\,\gamma \in H(\Delta_0(p))\,,\,\ell_\gamma < 5\} = \\
&\#\{\gamma \in \Delta_0(p)\,|\,\gamma \text{ primitive, hyperbolic, } \ell_\gamma < 5\}/\Delta_0(p)\text{−conjugacy} \leq \\
&\#\{\gamma \in \Delta_0(p)\,|\,\gamma \text{ primitive, hyperbolic, } \ell_\gamma < 5\}/\Delta_0(p_0p)\text{−conjugacy.}
\end{aligned}$$

We introduce the sets

$$\mathcal{C}(p) = \{\gamma \in \Delta_0(p) \,|\, \gamma \text{ primitive, hyperbolic, } \ell_\gamma < 5\}/\Delta_0(p_0p)\text{−conjugacy},$$

$$\mathcal{C}'(p_0p) = \{\gamma' \in \Delta_0(p_0p) \,|\, \gamma' \text{ hyperbolic, } \ell_{\gamma'} < 5(p_0+1)\}/\Delta_0(p_0p)\text{−conjugacy}.$$

As in the proof of part (c), we find for any $\gamma \in \Delta_0(p)$ a minimal $d \in \mathbb{N}$, $1 \leq d \leq p_0+1$, such that $\gamma' = \gamma^d \in \Delta_0(p_0p)$; note that for $\gamma \in \Delta_0(p)$ with $\ell_\gamma < 5$, we have $\ell_{\gamma'} < 5d \leq 5(p_0+1)$. By associating the $\Delta_0(p_0p)$-conjugacy class of $\gamma \in \Delta_0(p)$ (γ primitive, hyperbolic, $\ell_\gamma < 5$) to the $\Delta_0(p_0p)$-conjugacy class of $\gamma' = \gamma^d \in \Delta_0(p_0p)$ (γ' hyperbolic, $\ell_{\gamma'} < 5(p_0+1)$), we obtain a well-defined map

$$\varphi : \mathcal{C}(p) \longrightarrow \mathcal{C}'(p_0p)\,.$$

Let now $[\gamma_1], [\gamma_2] \in \mathcal{C}(p)$ be such that $\varphi([\gamma_1]) = \varphi([\gamma_2])$, i.e., there exists $d_1, d_2 \in \mathbb{N}$, $1 \leq d_1, d_2 \leq p_0+1$, and $\delta \in \Delta_0(p_0p)$ such that

$$\gamma_1^{d_1} = \delta\gamma_2^{d_2}\delta^{-1}\,.$$

Since γ_1, γ_2 are hyperbolic elements, there exists $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\alpha\gamma_1^{d_1}\alpha^{-1} = \begin{pmatrix} e^{\ell} & 0 \\ 0 & e^{-\ell} \end{pmatrix} = \alpha\left(\delta\gamma_2^{d_2}\delta^{-1}\right)\alpha^{-1}$$

with $\ell \in \mathbb{R}_{>0}$, i.e., we have

$$\gamma_1 = \alpha^{-1}\begin{pmatrix} e^{\ell/d_1} & 0 \\ 0 & e^{-\ell/d_1} \end{pmatrix}\alpha\,, \quad \delta\gamma_2\delta^{-1} = \alpha^{-1}\begin{pmatrix} e^{\ell/d_2} & 0 \\ 0 & e^{-\ell/d_2} \end{pmatrix}\alpha\,.$$

This shows that γ_1 and $\delta\gamma_2\delta^{-1}$ commute in $\Delta_0(p)$, i.e.,

$$\delta\gamma_2\delta^{-1} \in \mathrm{Cent}_{\Delta_0(p)}(\gamma_1)\,.$$

Since γ_1 is primitive, it generates its own centralizer, i.e., $\delta\gamma_2\delta^{-1} = \gamma_1^n$ with $n \in \mathbb{Z}$. But since $\delta\gamma_2\delta^{-1}$ is also primitive, we must have $n = \pm 1$. This proves $[\gamma_1] = [\gamma_2^{\pm 1}]$, i.e., the map φ is two-to-one. From this we immediately deduce the estimate

$$N^{[0,5)}_{\mathrm{geo},X_0(p)} \leq \#\mathcal{C}(p) \leq 2 \cdot \#\mathcal{C}'(p_0p)$$

for all $p \in \mathcal{B}_2(p_0)$. Introducing the set

$$\mathcal{C}''(p_0) = \{\gamma'' \in \Delta_0(p_0) \,|\, \gamma'' \text{ hyperbolic, } \ell_{\gamma''} < 5(p_0+1)\}/\Delta_0(p_0)\text{−conjugacy},$$

we have the obvious map

$$\varphi' : \mathcal{C}'(p_0p) \longrightarrow \mathcal{C}''(p_0)$$

given by associating the $\Delta_0(p_0p)$-conjugacy class of $\gamma' \in \Delta_0(p_0p)$ (γ' hyperbolic, $\ell_{\gamma'} < 5(p_0+1)$) to the $\Delta_0(p_0)$-conjugacy class of γ' viewed as an element of $\Delta_0(p_0)$. Since $[\Delta_0(p_0) : \Delta_0(p_0p)] = p+1$, at most $(p+1)$ $\Delta_0(p_0p)$-conjugacy classes collapse to a single $\Delta_0(p_0)$-conjugacy class, i.e., φ' maps at most $p+1$

elements of $\mathcal{C}'(p_0p)$ to the same element of $\mathcal{C}''(p_0)$. Therefore, we obtain the estimate

$$N^{[0,5)}_{\mathrm{geo},X_0(p)} \leq 2 \cdot \#\mathcal{C}'(p_0p) \leq 2(p+1) \cdot \#\mathcal{C}''(p_0)\,.$$

Since the set $\mathcal{C}''(p_0)$ depends solely on p_0 and since the set $\mathcal{B}_1(p_0)$ is finite, we arrive at the bound

$$N^{[0,5)}_{\mathrm{geo},X_0(N)} = O\left(g_{X_0(N)}\right)$$

for any $N \in \mathcal{B}(p_0)$, with an implied constant that depends solely on p_0. Finally, in general, and in particular for $N \in \mathcal{B}(p_0)$, it is well-known (see, e.g., [9], p. 45) that

$$\#\{\gamma \in \Delta_0(N)\,|\,\gamma \text{ hyperbolic, } \ell_\gamma < 5\}/\Delta_0(N)\text{−conjugacy} = \sum_{n=1}^{\infty} N^{[0,5/n)}_{\mathrm{geo},X_0(N)}\,.$$

But from part (c), we know that $N^{[0,5/n)}_{\mathrm{geo},X_0(N)} = 0$ provided $5/n < c_3$, i.e., we have $n \leq 5/c_3$ in the above sum. Therefore, we find

$$\begin{aligned} &\#\{\gamma \in \Delta_0(N)\,|\,\gamma \text{ hyperbolic, } \ell_\gamma < 5\}/\Delta_0(N)\text{−conjugacy} \leq \\ &\left\lceil \frac{5}{c_3} \right\rceil \cdot N^{[0,5)}_{\mathrm{geo},X_0(N)} = O\left(g_{X_0(N)}\right) \end{aligned} \tag{42}$$

for any $N \in \mathcal{B}(p_0)$, with an implied constant that depends solely on p_0.
In order to complete the proof of part (d), let now $X_0(N)$ be any modular curve with $g_{X_0(N)} > 1$. By definition, we have

$$\begin{aligned} &N^{[0,5)}_{\mathrm{geo},X_0(N)} = \\ &\#\{\gamma \in \Delta_0(N)\,|\,\gamma \text{ primitive, hyperbolic, } \ell_\gamma < 5\}/\Delta_0(N)\text{−conjugacy.} \end{aligned}$$

Given N, choose $N' \in \mathcal{B}(p_0)$, so that $X_0(N)$ is a finite cover of $X_0(N')$. We then associate the $\Delta_0(N)$-conjugacy class of $\gamma \in \Delta_0(N)$ (γ primitive, hyperbolic, $\ell_\gamma < 5$) to the $\Delta_0(N')$-conjugacy class of γ viewed as an element of $\Delta_0(N')$. Since at most $\deg(X_0(N)/X_0(N'))$ $\Delta_0(N)$-conjugacy classes collapse to a single $\Delta_0(N')$-conjugacy class, we find by arguing as before

$$\begin{aligned} &N^{[0,5)}_{\mathrm{geo},X_0(N)} \leq \deg(X_0(N)/X_0(N')) \\ &\times \#\{\gamma' \in \Delta_0(N')\,|\,\gamma' \text{ hyperbolic, } \ell_{\gamma'} < 5\}/\Delta_0(N')\text{−conjugacy.} \end{aligned}$$

By equation (42), we conclude

$$N^{[0,5)}_{\mathrm{geo},X_0(N)} = \deg(X_0(N)/X_0(N')) \cdot O\left(g_{X_0(N')}\right),$$

where the implied constant depends solely on p_0. Since $\deg(X_0(N)/X_0(N')) \cdot g_{X_0(N')} = O(g_{X_0(N)})$ with an implied constant which is universal, the proof of part (d) is complete. □

5.4. **Proposition.** *Let $N > N_0$ be such that $X_0(N)$ has genus $g_{X_0(N)} > 1$. Then, we have the bound*

$$d_{\sup,X_0(N)} = O(1)\,,$$

where the implied constant is independent of N.

Proof. For $n \in \mathbb{N}$, let $Y_0(n) = \Gamma_0(n)\backslash\mathbb{H}$, so that $X_0(n)$ is (isomorphic to) the compactification of $Y_0(n)$ by adding the cusps and re-uniformizing at the elliptic fixed points. If n_1 is a divisor of n_2, denote by $\pi_{n_2,n_1} : X_0(n_2) \longrightarrow X_0(n_1)$ the natural projection. For $0 < \varepsilon < 1$, let

$$B(\varepsilon) = \{w \in \mathbb{C} \mid |w| < \varepsilon\}$$

be equipped with the complete hyperbolic metric

$$\mu_{\mathrm{hyp},B(\varepsilon)}(w) = \frac{i}{2} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2}\,.$$

Denote by $X_0'(1)$ the Riemann surface obtained from $X_0(1)$ by removing neighborhoods centered at the three points corresponding to the unique cusp and the two elliptic fixed points of $Y_0(1)$, respectively. Let $X_0'(N) = \pi_{N,1}^{-1}(X_0'(1))$; we may assume that

$$X_0'(N) = X_0(N) \setminus \bigcup_{k=1}^{s} U_k$$

such that the neighborhoods U_k are isometric to the complex disc $B(\varepsilon)$.
In course of this proof, we will use the hyperbolic metric on $X_0(N)$, resp. on $Y_0(N)$; we will distinguish them by denoting them by $\mu_{\mathrm{hyp},X_0(N)}$, resp. $\mu_{\mathrm{hyp},Y_0(N)}$ (which is slightly different from our previous notation, but will be used in this proof alone). For $x \in \bigcup_{k=1}^{s} U_k$, we now have

$$\mu_{\mathrm{hyp},X_0(N)}(x) \geq \frac{i}{2}\,\mathrm{d}z(x) \wedge \mathrm{d}\bar{z}(x)\,,$$

which leads to the estimate

$$\frac{g_{X_0(N)} \cdot \mu_{\mathrm{can},X_0(N)}(x)}{\mu_{\mathrm{hyp},X_0(N)}(x)} \leq \sum_{j=1}^{g_{X_0(N)}} |f_j(z(x))|^2\,.$$

Since the functions $f_j(z(x))$ ($j = 1, \ldots, g_{X_0(N)}$) are bounded and holomorphic on the neighborhoods U_k ($k = 1, \ldots, s$), the functions $|f_j(z(x))|^2$ are subharmonic on U_k, as is the sum of these functions (see, e.g., [27], p. 362). By the strong maximum principle for subharmonic functions (see, e.g., [8], p. 15, Theorem 2.2), we then have for $k = 1, \ldots, s$

$$\sup_{x \in U_k} \left(\sum_{j=1}^{g_{X_0(N)}} |f_j(z(x))|^2 \right) \leq \sup_{x \in \partial U_k} \left(\sum_{j=1}^{g_{X_0(N)}} |f_j(z(x))|^2 \right).$$

In the given local coordinate, the conformal factor for the hyperbolic metric is constant on ∂U_k, thus we have shown that

$$\sup_{x\in U_k}\left(\frac{g_{X_0(N)}\cdot\mu_{\mathrm{can},X_0(N)}(x)}{\mu_{\mathrm{hyp},X_0(N)}(x)}\right)=O_\varepsilon\left(\sup_{x\in\partial U_k}\left(\frac{g_{X_0(N)}\cdot\mu_{\mathrm{can},X_0(N)}(x)}{\mu_{\mathrm{hyp},X_0(N)}(x)}\right)\right).$$

Therefore, in order to prove the proposition, it suffices to show

$$\sup_{x\in X_0'(N)}\left(\frac{g_{X_0(N)}\cdot\mu_{\mathrm{can},X_0(N)}(x)}{\mu_{\mathrm{hyp},X_0(N)}(x)}\right)=O(1)$$

with an implied constant that is independent of N. Recalling that $\mu_{\mathrm{can},X_0(N)}$ on $X_0'(N)$ equals $\mu_{\mathrm{can},Y_0(N)}$ on $Y_0'(N)=Y_0(N)\setminus\bigcup_{k=1}^s U_k$, we can consider the formal identity

$$\frac{g_{X_0(N)}\cdot\mu_{\mathrm{can},X_0(N)}(x)}{\mu_{\mathrm{hyp},X_0(N)}(x)}=\frac{g_{X_0(N)}\cdot\mu_{\mathrm{can},Y_0(N)}(x)}{\mu_{\mathrm{hyp},Y_0(N)}(x)}\cdot\frac{\mu_{\mathrm{hyp},Y_0(N)}(x)}{\mu_{\mathrm{hyp},X_0(N)}(x)}\tag{43}$$

on the set $X_0'(N)=Y_0'(N)$. The argument given in [6], [16], or [17] proves a sup-norm bound for the ratio of the canonical metric by the hyperbolic metric through compact covers; however, the argument is adapted easily to towers of non-compact surfaces when restricting attention to compact subsets, such as the subsets $Y_0'(N)$. Thus, the first factor on the right-hand side of (43) is bounded through covers, with a bound depending solely on the base $Y_0(1)$, i.e., independent of N. As for the second factor on the right-hand side of (43), we argue as follows. Put

$$F(N)=\sup_{x\in Y_0'(N)}\frac{\mu_{\mathrm{hyp},Y_0'(N)}(x)}{\mu_{\mathrm{hyp},X_0'(N)}(x)},$$

where $\mu_{\mathrm{hyp},X_0'(N)}=\mu_{\mathrm{hyp},X_0(N)}|_{X_0'(N)}$, and $\mu_{\mathrm{hyp},Y_0'(N)}=\mu_{\mathrm{hyp},Y_0(N)}|_{Y_0'(N)}$. The quantity $F(N)$ is easily shown to be finite, since $\mu_{\mathrm{hyp},X_0(N)}$ is non-vanishing everywhere on the compact Riemann surface $X_0(N)$, and $\mu_{\mathrm{hyp},Y_0(N)}$ is non-vanishing on $Y_0(N)$ and decaying at the cusps of $Y_0(N)$. Let then p_0 be as in Definition 5.1, and $p\in\mathcal{B}_2(p_0)$. Since $X_0'(p_0p)$ is an unramified cover of $X_0'(p)$, resp. $Y_0'(p_0p)$ is an unramified cover of $Y_0'(p)$, we have (denoting both covering maps by $\pi'_{p_0p,p}$)

$$\pi'^{*}_{p_0p,p}\left(\mu_{\mathrm{hyp},X_0'(p)}\right)=\mu_{\mathrm{hyp},X_0'(p_0p)},\ \text{resp. }\pi'^{*}_{p_0p,p}\left(\mu_{\mathrm{hyp},Y_0'(p)}\right)=\mu_{\mathrm{hyp},Y_0'(p_0p)},$$

hence $F(p_0p)=F(p)$ for all $p\in\mathcal{B}_2(p_0)$. Symmetrically, $X_0'(p_0p)$ is an unramified cover of $X_0'(p_0)$, resp. $Y_0'(p_0p)$ is an unramified cover of $Y_0'(p_0)$, which analoguously implies (denoting both covering maps by π'_{p_0p,p_0})

$$\pi'^{*}_{p_0p,p_0}\left(\mu_{\mathrm{hyp},X_0'(p_0)}\right)=\mu_{\mathrm{hyp},X_0'(p_0p)},\ \text{resp. }\pi'^{*}_{p_0p,p_0}\left(\mu_{\mathrm{hyp},Y_0'(p_0)}\right)=\mu_{\mathrm{hyp},Y_0'(p_0p)},$$

hence $F(p_0p)=F(p_0)$ for all $p\in\mathcal{B}_2(p_0)$. Summarizing, we have $F(p)=F(p_0)$ for all $p\in\mathcal{B}_2(p_0)$. Since the set $\mathcal{B}_1(p_0)$ is finite, we have

$$c=\sup_{N\in\mathcal{B}(p_0)}\{F(N)\}=\sup_{N\in\mathcal{B}_1(p_0)}\{F(N),F(p_0)\}<\infty,$$

which just depends on p_0. It remains to bound $F(N)$ for any N such that $X_0(N)$ is a modular curve with $g_{X_0(N)} > 1$. Given such an N, choose $N' \in \mathcal{B}(p_0)$, so that $X_0(N)$ is a finite cover of $X_0(N')$. Noting that $X_0'(N)$, resp. $Y_0'(N)$ are unramified covers of $X_0'(N')$, resp. $Y_0'(N')$ of the same degree, we show as above that $F(N) = F(N')$. Since $F(N') \leq c$, we find $F(N) \leq c$ with c depending solely on p_0, hence being independent of N. This completes the proof of the proposition. □

5.5. **Proposition.** *Let $N > N_0$ be such that $X_0(N)$ has genus $g_{X_0(N)} > 1$. Then, we have*

$$C_{\mathrm{Hub},X_0(N)} = O(g_{X_0(N)})\,,$$

where the implied constant is universal, i.e., independent of N.

Proof. Before entering into the proof we begin with the following general observation. Let X_1 be a finite isometric cover of the compact Riemann surface X_0 of genus $g_{X_0} > 1$. As usual, if $\lambda_{X_1,j}$ is an eigenvalue for the hyperbolic Laplacian on X_1 satisfying $\lambda_{X_1,j} \geq 1/4$, we write $\lambda_{X_1,j} = 1/4 + r_{X_1,j}^2$ with $r_{X_1,j} \geq 0$. For $r \geq 0$, we put

$$N_{X_1}(r) = \#\{r_{X_1,j} \mid 0 \leq r_{X_1,j} \leq r\}.$$

Similarly, we can define $N_{X_0,\psi}(r)$, if ψ is a finite dimensional, unitary representation of the fundamental group $\pi_1(X_0)$ of X_0. From [32], Theorem 6.2.2 (see also [15], Lemma 3.2 (e)), we recall that the system of functions $N_{X_1}(r)$ and $\{N_{X_0,\psi}(r)\}$ satisfies the additive Artin formalism, i.e.,

$$N_{X_1}(r) = \sum_{\psi} \mathrm{mult}(\psi) \cdot N_{X_0,\psi}(r)\,,$$

where the sum is taken over all irreducible representations ψ occurring with multiplicity $\mathrm{mult}(\psi)$ in the representation $\mathrm{ind}_{\pi_1(X_1)}^{\pi_1(X_0)}(\mathbf{1})$.

After these preliminary remarks, we start with the proof of Proposition 5.5. For this, we let p_0 be as in Definition 5.1, and $p \in \mathcal{B}_2(p_0)$. Since $X_0(p_0p)$ is a finite isometric cover of $X_0(p_0)$, we have by the additive Artin formalism

$$N_{X_0(p_0p)}(r) = \sum_{\psi} \mathrm{mult}(\psi) \cdot N_{X_0(p_0),\psi}(r)\,.$$

Now, by [15], Lemma 3.3, there is a constant A_{p_0} depending solely on p_0 such that

$$|N_{X_0(p_0),\psi}(r)| \leq A_{p_0} \cdot \mathrm{rk}(\psi) \cdot r^2\,.$$

Using the relation $\sum_{\psi} \mathrm{mult}(\psi) \cdot \mathrm{rk}(\psi) = \deg(X_0(p_0p)/X_0(p_0)) = p+1$, we find

$$N_{X_0(p_0p)}(r) \leq A_{p_0} \sum_{\psi} \mathrm{mult}(\psi) \cdot \mathrm{rk}(\psi) \cdot r^2 = A_{p_0} \cdot (p+1) \cdot r^2\,.$$

On the other hand, viewing $X_0(p_0p)$ as a finite isometric cover of $X_0(p)$, we get the trivial estimate

$$N_{X_0(p)}(r) \leq N_{X_0(p_0p)}(r)\,,$$

since every eigenfunction on $X_0(p)$ lifts to an eigenfunction on $X_0(p_0p)$ with the same eigenvalue. Combining the last two inequalities yields the crucial bound

$$N_{X_0(p)}(r) \leq A_{p_0} \cdot (p+1) \cdot r^2\,. \tag{44}$$

We now discuss how the bound (44) leads to a bound of the Huber constant $C_{\mathrm{Hub},X_0(p)}$ for $p \in \mathcal{B}_2(p_0)$. For this we analyze the proof of the prime geodesic theorem on $X_0(p)$ as given in [5], pp. 295–300, which we now review.

Let $G(T) = \pi_{X_0(p)}(u)$ with $T = \log(u)$ be the prime geodesic counting function. Let $\varphi(x)$ be a non-negative C^∞-function with support on $[-1,+1]$ with L^1-norm equal to one. Let $\varepsilon > 0$, to be chosen later, let $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$, and let $I_T(x)$ be the indicator function of $[-T,+T]$. We define

$$g_T^\varepsilon(x) = 2\cosh(x/2)(I_T * \varphi_\varepsilon)(x),$$

which is a valid test function for the Selberg trace formula whose Fourier transform is denoted by $h_T^\varepsilon(r)$. If we define

$$H_\varepsilon(T) = \sum_{\gamma\in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell_\gamma}{e^{n\ell_\gamma/2} - e^{-n\ell_\gamma/2}}\, g_T^\varepsilon(\ell_\gamma),$$

the Selberg trace formula yields

$$H_\varepsilon(T) = \sum_{0\leq\lambda_{X_0(p),j}<1/4} h_T^\varepsilon(s_{X_0(p),j}) + \int_0^\infty h_T^\varepsilon(r)\,\mathrm{d}N_{X_0(p)}(r)\,. \tag{45}$$

By taking $\varepsilon = e^{-T/4}$, it is shown on p. 298 in [5] that

$$h_T^\varepsilon(s_{X_0(p),j}) = E_T(s_{X_0(p),j}) + O\left(\varepsilon \cdot e^{s_{X_0(p),j}T}\right),$$

where $E_T(x) = e^{Tx}/x$.

Since $1/2 < s_{X_0(p),j} \leq 1$, and $N^{[0,1/4)}_{\mathrm{ev},X_0(p)} = O(g_{X_0(p)}) = O(p+1)$ by Proposition 5.3 (b), this leads to

$$\sum_{0\leq\lambda_{X_0(p),j}<1/4} h_T^\varepsilon(s_{X_0(p),j}) = \sum_{0\leq\lambda_{X_0(p),j}<1/4} E_T(s_{X_0(p),j}) + (p+1)\cdot O(e^{3T/4})\,, \tag{46}$$

where the implied constant is universal. Continuing with the argument on p. 299, together with our bound (44), we find that

$$\int_0^\infty h_T^\varepsilon(r)\,\mathrm{d}N_{X_0(p)}(r) = (p+1)\cdot O_{p_0}(e^{3T/4})\,, \tag{47}$$

where the implied constant depends solely on p_0. Substituting (46) and (47) into (45) yields

$$H_\varepsilon(T) = \sum_{0\leq\lambda_{X_0(p),j}<1/4} E_T(s_{X_0(p),j}) + (p+1)\cdot O_{p_0}(e^{3T/4})\,,$$

where the implied constant depends solely on p_0.

Let

$$H(T) = \sum_{\substack{\gamma\in H(\Gamma),n\geq 1\\ n\ell_\gamma\leq T}} \frac{\ell_\gamma}{e^{n\ell_\gamma/2}-e^{-n\ell_\gamma/2}}\,,$$

one has

$$H_\varepsilon(T-\varepsilon)\leq H(T)\leq H_\varepsilon(T+\varepsilon),$$

which follows easily from the definition of $g_T^\varepsilon(x)$. Using these bounds together with the elementary estimates

$$E_{T\pm\varepsilon}(s_{X_0(p),j}) = E_T(s_{X_0(p),j}) + O(e^{3T/4})\,,$$

we get

$$\sum_{0\leq\lambda_{X_0(p),j}<1/4} E_{T\pm\varepsilon}(s_{X_0(p),j}) = \sum_{0\leq\lambda_{X_0(p),j}<1/4} E_T(s_{X_0(p),j}) + N^{[0,1/4)}_{\mathrm{ev},X_0(p)}\cdot O(e^{3T/4})\,,$$

where the implied constant is universal. Using Proposition 5.3 (b) again, we arrive at the bound

$$H(T) = \sum_{0\leq\lambda_{X_0(p),j}<1/4} E_T(s_{X_0(p),j}) + (p+1)\cdot O_{p_0}(e^{3T/4})\,, \tag{48}$$

where the implied constant depends solely on p_0.

The prime geodesic theorem, i.e., the asymptotic behavior of the function $G(T)$, can now be derived applying standard methods from (48) (see, e.g., [5], pp. 296–297, for a detailed proof). In order to arrive at the assertion

$$\pi_{X_0(p)}(u) - \sum_{0\leq\lambda_{X_0(p),j}<1/4} \mathrm{li}(u^{s_{X_0(p),j}}) = (p+1)\cdot O_{p_0}\left(u^{3/4}(\log(u))^{-1}\right),$$

one needs to also use Proposition 5.3 (b) in the derivation of the asymptotics of $G(T)$ from (48). Finally, since $u^{3/4}(\log(u))^{-1}\leq u^{3/4}(\log(u))^{-1/2}$, we conclude that

$$C_{\mathrm{Hub},X_0(p)} = O(p+1) = O(g_{X_0(p)})$$

for any $p\in\mathcal{B}_2(p_0)$, with an implied constant that depends solely on p_0. Since the set $\mathcal{B}_1(p_0)$ is finite, we end up with the estimate

$$C_{\mathrm{Hub},X_0(N)} = O(g_{X_0(N)})$$

for any $N\in\mathcal{B}(p_0)$, again with an implied constant that depends solely on p_0.

Finally, given any modular curve $X_0(N)$ with $g_{X_0(N)} > 1$, choose $N' \in \mathcal{B}(p_0)$, so that $X_0(N)$ is a finite cover of $X_0(N')$. Then, inequality (15) states that

$$C_{\mathrm{Hub},X_0(N)} \leq \deg(X_0(N)/X_0(N')) \cdot C_{\mathrm{Hub},X_0(N')} \,.$$

Since we have shown above that $C_{\mathrm{Hub},X_0(N')} = O(g_{X_0(N')})$ with an implied constant that depends solely on p_0, and since $\deg(X_0(N)/X_0(N')) \cdot g_{X_0(N')} = O(g_{X_0(N)})$ with an implied constant that is universal, the proof of the proposition is now complete. □

5.6. **Theorem.** *Let $N > N_0$ be such that $X_0(N)$ has genus $g_{X_0(N)} > 1$. Then, we have*

$$\delta_{\mathrm{Fal}}(X_0(N)) = O(g_{X_0(N)})\,,$$

where the implied constant is universal, i.e., independent of N.

Proof. Beginning with Theorem 4.5, we follow the method of proof of Corollary 4.6 by citing results from the present section, namely Propositions 5.3, 5.4, and 5.5 to bound the six geometric invariants, aside from the genus $g_{X_0(N)}$ appearing in Theorem 4.5. □

5.7. **Remark.** In the finite number of cases when $X_0(N)$ is not hyperbolic, Faltings's delta function $\delta_{\mathrm{Fal}}(X_0(N))$ can be explicitly evaluated. If $X_0(N)$ has genus zero, then Faltings's delta function is simply a universal constant. If $X_0(N)$ has genus one, then Faltings's delta function is expressed in terms of the Dedekind delta function, the unique holomorphic cusp form of weight 12 with respect to $\mathrm{PSL}_2(\mathbb{Z})$ (see [7]).

5.8. **Remark.** The analysis carried out in the present section applies to establish Theorem 5.6 for other families of modular curves, namely $\{X_1(N)\}$ and $\{X(N)\}$.

6. Arithmetic implications

6.1. **Faltings height of the Jacobian of $X_0(N)$.** In this section, we let N be a squarefree natural number such that 2, 3 do not divide N. We then let $\mathcal{X}_0(N)/\mathbb{Z}$ denote a minimal regular model of the modular curve $X_0(N)/\mathbb{Q}$. In [1], A. Abbes and E. Ullmo, computed the arithmetic self-intersection number of the relative dualizing sheaf $\overline{\omega}_{\mathcal{X}_0(N)}$ on $\mathcal{X}_0(N)$ equipped with the Arakelov metric. They came up with the following upper bound (see [1], Théorème B,

p. 3)

$$\overline{\omega}^2_{\mathcal{X}_0(N)} \leq -8\pi \cdot \frac{g_{X_0(N)}-1}{\mathrm{vol}_{\mathrm{hyp}}(X_0(N))} \cdot \lim_{s\to 1}\left(\frac{Z'_{\Gamma_0(N)\backslash\mathbb{H}}}{Z_{\Gamma_0(N)\backslash\mathbb{H}}}(s) - \frac{1}{s-1}\right) +$$

$$g_{X_0(N)} \sum_{p|N} \frac{p+1}{p-1}\log(p) + 2g_{X_0(N)}\log(N) + o(g_{X_0(N)}\log(N))\,.$$

Using [24], Corollaire 1.4, p. 649 (see also [14], section 5.3), in combination with a corresponding lower bound for $\overline{\omega}^2_{\mathcal{X}_0(N)}$ (see [1], Proposition C), one then finds

$$\overline{\omega}^2_{\mathcal{X}_0(N)} = 3g_{X_0(N)}\log(N) + o(g_{X_0(N)}\log(N))\,. \tag{49}$$

Using Noether's formula, one obtains the following formula for the Faltings height $h_{\mathrm{Fal}}(J_0(N))$ of the Jacobian $J_0(N)/\mathbb{Q}$ of the modular curve $X_0(N)$

$$12\cdot h_{\mathrm{Fal}}(J_0(N)) = \overline{\omega}^2_{\mathcal{X}_0(N)} + \sum_{p|N}\delta_p\log(p) + \delta_{\mathrm{Fal}}(X_0(N)) - 4g_{X_0(N)}\log(2\pi)\,; \tag{50}$$

here δ_p denotes the number of singular points in the special fiber of $\mathcal{X}_0(N)$ over $\mathbb{F}_p$. This leads to the following asymptotic behavior of the Faltings height of the Jacobian of $X_0(N)$.

6.2. **Theorem.** *With the above notations, we have*

$$h_{\mathrm{Fal}}(J_0(N)) = \frac{g_{X_0(N)}}{3}\log(N) + o(g_{X_0(N)}\log(N))\,.$$

Proof. The claim follows immediately from (50) using (49) and Theorem 5.6. □

6.3. **Remark.** If $E/\mathbb{Q}$ is a semi-stable elliptic curve of conductor N, one conjectures (see also [31], Conjecture 1.4)

$$h_{\mathrm{Fal}}(E) \leq a\cdot\frac{h_{\mathrm{Fal}}(J_0(N))}{g_{X_0(N)}} \tag{51}$$

with an absolute constant $a > 0$. Assuming the validity of the conjectured inequality (51) with constant $a = 3/2$, one can derive Szpiro's conjecture by means of Theorem 6.2 as in [31], i.e.,

$$\Delta_E \leq c(\varepsilon)\cdot N^{6+\varepsilon}$$

for the minimal discriminant Δ_E of E (note that in [31] it was speculated that one could take the value 1 for the constant a).

6.4. **Congruences of modular forms.** We start by mentioning that Theorem 5.6 improves the bounds for $\delta_{\mathrm{Fal}}(X_0(N))$ given in [31], Corollaire 1.3, namely

$$-4g_{X_0(N)}\log(N)+o(g_{X_0(N)}\log(N))\leq\delta_{\mathrm{Fal}}(X_0(N))\leq$$
$$2g_{X_0(N)}\log(N)+o(g_{X_0(N)}\log(N))\,. \tag{52}$$

Furthermore, Theorem 6.2 improves the bounds for the Faltings height of the Jacobian of $X_0(N)$ given in [31], Théorème 1.2, namely

$$-Bg_{X_0(N)}\leq h_{\mathrm{Fal}}(J_0(N))\leq\frac{g_{X_0(N)}}{2}\log(N)+o(g_{X_0(N)}\log(N))\,; \tag{53}$$

here $B>0$ is an absolute constant. The latter upper bound was obtained by means of the formula (see [31], Théorème 1.1)

$$h_{\mathrm{Fal}}(J_0(N))=\frac{1}{2}\log|\delta_{\mathbb{T}}|-\frac{1}{2}\log|\det(M_N)|-\log(\alpha)\,, \tag{54}$$

in which the Faltings height of the Jacobian of $X_0(N)$ is expressed in terms of a suitably defined discriminant $\delta_{\mathbb{T}}$ of the Hecke algebra $\mathbb{T}$ of $J_0(N)$, the matrix M_N of all possible Petersson inner products of a certain basis of eigenforms of weight 2 for $\Gamma_0(N)$, and a suitable natural number α with support contained in the support of $2N$. In order to obtain the upper bound in (53), E. Ullmo established the bounds

$$\begin{aligned}\log|\delta_{\mathbb{T}}| &\leq 2g_{X_0(N)}\log(N)+o(g_{X_0(N)}\log(N))\,,\\ -\log|\det(M_N)| &\leq -g_{X_0(N)}\log(N)+o(g_{X_0(N)}\log(N))\,.\end{aligned}$$

The lower bound in (53) is due to unpublished work of J.-B. Bost. Combining equation (50) with the asymptotics (49) and the estimates (53), one immediately derives the bounds (52) for $\delta_{\mathrm{Fal}}(X_0(N))$.

6.5. **Theorem.** *With the above notations, we have*

$$\log|\delta_{\mathbb{T}}|\geq\frac{5}{3}g_{X_0(N)}\log(N)+o(g_{X_0(N)}\log(N))\,. \tag{55}$$

Proof. Using (54) in combination with Theorem 6.2, we get

$$\frac{1}{2}\log|\delta_{\mathbb{T}}|-\frac{1}{2}\log|\det(M_N)|-\log(\alpha)=\frac{g_{X_0(N)}}{3}\log(N)+o(g_{X_0(N)}\log(N))\,.$$

The claim now follows immediately from the upper bound for $-\log|\det(M_N)|$ given above. □

6.6. **Remark.** The lower bound given in Theorem 6.5 improves the lower bound

$$\log|\delta_{\mathbb{T}}|\geq g_{X_0(N)}\log(N)+o(g_{X_0(N)}\log(N))$$

given in [31], Théorème 1.2. Since the fundamental invariant $\delta_{\mathbb{T}}$ controls congruences between modular forms, the lower bound (55) thus improves the lower bound for the minimal number of such congruences.

7. Appendix I: Comparing canonical and hyperbolic metrics

In the proof of Proposition 3.7 we used the explicit relation

$$\mu_{\mathrm{can}}(x) = \mu_{\mathrm{shyp}}(x) + \frac{1}{2g_X}\left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;x)\mathrm{d}t\right)\mu_{\mathrm{hyp}}(x).$$

The purpose of this appendix is to prove this identity, rather than referring to [18] or [19], thus making the present article more self-contained. Our approach uses analytic aspects of the Arakelov theory for algebraic curves.

7.1. **Proposition.** *With the above notations, we have the following equality of forms on X*

$$g_X\mu_{\mathrm{can}}(x) = \mu_{\mathrm{shyp}}(x) + \frac{1}{2}\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}});$$

here Ω^1_X denotes the canonical line bundle on X.

Proof. By choosing $\mu_1 = \mu_{\mathrm{shyp}}$ and $\mu_2 = \mu_{\mathrm{can}}$, the identity in Lemma 3.3 can be rewritten as

$$g_{\mathrm{hyp}}(x,y) - g_{\mathrm{can}}(x,y) = \phi(x) + \phi(y)\,, \tag{56}$$

where

$$\phi(x) = \int_X g_{\mathrm{hyp}}(x,\zeta)\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X\int_X g_{\mathrm{hyp}}(\xi,\zeta)\mu_{\mathrm{can}}(\zeta)\mu_{\mathrm{can}}(\xi)\,.$$

Taking $\mathrm{d}_x\mathrm{d}^c_x$ in relation (56), we get the equation

$$\mu_{\mathrm{shyp}}(x) - \mu_{\mathrm{can}}(x) = \mathrm{d}_x\mathrm{d}^c_x\phi(x)\,. \tag{57}$$

On the other hand, we have by definition

$$\log\|\mathrm{d}z(x)\|^2_{\mathrm{hyp,res}} = \lim_{y\to x}\left(g_{\mathrm{hyp}}(x,y) + \log|z(x)-z(y)|^2\right),$$
$$\log\|\mathrm{d}z(x)\|^2_{\mathrm{can,res}} = \lim_{y\to x}\left(g_{\mathrm{can}}(x,y) + \log|z(x)-x(y)|^2\right).$$

From this we deduce, again using (56),

$$\log\|\mathrm{d}z(x)\|^2_{\mathrm{hyp,res}} - \log\|\mathrm{d}z(x)\|^2_{\mathrm{can,res}} = \lim_{y\to x}\left(g_{\mathrm{hyp}}(x,y) - g_{\mathrm{can}}(x,y)\right) = 2\phi(x)\,. \tag{58}$$

Now, taking $-\mathrm{d}_x\mathrm{d}^c_x$ of equation (58), yields

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}}) - \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{can,res}}) = -2\mathrm{d}_x\mathrm{d}^c_x\phi(x)\,. \tag{59}$$

Combining equations (57) and (59) leads to

$$2(\mu_{\mathrm{shyp}}(x) - \mu_{\mathrm{can}}(x)) = \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{can,res}}) - \mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}})\,. \tag{60}$$

Recalling

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{can,res}}) = (2g_X - 2)\mu_{\mathrm{can}}(x),$$

we derive from (60)

$$\mu_{\mathrm{shyp}}(x) - \mu_{\mathrm{can}}(x) = \frac{2g_X - 2}{2}\mu_{\mathrm{can}}(x) - \frac{1}{2}\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}}),$$

which proves the proposition. □

7.2. **Proposition.** *With the above notations, we have the following formula for the first Chern form of Ω^1_X with respect to $\|\cdot\|_{\mathrm{hyp,res}}$*

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}}) = \frac{1}{2\pi}\mu_{\mathrm{hyp}}(x) + \left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;x)\mathrm{d}t\right)\mu_{\mathrm{hyp}}(x).$$

Proof. Our proof involves analysis similar to the proof of Lemma 3.6. By our definitions, we have for $x \in X$

$$\mathrm{c}_1(\Omega^1_X, \|\cdot\|_{\mathrm{hyp,res}}) = -\mathrm{d}_x\mathrm{d}_x^c \log\|\mathrm{d}z(x)\|^2_{\mathrm{hyp,res}} =$$

$$-\mathrm{d}_x\mathrm{d}_x^c \lim_{y\to x}(g_{\mathrm{hyp}}(x,y) + \log|z(x)-z(y)|^2) =$$

$$-\mathrm{d}_x\mathrm{d}_x^c \lim_{y\to x}\left(4\pi\int_0^\infty\left(K_{\mathrm{hyp}}(t;x,y) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t + \log|z(x)-z(y)|^2\right) =$$

$$-\mathrm{d}_z\mathrm{d}_z^c \lim_{y\to x}\left(4\pi\int_0^\infty K_{\mathbb{H}}(t;z(x),z(y))\mathrm{d}t + \log|z(x)-z(y)|^2\right)$$

$$-\mathrm{d}_z\mathrm{d}_z^c \lim_{y\to x}\left(4\pi\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z(x),\gamma z(y)) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\right).$$

Using the formula for the Green's function $g_{\mathbb{H}}(x,y)$ on $\mathbb{H}$, we obtain for the first summand in the latter sum

$$\begin{aligned}
A &= -\mathrm{d}_z\mathrm{d}_z^c \lim_{y\to x}\left(4\pi\int_0^\infty K_{\mathbb{H}}(t;z(x),z(y))\,\mathrm{d}t + \log|z(x)-z(y)|^2\right)\\
&= -\mathrm{d}_z\mathrm{d}_z^c \lim_{y\to x}\left(g_{\mathbb{H}}(z(x),z(y)) + \log|z(x)-z(y)|^2\right)\\
&= -\mathrm{d}_z\mathrm{d}_z^c \log|z(x)-\bar{z}(x)|^2 = -\frac{2i}{2\pi}\,\partial_z\,\bar{\partial}_z \log(z(x)-\bar{z}(x))\\
&= \frac{i}{\pi}\partial_z\frac{\mathrm{d}\bar{z}(x)}{z(x)-\bar{z}(x)} = -\frac{i}{\pi}\cdot\frac{\mathrm{d}z(x)\wedge\mathrm{d}\bar{z}(x)}{(z(x)-\bar{z}(x))^2}\\
&= -\frac{i}{\pi}\cdot\frac{\mathrm{d}z(x)\wedge\mathrm{d}\bar{z}(x)}{(2i\mathrm{Im}(z(x)))^2} = \frac{1}{2\pi}\cdot\mu_{\mathrm{hyp}}(x).
\end{aligned}$$

For the second summand we obtain

$$\begin{aligned} B &= -\mathrm{d}_z\mathrm{d}_z^c \lim_{y\to x}\left(4\pi\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z(x),\gamma z(y)) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\right) \\ &= -4\pi\mathrm{d}_z\mathrm{d}_z^c\int_0^\infty\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z(x),\gamma z(x)) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t\,. \end{aligned}$$

Since the latter integral converges absolutely, we are allowed to interchange differentiation and integration; this gives

$$\begin{aligned} B &= -4\pi\int_0^\infty \mathrm{d}_z\mathrm{d}_z^c\left(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} K_{\mathbb{H}}(t;z(x),\gamma z(x)) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mathrm{d}t \\ &= -4\pi\int_0^\infty \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}} \mathrm{d}_z\mathrm{d}_z^c\, K_{\mathbb{H}}(t;z(x),\gamma z(x))\mathrm{d}t\,. \end{aligned}$$

The claimed formula then follows, since $K_{\mathbb{H}}(t;z(x),z(x))$ is independent of x, and recalling the identity (under our normalization of the Laplacian, as stated in (7))

$$\mathrm{d}_x\mathrm{d}_x^c f(x) = -(4\pi)^{-1}\Delta_{\mathrm{hyp}} f(x)\mu_{\mathrm{hyp}}(x)\,, \tag{61}$$

for any smooth function f on X. □

7.3. **Theorem.** *With the above notations, we have, for all $x \in X$, the formula*

$$\mu_{\mathrm{can}}(x) = \mu_{\mathrm{shyp}}(x) + \frac{1}{2g_X}\left(\int_0^\infty \Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;x)\mathrm{d}t\right)\mu_{\mathrm{hyp}}(x).$$

Proof. We simply have to combine Propositions 7.1 and 7.2, and to use that

$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{1}{4\pi} = \frac{g_X}{\mathrm{vol}_{\mathrm{hyp}}(X)}\,.$$

□

8. Appendix II: The Polyakov formula

We shall work from the article [26]. Let us begin using the notation in that article, then in the end indicate the changes needed to conform with other conventions.

Let us consider two metrics, whose area forms are written as $\mathrm{d}A_0$ and $\mathrm{d}A_1$. In a local coordinate z on the Riemann surface X, setting $z = x + iy$, let us write

$$\mathrm{d}A_0(z) = e^{2\rho_0(z)} \cdot \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}\,,$$

$$\mathrm{d}A_1(z) = e^{2\rho_1(z)} \cdot \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

If we then write $\mathrm{d}A_1 = e^{2\varphi}\mathrm{d}A_0$ (see [26], p. 155, formula (1.11)), we then have $\varphi = \rho_1 - \rho_0$. The convention for the Laplacian is established in [26], p. 154, formula (1.1). In the above coordinates, we have

$$\Delta_0(z) = e^{-2\rho_0(z)} \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\,, \tag{62}$$

$$\Delta_1(z) = e^{-2\rho_1(z)} \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\,.$$

The Gauss curvature K_0 is then

$$K_0 = -\Delta_0\rho_0.$$

Note that if $\mathrm{d}A_0$ is the standard hyperbolic metric, then $e^{2\rho_0} = y^{-2}$, so $\rho_0 = -\log(y)$, and it is easy to show that $K_0 = -1$ as expected.

The Polyakov formula is proved in [26], p. 156, and stated as formula (1.13); it says

$$\log\left(\frac{\det'\Delta_\varphi}{A_\varphi}\right) = -\frac{1}{6\pi}\left(\frac{1}{2}\int\limits_X |\nabla_0\varphi|^2\,\mathrm{d}A_0 + \int\limits_X K_0\varphi\mathrm{d}A_0\right) + C\,.$$

If we take $\rho_1 = \rho_0$, then $\varphi = 0$, so we get

$$C = \log\left(\frac{\det'\Delta_0}{A_0}\right).$$

Therefore, in obvious notation, we find

$$\log\left(\frac{\det'\Delta_1}{A_1}\right) - \log\left(\frac{\det'\Delta_0}{A_0}\right) = -\frac{1}{6\pi}\left(\frac{1}{2}\int\limits_X |\nabla_0\varphi|^2\,\mathrm{d}A_0 + \int\limits_X K_0\varphi\mathrm{d}A_0\right).$$

Let us work with the right-hand side. Recall that, with the above notational conventions, we have, for any smooth f, the formula $\Delta(f)\mathrm{d}A = 4\pi\mathrm{dd}^c(f)$, for any metric. (Note: The normalization of the Laplacian in [26] as stated in (62) does not include the minus sign as in our normalization, see (7); as a result, the formula relating dd^c to the Laplacian of [26] does not contain the minus sign appearing in (61).) Therefore, if we integrate by parts, we have

$$\frac{1}{2}\int\limits_X |\nabla_0\varphi|^2\,\mathrm{d}A_0 = -\frac{1}{2}\int\limits_X \varphi\Delta_0\varphi\mathrm{d}A_0 = -2\pi\int\limits_X \varphi\mathrm{dd}^c\varphi\,.$$

Also, we have

$$\int_X K_0\varphi \mathrm{d}A_0 = -\int_X \varphi\Delta_0\rho_0 \mathrm{d}A_0 = -4\pi\int_X \varphi \mathrm{dd}^c\rho_0\,.$$

Therefore, we find

$$\begin{aligned}\log\left(\frac{\det'\Delta_1}{A_1}\right) - \log\left(\frac{\det'\Delta_0}{A_0}\right) &= -\frac{1}{6\pi}\left(-2\pi\int_X \varphi\mathrm{dd}^c\varphi - 4\pi\int_X \varphi\mathrm{dd}^c\rho_0\right)\\ &= \frac{1}{3}\cdot\int_X \varphi\left(\mathrm{dd}^c\varphi + 2\mathrm{d}d^c\rho_0\right)\,.\end{aligned}$$

However, since $\varphi = \rho_1 - \rho_0$, this becomes

$$\log\left(\frac{\det'\Delta_1}{A_1}\right) - \log\left(\frac{\det'\Delta_0}{A_0}\right) = \frac{1}{3}\cdot\int_X \varphi\left(\mathrm{dd}^c\rho_0 + \mathrm{dd}^c\rho_1\right)\,.$$

Let us now fit this into our notation. Since $\mathrm{d}A_1 = e^{2\rho_1}\frac{i}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}$, we have $\mathrm{c}_1(\Omega^1_X, \|\cdot\|_1) = \mathrm{dd}^c(2\rho_1)$. Similarly, $\mathrm{c}_1(\Omega^1_X, \|\cdot\|_0) = \mathrm{dd}^c(2\rho_0)$, so then

$$\mathrm{dd}^c\rho_0 + \mathrm{dd}^c\rho_1 = \frac{1}{2}\left(\mathrm{c}_1(\Omega^1_X, \|\cdot\|_1) + \mathrm{c}_1(\Omega^1_X, \|\cdot\|_0)\right)\,.$$

In our notation, we write $\mu_1 = e^{\phi}\mu_0$, so then $\phi = 2\varphi$. Therefore, we get

$$\log\left(\frac{\det'\Delta_1}{A_1}\right) - \log\left(\frac{\det'\Delta_0}{A_0}\right) = \frac{1}{3}\cdot\int_X \varphi\left(\mathrm{dd}^c\rho_0 + \mathrm{dd}^c\rho_1\right) =$$

$$\frac{1}{6}\cdot\int_X \phi\cdot\frac{1}{2}\left(\mathrm{c}_1(\Omega^1_X, \|\cdot\|_1) + \mathrm{c}_1(\Omega^1_X, \|\cdot\|_0)\right)\,.$$

Now consider the special case when $\mu_0 = \mu_{\mathrm{hyp}}$ is the hyperbolic metric, with Gauss curvatuve equal to -1. Equivalent to the statement $K_0 = -1$ is the statement that $\mathrm{c}_1(\Omega^1_X, \|\cdot\|_0) = (2g_X - 2)\mu_{\mathrm{shyp}}$. If μ_1 is the Arakelov metric, then $\mathrm{c}_1(\Omega^1_X, \|\cdot\|_1) = (2g_X - 2)\mu_{\mathrm{can}}$, where μ_{can} is the canonical metric. If we write $\mu_{\mathrm{Ar}} = e^{\phi_{\mathrm{Ar}}}\mu_{\mathrm{hyp}}$, then the above identity becomes

$$\log\left(\frac{\det'\Delta_{\mathrm{Ar}}}{A_{\mathrm{Ar}}}\right) - \log\left(\frac{\det'\Delta_{\mathrm{hyp}}}{A_{\mathrm{hyp}}}\right) = \frac{g_X - 1}{6}\cdot\int_X \phi_{\mathrm{Ar}}\left(\mu_{\mathrm{can}} + \mu_{\mathrm{shyp}}\right)\,.$$

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