Star products of Green's currents and automorphic forms

J. Jorgenson and J. Kramer *

1 Introduction

1.1. In the previous article [11], archimedian heights of hermitian line bundles on families of polarized, n-dimensional, abelian varieties were computed: Letting Y denote the moduli space of such varieties and $f : X \longrightarrow Y$ the corresponding universal family, the archimedian height of the hermitian line bundle $\overline{L} = (L, \|\cdot\|_{\mathrm{trans}})$ on X corresponding to the given polarization and being equipped with the translation invariant metric $\|\cdot\|_{\mathrm{trans}}$, is given as an integral of the $(n+1)$-fold star product of the Green's currents $g_j = -\log \|s_j\|^2_{\mathrm{trans}}$ along the fibers of f, where $s_1, ..., s_{n+1}$ are generically chosen global sections of L. An analysis of this integral then showed that it can be calculated as the logarithm of the Petersson norm of a Siegel modular form, which turned out to be given as a finite product of the global sections s_j evaluated along certain torsion sections of X. The same type of Siegel modular forms occurs in the recent paper [21] of K.-I. Yoshikawa, in course of his computation of the Quillen metric of the determinant line bundle $\det f_*(L)$. The coincidence of these two results is a consequence of the arithmetic Riemann-Roch Theorem applied in degree one to an integral model of the fibration $f : X \longrightarrow Y$.

1.2. A detailed analysis of the results obtained in the setting of abelian fibrations in [11] showed that the proofs could be modified in such a way that they do not any longer depend on the specific setting of abelian fibrations and, therefore, extend to the following, quite general situation: We let $f : X \longrightarrow Y$ be *any* family of smooth, projective, n-dimensional, complex varieties over some base variety Y, and $\overline{L} = (L, \|\cdot\|)$ a line bundle on X equipped with a smooth, hermitian metric $\|\cdot\|$. To these data we associate a hermitian line bundle $\overline{M}(\overline{L}) = (M(L), \|\cdot\|)$ on Y such that $M(L)$ can be characterized by the property that the first Chern class $c_1(M(L))$ of $M(L)$ equals the push-forward $f_*(c_1(L)^{n+1})$ of the $(n+1)$-st power of the first Chern class $c_1(L)$ of L by f. We emphasize that R. Elkik in a series of two papers [6], [7], based on P. Deligne's original note [3], gave an alternative construction of the hermitian line bundle $\overline{M}(\overline{L})$. Under some mild, additional hypotheses, e.g., if Y is quasi-projective and L is f-ample, we then show as the first result of this paper (Theorem 4.7) that for generically chosen global sections $s_1, ..., s_{n+1}$ of L, the integral of the $(n+1)$-fold star product of the Green's currents $g_j = -\log \|s_j\|^2$ along the fibers of f is given by $-\log \|t\|^2$ for a certain global section t of $M(L)$. As a second result (Theorem 4.12) we then prove that the global section t is essentially given as the norm of the restriction of s_{n+1} to the cycle obtained by intersecting $\mathrm{div}(s_1), ..., \mathrm{div}(s_n)$ down to Y provided that the base variety Y has a compactification $\overline{Y}$ satisfying $\mathrm{codim}(\overline{Y} \setminus Y) \geq 2$.

*The first author acknowledges support from NSF grant DMS-96-22535; the second author would like to express his thanks to G. Wüstholz and A.-S. Sznitman for the invitation to the FIM at the ETH in Zurich (Switzerland), where part of this note has been written.

1.3. It is now of special interest to apply the formalism described in 1.2 to specific families $f : X \longrightarrow Y$. It is not too surprising that particularly interesting examples are obtained, if the base variety Y is the moduli space associated to some moduli problem and, as such, given as the quotient of a symmetric space by an arithmetic subgroup Γ. The line bundle $M(L)$ (constructed for a natural choice of L) then becomes the line bundle of certain Γ-automorphic forms on Y. For example, by choosing Y to be the moduli space of polarized, n-dimensional, abelian varieties, we are led back to the results obtained in [11]. Another interesting example is obtained by choosing Y to be the moduli space of Enriques surfaces with a polarization of degree two; the global section t then equals up to a non-zero constant some power of Borcherds's Φ-function on $\mathrm{SO}(2, 10)$ introduced in [2]. It is important to mention that K.-I. Yoshikawa in his computations of the analytic torsion of the line bundle corresponding to the degree two polarization and being equipped with the Calabi-Yau metric, was also led to Borcherds's Φ-function (cf. [22]). Again, this coincidence could be explained by an application of the arithmetic Riemann-Roch Theorem provided that a model of the universal family of Enriques surfaces over the integers existed, which is most likely true, but seems not to be established yet. On a more speculative side, we finally mention that our integral representation of the logarithm of norms of automorphic forms could be useful in finding analogues of Kronecker type limit formulae in higher dimensions; in this respect we also refer to the recent paper [15] of U. Kühn.

1.4. This paper is organized as follows: After giving the basic notations in the next section, in particular a choice of local, holomorphic coordinates, we present in the third section the local results needed to establish our main identities. This section is similar to section 3 in [11] up to the proof of Proposition 3.4, although no analogue of the real analytic α, β-coordinates, used in the setting of abelian varieties, is needed. In fact, the consequent use of the local, holomorphic coordinates allows us to prove in generality a result (Proposition 3.5), which in the previous paper [11], Remark 4.6, could only be proven by using a strong result of A. Silverberg. In the fourth section, after constructing the line bundle $M(L)$ together with its hermitian metric, we are in position to state and prove the two main results (Theorems 4.7 and 4.12). In the last section, several examples are discussed with a specific emphasis on the example with Y being the moduli space of Enriques surfaces with a polarization of degree two, which seems to be of some significance as already pointed out above.

1.5. *Acknowledgement:* We would like to thank J.-B. Bost, G. Hein, S. Kudla, S. Rallis for inspiring discussions and K.-I. Yoshikawa for sending his preprints [21], [22]. In addition, we would also like to thank J. Wahl and the referee for several important remarks whose corrections led to a substantial improvement of this paper.

2 Notations

2.1. Let $f : X \longrightarrow Y$ be a smooth, projective morphism of relative dimension n between smooth, complex varieties. Furthermore, let $\overline{L} := (L, \|\cdot\|)$ be a holomorphic line bundle on X equipped with a smooth, hermitian metric $\|\cdot\|$. If s is a global section of L, we denote by g the Green's current associated to s, namely

$$g = -\log \|s\|^2.$$

Letting ω denote the first Chern form $c_1(\overline{L})$ of $\overline{L}$, we recall the dd^c-equation

$$dd^c g + \delta_{\mathrm{div}(s)} = \omega.$$

Finally, for $y \in Y$, we put $X_y := f^{-1}(y)$, $\overline{L}_y := \overline{L}|_{X_y}$, and $\omega_y := \omega|_{X_y}$.

2.2. In the sequel, we will fix an open covering $\mathcal{U} := \{U_\alpha\}$ of Y. Choosing $\mathcal{U}$ sufficiently fine, we are able to find for each α finite, open coverings

$$f^{-1}(U_\alpha) = \bigcup_a V_{\alpha,a}$$

such that within each open set $V_{\alpha,a}$ we have local, holomorphic coordinates (y,z) satisfying $f(y,z) = y$ with y being a local, holomorphic coordinate of U_α. The collection $\mathcal{V} := \{V_{\alpha,a}\}$ gives rise to an open covering of X which will also be fixed in the sequel.
In addition to the open covering $\mathcal{V} = \{V_{\alpha,a}\}$, we need to introduce a subordinate covering $\mathcal{W} := \{W_{\alpha,a}\}$, whose elements are not necessarily open, but pairwise disjoint for fixed α; the covering $\mathcal{W} = \{W_{\alpha,a}\}$ is constructed as follows: For fixed α, we first consider the finite, open covering $f^{-1}(U_\alpha) = \bigcup_a V_{\alpha,a}$ and we enumerate its elements by $V_{\alpha,1}, ..., V_{\alpha,N}$. Then, we define

$$W_{\alpha,1} := V_{\alpha,1}$$

and for $j = 2, ..., N$

$$W_{\alpha,j} := V_{\alpha,j} \setminus \bigcup_{k=1}^{j-1} V_{\alpha,k}.$$

In this way, we obtain by means of the sets $W_{\alpha,1}, ..., W_{\alpha,N}$ a covering of $f^{-1}(U_\alpha)$ by pairwise disjoint sets satisfying $W_{\alpha,j} \subseteq V_{\alpha,j}$ $(j = 1, ..., N)$. Finally, letting vary α, we obtain the desired covering $\mathcal{W} = \{W_{\alpha,a}\}$ subordinate to $\mathcal{V} = \{V_{\alpha,a}\}$. We note that, if $U_\alpha \cap U_\beta \neq \emptyset$ for fixed α, β, the set of all possible intersections $\{W_{\alpha,a} \cap W_{\beta,b}\}$ gives rise to a covering of $f^{-1}(U_\alpha \cap U_\beta)$ by pairwise disjoint sets.

2.3. Without loss of generality, we may assume that L trivializes on $V_{\alpha,a}$; we then denote by $\{\varphi_{\alpha,a;\beta,b}\}$ the holomorphic transition functions of L with respect to $\mathcal{V}$ and by $\rho_{\alpha,a} : V_{\alpha,a} \longrightarrow \mathbb{R}_{>0}$ the smooth, positive functions defining the hermitian metric $\|\cdot\|$, i.e.,

$$\|s(y,z)\|^2 = \frac{|s(y,z)|^2}{\rho_{\alpha,a}(y,z)},$$

for any global section s of L.

3 Local results

3.1. In this section we frequently refer to results in [11], section 3. In [11] we treated the case of families $f : X \longrightarrow Y$ of polarized, n-dimensional, abelian varieties (cf. also 1.1 and 5.1). Since the considerations in [11], section 3, are of purely local nature and do not depend on the specific setting, they can also be applied to the more general fibrations $f : X \longrightarrow Y$ considered in this paper.
For the subsequent, local computations we make the following conventions: We fix $U_\alpha \in \mathcal{U}$ and $y_0 \in U_\alpha$. Apart from Proposition 3.5, we assume throughout this section that there exist $(n+1)$ global sections $s_1, ..., s_{n+1}$ of L whose restrictions to X_y have properly intersecting divisors for all $y \in U_\alpha$. As in [11], section 3, we put $\Theta_{k,y} := \mathrm{div}(s_k)|_{X_y}$, $D_{k,y} := \Theta_{1,y} \cdot ... \cdot \Theta_{k,y}$, and $D_{0,y} := X_y$.
Without loss of generality, we may assume that X_{y_0} can locally, i.e., inside of the non-empty intersections $X_{y_0} \cap V_{\alpha,a}$, be identified with $\mathbb{C}^n$ with coordinates $w = (w_1, ..., w_n)$ and that,

possibly after a resolution of singularities, the non-empty intersections $D_{k,y_0} \cap V_{\alpha,a}$ can be identified with the cycle described by the equations $w_1 = ... = w_k = 0$; we denote these cycles by $D_{k,y_0;a}$. As in [11], section 3, we define for sufficiently small $\varepsilon > 0$ the tubular neighbourhood $D_{k;a}^{(\varepsilon)}$ around $D_{k,y_0;a}$ by

$$D_{k;a}^{(\varepsilon)} := \{w = (w_1, ..., w_n) \in \mathbb{C}^n |\ |w_1|^2 + ... + |w_k|^2 < \varepsilon^2\}.$$

If the index a is such that $D_{k,y_0} \cap V_{\alpha,a} = \emptyset$, we let $D_{k,y_0;a}$ and $D_{k;a}^{(\varepsilon)}$ denote the empty set. By abuse of notation, we denote the differential form on $\mathbb{C}^n$ induced by ω_{y_0} via the above identification again by ω_{y_0}.
We now let vary y through U_α. Eventually by choosing a refinement of $\mathcal{U}$, we may assume that for each index a the intersection $D_{k,y} \cap V_{\alpha,a}$ is non-empty, if and only if $D_{k,y_0} \cap V_{\alpha,a} \neq \emptyset$. Then, as in the considerations of the previous paragraph, we get identifications of the non-empty intersections $X_y \cap V_{\alpha,a}$ with (the same) $\mathbb{C}^n$ such that the non-empty intersections $D_{k,y} \cap V_{\alpha,a}$ can be identified with the cycle given by the equations $w_1 = p_1(y), ..., w_k = p_k(y)$ for suitable holomorphic functions $p_1(y), ..., p_k(y)$ satisfying $p_1(y_0) = ... = p_k(y_0) = 0$; we denote this cycle by $D_{k,y;a}$. If the index a is such that $D_{k,y} \cap V_{\alpha,a} = \emptyset$, we let $D_{k,y;a}$ denote the empty set. Eventually by refining the open covering $\mathcal{U}$ again and by choosing $\varepsilon > 0$ suitably, we may furthermore assume that $D_{k,y;a}$ varies within the thickening $D_{k;a}^{(\varepsilon)}$.
By means of the distribution functions (cf. [9], p. 80)

$$f_{\Theta_{k,y;a}}(w) := \log |w_k - p_k(y)|^2 \ \ (k = 1, ..., n), \text{ resp.}$$

$$f_{D_{k,y;a}}(w) := -\frac{4\pi}{(2k-2)\gamma_{2k}} \cdot (|w_1 - p_1(y)|^2 + ... + |w_k - p_k(y)|^2)^{1-k} \ \ (k = 2, ..., n),$$

with γ_{2k} denoting the volume of the unit sphere in $\mathbb{C}^k$, we derive as in [11]

3.2. Lemma. *With the above notations and $\tilde{g}_k(y,w) := -\log |s_k(y,w)|^2$, we have for $y \in U_\alpha$ and $k = 2, ..., n+1$:*

$$(i) \quad \int_{D_{k-1;a}^{(\varepsilon)}} \tilde{g}_k(y,w) \wedge \delta_{D_{k-1,y;a}} \wedge \omega_{y_0}^{n+1-k} = \int_{D_{k-1;a}^{(\varepsilon)}} \tilde{g}_k(y,w) \wedge d_w d_w^c f_{D_{k-1,y;a}}(w) \wedge \omega_{y_0}^{n-1},$$

$$(ii) \quad d_y d_y^c \int_{D_{k-1;a}^{(\varepsilon)}} f_{D_{k-1,y;a}}(w) \wedge \delta_{\Theta_{k,y;a}} \wedge \omega_{y_0}^{n-1} = d_y d_y^c \int_{D_{k;a}^{(\varepsilon)}} f_{D_{k,y;a}}(w) \wedge \omega_{y_0}^{n}.$$

Proof. The proofs are the same as the ones given in [11], Lemmata 3.7 and 3.8. □

3.3. Proposition. *In addition to the above hypotheses, define for $y \in U_\alpha$ and $k = 1, ..., n+1$,*

$$v_{k;\alpha}(y) := \sum_a \int_{X_y \cap W_{\alpha,a}} \log \rho_{\alpha,a}(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k}.$$

Then, with the above notations, we have the following variational formulae for $y \in U_\alpha$:
(a)

$$d_y d_y^c \left(\int_{X_y} g_1(y,z) \wedge \omega_y^n - v_{1;\alpha}(y) \right) = -d_y d_y^c \sum_a \int_{\Theta_{1;a}^{(\varepsilon)} \cap W_{\alpha,a}} f_{\Theta_{1,y;a}}(w) \wedge \omega_{y_0}^n.$$

(b) For $k=2,...,n$, we have

$$d_y d_y^c \left(\int_{X_y} g_k(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k} - v_{k;\alpha}(y) \right) =$$

$$d_y d_y^c \sum_a \int_{D^{(\varepsilon)}_{k-1;a} \cap W_{\alpha,a}} f_{D_{k-1,y;a}}(w) \wedge \omega_{y_0}^n - d_y d_y^c \sum_a \int_{D^{(\varepsilon)}_{k;a} \cap W_{\alpha,a}} f_{D_{k,y;a}}(w) \wedge \omega_{y_0}^n + O_y(\varepsilon).$$

(c)

$$d_y d_y^c \left(\int_{X_y} g_{n+1}(y,z) \wedge \delta_{D_{n,y}} - v_{n+1;\alpha}(y) \right) = d_y d_y^c \sum_a \int_{D^{(\varepsilon)}_{n;a} \cap W_{\alpha,a}} f_{D_{n,y;a}}(w) \wedge \omega_{y_0}^n + O_y(\varepsilon).$$

Proof. By means of the equality

$$\int_{X_y} g_k(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k} - v_{k;\alpha}(y) = \sum_a \int_{D^{(\varepsilon)}_{k-1;a} \cap W_{\alpha,a}} \tilde{g}_k(y,w) \wedge \delta_{D_{k-1,y;a}} \wedge \omega_{y_0}^{n+1-k},$$

the proof of the proposition runs along the same lines as the proof of Proposition 3.9 in [11] after an application of Lemma 3.2 to the integral on the right-hand side. The main difference with the set-up in [11], Proposition 3.9, is that the sets $W_{\alpha,a}$ in the latter case are products of subcubes in X_{y_0} with neighbourhoods in Y, whereas here they are more general sets. However, Stokes' theorem can also be applied in this more general setting and the boundary integral calculations can therefore be carried out as in the proof of [11], Proposition 3.9. □

3.4. Proposition. *In addition to the above hypotheses, define for $y \in U_\alpha$ and $m \in \mathbb{N}$,*

$$V_{m;\alpha}(y) := \sum_{k=1}^{m} v_{k;\alpha}(y).$$

Then, the following variational formula holds for all $y \in U_\alpha$:

$$d_y d_y^c \left(\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) - V_{n+1;\alpha}(y) \right) = 0. \tag{1}$$

Proof. Using the formula

$$g_1(y,z) * ... * g_{n+1}(y,z) = \sum_{k=1}^{n+1} g_k(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k}$$

for the $(n+1)$-fold star product (cf. [18], chapter 2, and [11], Lemma 3.2), the claim becomes an immediate consequence of Proposition 3.3. □

3.5. Proposition. *In contrast to the hypotheses of this section, we assume here that there exist $(n+1)$ global sections $s_1,...,s_{n+1}$ of L whose restrictions to X_y have properly intersecting divisors for all y in U_α except for $y = y_0$, where only the first n restrictions have properly intersecting divisors. Then, the following variational formula holds for all $y \in U_\alpha$:*

$$d_y d_y^c \left(\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) - \sum_{z(y) \in D_{n,y}} g_{n+1}(y,z(y)) - V_{n;\alpha}(y) \right) = 0. \tag{2}$$

Proof. We start by noting that the n-fold intersection $D_{n,y}$ consists of finitely many points $\{z(y)\}$ for all $y \in U_\alpha$ and that the difference

$$\int_{X_y} g_1(y,z) * \ldots * g_{n+1}(y,z) - \sum_{z(y)\in D_{n,y}} g_{n+1}(y,z(y)) \tag{3}$$

is a priori only defined for $y \neq y_0$. But using the formula for the star product, we can rewrite (3) for $y \neq y_0$ as

$$\sum_{k=1}^{n} \int_{X_y} g_k(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k},$$

which immediately implies that (3) is also well-defined and hence bounded for $y = y_0$. Proposition 3.4 together with the equality

$$d_y d_y^c \sum_{z(y)\in D_{n,y}} (g_{n+1}(y,z(y)) - \log \rho_{\alpha,a}(y,z(y))) = -d_y d_y^c \sum_{z(y)\in D_{n,y}} \log |s_{n+1}(y,z(y))|^2 = 0$$

now shows that the claimed formula (2) holds true for $y \neq y_0$. Since the difference (3) is bounded as observed in the beginning of the proof, the claimed formula (2) extends to hold true for $y = y_0$ by the Riemann removable singularity theorem. □

4 The main identities

4.1. Lemma. *In addition to the above hypotheses, assume that the open covering $\mathcal{V}$ is chosen such that all the possible intersections $W_{\alpha,a} \cap W_{\beta,b}$ for the covering $\mathcal{W}$ are simply connected. Then, for $k = 1, \ldots, n+1$, define the functions $\psi_{k;\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$ by*

$$\psi_{k;\alpha,\beta}(y) := \exp\left(\sum_{a,b} \int_{X_y \cap W_{\alpha,a} \cap W_{\beta,b}} \log \varphi_{\alpha,a;\beta,b}(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k} \right).$$

The collection of functions $\{\psi_{k;\alpha,\beta}\}$ are holomorphic transition functions with respect to $\mathcal{U}$.

Proof. Since the intersections $W_{\alpha,a} \cap W_{\beta,b}$ are assumed to be simply connected, the functions $\log \varphi_{\alpha,a;\beta,b}(y,z)$ are well-defined and holomorphic on $W_{\alpha,a} \cap W_{\beta,b}$ (taking the principal logarithm). Therefore, the functions $\psi_{k;\alpha,\beta}$ are holomorphic on $U_\alpha \cap U_\beta$. Finally, the 1-cocycle relation

$$\varphi_{\alpha,a;\beta,b}(y,z) \cdot \varphi_{\beta,b;\gamma,c}(y,z) = \varphi_{\alpha,a;\gamma,c}(y,z) \quad ((y,z) \in V_{\alpha,a} \cap V_{\beta,b} \cap V_{\gamma,c})$$

immediately leads to the relation

$$\psi_{k;\alpha,\beta}(y) \cdot \psi_{k;\beta,\gamma}(y) = \psi_{k;\alpha,\gamma}(y),$$

which shows that the collection of functions $\{\psi_{k;\alpha,\beta}\}$ satisfies the properties of holomorphic transition functions with respect to $\mathcal{U}$. □

4.2. Definition. The holomorphic line bundle on Y determined by the holomorphic transition functions $\{\psi_{k;\alpha,\beta}\}$ of Lemma 4.1 is denoted by $M_k(L)$. Furthermore, we put $M(L) := \bigotimes_{k=1}^{n+1} M_k(L)$; its holomorphic transition functions are given by $\psi_{\alpha,\beta} := \prod_{k=1}^{n+1} \psi_{k;\alpha,\beta}$.

4.3. Lemma. *Under the same hypotheses as in Lemma 4.1, define for $k = 1, ..., n+1$ the functions $\sigma_{k;\alpha} : U_\alpha \longrightarrow \mathbb{R}_{>0}$ by $\sigma_{k;\alpha}(y) := \exp(v_{k;\alpha}(y))$. Then, the functions $\sigma_{k;\alpha}$ are smooth, positive functions satisfying*

$$\sigma_{k;\alpha}(y) \cdot |\psi_{k;\alpha,\beta}(y)|^{-2} = \sigma_{k;\beta}(y) \quad (y \in U_\alpha \cap U_\beta).$$

Proof. By the definition of the function $v_{k;\alpha}$, we have the formula

$$\sigma_{k;\alpha}(y) = \exp\left(\sum_a \int_{X_y \cap W_{\alpha,a}} \log \rho_{\alpha,a}(y,z) \wedge \delta_{D_{k-1,y}} \wedge \omega_y^{n+1-k}\right).$$

Taking logarithms, integrating and exponentiating the equality

$$\rho_{\alpha,a}(y,z) \cdot |\varphi_{\alpha,a;\beta,b}(y,z)|^{-2} = \rho_{\beta,b}(y,z) \quad ((y,z) \in V_{\alpha,a} \cap V_{\beta,b}),$$

the proof of the lemma follows after recalling the definition of the transition functions $\psi_{k;\alpha,\beta}$. □

4.4. Definition. The smooth, positive functions $\{\sigma_{k;\alpha}\}$ of Lemma 4.3 define a smooth, hermitian metric $\|\cdot\|$ on $M_k(L)$; we put $\overline{M}_k(\overline{L}) := (M_k(L), \|\cdot\|)$. Furthermore, we put $\overline{M}(\overline{L}) := \bigotimes_{k=1}^{n+1} \overline{M}_k(\overline{L})$; here the smooth, hermitian metric is determined by $\sigma_\alpha := \prod_{k=1}^{n+1} \sigma_{k;\alpha}$.

4.5. Definition. (1) For any k global sections $s_1, ..., s_k$ of L we set

(a) $\mathcal{S}(s_1, ..., s_k) := \{y \in Y \mid \mathrm{div}(s_1)|_{X_y}, ..., \mathrm{div}(s_k)|_{X_y}$ intersect properly on $X_y\}$,

(b) $\mathcal{T}(s_1, ..., s_k) := Y \setminus \mathcal{S}(s_1, ..., s_k)$.

(2) We call $(n+1)$ global sections $s_1, ..., s_{n+1}$ of L to be *in general position*, if

(a) $\mathrm{codim}(\mathcal{T}(s_1, ..., s_{n+1})) \geq 1$,

(b) $\mathrm{codim}(\mathcal{T}(s_{i_1}, ..., s_{i_n})) \geq 2$ for any choice of n indices $1 \leq i_1 < ... < i_n \leq n+1$.

4.6. Theorem. *With the notations of 2.1 the following holds true:*

(a) If, in addition, L is f-ample and Y is quasi-projective, then $(n+1)$ global sections $s_1, ..., s_{n+1}$ of L in general position exist, eventually after replacing L by $L^{\otimes m_1} \otimes f^*(K^{\otimes m_2})$ with $m_1, m_2 \gg 0$ and K an ample line bundle on Y.

(b) If, in addition, L is f-ample and $Y = Y' \setminus D$ with Y' a complex, quasi-projective variety and D an ample divisor on Y', then there exist $(n+1)$ global sections $s_1, ..., s_{n+1}$ of L satisfying $\mathcal{T}(s_1, ..., s_{n+1}) = \emptyset$, eventually after replacing L by $L^{\otimes m_1}$ for $m_1 \gg 0$; in particular, the global sections $s_1, ..., s_{n+1}$ are in general position.

Proof. (a) The proof of this statement follows from the arguments given in [11], section 4.
(b) We first show how Y' can be replaced by a suitable, complex, quasi-projective variety $\tilde{Y}$, which also contains Y as an open dense subset and which, in addition, has the property that the family $f : X \longrightarrow Y$ with f smooth and projective extends to a family $\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ with $\tilde{f}$ flat (not necessarily smooth) and projective: Eventually after replacing L by $L^{\otimes m_1}$ for

$m_1 \gg 0$, we may assume that L is relatively very ample. Therefore, we have an embedding $X \longrightarrow \mathbb{P}^N_Y = \mathbb{P}^N \times_{\mathbb{C}} Y$, and hence obtain the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \mathbb{P}^N_Y \arrow[r] \arrow[d] & \mathbb{P}^N_{Y'} \arrow[d] \\ & Y \arrow[r] & Y' \end{tikzcd}$$

Let now $Hilb^P_{\mathbb{P}^N}(Y')$ denote the Hilbert scheme over Y' parametrizing closed subschemes in $\mathbb{P}^N_{Y'}$, which are flat over Y' and have fixed Hilbert polynomial $P(\nu) = \mathrm{rk} f_*(\mathcal{O}_{\mathbb{P}^N_{Y'}}(\nu) \otimes \mathcal{O}_X)$; it is well-known that $Hilb^P_{\mathbb{P}^N}(Y')$ is projective over Y' (cf. [16], pp. 21-22). By means of the above diagram, we see that the family $f : X \longrightarrow Y$ gives rise to a morphism $\beta : Y \longrightarrow Hilb^P_{\mathbb{P}^N}(Y')$, which extends to a rational map $\beta' : Y' \longrightarrow Hilb^P_{\mathbb{P}^N}(Y')$, since Y is an open dense subset of Y'. By blowing up Y' along the locus of indeterminacy of β', which is contained in the support $\mathrm{supp}(D)$ of D, we obtain a complex, quasi-projective variety $\tilde{Y}$ together with a birational map $\gamma : \tilde{Y} \longrightarrow Y'$ such that the rational map β' extends to a morphism $\tilde{\beta} : \tilde{Y} \longrightarrow Hilb^P_{\mathbb{P}^N}(Y')$, i.e., we have the following commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[d, "\gamma"'] \arrow[dr, "\tilde{\beta}"] & \\ Y' \arrow[r, dotted, "\beta'"] & Hilb^P_{\mathbb{P}^N}(Y') \end{tikzcd}$$

We note that by taking $\tilde{D}$ to be $\gamma^{-1}(D)$ plus a suitable linear combination of the exceptional divisors on $\tilde{Y}$, we obtain an ample divisor on $\tilde{Y}$ satisfying $\gamma(\tilde{D}) = D$ and $Y = \tilde{Y} \setminus \tilde{D}$. The existence of the morphism $\tilde{\beta}$ now implies the existence of the desired family $\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ with $\tilde{f}$ flat and projective extending the original family, observing that Y is also an open dense subset of $\tilde{Y}$.

Let now $\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ be a flat family extending the given family $f : X \longrightarrow Y$ as described before. The above construction then also shows that the relatively very ample line bundle L on X (note that we replaced L by a suitable tensor-power of it so that it becomes relatively very ample) extends to a relatively very ample line bundle $\tilde{L}$ on $\tilde{X}$. After eventually replacing L by $L^{\otimes m_1}$ for $m_1 \gg 0$, we may assume that the vector bundle $\tilde{E} := \tilde{f}_* \tilde{L}$ has rank $\mathrm{rk}\tilde{E} > \dim Y + n$. Furthermore, using the ampleness of $\tilde{D}$ and eventually replacing $\tilde{E}$ by $\tilde{E} \otimes \mathcal{O}_{\tilde{Y}}(m_2 \tilde{D})$ for $m_2 \gg 0$, we may assume that $\tilde{E}$ is generated by global sections. Assuming that all the fibers of $\tilde{f}$ are irreducible and using the inequality $\mathrm{rk}\tilde{E} > \dim Y + n$, one easily establishes the existence of $\tilde{s}_1 \in H^0(\tilde{Y}, \tilde{E})$ with vanishing locus $V(\tilde{s}_1) = \emptyset$. If reducible fibers exist, the existence of the global section $\tilde{s}_1$ can be guaranteed by imposing the extra condition that $\tilde{s}_1$ does not vanish at the points of the Hilbert scheme parametrizing the irreducible components of the reducible fibers; since this is a finite scheme over $\tilde{Y}$, this can be achieved by eventually increasing the rank of $\tilde{E}$. Replacing now $\tilde{X}$ by the intersection $\tilde{X}_1 := \tilde{X} \cap \mathrm{div}(\tilde{s}_1)$, which is a flat family of relative dimension $n-1$ over $\tilde{Y}$, and $\tilde{E}$ by the quotient $\tilde{E}_1 := \tilde{E}/\tilde{s}_1 \cdot \mathcal{O}_{\tilde{Y}}$, which satisfies $\mathrm{rk}\tilde{E}_1 = \mathrm{rk}\tilde{E} - 1 > \dim Y + (n-1)$, we can show in the same way that there exists $\tilde{s}_2 \in H^0(\tilde{Y}, \tilde{E}_1)$ with vanishing locus $V(\tilde{s}_2) = \emptyset$; we note that $\tilde{s}_2$ lifts to a global section of $\tilde{E}$, which we denote again by $\tilde{s}_2$. Inductively, after $n-1$ steps, we construct in this way a flat family

$\tilde{f}_{n-1} : \tilde{X}_{n-1} \longrightarrow \tilde{Y}$ of relative dimension one over $\tilde{Y}$ and n global sections $\tilde{s}_1, ..., \tilde{s}_n \in H^0(\tilde{X}, \tilde{L})$ satisfying $\mathcal{T}(\tilde{s}_1, ..., \tilde{s}_n) = \emptyset$; we note that $\tilde{X}_{n-1} = \tilde{X} \cap \operatorname{div}(\tilde{s}_1) \cap ... \cap \operatorname{div}(\tilde{s}_{n-1})$.
Finally, putting $\tilde{X}_n := \tilde{X} \cap \operatorname{div}(\tilde{s}_1) \cap ... \cap \operatorname{div}(\tilde{s}_n)$, we obtain the finite morphism $\tilde{f}_n : \tilde{X}_n \longrightarrow \tilde{Y}$. Furthermore, we let $\tilde{L}_n$ denote the restriction of $\tilde{L}$ to $\tilde{X}_n$; by our ampleness assumptions, we may suppose that global sections of $\tilde{L}_n$ lift to global sections of $\tilde{L}$. Let now $\tilde{L}_n = \mathcal{O}_{\tilde{X}_n}(\tilde{D}_n)$ and $D' = \tilde{f}_n^{-1}(\tilde{D})$. Since the morphism $\tilde{f}_n$ is finite and $\tilde{D}$ is ample, D' is also ample, whence there exists $m_2 \gg 0$ such that $m_2 \cdot D'$ is linearly equivalent to $\tilde{D}_n$ plus some effective divisor. This shows the existence of $\tilde{s}_{n+1} \in H^0(\tilde{X}_n, \tilde{L}_n)$ such that the support of $\operatorname{div}(\tilde{s}_{n+1})$ is contained in $\operatorname{supp}(m_2 \cdot D')$. Denoting the lift of $\tilde{s}_{n+1}$ to $\tilde{L}$ again by $\tilde{s}_{n+1}$, we end up with $\tilde{s}_1, ..., \tilde{s}_{n+1} \in H^0(\tilde{X}, \tilde{L})$ satisfying $\mathcal{T}(\tilde{s}_1, ..., \tilde{s}_{n+1}) \subseteq \operatorname{supp}(\tilde{D})$. Denoting the restrictions of $\tilde{s}_j$ to X by s_j, we finally obtain $s_1, ..., s_{n+1} \in H^0(X, L)$ satisfying $\mathcal{T}(s_1, ..., s_{n+1}) = \emptyset$. $\square$

4.7. Theorem. *With the above notations, assume that $\overline{L}$ has $(n+1)$ global sections $s_1, ..., s_{n+1}$ in general position. Then, there exists a non-trivial global section t of $\overline{M}(\overline{L})$ such that the equality*

$$\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) = -\log \|t(y)\|^2$$

holds for all $y \in \mathcal{S}(s_1, ..., s_{n+1})$.

Proof. First, we note that by means of Proposition 3.4 there exists for every sufficiently small, open, simply connected subset $U \subseteq \mathcal{S}(s_1, ..., s_{n+1})$, $U \subseteq U_\alpha$ for some α, a non-vanishing, holomorphic function f_U on U satisfying

$$\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) - V_{n+1;\alpha}(y) + \log |f_U(y)|^2 = 0.$$

We are now going to extend this equality to every sufficiently small, open subset of Y away from the 2-codimensional subset $\mathcal{T}(s_1, ..., s_n)$. For this purpose, let $\mathcal{D} \subseteq \mathcal{S}(s_1, ..., s_n)$, $\mathcal{D} \subseteq U_\alpha$ for some α, be an embedded, complex 1-dimensional disk satisfying $\mathcal{D}^* \subseteq \mathcal{S}(s_1, ..., s_{n+1})$, where $\mathcal{D}^*$ denotes $\mathcal{D}$ minus its origin; let $y_0 \in Y$ correspond to the origin of $\mathcal{D}$. By arguing as in Proposition 3.5 with $\mathcal{D}$ in place of U_α, we conclude that there exists a non-vanishing, holomorphic function $f_{\mathcal{D}}$ on $\mathcal{D}$ such that the equality

$$\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) - \sum_{z(y) \in D_{n,y}} g_{n+1}(y, z(y)) - V_{n;\alpha}(y) + \log |f_{\mathcal{D}}(y)|^2 = 0 \qquad (4)$$

holds for all $y \in \mathcal{D}$. Let now $\tilde{y}$ be a local coordinate on $\mathcal{D}$ and let ν denote the number of points (counting multiplicities) in the intersection of D_{n,y_0} with Θ_{n+1,y_0} at the point $y_0 \in Y$ (corresponding to $\tilde{y} = 0$). We then find a non-vanishing, holomorphic function $\tilde{f}_{\mathcal{D}}$ on $\mathcal{D}$ such that the equality

$$\sum_{z(\tilde{y}) \in D_{n,\tilde{y}}} (g_{n+1}(\tilde{y}, z(\tilde{y})) - \log \rho_{\alpha,a}(\tilde{y}, z(\tilde{y}))) + \nu \log |\tilde{y}|^2 + \log |\tilde{f}_{\mathcal{D}}(\tilde{y})|^2 = 0 \qquad (5)$$

holds for all $\tilde{y} \in \mathcal{D}$. Adding (4) to (5) shows that there exists a holomorphic extension of f_U to *any* disk of the type considered above. The Cauchy integral formula for several variables (cf. [14], p. 5) then shows that for any $U_\alpha \in \mathcal{U}$, there exists a holomorphic function t_α defined on $U_\alpha \cap \mathcal{S}(s_1, ..., s_n)$ and satisfying

$$\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) - \log \sigma_\alpha(y) + \log |t_\alpha(y)|^2 = 0.$$

It is now easy to check that the collection of holomorphic functions $\{t_\alpha\}$ glue together by means of the transition functions $\{\psi_{\alpha,\beta}\}$ to give rise to a global section of the line bundle $M(L)$ restricted to $\mathcal{S}(s_1,...,s_n)$. Since $\operatorname{codim}(Y \setminus \mathcal{S}(s_1,...,s_n)) \geq 2$ by the choice of $s_1,...,s_{n+1}$, this global section extends to the whole of Y giving rise to a global section t of the line bundle $M(L)$ on Y. By construction, t satisfies the equality

$$\int_{X_y} g_1(y,z) * ... * g_{n+1}(y,z) = -\log \|t(y)\|^2$$

for all $y \in \mathcal{S}(s_1,...,s_{n+1})$, as claimed. □

4.8. Remark. *Characterization of the line bundle $M(L)$.*
(a) In addition to the hypotheses made in 2.1, assume that the line bundle L is f-ample and Y is quasi-projective. Then, according to Theorem 4.6(a), L has $(n+1)$ global sections $s_1,...,s_{n+1}$ in general position (eventually after twisting L suitably). Since we know from the proof of Theorem 4.6(a) that the divisors $\operatorname{div}(s_1),...,\operatorname{div}(s_{n+1})$ intersect properly on X, we conclude that $g_1 * ... * g_{n+1}$ is a well-defined Green's current on X associated to the $(n+1)$-codimensional cycle

$$D_{n+1} = \operatorname{div}(s_1) \cdot ... \cdot \operatorname{div}(s_{n+1}),$$

and we have

$$dd^c(g_1 * ... * g_{n+1}) + \delta_{D_{n+1}} = c_1(\overline{L})^{n+1}. \tag{6}$$

Applying f_* to equation (6) and observing that f_* commutes with dd^c and δ, we get

$$d_y d_y^c \left(f_*(g_1 * ... * g_{n+1})\right) + \delta_{f_*(D_{n+1})} = f_*\left(c_1(\overline{L})^{n+1}\right).$$

Therefore, the quantity $f_*(g_1*...*g_{n+1})$ is the Green's current, i.e., Green's function, associated to the divisor $f_*(D_{n+1})$ on Y. Noting that $f_*(g_1 * ... * g_{n+1})$ is nothing but the integral of the star product $g_1 * ... * g_{n+1}$ along the fibers of X/Y, a comparison with Theorem 4.7 shows that the line bundle $\mathcal{O}_Y(f_*(D_{n+1}))$ associated to the divisor $f_*(D_{n+1})$ is isomorphic to $M(L)$. In particular, this proves the relation of first Chern classes

$$c_1(M(L)) = f_*\left(c_1(L)^{n+1}\right),$$

viewed as an equality in the first Chow group $CH^1(Y)$ of Y. Using the classical Riemann-Roch Theorem in degree one, it is possible to give a more explicit description of $M(L)$ in terms of the given data. This is particularly simple, if there exists a vector bundle E on Y having the property that the relative tangent bundle $T_{X/Y}$ satisfies $T_{X/Y} = f^*E$, i.e., if the tangent bundle of the fibers X_y is trivial. By our hypotheses on the fibration $f : X \longrightarrow Y$, we then recall from [8], III.7.8.8, that there is an isomorphism $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, whence we derive by means of the projection formula for tensor products

$$f_*T_{X/Y} = f_*f^*E = f_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} f^*E) \cong f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} E \cong E.$$

By means of the classical Riemann-Roch Theorem we now obtain in the Chow ring of Y

$$ch(f_*L) = f_*\left(ch(L) \cdot Td(T_{X/Y})\right) = f_*\left(ch(L)\right) \cdot Td(E) =$$

$$\left(\frac{f_*(c_1(L)^n)}{n!} + \frac{f_*(c_1(L)^{n+1})}{(n+1)!} + ...\right) \cdot \left(1 + \frac{c_1(E)}{2} + ...\right).$$

Comparing the summands of degree one in the above equation, we are led to

$$c_1(\det f_*L) = \frac{1}{(n+1)!} \cdot f_*\left(c_1(L)^{n+1}\right) + \frac{\deg L}{2 \cdot n!} \cdot c_1(\det E),$$

where $\deg L = f_*\left(c_1(L)^n\right) \in \mathbb{Z}$ is the degree of L, which we assume for simplicity to be even (this can be achieved by replacing L by $L^{\otimes 2}$). From the last equation we then find in $CH^1(Y)$ the equality of first Chern classes

$$\begin{aligned} c_1(M(L)) = f_*\left(c_1(L)^{n+1}\right) &= (n+1)! \cdot c_1(\det f_*L) - (n+1)/2 \cdot \deg L \cdot c_1(\det E) \\ &= c_1\left(\det(f_*L)^{\otimes(n+1)!} \otimes \det\left(f_*T_{X/Y}\right)^{\otimes -(n+1)/2 \cdot \deg L}\right). \end{aligned}$$

Since the first Chern class $c_1(\cdot)$ induces an isomorphism between the Picard group $Pic(Y)$ of Y and the first Chow group $CH^1(Y)$ of Y, the above equality shows that there is an isomorphism of holomorphic line bundles

$$M(L) \cong \det(f_*L)^{\otimes(n+1)!} \otimes \det\left(f_*T_{X/Y}\right)^{\otimes -(n+1)/2 \cdot \deg L}.$$

(b) In addition to the hypotheses made in 2.1, assume that L is f-ample and $Y = Y' \setminus D$ with Y' a complex, quasi-projective variety and D an ample divisor on Y' of the form $D = \sum_{j=1}^r D_j$, where the divisors $D_1, ..., D_r$ are irreducible. Then, according to Theorem 4.6(b), L has $(n+1)$ global sections $s_1, ..., s_{n+1}$ satisfying $\mathcal{T}(s_1, ..., s_{n+1}) = \emptyset$ (eventually after twisting L suitably) and, hence, the global section t of $M(L)$ constructed in Theorem 4.7 does not vanish on the whole of Y. By construction, the Zariski closure $\overline{\operatorname{div}(t)}$ of $\operatorname{div}(t)$ in Y' is given as an integral linear combination $\sum_{j=1}^r m_j D_j$, which leads to the equality $M(L) = \mathcal{O}_{Y'}(\sum_{j=1}^r m_j D_j)|_Y$. If, in addition, all the multiplicities m_j are equal, say $m_j = m$ $(j = 1, ..., r)$, we find $M(L) = \mathcal{O}_{Y'}(mD)|_Y$, where m can be assumed to be positive by our ampleness assumptions made throughout the constructions; in this case we also see that the global section t can be extended by zero along D to the whole of Y'.

4.9. The line bundles $M_{i_1,\ldots,i_n}(L)$. With the above notations, assume that the hypotheses of Theorem 4.6(a) are satisfied such that L has $(n+1)$ global sections $s_1, ..., s_{n+1}$ in general position. For any choice of n indices $1 \leq i_1 < ... < i_n \leq n+1$, let i_{n+1} be the index different from $i_1, ..., i_n$ and denote by $D_{i_1,\ldots,i_n}$ the intersection of the n divisors $\operatorname{div}(s_{i_1}), ..., \operatorname{div}(s_{i_n})$ on X. From the proof of Theorem 4.6(a) we know that $\operatorname{div}(s_{i_1}), ..., \operatorname{div}(s_{i_n})$ intersect properly on X, which shows that $D_{i_1,\ldots,i_n}$ is a well-defined, n-codimensional cycle on X; we denote the restriction of the morphism $f : X \longrightarrow Y$ to $D_{i_1,\ldots,i_n}$ by $f_{i_1,\ldots,i_n}$. Furthermore, we denote the restriction of $D_{i_1,\ldots,i_n}$ to $\mathcal{S}(s_{i_1}, ..., s_{i_n})$ by $D'_{i_1,\ldots,i_n}$; the restriction of $f_{i_1,\ldots,i_n}$ to $D'_{i_1,\ldots,i_n}$ is then a finite morphism, which we denote by

$$f'_{i_1,\ldots,i_n} : D'_{i_1,\ldots,i_n} \longrightarrow \mathcal{S}(s_{i_1}, ..., s_{i_n}).$$

Finally, we denote the restriction of the line bundle L to $D'_{i_1,\ldots,i_n}$ by $L'_{i_1,\ldots,i_n}$. Since $f'_{i_1,\ldots,i_n}$ is a finite morphism, the higher direct images of $L'_{i_1,\ldots,i_n}$, resp. $\mathcal{O}_{D'_{i_1,\ldots,i_n}}$, by $f'_{i_1,\ldots,i_n}$ vanish; therefore, the direct images

$$f'_{i_1,\ldots,i_n *}\left(L'_{i_1,\ldots,i_n}\right), \text{ resp. } f'_{i_1,\ldots,i_n *}\left(\mathcal{O}_{D'_{i_1,\ldots,i_n}}\right),$$

are vector bundles on $\mathcal{S}(s_{i_1},...,s_{i_n})$. With this in mind, we define the following line bundle on $\mathcal{S}(s_{i_1},...,s_{i_n})$

$$M'_{i_1,...,i_n}(L) := \det\left(f'_{i_1,...,i_n *}\left(L'_{i_1,...,i_n}\right)\right) \otimes \det\left(f'_{i_1,...,i_n *}\left(\mathcal{O}_{D'_{i_1,...,i_n}}\right)\right)^{\otimes(-1)} .$$

Since $\operatorname{codim}(\mathcal{T}(s_{i_1},...,s_{i_n})) \geq 2$ by the choice of the global sections $s_1,...,s_{n+1}$, the line bundle $M'_{i_1,...,i_n}(L)$ extends to give rise to a line bundle on Y, which we denote by $M_{i_1,...,i_n}(L)$.

4.10. Lemma. *With the above notations, assume that the hypotheses of Theorem 4.6(a) are satisfied such that L has $(n+1)$ global sections $s_1,...,s_{n+1}$ in general position. Then, the norm*

$$\mathbf{N}_{D'_{i_1,...,i_n}/\mathcal{S}(s_{i_1},...,s_{i_n})}(s_{i_{n+1}})$$

of the global section $s_{i_{n+1}}$ (restricted to $D'_{i_1,...,i_n}$) with respect to the finite morphism $f'_{i_1,...,i_n}$ defines a global section of $M_{i_1,...,i_n}(L)$.

Proof. After restriction to $D'_{i_1,...,i_n}$, the global section $s_{i_{n+1}}$ of L gives rise to a global section of $L'_{i_1,...,i_n}$ and, hence, defines a homomorphism from $\mathcal{O}_{D'_{i_1,...,i_n}}$ to $L'_{i_1,...,i_n}$. After taking the push-forward by means of $f'_{i_1,...,i_n}$ and then taking determinants, we obtain a homomorphism

$$\det\left(f'_{i_1,...,i_n *}\left(\mathcal{O}_{D'_{i_1,...,i_n}}\right)\right) \longrightarrow \det\left(f'_{i_1,...,i_n *}\left(L'_{i_1,...,i_n}\right)\right) ,$$

which shows that the norm

$$\mathbf{N}_{D'_{i_1,...,i_n}/\mathcal{S}(s_{i_1},...,s_{i_n})}(s_{i_{n+1}})$$

defines a homomorphism from $\mathcal{O}_{\mathcal{S}(s_{i_1},...,s_{i_n})}$ to $M'_{i_1,...,i_n}(L)$ and, therefore, is a global section of $M'_{i_1,...,i_n}(L)$. Since $\operatorname{codim}(\mathcal{T}(s_{i_1},...,s_{i_n})) \geq 2$ by the choice of the global sections $s_1,...,s_{n+1}$, we find that the norm in question extends to the whole of Y and gives rise to a well-defined, global section of $M_{i_1,...,i_n}(L)$. □

4.11. Proposition. *With the above notations, assume that the hypotheses of Theorem 4.6(a) are satisfied such that L has $(n+1)$ global sections $s_1,...,s_{n+1}$ in general position. Then, for any choice of n indices $1 \leq i_1 < ... < i_n \leq n+1$, we have isomorphisms*

$$M(L) \cong M_{i_1,...,i_n}(L),$$

in particular, the isomorphism class of $M_{i_1,...,i_n}(L)$ does not depend on the choice of indices.

Proof. We begin by recalling from Remark 4.8(a) that the line bundle $M(L)$ is characterized in the first Chow group $CH^1(Y)$ by the property

$$c_1(M(L)) = f_*\left(c_1(L)^{n+1}\right) .$$

We now compute the first Chern class $c_1(M_{i_1,...,i_n}(L))$ of $M_{i_1,...,i_n}(L)$ in $CH^1(Y)$: For this we define X' to be the restriction of the family X to $\mathcal{S}(s_{i_1},...,s_{i_n})$ and we denote the restriction of f to X' by f'. We then obtain the commutative diagram

$$\begin{tikzcd}
D'_{i_1,...,i_n} \arrow[r, "j_{i_1,...,i_n}"] \arrow[d, "f'_{i_1,...,i_n}"'] & X' \arrow[d, "f'"] \\
\mathcal{S}(s_{i_1},...,s_{i_n}) \arrow[r, "\mathrm{id}"] & \mathcal{S}(s_{i_1},...,s_{i_n})
\end{tikzcd}$$

with $j_{i_1,...,i_n}$ denoting the natural embedding. Letting L' denote the restriction of L to X', observing the equality $L'_{i_1,...,i_n} = j^*_{i_1,...,i_n}(L')$ and setting $\mathcal{F} := j_{i_1,...,i_n*}(\mathcal{O}_{D'_{i_1,...,i_n}})$, we compute by means of the projection formula

$$f'_{i_1,...,i_n*}\left(L'_{i_1,...,i_n}\right) = f'_*(L' \otimes \mathcal{F}).$$

Therefore, we obtain

$$M'_{i_1,...,i_n}(L) = \det(f'_*(L' \otimes \mathcal{F})) \otimes \det(f'_*(\mathcal{F}))^{\otimes(-1)},$$

i.e., we have

$$c_1\left(M'_{i_1,...,i_n}(L)\right) = c_1(f'_*(L' \otimes \mathcal{F})) - c_1(f'_*(\mathcal{F})).$$

We now compute the right-hand side of the above equation using the Riemann-Roch Theorem twice. First, using the Riemann-Roch Theorem with respect to the smooth morphism $f' : X' \longrightarrow \mathcal{S}(s_{i_1}, ..., s_{i_n})$, we obtain

$$\begin{aligned}
&ch(f'_*(L' \otimes \mathcal{F})) - ch(f'_*(\mathcal{F})) = \\
&f'_*\left(ch(L' \otimes \mathcal{F}) \cdot Td(T_{f'})\right) - f'_*\left(ch(\mathcal{F}) \cdot Td(T_{f'})\right) = \\
&f'_*\left((ch(L') - 1) \cdot ch(\mathcal{F}) \cdot Td(T_{f'})\right),
\end{aligned}$$

where $T_{f'}$ denotes the relative tangent bundle associated to $f' : X' \longrightarrow \mathcal{S}(s_{i_1}, ..., s_{i_n})$. In order to compute $ch(\mathcal{F})$, we use the Riemann-Roch Theorem with respect to the closed immersion $j_{i_1,...,i_n} : D'_{i_1,...,i_n} \hookrightarrow X'$. Letting $N_{D'_{i_1,...,i_n}/X'}$ denote the normal bundle of $D'_{i_1,...,i_n}$ in X', we obtain

$$\begin{aligned}
&ch(\mathcal{F}) = ch\left(j_{i_1,...,i_n*}\left(\mathcal{O}_{D'_{i_1,...,i_n}}\right)\right) = j_{i_1,...,i_n*}\left(ch\left(\mathcal{O}_{D'_{i_1,...,i_n}}\right) \cdot Td\left(N_{D'_{i_1,...,i_n}/X'}\right)^{-1}\right) = \\
&j_{i_1,...,i_n*}((1 + ...) \cdot (1 + ...)) = j_{i_1,...,i_n*}\left([D'_{i_1,...,i_n}]\right) + ... = c_1(L')^n + ... ,
\end{aligned}$$

where $[D'_{i_1,...,i_n}] \in CH^0(D'_{i_1,...,i_n})$ denotes the fundamental cycle and the dots refer to contributions of higher degree. Taking the two Riemann-Roch computations together, we arrive at

$$\begin{aligned}
&ch(f'_*(L' \otimes \mathcal{F})) - ch(f'_*(\mathcal{F})) = \\
&f'_*\left((c_1(L') + ...) \cdot (c_1(L')^n + ...) \cdot (1 + ...)\right) = \\
&f'_*\left(c_1(L')^{n+1} + ...\right),
\end{aligned}$$

i.e., we have

$$c_1(M'_{i_1,...,i_n}(L)) = f'_*\left(c_1(L')^{n+1}\right).$$

Since $\operatorname{codim}(\mathcal{T}(s_{i_1}, ..., s_{i_n})) \geq 2$ by the choice of the global sections $s_1, ..., s_{n+1}$, the homomorphism from $CH^1(Y)$ to $CH^1(\mathcal{S}(s_{i_1}, ..., s_{i_n}))$ obtained by restriction is an isomorphism; therefore, we derive from the above equation

$$c_1(M_{i_1,...,i_n}(L)) = f_*\left(c_1(L)^{n+1}\right).$$

This shows

$$c_1(M_{i_1,...,i_n}(L)) = c_1(M(L)),$$

which proves that the line bundles $M(L)$ and $M_{i_1,...,i_n}(L)$ are isomorphic as claimed. □

4.12. Theorem. *With the above notations, assume that the hypotheses of Theorem 4.6(a) are satisfied such that L has $(n+1)$ global sections $s_1,...,s_{n+1}$ in general position, and assume that Y has a compactification $\overline{Y}$ with the property* $\operatorname{codim}(\overline{Y}\setminus Y)\geq 2$. *Then, the global section t constructed in Theorem 4.7 is given by*

$$t=\zeta_{i_1,...,i_n}\cdot\mathbf{N}_{D'_{i_1,...,i_n}/\mathcal{S}(s_{i_1},...,s_{i_n})}(s_{i_{n+1}}),$$

where $\zeta_{i_1,...,i_n}$ is a non-zero constant (depending on the choice of $s_1,...,s_{n+1}$).

Proof. To simplify notations, we set

$$t_{i_1,...,i_n}:=\mathbf{N}_{D'_{i_1,...,i_n}/\mathcal{S}(s_{i_1},...,s_{i_n})}(s_{i_{n+1}}),$$

which is by Lemma 4.10 a global section of $M_{i_1,...,i_n}(L)$. Since there is an isomorphism $M(L)\otimes M_{i_1,...,i_n}(L)^{\otimes(-1)}\cong\mathcal{O}_Y$ for any choice of n indices $1\leq i_1<...<i_n\leq n+1$ by Proposition 4.11, the quotient $t/t_{i_1,...,i_n}$ is a rational function on Y.

We are now going to show that this rational function is holomorphic on Y: For this we first note that locally, for sufficiently small, open subsets U of $\mathcal{S}(s_{i_1},...,s_{i_n})$, the global section $t_{i_1,...,i_n}$ is given as the finite product

$$t_{i_1,...,i_n}(y)=\prod_{z(y)\in D_{i_1,...,i_n;y}}s_{i_{n+1}}(y,z(y))\qquad(y\in U);$$

here $D_{i_1,...,i_n;y}$ denotes the restriction of $D_{i_1,...,i_n}$ (or $D'_{i_1,...,i_n}$) to the fiber X_y. This allows us to define

$$\|t_{i_1,...,i_n}(y)\|^2:=\prod_{z(y)\in D_{i_1,...,i_n;y}}\|s_{i_{n+1}}(y,z(y))\|^2\qquad(y\in U),$$

where the hermitian metric on the right-hand side is the one given for the hermitian line bundle $\overline{L}$. Let now $\mathcal{D}\subseteq\mathcal{S}(s_{i_1},...,s_{i_n})$ be an embedded disk satisfying $\mathcal{D}^*\subseteq\mathcal{S}(s_1,...,s_{n+1})$, where $\mathcal{D}^*$ denotes $\mathcal{D}$ minus its origin. Taking into account the commutativity of the star product, we then find by Proposition 3.5 (cf. also the proof of Theorem 4.7, up to a permutation of the indices) that the difference

$$\begin{aligned}&-\log\|t(y)\|^2+\log\|t_{i_1,...,i_n}(y)\|^2=\\&\int_{X_y}g_{i_1}(y,z)*...*g_{i_{n+1}}(y,z)+\sum_{z(y)\in D_{i_1,...,i_n;y}}\log\|s_{i_{n+1}}(y,z(y))\|^2=\\&\sum_{k=1}^{n}\int_{X_y}g_{i_k}(y,z)\wedge\delta_{D_{i_1,...,i_{k-1};y}}\wedge\omega_y^{n+1-k}\end{aligned}$$

is well-defined for all $y\in\mathcal{D}$; here $D_{i_1,...,i_{k-1};y}$ denotes the restriction of the cycle $\operatorname{div}(s_{i_1})\cdot...\cdot\operatorname{div}(s_{i_{k-1}})$ $(k=1,...,n)$ to the fiber X_y $(y\in\mathcal{D})$. This proves that the rational function $t/t_{i_1,...,i_n}$ is holomorphic on $\mathcal{S}(s_{i_1},...,s_{i_n})$. Since we have $\operatorname{codim}(\mathcal{T}(s_{i_1},...,s_{i_n}))\geq 2$ by our choice of global sections, we conclude by the Riemann-Hartog Theorem that $t/t_{i_1,...,i_n}$ extends to a holomorphic function on the whole of Y. Furthermore, since $\operatorname{codim}(\overline{Y}\setminus Y)\geq 2$, the rational function $t/t_{i_1,...,i_n}$ extends for the same reason in fact to a holomorphic function on $\overline{Y}$ and, hence, has to be constant. □

4.13. Remark. Assume that the hypotheses of Theorem 4.6(b) are satisfied such that L has $(n+1)$ global sections $s_1, ..., s_{n+1}$ satisfying $\mathcal{T}(s_1, ..., s_{n+1}) = \emptyset$, and assume that Y' has a compactification $\overline{Y'}$ with the property $\mathrm{codim}(\overline{Y'} \setminus Y') \geq 2$. Moreover, assume that the smooth family $f : X \longrightarrow Y$ together with the f-ample, hermitian line bundle $\overline{L}$ extends to a *smooth* family $f' : X' \longrightarrow Y'$ together with a f'-ample, hermitian line bundle $\overline{L'}$. Then, the proof of Theorem 4.6(b) shows that the global sections $s_1, ..., s_{n+1}$ are obtained by restriction to Y of global sections $s'_1, ..., s'_{n+1}$ of L', which are in general position. Now, we are in position to apply Theorem 4.12 over Y'. After restriction to Y, we obtain that there is a non-zero constant $\zeta_{i_1,...,i_n}$ (depending on the choice of $s_1, ..., s_{n+1}$) such that the global section t constructed in Theorem 4.7 is given by

$$t = \zeta_{i_1,...,i_n} \cdot \mathbf{N}_{D_{i_1,...,i_n}/Y}(s_{i_{n+1}}),$$

since the morphism $f_{i_1,...,i_n} : D_{i_1,...,i_n} \longrightarrow Y$ is finite by our choice of global sections.

4.14. Remark. The hypothesis made in Theorem 4.12 that there exists a compactification $\overline{Y}$ of Y satisfying $\mathrm{codim}(\overline{Y} \setminus Y) \geq 2$, is for example fulfilled, if Y is the quotient of a symmetric space H by an arithmetic subgroup Γ and $H = G(\mathbb{R})/K$, where G is a semi-simple Lie group of $\mathbb{R}$-rank bigger than one and K a maximal compact subgroup of $G(\mathbb{R})$; in this case, $\overline{Y}$ is given by the Baily-Borel compactification of Y.

5 Examples

5.1. Abelian varieties. Let Y be the moduli space of polarized, n-dimensional abelian varieties, X the corresponding universal abelian variety over Y and $\overline{L}$ the hermitian line bundle on X corresponding to the given polarization, equipped with the (up to scaling) unique translation invariant metric. In [11] it is shown that the hermitian line bundle $\overline{M}(\overline{L})$ occuring in Theorem 4.7 equals the line bundle of Siegel modular forms equipped with the Petersson metric, and, hence, the global section t is a Siegel modular form of a certain weight, which can be explicitly given. If $n \geq 2$, Theorem 4.12 applies and leads in fact to a (global) finite product expansion of t in terms of Siegel-Jacobi forms evaluated along certain torsion sections. For a detailed exposition of this example, in particular the explicit formulas given in the cases $n = 1, 2, 3$, we refer the interested reader to the article [11]. As already mentioned in the introduction, we emphasize the coincidence of these results with the ones obtained by K.-I. Yoshikawa in his computation of the Quillen metric of the determinant line bundle $\det f_*(L)$ (cf. [21]), which can be explained by an application of the arithmetic Riemann-Roch Theorem in degree one to a model of the fibration $f : X \longrightarrow Y$ over the integers.

5.2. Algebraic curves over function fields. Let Y' be a smooth, projective curve of genus g' and $\mathcal{M}_{g,N}$ the moduli space of smooth, projective curves of genus $g > 1$ with a Jacobi structure of level N, i.e., with a fixed isomorphism from the N-torsion of the Jacobian to $(\mathbb{Z}/N\mathbb{Z})^{2g}$; if $N \geq 3$, $\mathcal{M}_{g,N}$ is a smooth, quasi-projective variety (cf. [4], p. 108). Assuming that there is a holomorphic embedding $Y' \hookrightarrow \mathcal{M}_{g,N}$, we obtain, by restriction, a family $f' : X' \longrightarrow Y'$ of smooth, projective curves of genus g (with a Jacobi structure of level N) over Y'. The canonical line bundle on the fibers X'_y ($y \in Y'$) has a (up to scaling) unique hermitian metric, which corresponds to the hyperbolic metric on X'_y having constant negative curvature and being compatible with the complex structure of X'_y; the existence of this metric is a direct consequence of the uniformization theorem. This family of hermitian line bundles gives rise to a hermitian line bundle $\overline{L'}$ on X'; the smoothness of the hermitian metric on $\overline{L'}$ is based on classical results in Teichmüller theory (cf. [20], p. 969).

We now let $P \in Y'$ be a point and consider the complex, quasi-projective variety $Y = Y' \setminus P$ together with the family $f : X \longrightarrow Y$ with $X = X'|_Y$ and the hermitian line bundle $\overline{L} = \overline{L'}|_Y$. By Theorem 4.6(b), eventually after replacing L by $L^{\otimes m_1}$ for $m_1 \gg 0$, we find two global sections s_1, s_2 of L satisfying $\mathcal{T}(s_1, s_2) = \emptyset$, i.e., $\mathcal{S}(s_1, s_2) = Y$. From Theorem 4.7, we then derive for all $y \in Y$

$$\int_{X_y} g_1(y,z) * g_2(y,z) = -\log \|t(y)\|^2,$$

where t is a global section of the line bundle $M(L)$, whose divisor has, by Remark 4.8, Zariski closure $\overline{\mathrm{div}(t)} = mP$ for a suitable positive integer m, i.e., $M(L) = \mathcal{O}_{Y'}(mP)|_Y$. An application of Remark 4.13 leads, up to a constant, to a representation of t as the norm of s_1 (resp. s_2) restricted to $\mathrm{div}(s_2)$ (resp. $\mathrm{div}(s_1)$) down to Y. Since the global sections s_1, s_2 are given in terms of classical Riemann theta functions and t in terms of theta constants on the corresponding Jacobians, this leads to theta relations on the curve Y'.

5.3. Enriques surfaces. We begin by summarizing the essential facts about Enriques surfaces and their moduli spaces needed for the discussion of this example; for more details and proofs of these facts, we refer to the monograph [1] and to the articles [2] and [5]:

An Enriques surface Z is a complex, non-singular, projective surface such that the canonical line bundle K_Z is non-trivial, but its square $K_Z^{\otimes 2}$ is trivial, and such that the cohomology group $H^1(Z, \mathcal{O}_Z)$ is trivial. The universal cover $\tilde{Z}$ of Z is a two-fold cover by a $K3$-surface, which is defined as a complex, non-singular, compact surface with trivial canonical line bundle $K_{\tilde{Z}}$ and trivial cohomology group $H^1(\tilde{Z}, \mathcal{O}_{\tilde{Z}})$. The $K3$-surface $\tilde{Z}$ admits a fixed-point free involution ι such that $Z = \tilde{Z}/\langle\iota\rangle$. Every Enriques surface Z admits a polarization of degree two; we let L_Z denote the line bundle corresponding to this polarization. Furthermore, every Enriques surface Z admits a (up to scaling) unique Calabi-Yau metric whose curvature form defines a cohomology class which is equal to the first Chern class of L_Z (here viewed as a cohomology class in contrast to the preceding part of the paper). We equip L_Z with the (up to scaling) unique hermitian metric such that the corresponding first Chern form equals the curvature form of the Calabi-Yau metric.

Let now Y_{Enr} be the coarse moduli space of Enriques surfaces with a polarization of degree two. It is known that $Y_{\mathrm{Enr}} = Y'_{\mathrm{Enr}} \setminus D_{\mathrm{Enr}}$ with $Y'_{\mathrm{Enr}} = \Gamma_{\mathrm{Enr}} \backslash \mathbb{H}_{2,10}$, where $\mathbb{H}_{2,10}$ is the symmetric space associated to the connected component of the identity of the special orthogonal group $\mathrm{SO}(2,10)$, Γ_{Enr} is an arithmetic subgroup and D_{Enr} is a divisor. It is shown in [2] that D_{Enr} is ample and irreducible, that $M_{\mathrm{Enr}} := \mathcal{O}_{Y'_{\mathrm{Enr}}}(D_{\mathrm{Enr}})$ is isomorphic to the line bundle of automorphic forms of weight four on $\mathbb{H}_{2,10}$ with respect to Γ_{Enr} and that Borcherds's Φ-function is a global section of M_{Enr} with divisor D_{Enr} (cf. [2], p. 708).

Since Y_{Enr} is a stack, there is no universal family over it. By going to a suitable normal subgroup Γ of finite index in Γ_{Enr}, we obtain a fine moduli space Y of Enriques surfaces with a polarization of degree two and some additional structure, which is unimportant for the present discussion. We then find $Y = Y' \setminus D$, where $Y' = \Gamma \backslash \mathbb{H}_{2,10}$ and $D = \pi^{-1}(D_{\mathrm{Enr}})$ with the finite morphism $\pi : Y' \longrightarrow Y'_{\mathrm{Enr}}$; we note that the divisor D is of the form $D = \sum_{j=1}^{r} D_j$ with $D_1, ..., D_r$ irreducible and that the covering group $G := \Gamma_{\mathrm{Enr}}/\Gamma$ of Y' over Y'_{Enr} acts on D by permuting the irreducible components $D_1, ..., D_r$. Over the fine moduli space Y there exists a universal family $f : X \longrightarrow Y$ of Enriques surfaces with a polarization of degree two and the (unimportant) extra structure together with a universal line bundle L corresponding to the given polarization, which carries the (up to scaling) unique, smooth, hermitian metric $\|\cdot\|$ determined by the Calabi-Yau metric; we note that G acts in a natural way on L. As usual, we write $\overline{L} = (L, \|\cdot\|)$.

By Theorem 4.6(b), eventually after replacing L by $L^{\otimes m_1}$ for $m_1 \gg 0$, we find three global sections s_1, s_2, s_3 of L satisfying $\mathcal{T}(s_1, s_2, s_3) = \emptyset$, i.e., $\mathcal{S}(s_1, s_2, s_3) = Y$. From Theorem 4.7, we then derive for all $y \in Y$

$$\int_{X_y} g_1(y,z) * g_2(y,z) * g_3(y,z) = -\log \|t(y)\|^2,$$

where t is a global section of the line bundle $M(L)$, whose divisor has, by Remark 4.8(b), Zariski closure $\overline{\operatorname{div}(t)} = \sum_{j=1}^{r} m_j D_j$ for suitable integers $m_1, ..., m_r$, i.e., $M(L) = \mathcal{O}_{Y'}(\sum_{j=1}^{r} m_j D_j)|_Y$. By means of the G-action on L, we obtain a G-action on the Green's currents $g_j = -\log \|s_j\|^2$ and, hence, a G-action on the global section t, which we denote by $\gamma \circ t$ ($\gamma \in G$). By setting

$$t_{\mathrm{Enr}} := \prod_{\gamma \in G} \gamma \circ t,$$

we see that t_{Enr} descends to a global section on Y_{Enr} satisfying $\overline{\operatorname{div}(t_{\mathrm{Enr}})} = m D_{\mathrm{Enr}}$, where m is a multiple of the sum $\sum_{j=1}^{r} m_j$. Hence, we finally deduce that $t_{\mathrm{Enr}} = c \cdot \Phi^m|_Y$ for a suitable non-zero constant c.

We are left to describe the smooth, hermitian metric on $M(L)$: For this purpose, as mentioned at the beginning, we take into account that the universal family of Enriques surfaces X can equivalently be interpreted as a family $\tilde{X}_y$ ($y \in Y$) of $K3$-surfaces equipped with an involution ι and some extra structure; the pull-back of the line bundle L_y corresponding to the degree two polarization of X_y to $\tilde{X}_y$ is denoted by $\tilde{L}_y$. Let now $\mathcal{D} \subseteq Y'$ be an embedded disk satisfying $\mathcal{D}^* \subseteq Y$, where $\mathcal{D}^*$ denotes $\mathcal{D}$ minus its origin 0. By [12], Theorem 20, we know that as y approaches 0, the surface $\tilde{X}_y$ converges to a complex, non-singular, compact surface $\tilde{X}_0$, while the line bundle $\tilde{L}_y$ converges to a line bundle $\tilde{L}_0$, which has vanishing intersection with a finite number of (-2)-curves $E_{0,j}$ on $\tilde{X}_0$, in fact, at most 16 such curves (cf. also [13], section 6). Furthermore, it is shown in [12], Theorem 21, that the smooth, hermitian metric on $\tilde{L}_y$ (obtained by pulling back the smooth, hermitian metric on L_y) converges to a smooth, hermitian metric on $\tilde{L}_0$ away form the (-2)-curves $E_{0,j}$, where the metric tends polynomially to zero with respect to the absolute value of the local coordinate of $\mathcal{D}$ (cf. also [12], Theorems 18 and 19). Summarizing, we end up with holomorphic extensions of the functions $\varphi_{\alpha,a;\beta,b}$ (defining the holomorphic transition functions of L), resp. with smooth, non-negative extensions of the functions $\rho_{\alpha,a}$ (defining the smooth, hermitian metric on L) over Y', which we denote for simplicity by φ', resp. ρ' (keeping in mind that φ', resp. ρ' is defined locally, over open subsets U' of Y').
By eventually modifying the choice of the global sections s_1, s_2, s_3, we may achieve that s_1, s_2, s_3 extend to global sections s_1', s_2', s_3' over Y' such that $\operatorname{div}(s_1')$ does generically not intersect the (-2)-curves $E_{0,j}$, i.e., $\operatorname{div}(s_1')$ meets the (-2)-curves $E_{0,j}$ only above a 2-codimensional subset E of Y' contained in $\operatorname{supp}(D)$. By means of such a choice of global sections and the functions φ', resp. ρ', we are able to extend the formulas defining the functions $\psi_{\alpha,\beta}$ (giving rise to the holomorphic transition functions of $M(L)$, cf. Definition 4.2), resp. defining the functions σ_α (giving rise to the smooth, hermitian metric on $M(L)$, cf. Definition 4.4) across Y' away from the 2-codimensional subset E. In this way, we see that the smooth, hermitian metric on $M(L)$ extends to a smooth, hermitian metric on the line bundle $\mathcal{O}_{Y'}(\sum_{j=1}^{r} m_j D_j)$ away from E. By previous arguments allowing us to discard 2-codimensional subsets, we conclude that the metric in question extends across E, and we obtain a smooth, hermitian metric on $\mathcal{O}_{Y'}(\sum_{j=1}^{r} m_j D_j)$;

we denote the smooth, positive functions defining this metric for simplicity by σ' (keeping in mind that σ' is defined locally, on open subsets U' of Y'). Finally, after descending from Y' to Y'_{Enr}, we end up with a smooth, hermitian metric $\|\cdot\|$ on $M_{\mathrm{Enr}}^{\otimes m}$.
We finally compare this metric with the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ on $M_{\mathrm{Enr}}^{\otimes m}$. For this purpose, it suffices to compare the pull-backs of the metrics $\|\cdot\|$ and $\|\cdot\|_{\mathrm{Pet}}$ to $\mathcal{O}_{Y'}(\sum_{j=1}^{r} m_j D_j)$; we denote these pull-back metrics again by $\|\cdot\|$, resp. $\|\cdot\|_{\mathrm{Pet}}$. Now we argue locally, as follows: Let $U' \subseteq Y'$ be a sufficiently small, open subset of Y' such that there exists an open subset $\tilde{U} \subseteq \mathbb{H}_{2,10}$ which is biholomorphically mapped onto $U' \subseteq Y'$ via the natural projection $\mathbb{H}_{2,10} \longrightarrow Y'$. For $\tilde{y} \in \tilde{U}$ corresponding to $y \in U'$, we denote by $X_{\tilde{y}}$ the corresponding Enriques surface (resp. the corresponding limiting surface, cf. above) with line bundle $L_{\tilde{y}}$. If h is now a (biholomorphic) automorphism of $\mathbb{H}_{2,10}$, h induces an isomorphism between the cohomology groups of $X_{\tilde{y}}$ and $X_{h(\tilde{y})}$, which maps the first Chern class of $L_{\tilde{y}}$ (again viewed as a cohomology class) to that of $L_{h(\tilde{y})}$; furthermore, since the Monge-Ampère equations show that the Calabi-Yau metric on $L_{h(\tilde{y})}$ (resp. the corresponding limiting metric, cf. above) is induced from the Calabi-Yau metric on $L_{\tilde{y}}$ (resp. the corresponding limiting metric, cf. above) by the action of h, we even find that the first Chern *forms* of $L_{\tilde{y}}$ and $L_{h(\tilde{y})}$ agree. Recalling the constructions of the preceding paragraph, we therefore obtain the equalities

$$d_{\tilde{y}} d^c_{\tilde{y}} \log \sigma'(\tilde{y}) = d_{\tilde{y}} d^c_{\tilde{y}} \log \sigma'(h(\tilde{y})) = d_{h(\tilde{y})} d^c_{h(\tilde{y})} \log \sigma'(h(\tilde{y})) \qquad (\tilde{y} \in \tilde{U});$$

here, by abuse of notation, we denote the function induced by σ' on $\tilde{U}$ and $h(\tilde{U})$ again by σ'. This shows that the curvature of our metric $\|\cdot\|$ is invariant under (biholomorphic) automorphims of $\mathbb{H}_{2,10}$, and, hence, $\|\cdot\|$ is a constant multiple of the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$.

In connection with this example we also refer to [19] for general considerations concerning the nature of the induced Weil-Petersson metric on the moduli space of Calabi-Yau varieties. As already mentioned in the introduction, we emphasize the coincidence of our result with the one obtained by K.-I. Yoshikawa in his computation of the Quillen metric of the determinant line bundle $\det f_*(L)$ (cf. [22]), which could again be explained by an application of the arithmetic Riemann-Roch Theorem in degree one, if this was possible in this context.

5.4. Open problems. To end this section, we list another set of examples, where the results of this paper apply, which needs further investigation:

- **Algebraic $K3$-surfaces.** Let $Y_{K3,d}$ be the moduli space of algebraic $K3$-surfaces with a polarization of degree d. It is known that $Y_{K3,d} = Y'_{K3,d} \setminus D_{K3,d}$ with $Y'_{K3,d} = \Gamma_{K3,d} \backslash \mathbb{H}_{2,19}$, where $\mathbb{H}_{2,19}$ is the symmetric space associated to the connected component of the identity of the special orthogonal group $\mathrm{SO}(2,19)$, $\Gamma_{K3,d}$ is an arithmetic subgroup and $D_{K3,d}$ is a not necessarily irreducible divisor; assume that d is such that $D_{K3,d}$ is ample. By going to a suitable finite cover Y of $Y_{K3,d}$, we obtain a universal family $f : X \longrightarrow Y$ (cf. [17]) together with a universal line bundle L corresponding to the given polarization, which carries a (up to scaling) unique Calabi-Yau metric. By proceeding now as in example 5.3, in particular applying Theorems 4.6(b) and 4.7, we end up, after descending back to $Y_{K3,d}$, with a distinguished global section $t_{K3,d}$ having the property that the support of $\overline{\mathrm{div}(t_{K3,d})}$ is contained in the support of $D_{K3,d}$. What remains to be investigated is the characterization of the line bundle $M(L)$ and its smooth, hermitian metric. It seems likely that one is led once again to a line bundle of distinguished automorphic forms on $\mathbb{H}_{2,19}$ with respect to $\Gamma_{K3,d}$ equipped with the Petersson metric; if this turns out to be true, the global section $t_{K3,d}$ would be of a particular interest.

- **Calabi-Yau varieties.** In [2], p. 700, a conjecture of I. Dolgachev is described, which asserts that the family of Calabi-Yau n-folds obtained by a resolution of double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position, can be parametrized by a tube domain H_{CY}. Going beyond Dolgachev's conjecture, one could speculate that one obtains a moduli space of such varieties in the form $Y_{CY} = Y'_{CY} \setminus D_{CY}$ with $Y'_{CY} = \Gamma_{CY} \backslash H_{CY}$, where Γ_{CY} is an arithmetic subgroup and D_{CY} an ample, irreducible divisor. Assuming the validity of this conjecture, one could employ the methods developped in this paper to the universal family X_{CY} over Y_{CY} (eventually, one would have to pass to a suitable finite cover of Y_{CY}) together with the universal line bundle L_{CY} (obtained by the pull-back of the family of $\mathcal{O}_{\mathbb{P}^n}(1)$'s adjusted by the exceptional divisors of the resolution), which is equipped with a (up to scaling) unique Calabi-Yau metric. For a suitable choice of global sections of L_{CY}, the integral of the star product of Green's currents under consideration would lead to a distinguished global section t_{CY} on Y_{CY} extending by zero to the whole of Y'_{CY}. The section t_{CY} would lead to a generalization of Borcherds's Φ-function.
- **Complete intersection varieties.** Let Y be the parameter space of complete intersection varieties in some projective space or, more generally, in some toric variety, X the corresponding universal family and L the line bundle associated to a generic hyperplane section, equipped with the natural metric induced from the ambient space (e.g., the Fubini-Study metric). It is reasonable to expect that, after a suitable choice of global sections $s_1, ..., s_{n+1}$ of L, the global section t of $M(L)$ determined in Theorem 4.7 can be related to the algebraic discriminant (resultant) of the family X. Such a result would be analogous to the realization of the classical Dedekind delta function as a discriminant of cubics. Theorem 4.12, if applicable, would then provide a representation for the algebraic discriminant as a certain norm. In the setting of elliptic curves equipped with a level-N structure, the resulting formula is stated in [10] as a generalization of the Jacobi triple product formula.

References

[1] *Barth, W.; Peters, C.; Van de Ven, A.:* Compact complex surfaces. Ergeb. Math. **4**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1984.

[2] *Borcherds, R.:* The moduli space of Enriques surfaces and the fake Monster Lie algebra. Topology **35** (1996), 699-710.

[3] *Deligne, P.:* Le déterminant de la cohomologie. Current Trends in Arithmetical Algebraic Geometry. Contemp. Math. **67** (1987), 93-178.

[4] *Deligne, P.; Mumford, D.:* The irreducibility of the space of curves of given genus. Publ. Math. I.H.E.S. **36** (1969), 75-109.

[5] *Dolgachev, I.:* On automorphisms of Enriques surfaces. Invent. Math. **76** (1984), 163-177.

[6] *Elkik, R.:* Fibrés d'intersections et intégrales de classes de Chern. Ann. Scient. Éc. Norm. Sup. **22** (1989), 195-226.

[7] *Elkik, R.:* Métriques sur les fibrés d'intersections. Duke Math. J. **61** (1990), 303-328.

[8] *Grothendieck, A.; Dieudonné, J.:* Éléments de géométrie algébrique. Publ. Math. I.H.E.S. **17** (1963), 1-91.

[9] *Hörmander, L.:* The analysis of linear partial differential operators I, 2nd edition. Grundlehren math. Wiss. **256**, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong 1990.

[10] *Jorgenson, J.:* Analytic torsion for line bundles on Riemann surfaces. Duke Math J. **62** (1991), 527-549.

[11] *Jorgenson, J.; Kramer, J.:* Towards the arithmetic degree of line bundles on abelian varieties. manuscripta math. **96** (1998), 335-370.

[12] *Kobayashi, R.:* Moduli of Einstein metrics on a $K3$-surface and degenerations of type I. Advanced Studies in Pure Mathematics **18** (1990), 257-311.

[13] *Kobayashi, R.; Todorov, A.:* Polarized period map for generalized $K3$-surfaces and the moduli of Einstein metrics. Tôhoku Math. Journ. **39** (1987), 341-363.

[14] *Krantz, S.:* Function theory of several complex variables. Wadsworth & Brooks/Cole Advanced Books & Software (2nd edition), Pacific Grove, California 1992.

[15] *Kühn, U.:* Generalized arithmetic intersection numbers. C. R. Acad. Sci. Paris **327** (1998), 283-288.

[16] *Mumford, D.; Fogarty, J.; Kirwan, F.:* Geometric invariant theory. Springer-Verlag (3rd enlarged edition), Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong-Barcelona-Budapest 1994.

[17] *Piatetski-Shapiro, I.; Shafarevich, I.:* A Torelli theorem for algebraic surfaces of $K3$-type. Izv. Akad. Nau. USSR **35** (1971), 503-571.

[18] *Soulé, C.; Abramovich, D.; Burnol, J.-F.; Kramer, J.:* Lectures on Arakelov Geometry. Cambridge University Press, Cambridge 1992.

[19] *Tian, G.:* Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Weil-Petersson metric. In: Math. Aspects of String Theory, ed. S.-T. Yau. World Scientific, Singapore (1988), 629-646.

[20] *Wolpert, S.:* On the Weil-Petersson geometry of the moduli space of curves. Amer. J. Math. **107** (1985), 969-997.

[21] *Yoshikawa, K.-I.:* Discriminant of theta divisors and Quillen metrics. Preprint 1998.

[22] *Yoshikawa, K.-I.:* Generalized Enriques surfaces and analytic torsion. Preprint 1998.

Jay Jorgenson
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078
U.S.A.
e-mail: jjorgen@littlewood.math.okstate.edu
and
Department of Mathematics
City College of New York
Convent Avenue at 138th Street
New York, NY 10031
U.S.A.

Jürg Kramer
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
D-10099 Berlin
Germany
e-mail: kramer@mathematik.hu-berlin.de