Index determination and Calculation of Consistent Initial Values for DAEs

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Abstract

The index definition of DAEs with properly stated leading term bases on a matrix sequence with suitably chosen projectors. A way of realization of this matrix sequence is presented, it includes the calculation of suitable projectors using generalized inverses of the sequence matrices. With the help of the matrix sequence elements (matrices, projectors) we calculate consistent initial values for at most index 2 DAEs by describing the hidden constraint. Examples are given.

Keywords: DAE, index determination, consistent initial values, generalized inverse, hidden constraint

MSC2000: 65L80, 65F20

1 Introduction

Most of the DAEs coming from applications have or are easy to transform to the following structure

$$A(x(t),t)(D(t)x(t))' + b(x(t),t) = 0, \quad t \in I, \tag{1.1}$$

where I describes the interval of interest. A linearization of (1.1) along a given function $x_\star \in C_D^1 = \{z : z \in C(I), Dz \in C^1(I)\}$ leads to

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{1.2}$$

with $A(t) := A(x_\star(t),t)$, $B(y,x,t) := (A(x,t)y + b(x,t))'_x$ and $B(t) := B((D(t)x_\star(t))', x_\star(t), t)$, where all coefficients are supposed to be continuous matrix functions $A(t) \in I\!R^{n\times m}, D(t) \in I\!R^{m\times n}$ and $B(t) \in I\!R^{n\times n}$ (see [Mär01]).
The coefficients $A(t)$ and $D(t)$ fulfil

Definition 1.1 *[Mär02] The leading term of (1.2) is stated properly if the coefficients $A(t)$ and $D(t)$ are well-matched in the sense that*

$$\ker A(t) \oplus \operatorname{im} D(t) = I\!R^m, \quad t \in I,$$

and there is a continuously differentiable projector $R(t) \in I\!R^{m\times m}$ such that $\operatorname{im} R(t) = \operatorname{im} D(t), \quad \ker R(t) = \ker A(t), \quad t \in I.$

For our further considerations we will drop the argument t.
To describe the structure of a DAE and to determine the index we form a sequence of matrices. For given coefficients A, D and B (A and D well-matched) we define

$$\begin{array}{rcl} G_0 & := & AD, \quad B_0 := B, \\ G_{i+1} & := & G_i + B_iQ_i = (G_i + W_iB_iQ_i)(I + G_i^- B_iQ_i), \\ B_{i+1} & := & (B_i - G_{i+1}D^-(DP_0\dots P_{i+1}D^-)'DP_0\dots P_{i-1})P_i, \end{array} \tag{1.3}$$

where Q_i denotes a projector function such that $\operatorname{im} Q_i = \ker G_i$, $P_i := I - Q_i$ and W_i is a projector function such that $\ker W_i = \operatorname{im} G_i$. D^- denotes the reflexive generalized inverse of D such that $D^-DD^- = D^-, \quad DD^-D = D, \quad DD^- = R$ and $D^-D = P_0$, and G_i^- is the reflexive generalized inverse of G_i with $G_i^-G_i = P_i$ and $G_iG_i^- = I - W_i$. Using the condition $W_jG_j = 0$ we can replace the determination of G_{i+1} by

$$G_{i+1} := (G_i + W_iB_0Q_i)(I + G_i^- B_iQ_i). \tag{1.4}$$

Definition 1.2 *[Mär02] An equation (1.2) with properly stated leading term is said to be a tractability index μ DAE on the interval I, $\mu \in \mathbb{N}$, if there is a continuous matrix function sequence (1.3) such that*

(a) G_i has constant rank r_i on I,

(b) the projector Q_i fulfils $Q_iQ_j = 0, \quad 0 \le j < i$,

(c) $Q_i \in C(I, I\!R^{n\times n})$, $DP_0\dots P_iD^- \in C^1(I, I\!R^{m\times m})$, $i \ge 0$,

(d) $0 \leq r_0 \leq \cdots \leq r_{\mu-1} < n$ *and* $r_\mu = n$.

Denoting $N_i := \ker G_i$, we know for DAEs with tractability index that we can choose the projectors Q_i in such a way that

$$N_i \cap N_{i+1} = 0, \quad \forall i \geq 0, \tag{1.5}$$

(see [Mär02]). In order to use Definition 1.2 for determining the index of a DAE (pointwise) numerically we have to choose the projectors Q_i such that

$$Q_i Q_j = 0, \quad 0 \leq j < i. \tag{1.6}$$

In the consequence, certain products of projectors also become projectors, e.g. $P_0 P_1$ etc.

The paper aims at designing an algorithm to realize the matrix sequence (1.3) numerically. As the main problem we have to create projectors Q_i with (1.6).

After analyzing a properly formulated DAE using the matrix sequence (1.3) we know all the necessary matrices and projectors to compute consistent initial values for properly formulated DAEs of at most index 2. In [EL01], [Est00] and [HL02] algorithms to compute consistent initial values basing on the classical tractability index are given and you find here a discussion about different ideas to compute consistent initial values in the past, too.

2 The index determination

2.1 The pseudo-inverse

For a matrix $Z \in \mathbb{R}^{m \times n}$ we call $Z^- \in \mathbb{R}^{n \times m}$ a reflexive (generalized) inverse iff it fulfils

$$ZZ^-Z = Z \quad \text{and} \tag{2.1}$$

$$Z^-ZZ^- = Z^-. \tag{2.2}$$

The products $ZZ^- \in \mathbb{R}^{m \times m}$ and $Z^-Z \in \mathbb{R}^{n \times n}$ are projectors with the same rank r_Z. Let $P \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ be given projectors with rank r_Z and $ZP = Z$ and $RZ = Z$.

Lemma 2.1 *With (2.1), (2.2) and the conditions*

$$\begin{aligned} Z^-Z &= P \quad \textit{and} && (2.3)\\ ZZ^- &= R && (2.4)\end{aligned}$$

the reflexive inverse Z^- is uniquely determined.

Proof: Let Y be a further matrix fulfilling (2.1), (2.2), (2.3) and (2.4).

$$\begin{aligned} Y &\overset{(2.2)}{=} YZY \overset{(2.1)}{=} YZZ^-ZY \overset{(2.4)}{=} YRZY \\ &\overset{(2.4)}{=} YR \overset{(2.4)}{=} YZZ^- \overset{(2.3)}{=} PZ^- \overset{(2.2)}{=} Z^-. \end{aligned}$$

q.e.d.

To represent the pseudo inverse Z^- we want to use a decomposition of

$$Z = U \begin{pmatrix} S & \\ & 0 \end{pmatrix} V^{-1}$$

with nonsingular matrices U, V and S. The pseudo-inverse is given by

$$Z^- = V \begin{pmatrix} S^{-1} & m_2 \\ m_1 & m_1 S m_2 \end{pmatrix} U^{-1} \tag{2.5}$$

with m_1 and m_2 being matrices of free parameters that fulfil

$$P = Z^-Z = V \begin{pmatrix} I & 0 \\ m_1 S & 0 \end{pmatrix} V^{-1}$$

and

$$R = ZZ^- = U \begin{pmatrix} I & Sm_2 \\ 0 & 0 \end{pmatrix} U^{-1}.$$

(For details and different constructions of Z^- see [Zie79]).

2.2 Check of the well-matched condition

A and D have to be well-matched (see Def. 1.2). This is important in view of the representation of the DAE as a hand-made subroutine, which may easily contain a programming error. From Def. 1.2 we obtain the relations

$$AD = ARD, \quad A = AR, \quad D = RD,$$

and

$$rank(A) = rank(D) = rank(AD). \tag{2.6}$$

Performing an SVD of A and D yields

$$A = U_A \begin{pmatrix} \Sigma_A & \\ & 0 \end{pmatrix} V_A^T,$$

$$D = U_D \begin{pmatrix} \Sigma_D & \\ & 0 \end{pmatrix} V_D^T.$$

We can check now that $rank(\Sigma_A) = rank(\Sigma_D)$.
For computing the matrix sequence, we need $G_0 := AD$. Using the decompositions we have

$$AD = U_A \begin{pmatrix} \Sigma_A & \\ & 0 \end{pmatrix} V_A^T U_D \begin{pmatrix} \Sigma_D & \\ & 0 \end{pmatrix} V_D^T \tag{2.7}$$

and, by $V_A^T U_D =: H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$, the relation (2.6) is fulfilled iff H_1 remains nonsingular. To compute the generalized inverse of D we use the relations $DD^- = R = A^- A$. We have

$$DD^- = U_D \begin{pmatrix} I & \Sigma_D m_{D_2} \\ 0 & 0 \end{pmatrix} U_D^T,$$

$$A^- A = V_A \begin{pmatrix} I & 0 \\ m_{A_1} \Sigma_A & 0 \end{pmatrix} V_A^T$$

and, with $U_D = V_A H$, we obtain the relation

$$H \begin{pmatrix} I & \Sigma_D m_{D_2} \\ 0 & 0 \end{pmatrix} H^T = \begin{pmatrix} I & 0 \\ m_{A_1} \Sigma_A & 0 \end{pmatrix}.$$

This fixes the free parameters

$$m_{D_2} = \Sigma_D^{-1} H_1^{-1} H_2 \tag{2.8}$$

and

$$m_{A_1} = H_3 H_1^{-1} \Sigma_A^{-1}.$$

2.3 The matrix sequence

To start the construction of the matrix sequence (1.3) we need the pseudo-inverses D^-, G_0^- and the projectors Q_0, W_0 simultaneously. The following relations have to be taken into account: $D^-D = I - Q_0$, $G_0^-G_0 = I - Q_0$ and $G_0G_0^- = I - W_0$. For $G_0 = AD$ (2.7) provides the representation

$$G_0 = U_A \begin{pmatrix} Z & \\ & 0 \end{pmatrix} V_D^T \text{ with } Z = \Sigma_A H_1 \Sigma_D.$$

With an SVD of $Z = U_Z \Sigma_0 V_Z^T$ we have the SVD of G_0 as

$$G_0 = \underbrace{U_A \begin{pmatrix} U_Z & \\ & I \end{pmatrix}}_{U_0} \begin{pmatrix} \Sigma_0 & \\ & 0 \end{pmatrix} \underbrace{\begin{pmatrix} V_Z^T & \\ & I \end{pmatrix} V_D^T}_{V_0^T}.$$

Using the SVD of D and G_0, the pseudo-inverses have the general representation

$$D^- = V_D \begin{pmatrix} \Sigma_D^{-1} & m_{D_2} \\ m_{D_1} & m_{D_1}\Sigma_D m_{D_2} \end{pmatrix} U_D^T,$$

and

$$G_0^- = V_0 \begin{pmatrix} \Sigma_0^{-1} & m_{0_2} \\ m_{0_1} & m_{0_1}\Sigma_0 m_{0_2} \end{pmatrix} U_0^T = V_D \begin{pmatrix} Z^{-1} & V_Z m_{0_2} \\ m_{0_1}U_Z^T & m_{0_1}\Sigma_0 m_{0_2} \end{pmatrix} U_A^T. \qquad (2.9)$$

For

$$I - W_0 = U_0 \begin{pmatrix} I & \Sigma_0 m_{0_2} \\ 0 & 0 \end{pmatrix} U_0^T,$$

this yields

$$I - Q_0 = V_0 \begin{pmatrix} I & 0 \\ m_{0_1}\Sigma_0 & 0 \end{pmatrix} V_0^T = V_D \begin{pmatrix} I & 0 \\ m_{0_1}U_Z^T Z & 0 \end{pmatrix} V_D^T$$

and

$$D^-D = V_D \begin{pmatrix} I & 0 \\ m_{D_1}\Sigma_D & 0 \end{pmatrix} V_D^T,$$

which gives $m_{D_1} = m_{0_1}U_Z^T Z \Sigma_D^{-1}$ i.e., by m_{0_1}, all parameters of D^- are fixed. Let us now assume that we have realized the matrix sequence up to G_i

such that $Q_iQ_j = 0$ for $j < i$. We have to construct G_{i+1} and a reflexive generalized inverse G_{i+1}^- with

$$G_{i+1}G_{i+1}^- = I - W_{i+1}, \quad G_{i+1}^-G_{i+1} = I - Q_{i+1} \tag{2.10}$$

and

$$Q_{i+1}Q_j = 0 \text{ for } j < i+1. \tag{2.11}$$

First we give a representation of G_{i+1}^-. From the matrix sequence we have

$$G_{i+1} = G_i + B_iQ_i = (G_i + W_iB_iQ_i)F_i$$

with the nonsingular matrix $F_i = I + G_i^-B_iQ_i$. For the sequence matrix G_i we have a decomposition

$$G_i = \mathcal{U}_i \begin{pmatrix} S_i & \\ & 0 \end{pmatrix} \mathcal{V}_i^{-1}$$

with $\mathcal{U}_i$, S_i and $\mathcal{V}_i$ nonsingular matrices with $\mathcal{U}_0 = U_0$, $S_0 = \Sigma_0$ and $\mathcal{V}_0 = V_0$. The other components are given by

$$G_i^- = \mathcal{V}_i \begin{pmatrix} S_i^{-1} & m_{i,2} \\ m_{i,1} & m_{i,1}S_im_{i,2} \end{pmatrix} \mathcal{U}_i^{-1},$$

$$W_i = \mathcal{U}_i \begin{pmatrix} 0 & -S_im_{i,2} \\ & I \end{pmatrix} \mathcal{U}_i^{-1} = \mathcal{U}_iT_{u,i}^{-1} \begin{pmatrix} 0 & \\ & I \end{pmatrix} \mathcal{U}_i^{-1}, \tag{2.12}$$

$$Q_i = \mathcal{V}_i \begin{pmatrix} 0 & \\ -m_{i,1}S_i & I \end{pmatrix} \mathcal{V}_i^{-1} = \mathcal{V}_i \begin{pmatrix} 0 & \\ & I \end{pmatrix} T_{l,i}^{-1}\mathcal{V}_i^{-1} \tag{2.13}$$

with the upper and lower triangle matrices

$$T_{u,i} := \begin{pmatrix} I & S_im_{i,2} \\ & I \end{pmatrix} \text{ and } T_{l,i} := \begin{pmatrix} I & \\ m_{i,1}S_i & I \end{pmatrix}.$$

Using the detailed structure of the different matrices we find

$$G_{i+1} = \mathcal{U}_iT_{u,i}^{-1} \left(\begin{pmatrix} S_i & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ & I \end{pmatrix} \underbrace{\mathcal{U}_i^{-1}B_i\mathcal{V}_i}_{\bar{B}_i} \begin{pmatrix} 0 & \\ & I \end{pmatrix} \right) T_{l,i}^{-1}\mathcal{V}_i^{-1}F_i.$$

If we structure $\bar{B}_i = \begin{pmatrix} b^i_{11} & b^i_{12} \\ b^i_{21} & b^i_{22} \end{pmatrix}$, we obtain the SVD of $b^i_{22} = \tilde{U}_{i+1} \begin{pmatrix} \Sigma_{i+1} & \\ & 0 \end{pmatrix} \tilde{V}^T_{i+1}$. Using this decomposition we have

$$G_{i+1} = \underbrace{\mathcal{U}_i T_{u,i}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{i+1} \end{pmatrix}}_{=:\mathcal{U}_{i+1}} \begin{pmatrix} S_i & & \\ & \Sigma_{i+1} & \\ & & 0 \end{pmatrix} \underbrace{\begin{pmatrix} I & \\ & \tilde{V}^T_{i+1} \end{pmatrix} T_{l,i}^{-1} \mathcal{V}_i^{-1} F_i}_{=:\mathcal{V}^{-1}_{i+1}}$$

and we define $S_{i+1} := \begin{pmatrix} S_i & \\ & \Sigma_{i+1} \end{pmatrix}$. The pseudo inverse of G_{i+1} is then given by

$$G^-_{i+1} = \mathcal{V}_{i+1} \begin{pmatrix} S^{-1}_{i+1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} \mathcal{U}^{-1}_{i+1}.$$

To use G^-_{i+1} for calculations we have to determine the free parameters $m_{i+1,1}$ and $m_{i+1,2}$ in such a way that (2.11) is fulfilled. From (2.13) we see that only $m_{i+1,1}$ influences Q_{i+1}, and from (2.12) that only $m_{i+1,2}$ influences W_{i+1}. Up to now we have no special conditions to the projectors W_j. What is a criterion for (2.11)?

From (2.10) we have $I - G^-_{i+1} G_{i+1} = Q_{i+1}$ and it follows that $Q_j - G^-_{i+1} G_{i+1} Q_j = 0$ has to be fulfilled for $j < i+1$, and using the structure of G_{i+1} we obtain

$$G^-_{i+1} B_j Q_j = Q_j, \quad j = 0, \ldots, i. \tag{2.14}$$

Are these conditions helpful for a determination of $m_{i+1,1}$?

With $Q_j = \mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} T^{-1}_{l,j} \mathcal{V}^{-1}_j$ condition (2.14) reads in detail, after multiplying by $\mathcal{V}_j T_{l,j}$ from the right,

$$\mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} = \mathcal{V}_{i+1} \begin{pmatrix} S^{-1}_{i+1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} \mathcal{U}^{-1}_{i+1} B_j \mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix}.$$

Introducing $\bar{U}_k := T^{-1}_{u,k-1} \begin{pmatrix} I & \\ & \tilde{U}_k \end{pmatrix}$ it follows that

$$\underbrace{\mathcal{V}^{-1}_{i+1} \mathcal{V}_j}_{=:\tilde{w}_j} \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} =$$
$$\begin{pmatrix} S^{-1}_{i+1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} \bar{U}^{-1}_{i+1} \ldots \bar{U}^{-1}_{j+2} \begin{pmatrix} I & \\ & \tilde{U}^T_{j+1} \end{pmatrix} T^{-1}_{u,j} \bar{B}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix}. \tag{2.15}$$

With $\tilde{w}_j =: \begin{pmatrix} w_j^{11} & w_j^{12} \\ w_j^{21} & w_j^{22} \end{pmatrix}$ we have for $j = 0, \dots, i$ the relation

$$\underbrace{\begin{pmatrix} w_j^{12} \\ w_j^{22} \end{pmatrix}}_{=:w_j} = \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1}S_{i+1}m_{i+1,2} \end{pmatrix} \underbrace{\bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix} T_{u,j}^{-1} \begin{pmatrix} b_{12}^j \\ b_{22}^j \end{pmatrix}}_{=:z_j}. \tag{2.16}$$

Let us have a look at the special structure of z_j.

$$\begin{aligned} z_j &= \bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix} \begin{pmatrix} b_{12}^j - S_j m_{j,2} b_{22}^j \\ b_{22}^j \end{pmatrix} \\ &= \bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} b_{12}^j - S_j m_{j,2} b_{22}^j \\ \begin{pmatrix} \Sigma_{j+1} & 0 \end{pmatrix} \tilde{V}_{j+1}^T \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \}n - r_{j+1}. \end{matrix} \end{aligned}$$

All factors in $\bar{U}_k^{-1}$ have the structure $\begin{pmatrix} I & \star \\ & \star \end{pmatrix}$, where the number of columns in $\begin{pmatrix} \star \\ \star \end{pmatrix}$ is less than or equal to $n - r_{j+1}$, which means that

$$z_j = \begin{pmatrix} b_{12}^j - S_j m_{j,2} b_{22}^j \\ \begin{pmatrix} \Sigma_{j+1} & 0 \end{pmatrix} \tilde{V}_{j+1}^T \\ 0 \end{pmatrix}. \tag{2.17}$$

This forms the following linear system

$$\underbrace{\begin{pmatrix} w_0 & \dots & w_i \end{pmatrix}}_{=:W} = \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1}S_{i+1}m_{i+1,2} \end{pmatrix} \underbrace{\begin{pmatrix} z_0 & \dots & z_i \end{pmatrix}}_{=:Z}. \tag{2.18}$$

Let us investigate the properties of Z in detail.

Lemma 2.2 *Let the DAE (1.2) be a DAE with tractability index μ, then the matrix $Z := \begin{pmatrix} z_0 & \dots & z_i \end{pmatrix}$ has full (column) rank for $i \geq 0$.*

Proof: Due to the tractability of (1.2) it holds that

$$\begin{aligned} 0 = N_k \cap N_{k+1} &= \ker G_k \cap (\ker G_k \cap \ker B_k Q_k) \\ &= \ker G_k \cap \ker B_k Q_k \quad \text{(see [Mär02])}. \end{aligned} \tag{2.19}$$

Using the decomposition of

$$G_k = \mathcal{U}_k \begin{pmatrix} S_k & \\ & 0 \end{pmatrix} \mathcal{V}_k^{-1} = \mathcal{U}_k \begin{pmatrix} S_k & \\ & 0 \end{pmatrix} T_{l,k}^{-1} \mathcal{V}_k^{-1}$$

and the structure of

$$B_k Q_k = B_k \mathcal{V}_k \begin{pmatrix} 0 & \\ & I \end{pmatrix} T_{l,k}^{-1} \mathcal{V}_k^{-1} = \mathcal{U}_k \begin{pmatrix} 0 & b_{12}^k \\ & b_{22}^k \end{pmatrix} T_{l,k}^{-1} \mathcal{V}_k^{-1},$$

$\ker G_k$ has the representation

$$\ker G_k = \left\{ v : v = \mathcal{V}_k T_{l,k} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, v_1 = 0 \right\}$$

and

$$\ker B_k Q_k = \left\{ v : v = \mathcal{V}_k T_{l,k} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} b_{12}^k \\ b_{22}^k \end{pmatrix} v_2 = 0 \right\}.$$

The condition (2.19) means now that $\begin{pmatrix} b_{12}^k \\ b_{22}^k \end{pmatrix}$ must have full column rank. It follows immediately that $z_k = \bar{U}_{i+1}^{-1} \dots \bar{U}_{k+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{k+1}^T \end{pmatrix} T_{u,k}^{-1} \begin{pmatrix} b_{12}^k \\ b_{22}^k \end{pmatrix}$ has full rank, too. Every component of Z has full rank. Now let us have a look at the rank of Z itself.
Having reached the level i of the matrix sequence, we can compute G_{i+1}, and we want to compute the nullspace projector Q_{i+1} with property (b) of Definition 1.2. We know that there exists a (reflexive) generalized inverse G_{i+1}^- with $I - G_{i+1}^- G_{i+1} = Q_{i+1}$ and (2.14). The projectors $Q_0, \dots, Q_i$ were chosen in such a way that

$$N_0 \cap \dots \cap N_i = \{0\}. \tag{2.20}$$

Consider a nontrivial linear combination of the columns of Z and let us assume that it is identically zero. With $\lambda := \big(\lambda_0, \dots, \lambda_i\big)^T$ and $\lambda_j := \big(\lambda_{j1}, \dots, \lambda_{jk_j}\big)^T$ and k_j being equal to the number of columns of z_j, which is identical with rank Q_j, we can reformulate

$$\begin{aligned}
0 &= Z\lambda = \sum_{j=0}^{i} z_j\lambda_j = \sum_{j=0}^{i} \begin{pmatrix} 0 & z_j \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_j \end{pmatrix} \\
&= \sum_{j=0}^{i} \left(0 \quad \bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{j+1}^{T} \end{pmatrix} T_{u,j}^{-1} \begin{pmatrix} b_{12}^{j} \\ b_{22}^{j} \end{pmatrix} \right) \begin{pmatrix} 0 \\ \lambda_j \end{pmatrix} \\
&= \sum_{j=0}^{i} \bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{j+1}^{T} \end{pmatrix} T_{u,j}^{-1} \mathcal{U}_j^{-1} B_j \underbrace{\mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} T_{u,j}^{-1} \mathcal{V}_j^{-1}}_{=Q_j} \underbrace{\mathcal{V}_j T_{u,j} \begin{pmatrix} 0 \\ \lambda_j \end{pmatrix}}_{=:v_j} \\
&= \sum_{j=0}^{i} \mathcal{U}_{i+1}^{-1} B_j Q_j v_j .
\end{aligned}$$

Multiplying the latter expression by $G_{i+1}^{-}\mathcal{U}_{i+1}$ leads to

$$\begin{aligned}
0 &= G_{i+1}^{-}\mathcal{U}_{i+1} \sum_{j=0}^{i} \mathcal{U}_{i+1}^{-1} B_j Q_j v_j \\
&= \sum_{j=0}^{i} G_{i+1}^{-} B_j Q_j v_j \quad \text{and using (2.14)} \\
&= \sum_{j=0}^{i} Q_j v_j . \qquad (2.21)
\end{aligned}$$

Because of (2.20) the addends $Q_j v_j \in N_j$ are independent. This means that every addend $Q_j v_j$ of (2.21) is zero, hence, $\lambda_j = 0$ due to the structure of v_j. This contradicts our assumption and Z has full (column) rank. q.e.d.

Let us consider the solution of the linear system (2.18)

$$W = \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} Z.$$

Using the structure of $Z = \begin{pmatrix} \tilde{Z} \\ 0 \end{pmatrix} \}n - r_{i+1}$ (see (2.17)) we can reformulate (2.18) as

$$W = \begin{pmatrix} S_{i+1}^{-1} & 0 \\ m_{i+1,1} & 0 \end{pmatrix} Z.$$

First we discover that the parameter $m_{i+1,2}$ does not influence the computation of $m_{i+1,1}$ and, second, we can represent a solution X of $W = XZ$ by $X = WZ^-$ with an arbitrary (generalized) reflexive inverse Z^-, since $Z^-Z = I$ is valid for full column rank matrices. The appropriate part of X in the left lower corner gives us a value for $m_{i+1,1}$. Which parameter set we select depends on the used inverse Z^-. If we look at a Householder decomposition of a full column rank matrix $Z = U \begin{pmatrix} R \\ 0 \end{pmatrix}$ with nonsingular R, the (generalized) reflexive inverse is given by

$$Z^- = \begin{pmatrix} R^{-1} & \tilde{m} \end{pmatrix} U^T \tag{2.22}$$

with the free parameter $\tilde{m}$.

3 Consistent Initial Values

We consider the initial value problem (1.1) with the initial condition

$$(DP_1)(x(t_0), t_0)(x(t_0) - \alpha) = 0, \tag{3.1}$$

with an arbitrary given vector α. Problems in using (3.1) will be discussed in Chapter 3.1.
Let us assume that

$$\mathbf{A1} : \operatorname{im} A(x,t) \quad \text{depends smoothly on } t \text{ only}, \tag{3.2}$$

i.e., that the projector W_0 depends on t only. Notice that all solutions of (1.1) lie in

$$M_0(t) := \{x \in D_f, W_0(t)b(x,t) = 0\}, \tag{3.3}$$

which describes the constraint of (1.1), where $D_f \subset \mathbb{R}^n$ is the domain of the left side of (1.1) . $M_0(t)$ does not depend on the special choice of W_0. We introduce the space S, which is closely related to the tangent space of $M_0(t)$ by

$$S(x,t) := \{z \in \mathbb{R}^n : W_0(t)B(y,x,t)z = 0\} = \{z \in \mathbb{R}^n : W_0(t)b'_x(x,t)z = 0\}. \tag{3.4}$$

It is well known that the condition $N(t) \cap S(x,t) = \{0\}$ characterizes the index 1 case. We are interested in the index 2 case, where

$$N(t) \cap S(x,t) = \operatorname{im} Q_0(t)Q_1(y,x,t) \tag{3.5}$$

(see [Tis96], [Est00]). A straightforward modification of the proof in [Est00] shows the validity of (3.5) also for DAEs with proper formulated leading term. Let us introduce a projector $T(x,t)$ onto $\operatorname{im} Q_0(t)Q_1(y,x,t)$ and $U := I - T$. We will assume that

$$\mathbf{A2} : N_0(t) \cap S(x,t) \text{ does not depend on } x. \tag{3.6}$$

The idea of the computation of consistent initial values bases on a description of the so-called hidden constraint $M_1(t)$ of the DAE, which is a subset of $M_0(t)$ and which is, in contrast to $M_0(t)$, filled with solutions in the index 2 case. Following the discussions in [EL01], [Est00] we investigate the DAE

$$(I - \hat{W_1}(t))f((Dx)',x,t) + W_1(x,t)\frac{d}{dt}(K_{W_1}W_0(t)f((Dx)',x,t)) = 0, \tag{3.7}$$

where $\hat{W}_1$ describes an only time depending projector with $W_1\hat{W}_1 = \hat{W}_1$, $\hat{W}_1 W_1 = W_1$ and $\hat{W}_1(I - W_0) = 0$, which exists locally. K_{W_1} describes a constant matrix with $W_1 K_{W_1} = W_1$, which possibly reduces the necessary differentiations. For a better description we set

$$f((Dx)',x,t) := A(x,t)(Dx)' + b(x,t). \tag{3.8}$$

Theorem 3.1 *The DAE (1.1) has tractability index 2 and let the assumptions* **A1, A2** *be fulfilled and* $D \in C^1$*, then (3.7) has index 1 and the same solutions as (1.1) if additionally (at least in one point)*

$$(\hat{W}_1 b)(x(t_0),t_0) = 0. \tag{3.9}$$

Proof: We want to apply Theorem 2.4.6 from [Est00]. For that purpose we have to transform (1.1) into the structure used in [Est00]. Using $D \in C^1$ we obtain from (1.1)

$$\underbrace{A(x,t)D(t)}_{\bar{A}(x,t)}(P_0x)' + \underbrace{A(x,t)D'P_0x + b(x,t)}_{\bar{b}(x,t)} = 0.$$

We see that $\bar{Q}_0(t) = Q_0(t)$ and $\bar{W}_0(t) = W_0(t)$. The matrices $\bar{G}_1$ used in [Est00] and G_1 differ only in terms of the structure $(A(x,t)w)'_x$. Because of $W_1 W_0 = W_1$ and $W_0 = W_0(t)$ we can use the same projectors $W_1 = \bar{W}_1$ and $\hat{W}_1$. q.e.d

Let us have a look at (3.7) after multiplying by $W_1(x,t)$:

$$W_1(x,t)\frac{d}{dt}(K_{W_1}W_0(t)f((Dx)',x,t)) = 0. \tag{3.10}$$

Recall (see [Est00]) that due to (3.6) the projector U does not depend on x and

$$\begin{aligned} W_1(x,t) &= W_1(U(t)x,t), \\ (W_0b)(x,t) &= (W_0b)(U(t)x,t) \end{aligned} \tag{3.11}$$

are valid. (3.10) reads

$$\begin{aligned} W_1(Ux,t)\frac{d}{dt}(K_{W_1}(W_0b)(Ux,t)) &= W_1(Ux,t)((W_0b)'_x(Ux)' + (W_0b)'_t) \\ &= W_1(B \underbrace{P_0(Ux)'}_{D^-((Dx)'-D'Ux)} +(W_0b)'_t) \end{aligned}$$

$$= W_1(BD^-((Dx)' - D'Ux) + (W_0b)'_t) = 0. \tag{3.12}$$

(3.12)describes the hidden constraint of the index 2 DAE

$$\begin{aligned} M_1(t) := \{ & x \in D_f : \exists y \quad A(x,t)y + b(x,t) = 0, \\ & W_1(x,t)(b'_x(x,t)D^-(y - D'Ux) + (W_0b)'_t(x,t)) = 0\}. \end{aligned}$$

If we collect all equations now, we have

$$\begin{array}{lrcl} \text{(3.7) times } (I - \hat{W}_1) & (I - \hat{W}_1(t))f((Dx)',x,t) &=& 0, \\ \text{(3.12)} & W_1(BD^-((Dx)' - D'Ux) + (W_0b)'_t) &=& 0, \\ \text{(3.9)} & \hat{W}_1f((Dx)',x,t_0) &=& 0, \\ \text{(3.1)} & (DP_1)(x(t_0),t_0)(x(t_0) - \alpha) &=& 0. \end{array}$$

Introducing the unknowns $y := R(Dx)'(t_0)$, $x = x(t_0)$ and consider the above equations in the point t_0, we obtain

$$\begin{aligned} f(y,x,t) &= 0, \\ W_1(BD^-(y - D'Ux) + (W_0b)'_t) &= 0, \\ (DP_1)(x,t_0)(x - \alpha) &= 0, \\ (I - R)y &= 0. \end{aligned} \tag{3.13}$$

We have to check the solvability of system (3.13).

Theorem 3.2 *Let* **A1** *and* **A2** *be valid and let the implication*

$$((DP_1)(x,t)(x-\alpha))'_x(I-Q_1\mathcal{G}_2^{-1}BP_0P_1)(I-P_0Q_1(x,t))z=0 \Rightarrow DP_1(x,t)z=0 \tag{3.14}$$

hold with $\mathcal{G}_2 = G_1 + B_0P_0Q_1$, then the system (3.13) has a full-rank Jacobian matrix in a neighborhood of the solution.

Remark:

1. A similar proof, not basing on the proper leading term formulation and using the so-called canonical projector, you may find in [EL01], [Est00].
2. Condition (3.14) looks a little bit different from [EL01]. This follows from using arbitrary projectors P_1. A discussion of (3.14) you can find in [EL01].

Proof: For $A := f'_y(y,x,t)$ and $B := f'_x(y,x,t)$ the Jacobian matrix of (3.13) reads

$$J = \begin{pmatrix} A & B \\ W_1BD^- & \{W_1(BD^-(y-D'Ux)+(W_0b)'_t)\}'_x \\ 0 & \{DP_1(x,t)(x-\alpha)\}'_x \\ I-R & 0 \end{pmatrix}.$$

To prove its full rank we consider a z fulfilling $Jz=0$. For $z=(z_y,z_x)^T$ we obtain the first equation:

$$Az_y + Bz_x = 0.$$

Multiplying it by $\mathcal{G}_2^{-1}$ gives

$$P_1D^-z_y + \mathcal{G}_2^{-1}BD^-DP_1z_x + Q_0z_x + Q_1z_x = 0.$$

The multiplication by DP_1, DQ_1 and Q_0 yields

$$\begin{aligned} DP_1D^-z_y + DP_1\mathcal{G}_2^{-1}BD^-DP_1z_x &= 0, && (3.15)\\ DQ_1\mathcal{G}_2^{-1}BD^-DP_1z_x + DQ_1z_x &= 0, && (3.16)\\ -Q_0Q_1D^-z_y + (Q_0\mathcal{G}_2^{-1}BD^-DP_1 + Q_0Q_1 + Q_0)z_x &= 0. && (3.17) \end{aligned}$$

The other equations provide

$$\begin{aligned} W_1BD^-z_y + \{W_1(BD^-(y-D'Ux)+(W_0b)'_t)\}'_xz_x &= 0, && (3.18)\\ \{DP_1(x-\alpha)\}'_xz_x &= 0, && (3.19)\\ (I-R)z_y &= 0. && (3.20) \end{aligned}$$

Because of its structure and (3.11) the second addend of (3.18) depends on $U(t)x$ only, i.e. that we can replace it by

$$W_1BD^-z_y + \{W_1(BD^-(y - D'Ux) + (W_0b)'_t)\}'_x Uz_x = 0 \tag{3.21}$$

From (3.16) we obtain

$$DQ_1 \underbrace{(Q_1\mathcal{G}_2^{-1}BP_0P_1 + I)z_x}_{\bar{z_x}} = 0 \tag{3.22}$$

and from (3.19) we derive

$$\begin{aligned} \{DP_1(x-\alpha)\}'_x z_x &= \{DP_1(x-\alpha)\}'_x (I - Q_1\mathcal{G}_2^{-1}BP_0P_1)(I + Q_1\mathcal{G}_2^{-1}BP_0P_1)z_x \\ & \text{and, finally, with (3.22)} \\ &= \{DP_1(x-\alpha)\}'_x (I - Q_1\mathcal{G}_2^{-1}BD^-DP_1)(I - P_0Q_1)\bar{z_x}. \end{aligned}$$

Using assumption (3.14) it follows that $DP_1\bar{z_x} = 0$, which implies that $DP_1z_x = 0$ and $DQ_1z_x = 0$. With (3.15) it follows that $DP_1D^-z_y = 0$. With (3.20) we obtain that $z_y = DQ_1D^-z_y$ and $z_x = Q_0z_x$. With (3.17) this leads to $Q_0Q_1D^-z_y = Q_0z_x$ and, finally, (3.21) implies $W_1BP_0Q_1D^-z_y = 0$, i.e., $DQ_1D^-z_y = 0$, which means that $Q_0z_x = 0$. q.e.d.

3.1 The initial condition and its practical use

Before we are looking for the solution of (3.13) let us discuss a difficulty related to the initial condition (3.1). To illustrate the problem we consider the trajectory prescribed path control problem [Bre83], [HL02]. The components of the eight dimensional unknown vector have a physical meaning:

$x(1)$ – altitude (H)
$x(2)$ – longitude (ψ)
$x(3)$ – geocentric latitude (λ)
$x(4)$ – magnitude of the relative velocity vector (V_R)
$x(5)$ – relative flight path angle (γ)
$x(6)$ – relative azimuth (A)
…and so on.

A projector for this problem has the structure

$$DP_1 = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & \Phi_1 & \Phi_2 & & \\ & & & & 0 & & & \\ & & & & & 0 & 0 & 0 \end{pmatrix} \tag{3.23}$$

with functions Φ_1, Φ_2, which depends on the solution. The fourth equation of $(DP_1)(x(t_0), t_0)(x(t_0) - \alpha) = 0$ reads for this example

$$(x_4(t_0) - \alpha_4) + \Phi_1(x(t_0))(x_5(t_0) - \alpha_5) + \Phi_2(x(t_0))(x_6(t_0) - \alpha_6) = 0,$$

i.e., only if the values α_5 and α_6 are chosen consistently (which is not known in advanced), it will be realized that $x_4(t_0) = \alpha_4$. How can we support an engineer to realize necessary values? We could replace (3.1) by another equation, but the projector DP_1 has the advantage that its rank gives us the degree of freedom of our system and we have solvability for system (3.13). The proposal is to mix (3.1) with an additional equation

$$z(x(t_0)) = 0, \quad z \in \mathbb{R}^{n_z}, \quad 0 \leq n_z \leq rank(DP_1), \tag{3.24}$$

which contains user's specifications.

4 Numerical realization with MATLAB

4.1 The index determination

To realize the matrix sequence (1.3) we need the matrices A, D and B. If the DAE is given by

$$f((d(x(t), t))', x(t), t) = 0, \tag{4.1}$$

the related matrices are

$$A := f'_y, \quad B := f'_x \text{ and } D := d'_x.$$

f'_y means the derivative of f with respect to the first argument. For the theoretical investigations we consider a quasilinear structure (1.1), but the algorithm is realized for more general equations (4.1).

The algorithm is realized in MATLAB. In the first step, the well-matched condition (see Def.1.1) of A and D is checked (see Chapter 2.2). The SVD of A and D is used to perform an SVD of the first matrix sequence element $G_0 = AD$. The free parameter $m_{0,1}$ in G_0^- (see (2.9)) is set to zero (but a parameter in the routines) and the second parameter set $m_{i,2}$, which influences the projector W_i, is set to zero, too. The algorithm follows the description given in Chapter 2.3. This means that we have to perform an SVD of the dimension $d_i = n - \text{rank } G_i$ in every step up to the condition that $d_i = 0$. The computation of G_{i+1}^- needs the solution of (2.18). We check the full rank condition of Z, which checks the tractability of the DAE (see Def.1.2). The pseudo-inverse of Z is computed by (2.22) with the free parameter $\tilde{m} = 0$. The calculation of the matrix sequence element B_{i+1} (1.3) contains a time differentiation. It is well known that numerical differentiation is a difficult sensitive procedure. But if we look to (1.4) we discover that the first derivative is needed for the determination of an index 3 DAE. Other concepts (e.g. using derivative arrays, see [KM98]) needs at that point at least a second derivative of the whole equation. That means that the presented concept can certainly decide with one differentiation only whether a DAE has index 3 or a higher one. But also for higher index equations the algorithm is useful, because we get an information about a time dependency of $(DP_0 ... P_i D^-)$. All differentiations (numerical approximation of A, B or D, time differentiations in the matrix sequence to compute B_i) are performed by the MATLAB routine *numjac*.

4.2 Calculation of consistent initial values

We have to solve the nonlinear system (3.13) extended by the user routine $z(x)$.

$$\begin{array}{rcl} f(y,x,t) & = & 0, \\ W_1(BD^-(y - D'Ux) + (W_0 b)'_t) & = & 0, \\ (I - R)y & = & 0, \\ z(x) & = & 0, \\ (DP_1)(x,t_0)(x - \alpha) & = & 0. \end{array} \tag{4.2}$$

We solve (4.2) by an adapted Newton method. The first difference to a classical Newton method is that we do not take into account the dependences of all projectors and matrices on the unknowns y and x. We could say that we fix their variables. By this trick we can use an available Jacobian matrix,

. Now (4.2) reads

$$
\begin{array}{rcl}
f(y,x,t) & = & 0, \\
W_1(x_\star,t_0)(B(y_\star,x_\star,t_0)D^-(y-D'Ux)+(W_0b)'_t(x,t_0)) & = & 0, \\
(I-R)y & = & 0, \\
z(x) & = & 0, \\
(DP_1)(x_\star,t_0)(x-\alpha) & = & 0.
\end{array}
\tag{4.3}
$$

Recall that D, R, U and W_0 depend on t only. Secondly we have to make a decision, which of the equations we will use because of the possible overdetermination by the last two equations. For that reason we combine the decomposition of the Jacobian matrix with the determination of linearly dependent equations, which are not used. The Jacobian matrix of (4.3) with respect to y and x is given by

$$
J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \tag{4.4}
$$

with

$$
J_1 = \begin{pmatrix} A & B \\ W_1BD^- & W_1(BD^-D'U+B'_t) \\ I-R & 0 \\ 0 & z'_x \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & DP_1 \end{pmatrix}.
$$

Using a Householder decomposition of $J_1^T =: \bar{Q}\bar{R}$ we obtain for J^T, the representation (for a clearer representation we will suppress the necessary column exchange in the formula)

$$
J^T = \bar{Q}\begin{pmatrix} \bar{R} & \bar{Q}^TJ_2^T \end{pmatrix}
$$

and by $\bar{R} = \begin{pmatrix} \tilde{R} & 0 \\ 0 & 0 \end{pmatrix}$ we split $C := \bar{Q}^TJ_2^T$ in the same way into $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$. A second decomposition of $C_2 =: \bar{\bar{Q}}\bar{\bar{R}}$ and using the structure of $\bar{\bar{R}} = \begin{pmatrix} \tilde{\bar{\bar{R}}} & 0 \end{pmatrix}$, yields

$$
J^T = \bar{Q}\begin{pmatrix} I & \\ & \bar{\bar{Q}} \end{pmatrix}\begin{pmatrix} \tilde{R} & 0 & C_{11} & C_{12} \\ 0 & 0 & \tilde{\bar{\bar{R}}} & 0 \end{pmatrix}.
$$

This decomposition gives us the possibility to check the rank conditions, to decide which of the equations of (4.2) are linearly dependent and can be

deleted, and, additionally, we have a decomposition to solve the linear systems in every Newton step. It should be stressed that the computed solution is not a least square solution, but a solution of a nonsingular quadratic linear system, because of the linearly independent equations only.
The structure of J_1 ensures that the additional user's equation is taking into account in every case.
The decision that the residuum of the used equations of (4.2) is smaller than a given tolerance is made after a new calculation of the matrices and projectors, ensuring that all equations are really considered in the same point y, x.

5 Examples

We will present a few examples that illustrate the algorithms to determine the index and to compute consistent initial values. At first we have to describe a problem by a specially structured MATLAB routine. The structure is similar to the description of ODEs in MATLAB. The algorithm and the example files are available under
http://www.mathematik.hu-berlin.de/∼ lamour/software.

The following examples are tested:

		Index	Dimension
1.	Example 2.1 from [Mär02]	3	3
2.	Classical mathematical pendulum	3	5
3.	Andrew's squeezing mechanism from [CWI]	3	27
4.	Aircraft from [CWI]	5	8
5.	Discharge pressure control from [HLR89]	2	7
6.	Robotic arm from [CWI]	5	8
7.	Electronic circuit [Tis01]	2	2
8.	Path following [Bre83]	2	8

The index of the examples was verified by the algorithm. One of the reasons for developing this algorithm was to compute projectors Q_i with (1.6). The following table summarizes the dimensions of the matrices Σ_j of the different levels, the compliance with the property (1.6), and the projector property $Q_j^2 - Q_j = 0$.

Ex.	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	$\max\limits_{\substack{0\le k\le index-1\\ j<k}}\|Q_kQ_j\|$	$\max\limits_{j}\|Q_j^2-Q_j\|$
1.	2	0	0	1			0	0
2.	4	0	0	1			4.8e-16	3.9e-16
3.	14	7	0	6			8.7e-12	1.1e-11
4.	6	1	0	0	0	1	6.7e-15	4.3e-15
5.	3	3	1				5.0e-15	7.9e-15
6.	6	0	0	1	0	1	6.8e-14	4.4e-14
7.	1	0	1				0	0
8.	6	0	2				9.7e-12	2.1e-11

To show the computation of consistent initial values we select example 8 with the initial values $y=0, x=(100000,0,0,12000,-2,50,2.6,-0.5)$. For an accuracy of 10^{-13} we obtain the solution

$$y=\begin{pmatrix}-2.082942931551142e+02\\ 4.017553236288956e-04\\ 4.017553236288956e-04\\ -3.459611347519115e+01\\ -1.490116635716069e-22\\ 1.490279137166063e-21\end{pmatrix}, x=\begin{pmatrix}1.000000000000000e+05\\ 0\\ 1.520496541764939e-28\\ 1.193498981952058e+04\\ -1.000000000000000e+00\\ 4.500000000000000e+01\\ 2.711276792141309e+00\\ -5.175036627397555e-02\end{pmatrix}.$$

To fulfil the request of an engineer we set, with the help of user's function z: $z(1)=x(4)-12000$. It results

$$y=\begin{pmatrix}-2.094288772474021e+02\\ 4.039436946700642e-04\\ 4.039436946700642e-04\\ -3.497826035277599e+01\\ -1.490065609701807e-22\\ 1.503225041635664e-21\end{pmatrix}, x=\begin{pmatrix}1.000000000000000e+05\\ 2.853257150217720e-21\\ 8.956711672724184e-21\\ 1.200000000000000e+04\\ -1.000000000000000e+00\\ 4.500000000000000e+01\\ 2.672870048053042e+00\\ -5.220958583823603e-02\end{pmatrix}.$$

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