Local Unitarity: A new approach to perturbative computations in Quantum Field Theories

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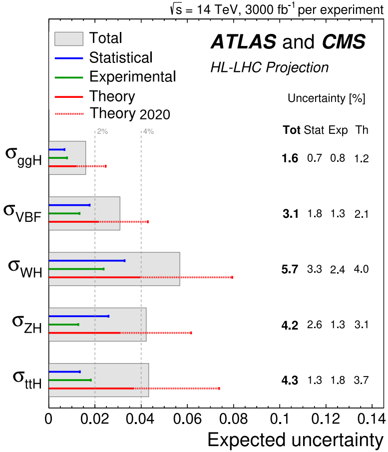
The European Strategy for Particle Physics

> *"The vision is to prepare a Higgs factory, followed by a future hadron collider with sensitivity to energy scales an order of magnitude higher than those of the LHC"*

Keeping up with experiment

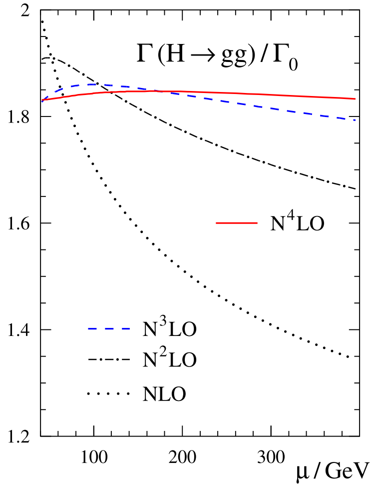
- High Luminosity LHC is expected to drastically reduce statistical uncertainty
- Theory uncertainty will be the **dominating contribution**
- We need to compute more higher-order corrections



[CERN-LPCC-2018-04]

The effect of higher-order corrections

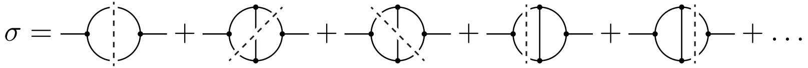
- Large reduction of uncertainty
- NLO cross sections are available with the push of a button
- Why no automation at NNLO?



[Herzog, Ruijl, Ueda, Vermaseren, Vogt, JHEP 2017]

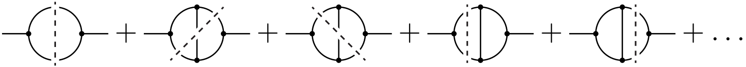
Cross sections

A cross-section can be represented as a sum of interference diagrams

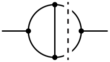
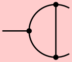


Cross sections

A cross-section can be represented as a sum of interference diagrams

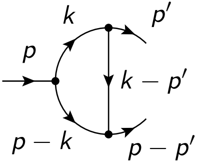
$\sigma =$ 

Each interference can be represented as integrals of amplitudes:

 $= \int d^3\vec{p}\,\delta(p^0 - |\vec{p}\,'| - |\vec{p} - \vec{p}\,'|) \cdot$  $\cdot \int d^4k$ 

Both the phase space integral and loop integral are divergent!

Challenges



$$= \int d^4 k \frac{1}{k^2 (k-p')^2 (p-k)^2}$$

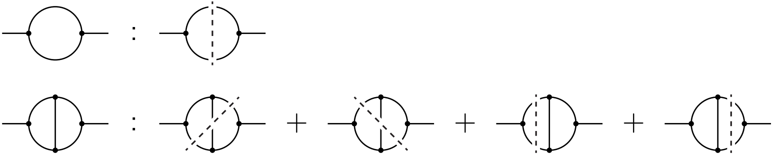
- This integral diverges when $p'^2 = 0$
- Workarounds to integrate require changing the space-time dimension or introducing a counterterm:

$$\int \underbrace{I - I_{CT}}_{\text{finite}} + \underbrace{\int I_{CT}}_{\text{easy}}$$

- Some integrals are elliptic functions that cannot be evaluated

Real-virtual cancellations

- Do not consider the amplitude as the fundamental component anymore, but elements of the cross section with the same *supergraph*: [Capatti, Hirschi, Kermanschah, Pelloni, Ruijl, JHEP 2020]



- Each of these collections is finite (KLN theorem)
- If we can make it locally finite, we can Monte Carlo integrate them

Local Unitarity

Wishes	Local Unitarity
Differential cross sections	
No restrictions on mass scales	
No reduction to master integrals	
No complicated function evaluations	
No IR counterterms for loop degrees of freedom	
No IR counterterms for real degrees of freedom	
Fully automated renormalization	
Integration in lower dimensions	
Method generic for any process to any order	

Local Unitarity

Wishes	Local Unitarity
Differential cross sections	✓
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No IR counterterms for real degrees of freedom	✓
Fully automated renormalization	✓
Integration in lower dimensions	✓
Method generic for any process to any order	✓

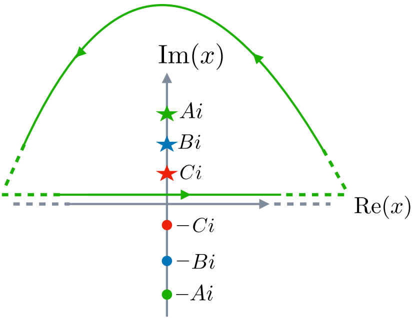
Amplitudes

- First we study the singularities of amplitudes
- Our treatment of amplitudes will guide the extension to cross sections
- Loop-Tree Duality (LTD): integrate out the loop energies analytically

Residue theorem

$$I = \int dx F(x)$$

$$F(x) = \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(x - Bi)(x + Bi)} \frac{1}{(x - Ci)(x + Ci)}$$



Cauchy: $R(x^*) \equiv Res(F, x = x^*)$

$$I = (-2\pi i) \left[R(Ai) + R(Bi) + R(Ci) \right]$$

Inverse propagator notation

$$(k+p_i)^2 - m_i^2 + i\epsilon = (k^0 + p_i^0)^2 - (\vec{k} + \vec{p_i})^2 - m_i^2 + i\epsilon$$

Inverse propagator notation

$$(k+p_i)^2 - m_i^2 + i\epsilon = (k^0+p_i^0)^2 - (\vec{k}+\vec{p}_i)^2 - m_i^2 + i\epsilon$$

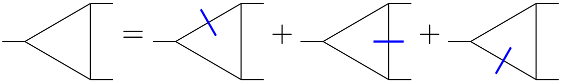
Factorizing:

$$\left(k^0 + p_i^0 + \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2 - i\epsilon}\right)\left(k^0 + p_i^0 - \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2 - i\epsilon}\right)$$

We call the spatial and mass part Δ:

$$\left(k^0 + p_i^0 + \Delta_i\right)\left(k^0 + p_i^0 - \Delta_i\right)$$

Residue theorem for propagators



We take the positive energy solution:

$$\delta(k^0 + p_i^0 - \Delta_i) \to k^0 = \Delta_i - p_i^0$$

Effect of δ on propagator i:

$$\left(k^0 + p_i^0 + \Delta_i\right) \to 2\Delta_i$$

Effect of δ on other propagators j:

$$\left(\Delta_i - p_i^0 + p_j^0 + \Delta_j\right)\left(\Delta_i - p_i^0 + p_j^0 - \Delta_j\right) = E_{ij} H_{ij}$$

Loop Tree Duality (LTD)

- LTD yields a sum of cut diagrams without loops [Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 2009]
- Expression at one-loop:

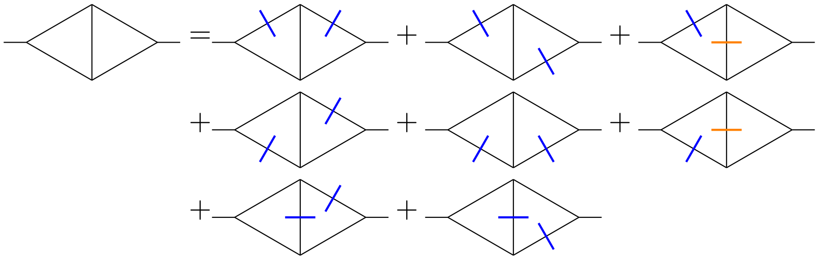
$$I = -\int d^4k \sum_{i}^{N} \tilde{\delta}(q_i^2) \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{q_j^2 - i0\eta(q_j - q_i)}$$

- Alternatively:

$$I = -\int d^3k \sum_{i}^{N} \frac{1}{2\Delta_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{\left(\Delta_i - p_i^0 + p_j^0 + \Delta_j\right)\left(\Delta_i - p_i^0 + p_j^0 - \Delta_j\right)}$$

Multi-Loop-Tree Duality

- Integrate the energies from all loop variables one by one [Capatti, Hirschi, Kermanschah, Ruijl PRL 2019; Bierenbaum, Catani, Draggiotis, Rodrigo JHEP 2010]
- Effect: cut all propagator combinations that leave no loops
- Due to the iterative procedure, sometimes the negative energy solution has to be taken (orange)



Singular structure

When is the inverse propagator 0?

$$E_{ij} \equiv \Delta_i - p_i^0 + p_j^0 + \Delta_j = 0$$
$$H_{ij} \equiv \Delta_i - p_i^0 + p_j^0 - \Delta_j = 0$$

- Δ_i is always ≥ 0
- E_{ij} is an ellipsoid
- H_{ij} is a hyperboloid

Hyperboloid cancellation

Every hyperboloid appears twice with opposite sign:

$$H_{ij} = \Delta_i - p_i^0 + p_j^0 - \Delta_j$$
$$H_{ji} = \Delta_j - p_j^0 + p_i^0 - \Delta_i$$

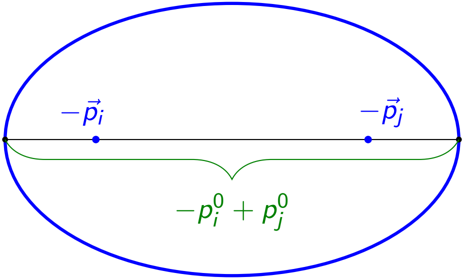
- The first surface appears when cutting i and the second appears in the cut of j
- All hyperboloids cancel!
- All singularities are ellipsoids and are thus bounded!
- This holds for any loops

Ellipsoids

$$E_{ij} = \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2 - i\epsilon} + \sqrt{(\vec{k}+\vec{p}_j)^2 + m_j^2 - i\epsilon} - p_i^0 + p_j^0 = 0$$

Massless case (and ignoring $i\epsilon$):

$$|\vec{k}+\vec{p}_i| + |\vec{k}+\vec{p}_j| - p_i^0 + p_j^0 = 0$$



Results

- Existence depends on external momenta
- It is possible to have configurations without ellipsoids

[PRL '19, Zeno Capatti, Dario Kermanschah, Valentin Hirschi, Ben Ruijl]

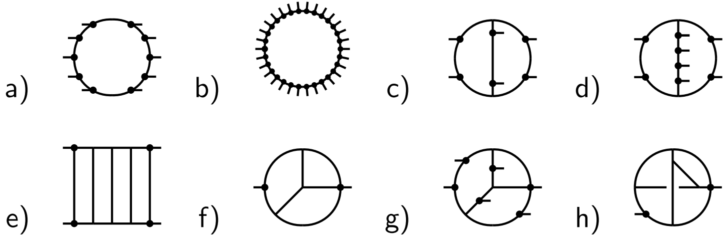
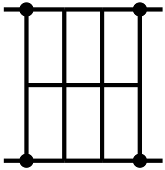


Table: All within 1% of the analytical result

Pushing the boundaries

6-loop four-point function:



- Analytic: $8.4044862640909 \cdot 10^{-19}$ [Basso, Dixon 2017]
- Numerical: $8.38533 \cdot 10^{-19} \pm 2.99674 \cdot 10^{-21}$

Ellipsoids II

$$\sqrt{(\vec{k}+\vec{p_i})^2+m_i^2-i\epsilon}+\sqrt{(\vec{k}+\vec{p_j})^2+m_j^2-i\epsilon}-p_i^0+p_j^0=0$$

- A deformation is only required on the ellipsoid surfaces
- The signature is simply:

$$\frac{-i\epsilon}{\sqrt{(\vec{k}+\vec{p})^2+m^2}}=-i\epsilon$$

- A valid contour deformation should always yield this signature

Deformation direction

- Deform: $k \to k - i\kappa(k)$
- Expanding ellipsoids equation to first order:

$$\sqrt{(\vec{k} - i\vec{\kappa} + \vec{p}_i)^2 + m_i^2} + \sqrt{(\vec{k} - i\vec{\kappa} + \vec{p}_j)^2 + m_j^2} - p_i^0 + p_j^0$$

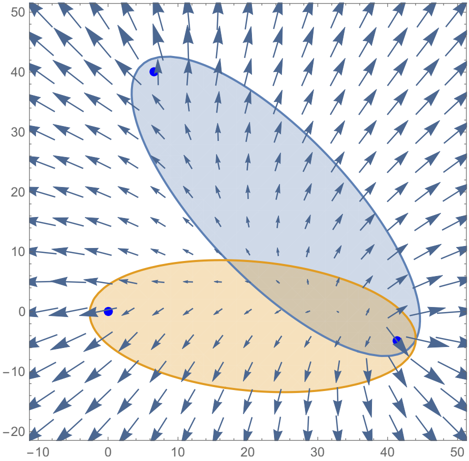
yields:

$$\vec{\kappa} \cdot \underbrace{\left(\frac{\vec{k} + \vec{p}_i}{\Delta_i} + \frac{\vec{k} + \vec{p}_j}{\Delta_j} \right)}_{\text{Normal vector}} > 0$$

- Positive projection on normal $\rightarrow$ point outward

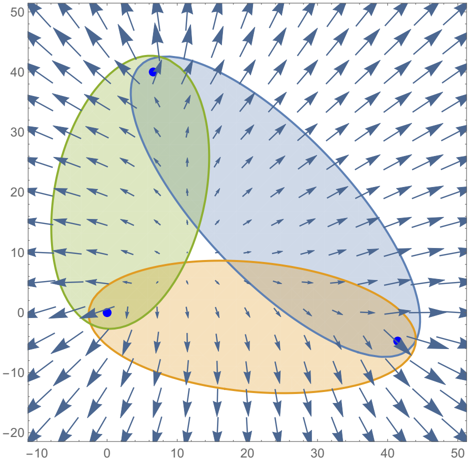
Deformation direction

- The normal is always a good deformation for one ellipsoid
- The sum of normals also works for intersections of two ellipsoids
- Deformation was proposed in [Buchta, Chachamis, Draggiotis, Rodrigo, JHEP 2017]



Normal vectors I

- In general the sum will not work
- Exponential dampening around the ellipsoids does not work without extensive finetuning



Normal vectors II

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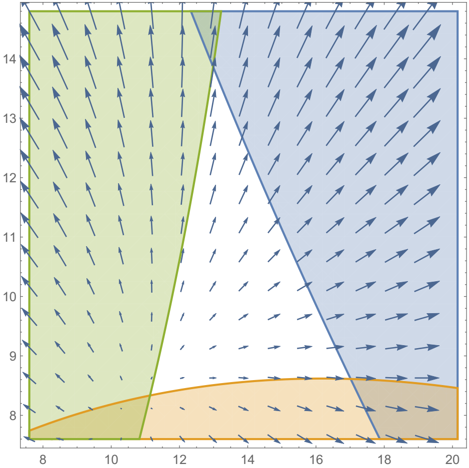
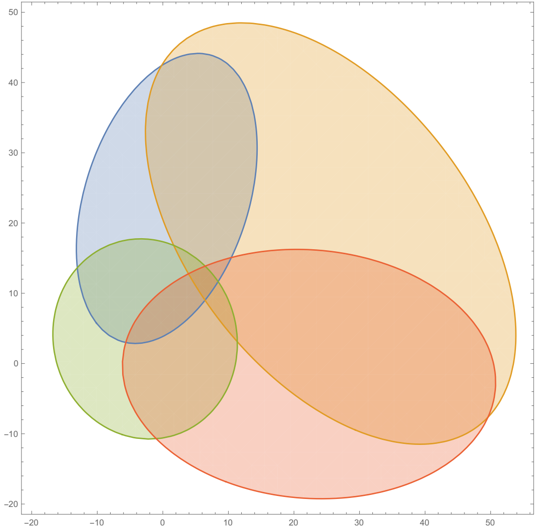
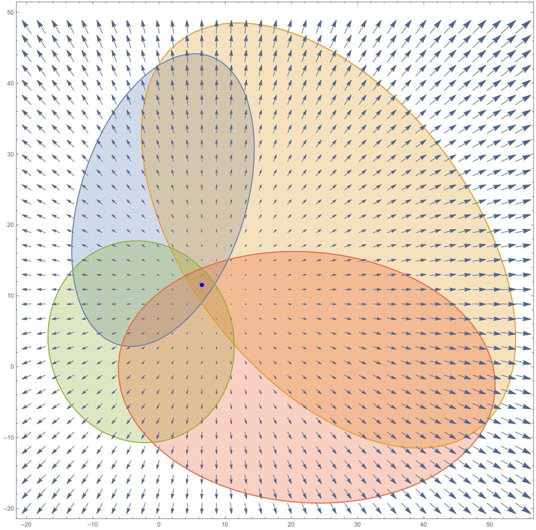


How to deform here?

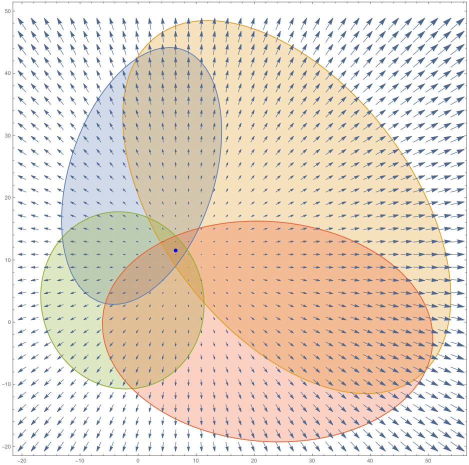


Deformation sources



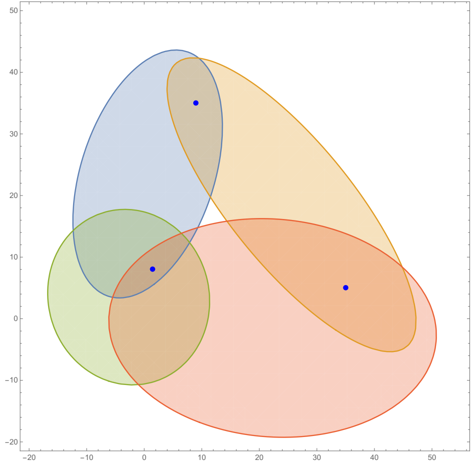
Deformation direction

- Any *source* inside all ellipsoids will produce a good vector!
- Finding a point of overlap is a convex optimization problem
- A solution is always found if it exists



Deformation direction

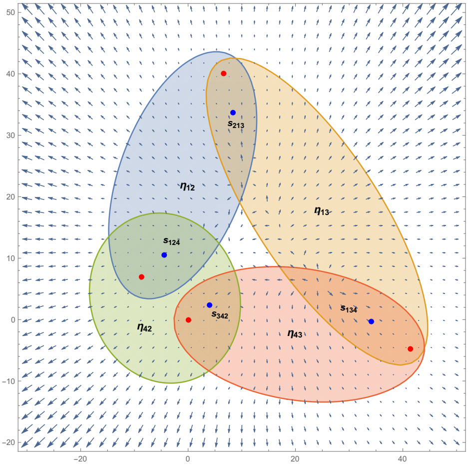
- Finding the maximal overlap structure is NP-hard
- Three sources needed for picture on the right



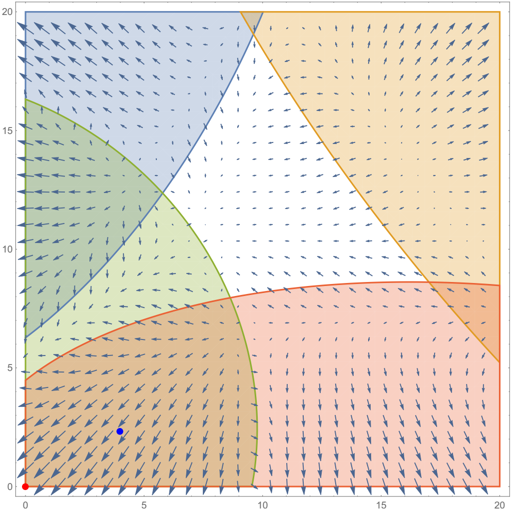
Deformation exclusion

- Deformation field is sum of fields multiplied by function $T(E_{ij})$ that is 0 on a surface E_{ij} and goes to 1 away from it
- Exclusion guarantees correct deformation

$$\vec{\kappa} = (\vec{k} - \vec{s}_{124})T(E_{13})T(E_{43}) + (\vec{k} - \vec{s}_{213})T(E_{42})T(E_{43}) + (\vec{k} - \vec{s}_{134})T(E_{42})T(E_{12}) + (\vec{k} - \vec{s}_{342})T(E_{12})T(E_{13})$$

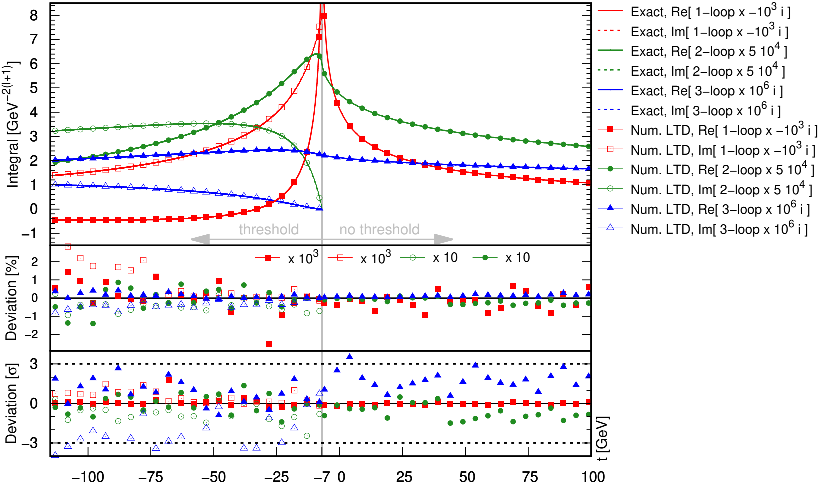


A general and valid deformation



[Capatti, Hirschi, Kermanschah, Pelloni, Ruijl, JHEP 2019]

A scan for multi-loop box topologies

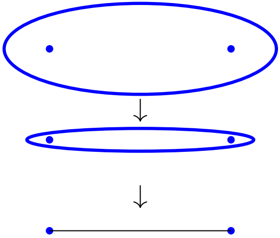


Pinched ellipsoids

- An ellipsoid is *pinched* when it has no interior
- No deformation possible
- *All* IR singularities are pinched ellipsoids
- Pinching not possible if there are masses
- Occurs when external momenta are on-shell:

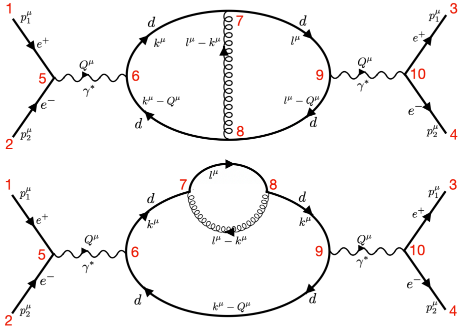
$$|\vec{k}+\vec{p}|+|\vec{k}|-p^0=0 \rightarrow$$
$$|\vec{k}+\vec{p}|+|\vec{k}|-|\vec{p}|=0$$

  when $\vec{k}=x\vec{p}$ with $0 \leq x \leq 1$
- Collinear singularity with soft singularities on the focal points

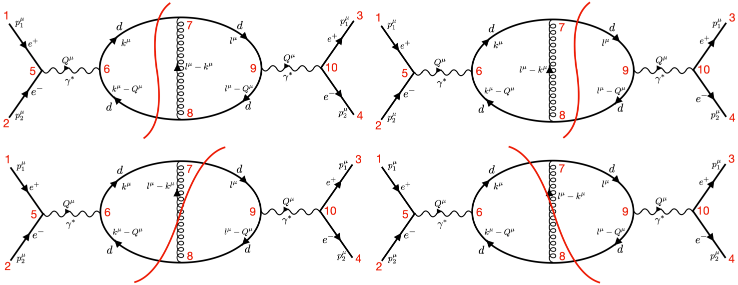


Local Unitarity

- The cross section is given by the sum of all Cutkosky cuts over all *supergraphs*
- Each cut supergraph is IR finite
- For NLO $e^+e^- \to d\bar{d}$ we have 2 supergraphs:

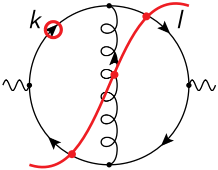


Double-triangle cuts



Cancelling singularities I

- Consider Cutkosky cuts with additional on-shell propagators:



- $k^0 \pm |\vec{k}| \to (|\vec{l}| + |\vec{l} - \vec{k}| + |\vec{k}|)\ (|\vec{l}| + |\vec{l} - \vec{k}| - |\vec{k}|) \xrightarrow{\lim} 2|k|$

Cancelling singularities I

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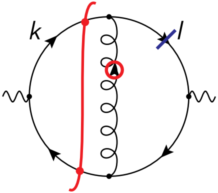
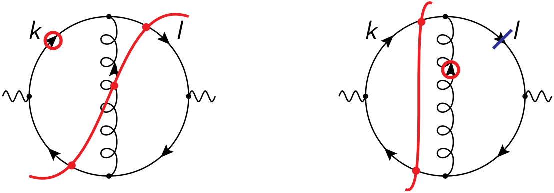


- $l^0 - k^0 \pm |\vec{l} - \vec{k}| \rightarrow$
$(|\vec{l}| - |\vec{k}| + |\vec{l} - \vec{k}|)\,(|\vec{l}| - |\vec{k}| - |\vec{l} - \vec{k}|) \xrightarrow{\lim} -2|\vec{l} - \vec{k}|$

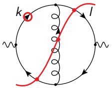
Cancelling singularities I

- Consider Cutkosky cuts with additional on-shell propagators:



- $k^0 \pm |\vec{k}| \to (|\vec{l}| + |\vec{l} - \vec{k}| + |\vec{k}|)\,(|\vec{l}| + |\vec{l} - \vec{k}| - |\vec{k}|) \xrightarrow{\lim} 2|k|$
- $l^0 - k^0 \pm |\vec{l} - \vec{k}| \to$
  $(|\vec{l}| - |\vec{k}| + |\vec{l} - \vec{k}|)\,(|\vec{l}| - |\vec{k}| - |\vec{l} - \vec{k}|) \xrightarrow{\lim} -2|\vec{l} - \vec{k}|$
- Exactly the same on-shell propagators (and $2\Delta_i$s), but with a relative sign

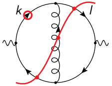
Cancelling singularities: observable functions



$$= N_3\delta(p^0 - |\vec{k} - \vec{p}| - |\vec{l} - \vec{k}| - |\vec{l}|)\mathcal{O}_3(-k + p, -l + k, l)$$

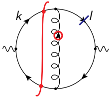
$$\xrightarrow{\lim} N\delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|)\, \mathcal{O}_3(-k + p, -l + k, l)\big|_{k^0 = |\vec{k}|}$$

Cancelling singularities: observable functions



$$= N_3 \delta(p^0 - |\vec{k} - \vec{p}| - |\vec{l} - \vec{k}| - |\vec{l}|) \mathcal{O}_3(-k + p, -l + k, l)$$

$$\xrightarrow{\lim} N \delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|) \; \mathcal{O}_3(-k + p, -l + k, l)\big|_{k^0 = |\vec{k}|}$$

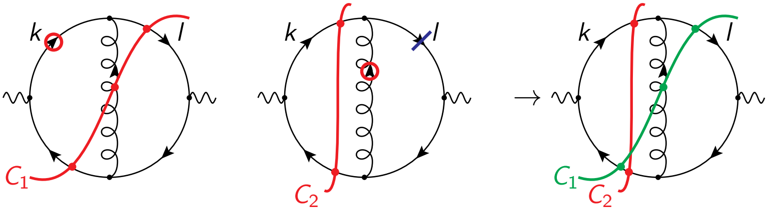


$$= N_2 \delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|) \mathcal{O}_2(k, -k + p)$$

$$\xrightarrow{\lim} -N \delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|) \; \mathcal{O}_2(k, -k + p)\big|_{l^0 - k^0 = -|\vec{l} - \vec{k}|}$$

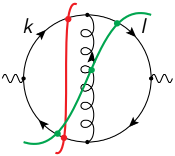
- Energy-conserving δs are the same on pinches
- Only cancels for IR-safe observables: $\mathcal{O}_3|_{\text{on pinch}} = \mathcal{O}_2|_{\text{on pinch}}$

Cancellations between Cutkosky cuts

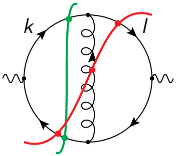


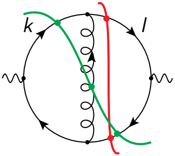
- Singularities lie on intersections of C_1 and C_2
- The intersection point of C_1 and C_2 is finite in $C_1 + C_2$
- The cancellations are pairwise (as with hyperboloids)

Cancellations between Cutkosky cuts

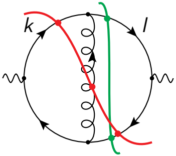


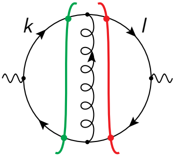
cancels



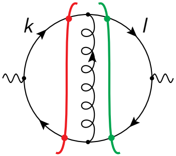


cancels





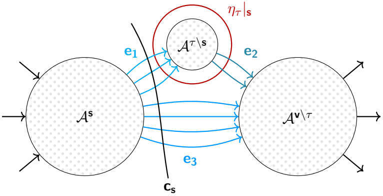
cancels



Non-pinched ellipsoid cancellation if the observable is 1!

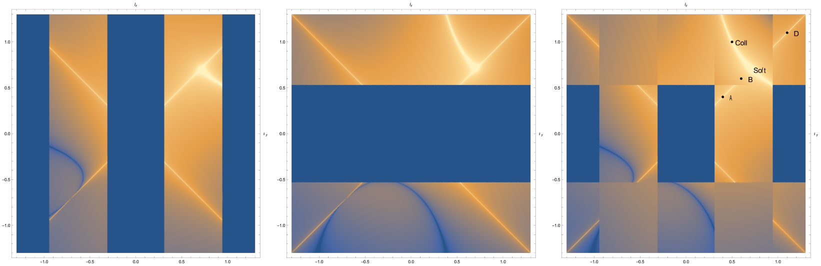
Cancellation pattern

Pairwise cancellation pattern holds for **any threshold** at **any order**



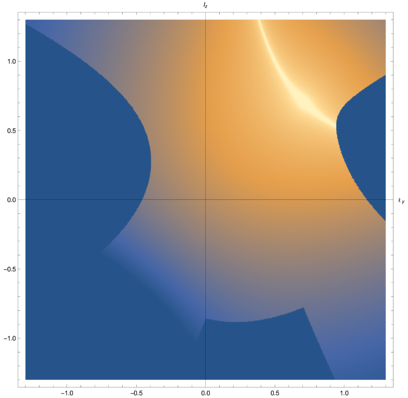
Results for Double-Triangle I

Semi-inclusive cross section for $(\vec{k}, \vec{l}) = ((0, k_y, \frac{1}{\sqrt{2}}), (0, \frac{1}{\sqrt{2}}, l_z))$

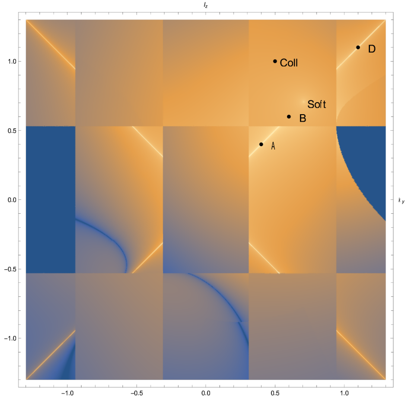


(a) Left virtual cut (b) Right virtual cut (c) Sum of virtual cuts

Results for Double-Triangle II

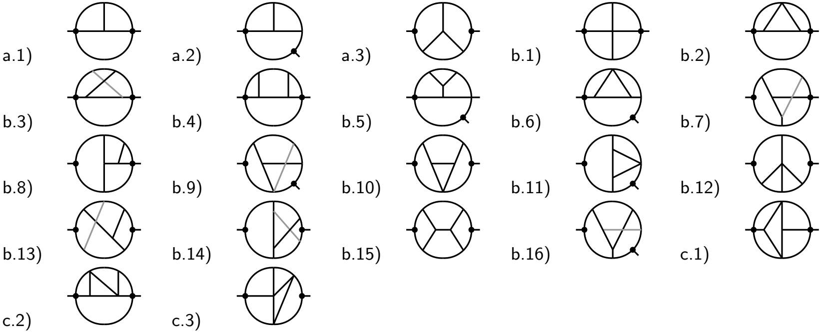


(a) Sum of 2 real cuts



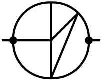
(b) Sum of all cuts

Results for scalar topologies



Results for scalar topologies

- Scalar diagrams have the worst IR divergences
- Example: [Forcer+R*, Herzog, Ruijl, Ueda, Vermaseren, JHEP 2018]



$$\xrightarrow{\text{optical theorem}} \frac{5\pi}{(16\pi)^5}\frac{441}{40}\zeta_7 = 1.77832 \cdot 10^{-9}$$

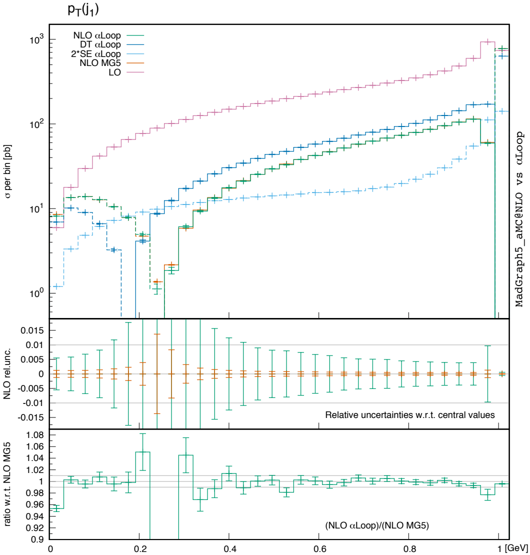
- LU with 1M samples: [Capatti, Hirschi, Kermanshah, Pelloni, Ruijl, JHEP 2020]

$$I = 1.7797(33) \cdot 10^{-9}$$
$$\Delta_\sigma = 0.42\sigma$$
$$\Delta_\% = 0.077\%$$

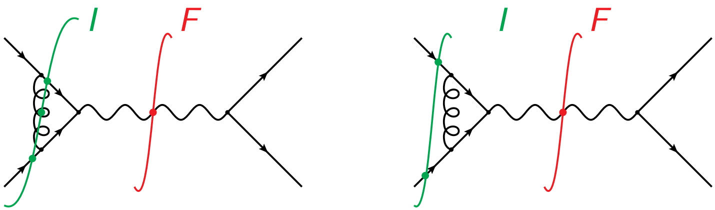
NLO correction to $e^+e^- \to Ht\bar{t}$

- 15 supergraphs Monte Carlo sampled over
- $\mathcal{O}(50)$ interference graphs (cuts)
- MG5_aMC@NLO: $-1.38400 \cdot 10^{-4} \pm 1.4 \cdot 10^{-7}$ [Alwall, Frederix, Frixione, Hirschi, Maltoni]
- Local Unitarity: $-1.38320 \cdot 10^{-4} \pm 5.9 \cdot 10^{-7}$

NLO correction to $e^+e^- \to d\bar{d}$



Initial-state radiation



- Initial-state infrared singularity cancellations to be studied
- Suggests extra particles in the initial state (as in the final state)
- Fixing the number of initial-state particles is a non-IR-safe observable
- Find connection to classical PDFs and PDF counterterms

Conclusion

The Local Unitarity approach

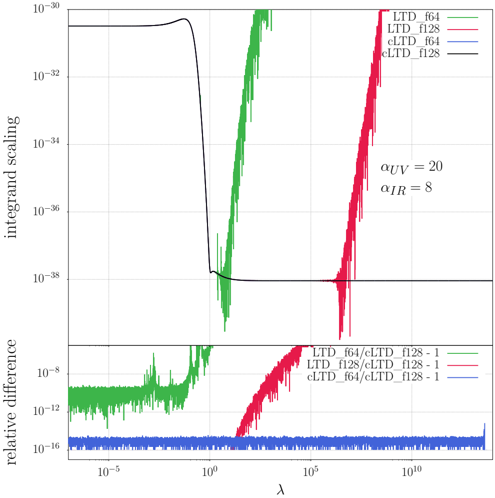
- locally realizes the KLN-theorem and regulates final-state radiation
- uses a contour deformation to regulate threshold singularities
- automatically renormalizes in $\overline{MS}$ using local BPHZ
- can be extended to initial-state radiation

Status:

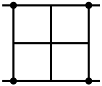
- Sample time around 1ms @ NNLO
- Verification at NLO
- Working on automatic on-shell renormalization @ NNLO
- Working on improving convergence rate

Thank you for your attention.

Manifestly Causal Loop-Tree Duality (cLTD)

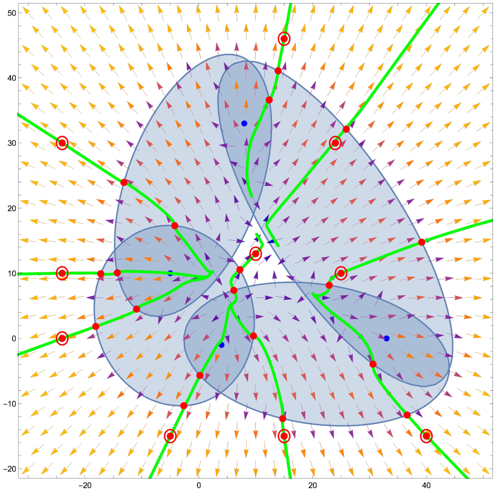


- An LTD term of



  scales as λ^0 and the sum as λ^{-8}
- The hyperboloids can be canceled analytically [Capatti, Kermanschah, Hirschi, Pelloni, BR '20]
- cLTD gives improved ultraviolet behaviour

Sampling on ellipsoids



- The deformation field gives us this map!
- Simply walk back along the vector field
- Can be solved using an ODE solver