Area B
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Traveling Efficiently with Mathematics

Ralf Borndörfer Olga Heismann Marika Karbstein
Christian Liebchen Rolf Möhring Andris Möller Werner Römisch

public transportation
line planning
timetabling
vehicle rotation planning
railway vehicle rotation planning
revenue management
multistage stochastic programming
line planning
timetabling
revenue management
railway vehicle rotation planning
vehicle rotation planning
rolling stock roster planning

1 Introduction

Public transportation, i.e., public transit, rail, and air traffic, gives rise to challenging *planning problems*. We distinguish *strategic planning problems* about network design, line planning, timetabling, and price planning that decide about the services that are being offered, *operational planning problems* of vehicle and crew scheduling and rostering about the cost minimal implementation of the service by a best possible allocation of resources, and *operations control problems* of vehicle and crew dispatching to monitor the execution of the plan and to counter deviations in real time. Solving these problems in a best possible way is crucial for the system's quality and efficiency, and mathematical optimization is the key to achieve that.

In the past 20 years, mathematical "optimizers" have been established as the industry standard in price planning in the airline industry, and in operational planning in public transit and air traffic. In particular, vehicle and crew scheduling problems, which can be modeled as multi-commodity flow and set packing or covering problems, and revenue management problems, which lead to multistage stochastic optimization problems, have been thoroughly investigated and can nowadays be solved for large-scale, industrial instances, see [7] for a survey and pointers to the literature. Such success stories show what mathematics can do for public transport. However: the bigger part of the planning process still lacks optimization support.

The vision of the TRAFFIC AND TRANSPORT DOMAIN OF EXPERTISE in MATHEON'S APPLICATION AREA B: NETWORKS is that the entire planning process in public transportation can be improved by mathematical optimization. "What constitutes a good [public transportation] network?" – this key question of AA-B summarizes this idea. MATHEON projects contributed to the following topics:

- *Project B15 (= B1 + B5 since phase II): Service design in public transport* has addressed the line planning and the timetabling problem.
- *Project B6: Origin destination control in airline revenue management by dynamic stochastic programming* has investigated a network wide approach to airline ticket sales maximization.
- *Project B22: Rolling stock roster planning* worked on the railway version of the vehicle scheduling problem.

These questions lead directly to an abstract *theory of network optimization*, that carries over between different applications:

hypergraph
Steiner connectivity problem
hypergraph assignment problem
path connectivity
timetabling
cycle basis
revenue management
multistage stochastic programming
stochastic programming
line planning
passenger routing

- *Hypergraph optimization*: Line planning and rolling stock roster planning lead to hypergraph generalizations of fundamental combinatorial optimization problems on Steiner trees and assignments, that model path connectivity and locally constrained network flows.
- *Combinatorial optimization*: Timetabling problems can only be solved by using strong integer programming formulations based on structural knowledge about short cycle bases of underlying scheduling graphs.
- *Stochastic optimization*: Network wide revenue management requires approximations of high-dimensional stochastic processes that open the application of multistage stochastic programming to large scale dynamic models.

Our contributions were awarded with national and international prizes, namely, the Diploma thesis and the Dissertation award of the German operations research society 2006, the INFORMS TSL Dissertation Award 2006, the Heureka prize 2008, the DMV Abitur prize 2008, and the ATMOS best paper award 2013. The projects also demonstrated the value of mathematical optimization for public transportation in industrial cooperations. Major breakthroughs were the implementation of the *first optimized timetable* at the Berlin subway in 2005 and the *first optimized line plan* for ViP Potsdam in 2010, see also the boxes on pages **??** and **??**, repectively.

The following subsections of this article discuss the results of the projects of MATHEON's Domain of Expertise Traffic and Transport. The exposition is subdivided into four subsections on line planning and path connectivity, timetabling and cycle bases, revenue management and scenario reduction, and rolling stock roster planning and hyper-assignments. We review each application, present the mathematical model and the solution approach, discuss the main theoretical contributions, and report on the impact that could be achieved.

2 Line Planning and Path Connectivity

Line planning is a fundamental problem in the design of a public transportation system. It consists in finding a set of lines in an infrastructure network and their frequencies of operation such that a given travel demand can be routed. There are two main objectives, namely, minimization of operation costs and minimization of travel and transfer times. Since the 1970s, ever more realistic optimization models have been developed, see [44, 28] for a survey. MATHEON's project B15 contributed to the integration of line planning and passenger routing, transfer handling, and dynamic line generation in order to optimize the line plan of Potsdam.

Basic model. A basic *integrated line planning and passenger routing model* (LPP) can be formulated as in Figure 1. It uses binary variables $x_{\ell,f}$ for the choice of line $\ell \in \mathcal{L}$ at frequency $f \in \mathcal{F}$ and continuous variables y_p for the number of passengers traveling on path $p \in \mathcal{P}$. Inequalities (i) stipulate a passenger flow equal to the demand d_{st} (number of passengers) for each pair of "origin-destination nodes" (OD-nodes) $(s,t) \in D$. Inequalities (ii) enforce sufficient transportation capacity on each arc. Inequalities (iii) ensure that a line is operated with at most

$$
\begin{array}{lll}
\text{(LPP)} & \min & \lambda \sum_{\ell\in\mathcal{L}} \sum_{f\in\mathcal{F}} c_{\ell,f}\, x_{\ell,f} \;+\; (1-\lambda) \sum_{p\in\mathcal{P}} \tau_p\, y_p \\
\text{(i)} & \sum_{p\in\mathcal{P}_{st}} y_p = d_{st} & \forall (s,t) \in \mathcal{D} \\
\text{(ii)} & \sum_{p:a\in p} y_p \le \sum_{\ell:a\in\ell} \kappa_{\ell,f}\, x_{\ell,f} & \forall a \in A \\
\text{(iii)} & \sum_{f\in\mathcal{F}} x_{\ell,f} \le 1 & \forall \ell \in \mathcal{L} \\
\text{(iv)} & \sum_{\ell:a\in\ell} \sum_{f\in\mathcal{F}} f\, x_{\ell,f} \le F_a & \forall a \in A \\
\text{(v)} & x_{\ell,f} \in \{0,1\} & \forall \ell \in \mathcal{L}, \forall f \in \mathcal{F} \\
\text{(vi)} & y_p \ge 0 & \forall p \in \mathcal{P}
\end{array}
$$

Figure 1: Integrated line planning and passenger routing model.

one frequency, while inequalities (iv) bound the sum of the frequencies of lines that can be operated on an individual arc. The objective minimizes line operating costs $c_{\ell,f}$ and passenger traveling times τ_p weighted by a parameter $\lambda \in [0,1]$. Varying λ, the trade-offs between the two objectives can be studied in terms of *Pareto curves*, see Figure 2 for an application at the city of Potsdam.

Dynamic line generation. The main innovation of model (LPP) was the consideration of all possible line routes to overcome static line pools. It is solved by a *branch-and-price algorithm* that iteratively constructs the needed line and passenger paths by solving so-called *pricing problems*. The pricing problem for the passenger path variables is a shortest path problem which can be solved in polynomial time via Dijkstra's algorithm. The pricing problem for the line path variables is a longest path problem and thus $\mathcal{NP}$-hard. If the lines have lengths $O(\log |V|)$ that are logarithmic in the number of nodes, lines can also be priced in polynomial time via randomized coloring [3, 28]. This freedom in line and passenger routing can drastically reduce line costs by up to 10% [5]. The model further allows to deal with more realistic passenger routing models, e. g., to avoid long detours or splittings of passengers routes [43].

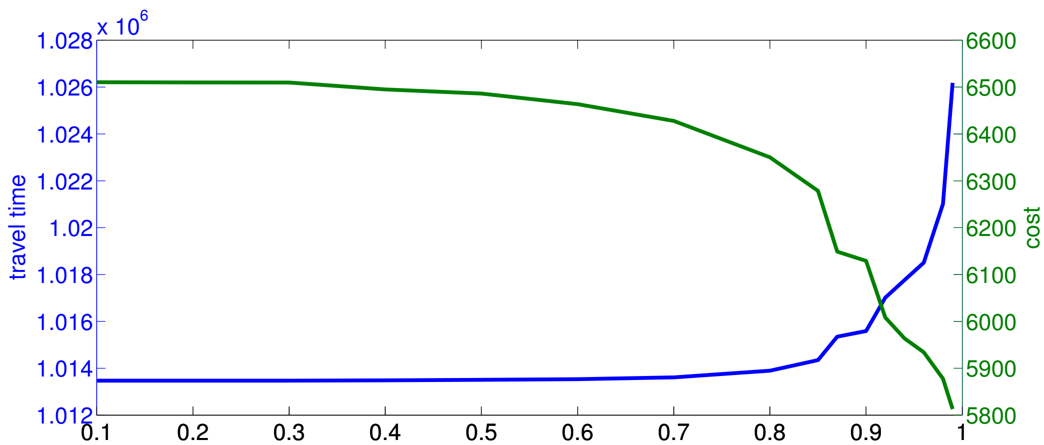

Figure 2: Cost vs travel time in Potsdam. Varying the weight parameter λ (x-axis) results in line plans with different costs (green) and travel times (blue). A value of $\lambda = 0.1$ almost ignores costs, a value of $\lambda = 1$ ignores travel time.

direct connection model for line planning
path connectivity
Steiner connectivity problem
hypergraph

Direct connections. The model (LPP) can also be extended to deal efficiently with *transfers* or, more precisely, with *direct connections*. To this purpose, the passenger flow variables y_p are split into two types $y_{p,0}$ and $y_{p,1}$ that account for the number of passengers on a transfer free connection on path p and for number of passengers transferring at least once on path p, respectively. These are linked via *direct connection capacity constraints* that ensure the correct passenger flow on direct connections. These constraints are related to *metric inequalities* and can be derived via a Benders decomposition. We showed that a certain explicit combinatorial subset of these constraints suffices to estimate the number of direct travelers in a surprisingly accurate way [11, 28]. In our computations, the geometric mean of the prediction error on a test set of 14 large-scale real-world instances was only 0.39% [28].

Configuration model. One of the reasons for the hardness of line planning is the gap between line capacities and demands. We proposed a concept to strengthen the capacity constraints by means of a novel *configuration model* that enumerates the set of possible line frequency *configurations* on each arc of the network. This replaces numeric capacities by a combinatorial choice of configurations. We showed that such an *extended formulation* implies general facet defining inequalities such a set cover, band, and mixed integer rounding inequalities for the standard formulation, and that it can be used to solve large-scale line planning instances [10].

Special network topologies. Another idea to cope with the complexity of line planning is to exploit the structure of *special network topologies*. We studied the case of the Quito Trolbús system, which consists of a trunk route and a number of feeder bus systems. Trolebús therefore has a simple tree topology, but it also features many degrees of freedom in line construction, e. g., express lines and uni-directional lines; these had not been considered in the mathematical line planning literature before. It turned out that the problem remains hard in general, however, some cases can be solved in polynomial time, e. g., if only "closed lines" are used [48]. Moreover, our computations indicated potential cost reductions of sometimes up to 50%.

ViP. We optimized the ViP (Verkehrsbetrieb Potsdam GmbH) line plan for 2010 within an industry project *Stadt+*. In fact, our optimization methods worked very well, and the difficulties in the project were mainly data issues: We had to define possible endpoints of new lines, add missing links, and model requirements such as a minimum cycle time for the tram, minimum frequency requirements for each station, or minimal and maximal lengths for lines with respect to travel time and distance. The final optimized line plan achieved substantial reductions in costs of around 4% and in perceived travel times of around 6% [13, 6], see also page **??**. ViP implemented this solution almost one-to-one. As far as we know, this is the first mathematically optimized line plan that was implemented in practice. In this way, project Stadt+ proved that line optimization methods are ready for practice.

Connectivity theory. The abstract combinatorial problem that underlies line planning is the *Steiner connectivity problem*. It generalizes the well known *Steiner tree problem* from the graphical to a hyper-graphical setting: Given a graph $G = (V, E)$, a subset $T \subseteq V$ of the nodes, and a set of paths $\mathcal{P}$, the Steiner connectivity problem asks for a minimum cost subset of paths from $\mathcal{P}$ connecting all nodes in T. Similar to the relevance of the Steiner tree problem in network design, the Steiner

Menger companion theorem for hypergraphs
timetabling

connectivity problem can be seen as the prototype of all problems where nodes are connected by installing capacities on *paths* which is exactly the case in line planning. Likewise, main results about complexity, approximation, integer programming formulations, and polyhedra can be generalized from the Steiner tree to the Steiner connectivity setting [12, 28].

Complexity and approximation. A relation of the Steiner connectivity problem to the undirected and the directed Steiner tree problem yields the $\mathcal{NP}$-hardness of the general problem and some *polynomially solvable cases*, e. g., if $|T|$ is constant. However, a major difference between the Steiner tree problem and the Steiner connectivity problem is the "spanning" case in which all nodes have to be connected: While the spanning tree problem is solvable in polynomial time, the *spanning set problem* is $\mathcal{NP}$-hard. The greedy algorithm, however, gives a *logarithmic approximation* for this case [9]. In general, we showed that the primal-dual approximation technique of Goemans and Williams can be extended to the Steiner connectivity problem. This yields constant factor approximations.

Polyhedral results. The investigation of the *Steiner connectivity polytope* is the basis for the development of cutting plane methods for the line planning problem. We investigated a canonical undirected cut formulation as well as an extended directed cut formulation. The *Steiner partition inequalities*, a fundamental class of facet defining inequalities for the Steiner tree problem, can be generalized to the Steiner connectivity setting. We also stated necessary and sufficient conditions for these inequalities to be *facet defining*. Our main algorithmic result is that (a super class of) the *Steiner partition inequalities can be separated in polynomial time* [12, 28]. In particular, the undirected cut formulation enriched by all Steiner partition inequalities is dominated by the extended directed cut formulation.

Min-max results. Properties and duality results on paths and cuts in graphs generalize to results on *connecting* and *disconnecting sets* in hypergraphs in the Steiner connectivity setting. In particular, a *Menger result* holds in the two-terminal case. Here, connecting and disconnecting sets give rise to a blocking pair of ideal matrices just like the incidence matrices of paths and cuts. Moreover, the LP relaxation of the cut formulation of the Steiner connectivity problem for two nodes is *totally dual integral* [9, 28]. It follows that not only Menger's Theorem holds for hypergraphs, which is folklore, but also that the associated *Menger companion Theorem* holds:

Theorem 1 (Menger Companion Theorem for Hypergraphs)**.** *The minimum cardinality of an st-hyperpath is equal to the maximum number of hyperedge-disjoint st-hypercuts.*

3 Timetabling and Cycle Bases

"The timetable is the essence of the service offered by any provider of public transport" (Jonathan Tyler, CASPT 2006). Indeed, the timetable has a major impact on both operating costs and on passenger comfort. Moreover, in a railway context, the timetable determines the use of the track capacity. Yet, depending on the context, there may be different perspectives on what actually is "the timetable": It can range from just a basic hourly pattern to a complete composition of all trips of a day including infrastructure capacity assignments, see [31] for a survey.

periodic event scheduling problem
node potential
arc tension
cycle basis
cycle basis

MATHEON's project B5 addressed the computation of a good basic (hourly) pattern, which is of course the key structure for the actual timetable, in particular in a non-railway context. Its key property is that any action within the network repeats periodically, say every hour. This structure is reflected well in the so-called *periodic event scheduling problem* (PESP) [45].

Periodic event scheduling model. The core of the PESP is a directed graph $D = (V, A)$. The vertices are events which repeat periodically after the constant period time T (e.g., $T = 60$ minutes), and which typically represent arrivals or departures of a line at particular stops. An arc $a \in A$ measures the time duration that passes from the occurrence π_i of its head event i until the occurrence π_j of its tail event j. A PESP constraint requires this time duration to be within some periodic interval $[\ell_a, u_a]_T$:

$$\ell_a \leq (\pi_j - \pi_i - \ell_a) \bmod T + \ell_a \leq u_a,$$

where only $u_a < \ell_a + T$ is of any interest. Bounds on the time duration between events may model minimum headway times, minimum transfer times, or upper bounds on the transfer waiting time of important directed transfers. An instance of the PESP is then specified by a constant period time T and an event-activity network (D, ℓ, u). In order to transfer this model into an integer linear program, we have to express the modulo operator of an arc a in terms of additional integer variables p_a. At the same time, typically a linear objective function is added in which time durations on transfer waiting times or turnaround times of vehicles are penalized.

Complexity. It is MAXSNP-hard to determine the maximum number of PESP-constraints that can be satisfied by the a timetable vector π. This explains why the `MIPLIB` 2003 contains two PESP instances, although they have only a relatively small number of rows and columns.

Node potentials and arc tensions. In several computational studies it turned out that an alternative IP formulation works much better [34]. Instead of expressing time values at the vertices, which can be interpreted as *node potentials*, consider the corresponding *periodic tension*

$$x_a := \pi_j - \pi_i + T \cdot p_a,$$

where $a = (i, j) \in A$ and p_a is the integer such that $\ell_a \leq \pi_j - \pi_i + T \cdot p_a \leq u_a$ in the initial PESP-constraint, if it exists. One can get rid of the node variables entirely by observing that an arc vector x is the periodic tension of some node potential vector π if and only if for every oriented circuit in the digraph D, the sum of the arc values is an integer multiple of the period time T [33]. In fact, it even suffices to require the sum along every oriented circuit of an integral cycle basis B to be an integer multiple of the period time T. With Γ denoting the arc-cycle incidence matrix of B, the starting point for the most efficient IP formulations of the PESP is the integer programming model on the left of Figure 3. Inequalities (i) and (ii) model the PESP-constraints, while equality (iii) ensures x to be indeed a periodic tension.

Cycle bases. Short integral *cycle bases* promise short running times for integer programs that arise in periodic timetabling. Recent studies reveal that state-of-strictly fundamental cycle basis

$$
\begin{array}{lll}
\text{(PESP-IP-}x\text{-}z\text{)} & \min & w^{\mathsf{T}}(x-\ell) \\
\text{(i)} & & x \le u \\
\text{(ii)} & & x \ge \ell \\
\text{(iii)} & & \Gamma^{\mathsf{T}} x - T z = 0 \\
\text{(iv)} & & z \in \mathbb{Z}^B \\
\text{(v)} & & x \in \mathbb{Z}^A
\end{array}
$$

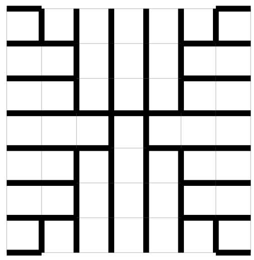

Figure 3: *Left:* Cycle-based integer programming formulation for the periodic event scheduling problem PESP as it typically arises in periodic timetabling problems in public transport. *Right:* Consider the following problem. Given the $N \times N$ square planar grid graph $G_{N,N}$. Find a spanning tree T such that the sum of the lengths of its induced fundamental circuits is as small as possible. A very good solution for $G_{8,8}$ is depicted – is it optimal?

the-art solvers for integer programs like `CPLEX` do not only profit from cycle bases of short length, but take even more advantage if the basis is structured *well*. In particular *strictly fundamental cycle bases*, i.e., bases which are induced by spanning trees, turn out to yield the shortest overall solution times. This motivates an in-depth investigation of strictly fundamental cycle bases and related combinatorial optimization problems in the popular area of graph spanners [37].

Short strictly fundamental cycle bases. Our goal was to capture the gap between the values of optimum strictly fundamental cycle bases and optimum weakly fundamental cycle bases [29]. Here, planar square grid graphs are a challenging benchmark, see the right of Figure 3 for an example. In [38], we significantly improved the constant factor in the known asymptotic bound on this gap [1]. Furthermore, we came up with a new combinatorial bound on this gap which is much stronger than the asymptotic bounds for small and medium sized grids ($n < 5000$). Finally, we also proved that each unweighted graph has a strictly fundamental cycle basis of length $O(n^2)$, hereby proving a 25-years old conjecture by Deo, Krishnamoorthy, and Prabhu [18]. In addition, we proved [20]

Theorem 2. *Every weighted graph (G, w) with total edge weight W admits a weakly fundamental cycle basis of length at most $W \cdot O(\log n \log \log n)$. Such a basis can be computed in polynomial time.*

Polyhedral results. New classes of valid inequalities for the PESP polytopes that arise in periodic timetabling were found in the second Chvátal closure [36]. They serve as the only known polyhedral certificate for the infeasibility of a very small timetabling instance and were instrumental in the development of a branch-and-cut algorithm that could solve the MIPLIB instance `timetab2` for the first time, and within less than one day on a standard PC.

Robustness. To support punctuality, the planning of public transport systems incorporates certain buffer times, that absorb occasional delays. Traditionally such buffers are distributed evenly and a priori over the network. This comes at the expense of operating costs and nominal travel times. As exact assessment of the robustness of timetables, on the other hand, has to take the delay management policy into account, which makes these problems PSPACE-hard. We could nevertheless develop a new concept of robustness that incorporates optimal buffer allocation

revenue management
stochastic programming
multistage stochastic programming
scenario tree

with respect to a simplified delay management policy into our periodic timetabling models. Numerical simulations within the DisKon software system of the numerical and applied mathematics group at the University of Göttingen proved to reduce the number of lost connections for passengers by up to 50% at the expense of at most 7% of nominal travel time [35].

ViP. The ViP (Verkehrsbetrieb Potsdam GmbH) timetable was optimized in an industry project. ViP explicitly requested to extend the timetable model to cover duty scheduling aspects, and finally attested that the extended model indeed yields good timetables [26, 32]. A theoretical comparison of periodic and non-periodic timetables showed that for a sufficiently long planning horizon, omitting the periodicity requirement cannot lead to timetables that require fewer vehicles [4].

BVG. An industrial cooperation with the Berlin underground division of the Berliner Verkehrsbetriebe (BVG) resulted in a notable achievement: In effect since December 12, 2004, the 2005 timetable of the Berlin underground is based on the results of mathematical programming techniques. It is the first such service concept that has been put into daily operation. This timetable offers shorter passenger waiting times – both at stops and at transfers – and even saves one train [39], see also page **??**.

4 Revenue Management and Scenario Reduction

Origin&Destination Revenue Management (O&D RM) is a stochastic optimization method to control the passenger booking process for flight tickets in order to maximize the total (expected) revenue. The idea is that low fare passengers typically book early while the willingness to pay higher fares tends to increase as the departure date comes closer. Hence, selling too many low fare tickets as well as keeping to many seats for higher fares which remain unsold results in revenue losses. O&D RM tries to forecast these demands and allocate "the right ticket contingents at the right prices".

O&D RM became standard in the airline industry during the last 20 years [47], but is currently done for each leg (=flight) separately. *Leg-based methods*, however, ignore that the revenue for different O&D itineraries using the same legs may vary when selling a seat to a longer distance itinerary or to several shorter parts of it depending on the demand. This means that there is a competitive demand on different itineraries in a flight network. Consequently, revenue management methods must include the whole flight network and a suitable approximation of the stochastic demand process for all itineraries, fare classes, and the whole booking horizon.

MATHEON's project B6 developed stochastic programming techniques for a *network wide treatment* of the airline O&D RM problem. Since recursive observations of the booking process should lead to new decisions, an O&D RM optimization model has to be *multistage*, as first proposed in [40], see [25, 16] for two-stage approaches. Correspondingly, the approximation of the (high-dimensional) passenger demand process has to be modeled in terms of a *scenario tree*. Structural and stability properties of this stochastic programming model were studied and a decomposition-based solution strategy was developed.

stochastic process
scenario tree
scenario reduction
stochastic programming
multistage stochastic programming

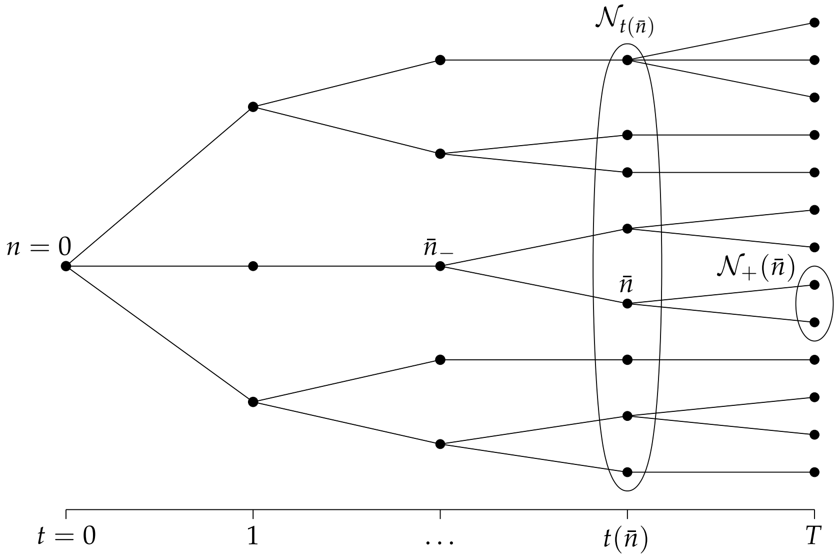

Figure 4: Illustration of a scenario tree.

Approximation of the booking process. As a first step toward the numerical solution of the O&D revenue management problem, the stochastic passenger demand is approximated by a process having a finite number of scenarios with given probabilities. To determine such scenarios, historical data of the model must be adjusted subject to a suitable demand model (unconstraining) and then drawn by resampling techniques from the records.

Scenario tree generation. Starting with a certain number of (individual) scenarios, the tree generation procedure described in [22] is used to produce a *scenario tree* whose probability distribution is close to the original one up to a prescribed tolerance. The closeness is measured in terms of the Kantorovich metric on the space of all probability measures. The whole procedure is based on recursive scenario reduction [19] on decreasing time horizons, i.e., by bundling scenarios which are close to each other on the relevant time horizon. This approach allowed for the first time the generation of higher-dimensional scenario trees by preserving statistical dependencies and properties. Scenario tree generation is available within the professional software package `GAMS-SCENRED`.

Figure 4 illustrates the construction. The optimization horizon is divided into T time periods $(t-1, t]$ $(t = 1, \dots, T)$, the scenario tree represents an approximation of a T-variate random vector. The set of all tree nodes is denoted by $\mathcal{N}$ while the set of nodes belonging to a certain time point t is denoted by $\mathcal{N}_t$. The notation $t(\bar{n})$ is used to specify the time point belonging to node $\bar{n}$. Each node $\bar{n} \in \mathcal{N}$ has a unique predecessor $\bar{n}_-$ (excepted the root node $n = 0$) as well as a nonempty set $\mathcal{N}_+(\bar{n})$ of succeeding nodes (excepted the leafs $n \in \mathcal{N}_T$). A path from the root node to some leaf is called a scenario. The decision at $t = 0$ is used to control the booking process.

Stochastic optimization model. The O&D revenue management problem can be modeled in terms of a very large scale mixed-integer multistage stochastic programming model [41], see Figure 5. Inputs and decisions are the stochastic passenger demand and the “protection levels” of booking classes, respectively, stages overbooking

$$
\begin{array}{lll}
\text{(ODRM)} & \max_{(P^n_{i,j,k})} \sum_{n=1}^{N} \pi^n \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[f^b_{i,j,k,t(n)} b^n_{i,j,k} - f^c_{i,j,k,t(n)} c^n_{i,j,k} \right] & \\
\text{(i)} & B^0_{i,j,k} = \bar{B}^0_{i,j,k} \geq 0, C^0_{i,j,k} = \bar{C}^0_{i,j,k} \geq 0 & \forall (i,j,k) \in I \times J \times K \\
\text{(ii)} & B^n_{i,j,k} = \min \left\{ \left\lfloor \frac{P^{n_-}_{i,j,k}}{1-\gamma^n_{i,j,k}} + \frac{1}{2} \right\rfloor, B^{n_-}_{i,j,k} + d^n_{i,j,k} \right\} & \forall (i,j,k) \in I \times J \times K, \forall n \in \mathcal{N} \setminus \{0\} \\
\text{(iii)} & C^n_{i,j,k} = \left\lfloor \gamma^n_{i,j,k} B^n_{i,j,k} + \frac{1}{2} \right\rfloor & \forall (i,j,k) \in I \times J \times K, \forall n \in \mathcal{N} \setminus \{0\} \\
\text{(iv)} & b^n_{i,j,k} = B^n_{i,j,k} - B^{n_-}_{i,j,k} & \forall (i,j,k) \in I \times J \times K, \forall n \in \mathcal{N} \setminus \{0\} \\
\text{(v)} & c^n_{i,j,k} = C^n_{i,j,k} - C^{n_-}_{i,j,k} & \forall (i,j,k) \in I \times J \times K, \forall n \in \mathcal{N} \setminus \{0\} \\
\text{(vi)} & \mathcal{C}_{l,m} \geq \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \delta^{l,m}_{i,j} P^n_{i,j,k} & \forall l \in L, \forall m \in M(l), \forall n \in \mathcal{N}_{T-1} \\
\text{where} & \delta^{l,m}_{i,j} := \begin{cases} 1, & \text{if fareclass j of ODI i belongs to compartment m of leg l} \\ 0, & \text{otherwise} \end{cases} & \\
\text{(vii)} & b^n_{i,j,k} \geq 0, c^n_{i,j,k} \geq 0 & \forall (i,j,k) \in I \times J \times K, \forall n \in \mathcal{N} \setminus \{0\} \\
\text{(viii)} & P^n_{i,j,k} \in \mathbb{Z} & \forall (i,j,k) \in I \times J \times K, \forall n \in \mathcal{N} \setminus \mathcal{N}_T
\end{array}
$$

Figure 5: Node representation of the airline network revenue management model.

refer to the data collection points of the booking horizon. The optimization goal consists in finding cumulative protection levels $P^n_{i,j,k}$ for each O&D itinerary i, each fare class j, each point of sale k and each node $n \in \mathcal{N} \setminus \mathcal{N}_T$ such that the expected revenue is maximized. Initial cumulative bookings and cancellations are accounted by equations (i). Equation (ii) states that the number of cumulative bookings $B^n_{i,j,k}$ at node $n \in \mathcal{N} \setminus \{0\}$ exhaust the contingent allocated by the protection levels, if possible, but is also restricted by the passenger demand $d^n_{i,j,k}$ during the time interval $(t_{(n)} - 1, t_{(n)}]$. Furthermore, cancellation rates $\gamma^n_{i,j,k}$ (percentages of the number of bookings) are taken into account by equation (ii). The cumulative cancellations $C^n_{i,j,k}$ are computed $\forall n \neq 0$ by equation (iii) while the bookings $b^n_{i,j,k}$ and cancellations $c^n_{i,j,k}$ during the corresponding time interval when traversing the scenario tree from node n_- to node n are represented by equation (iv) and equation (v), respectively. Inequality (vi) ensures that for each leg l and each compartment m its capacity is not exceeded by the corresponding protection levels at the end of the optimization horizon. Capacity constraints are modeled for the last stage only in order to allow for overbooking in earlier stages without additional efforts. Finally, (vii) and (viii) contain non-negativity and integrality conditions.

Stability properties. Within the framework of the MATHEON companion project C7 "Mean-risk optimization of electricity production in liberalized markets" stability results for multistage stochastic programs have been obtained [24, 23] that unveiled the role of filtration distances for tree generation and allowed to develop a forward variant for generating trees.

Computational results. Numerical computations were performed for two test problems: A network model consisting of a hub-and-spokes flight network containing 6 legs, 2 compartments per leg, 12 itineraries, 6 fare classes, and 14 data collection points. The demand process was approximated by about 100 scenarios leading to about 500.000 decision variables. The tree representation of the model was solved by a standard solver for mixed-integer linear programs (CPLEX) [41]. The second (still unpublished) network model is based on real-life data and contains 54 itineraries, 27 legs, 2 compartments per leg, 6–10 fare classes and 23 data collection points. The demand was approximated by about 100 scenarios leading to about 3.5 million decision variables. The model was solved by a decomposition approach based on Lagrangian relaxation of the capacity constraints. It decomposes the original optimization model into the successive solution of fast solvable small O&D subproblems. The computational experience justifies the multistage stochastic programming approach even for larger O&D flight networks.

railway vehicle rotation planning
rolling stock roster planning
vehicle rotation planning
regular vehicle rotation
hypergraph
hypergraph assignment problem

5 Railway Vehicle Rotations and Hypergraph Assignments

Vehicle rotation planning deals with the construction of rotations for individual units of rolling stock and, simultaneously, the composition of trains from these units. This is one of the basic planning problems in rail transport, and known to be notoriously hard, see [15] for a survey. MATHEON's project B22 focused on *(long distance) passenger transport*. Here, units of different types are arranged to form trains in "regular" sequences and orientations, i. e., the trains are composed "in the same way" every day of the week when they operate.

Regularity. We consider two types of regularity. *Operational regularity* stipulates that train turns should be regular, i. e., if train 4711 ends in Frankfurt and continues as 4712 on Monday, this should also be the case on Tuesday, Wednesday, etc., if possible. *Sequence regularity* aims at regular train compositions, i. e., train 4711 should consist of the same types of rolling stock, in the same sequence and orientation, every day of the week. This type of regularity is well known from car position indicators. Regularity makes the operation of a railway easier. It minimizes potential sources of disturbances by establishing every-day routines. Despite its significance, regularity has been investigated only recently in the optimization literature [46, 2].

Hypergraph assignments. The *hypergraph assignment problem (HAP)*, a hypergraph generalization of the fundamental *assignment problem*, can be used to describe the vehicle rotation planning problem in its simplest form. It consists of finding a perfect matching, called a *hyperassignment*, of minimum cost in what we call a *bipartite hypergraph*, i. e., we assign sets of vertices on one side of a bipartite hypergraph to sets of vertices on the other side. The right of Figure 6 gives an example of a bipartite hypergraph $G = (U, V, E)$ with $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3, e_4\}$, $e_1 = \{u_1, v_1\}$, $e_2 = \{u_1, u_2, v_1, v_2\}$, $e_3 = \{u_1, u_3, v_2, v_3\}$, $e_4 = \{u_3, v_3\}$. Vertices are circles, hyperedges have square labels. We assume w.l.o.g. that each hyperedge contains the same number of vertices on both sides. The hyperedges of the hyperassignment $\{e_2, e_4\}$ are drawn with thick lines. The hypergraph assignment problem can be formulated as a set partitioning problem, see the left of Figure 6. The objective minimizes the cost sum of the selected hyperedges for the cost function $c : E \to \mathbb{R}$. Equations (i) are the perfect matching constraints: There is exactly one incident hyperedge for every vertex in U and V. Constraints (ii) and (iii) are the non-negativity and integrality constraints, respectively.

hypergraph matching problem
Hall theorem for normal hypergraphs
partitioned hypergraph

$$
\begin{array}{lll}
\text{(HAP)} & \min & \sum_{x_e \in E} c(e)\, x_e \\
\text{(i)} & & \sum_{e \in \delta(v)} x_e = 1 \quad \forall v \in U \cup V \\
\text{(ii)} & & x_e \geq 0 \quad \forall e \in E \\
\text{(iii)} & & x_e \in \mathbb{Z} \quad \forall e \in E
\end{array}
$$

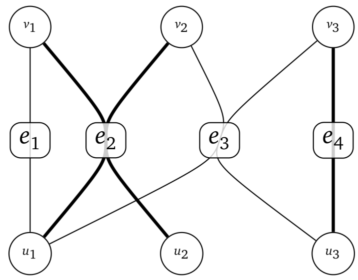

Figure 6: *Left:* The hypergraph assignment problem. *Right:* A bipartite hypergraph.

Hypergraph matching. Hall's Theorem gives a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. This result has been generalized to certain classes of hypergraphs, most notably, to balanced hypergraphs [17, 27]. We recently obtained a (yet unpublished) further generalization to an even broader class of normal hypergraphs:

Theorem 3 (Hall Theorem for Normal Hypergraphs)**.** *A normal hypergraph H has a perfect matching if and only if there exists a natural number N such that the hypergraph H^N that arises from H by "N-fold node multiplication" satisfies the Hall condition.*

Complexity. The assignment problem can be solved in polynomial time, e. g., with the famous Hungarian algorithm. The hypergraph assignment problem, however, is NP-hard. It is also APX-hard and the gap between the optimum IP solution and the corresponding LP relaxation as well as the sizes of basis matrix determinants can be arbitrarily large. All these complexity results hold even in very simple cases, e.g., if all hyperedges have head and tail size two.

Random hypergraph assignments. The analysis of random instances of the hypergraph assignment problem provides insights about the structure of "typical instances". We transfered the ideas underlying the analysis of random assignment problems and its generalizations [30] to random hypergraph assignments in partitioned hypergraphs of part size two. We proved that the expected value of a minimum cost hyperassignment which uses exactly half the possible maximum number of proper hyperedges lies between 0.3718 and 1.8310 if the vertex number tends to infinity and all hyperedge costs are exponentially i. i. d. with mean 1 [21].

Polyhedral results. The special structure of cliques in bipartite hypergraphs can be exploited to derive a strengthened *extended formulation* of the hypergraph assignment problem. A *clique* is a set of hyperedges which have a pairwise nonempty intersection. Associated with a clique is an inequality stipulating that only one hyperedge from the clique can be part of a hyperassignment. Adding clique inequalities to the IP formulation can significantly improve the root LP bound of the set partitioning formulation of the hypergraph assignment problem [8]. In general hypergraphs, clique inequalities are hard to separate as hyperedges involved in a clique can be distributed "globally". For *partitioned hypergraphs*, a special type of bipartite hypergraphs, however, we developed an extended formulation that implies all clique inequalities. This extended formulation has a polynomial number of extra variables, that describe the local structure of hyperassignments, i.e., (a super class of) clique inequalities for hyperassignment problems on partitioned hypergraphs can be separated in polynomial time. We also proved that every hypergraph assignment problem can be polynomially transformed into a hypergraph assignment problem on a partitioned hypergraph. The investigation of small hypergraph assignment polytopes led to a new class of inequalities that subsume half of the known facet classes. These inequalities are "clique generalizations" of the well-known odd set inequalities for the matching problem. They are related to (but different from) the general clique family inequalities of Pêcher and Wagler [42]; like these, they also hold for general set packing problems.

facet classification
railway vehicle rotation planning
regular vehicle rotation
rolling stock roster planning
vehicle rotation planning

Facet classification. The large number of facets in polytopes of already very small problem sizes impedes their classification and understanding. This does not only apply to hypergraph assignments, but to many combinatorial optimization problems. Fortunately, there are often symmetries in the combinatorial structure of the problem. These symmetries also hold for the polytope and can be used to group the facets. Due to the lack of a normal form the usual method to do this could not be used for the hypergraph assignment problem. We devised a facet classification algorithm that relies solely on the vertex-facet incidence structure of the polytope to group its facets in symmetry classes and showed its applicability to several combinatorial optimization problems. A normal form is not needed.

ICE rotation planning. The polyhedral analysis of the hypergraph assignment problem is the basis for the development of a branch-and-cut algorithm for the solution of railway vehicle rotation planning problems. Such an algorithm has been developed in an industry companion project with Deutsche Bahn. It combines our results on hyperassignments with sophisticated column generation and rapid branching strategies, large scale LP techniques, and special primal heuristics [14, 15]. In 2010, it became for the first time possible to solve strategic vehicle rotation planning problems for the entire ICE fleet of Deutsche Bahn over a planning horizon of one week. The roll-out within Deutsche Bahn's "Fahr- und Einsatz-Optimierung" (FEO) planning system is currently under way.

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