On distribution sensitivity in chance constrained programming

Werner Römisch and Rüdiger Schultz*

Abstract:

Chance constrained stochastic programs are studied as parametric programming problems with respect to the underlying probability distribution. Results on continuity properties of (local) optimal values and optimal solution sets are established resp. reviewed from [15].

Let us consider the following chance constrained model

$$\min\{f(x):\ x\in\mathbb{R}^m,\ \mu(\{z\in\mathbb{R}^s:\ x\in X(z)\})\geq p_o\} \tag{1}$$

where f is a real-valued function defined on $\mathbb{R}^m$, X is a set-valued mapping from $\mathbb{R}^s$ into $\mathbb{R}^m$, $p_o\in[0,1]$ is a prescribed probability level and μ is a probability distribution on $\mathbb{R}^s$. The multifunction X has e.g. the following shape $X(z):=\{x\in X_o,\ h_i(x,z)\geq 0,\ i\in I\}$ $(z\in\mathbb{R}^s)$, where $X_o\subset\mathbb{R}^m$ is nonempty and closed, I is a finite index set and h_i is a continuous real-valued function defined on $\mathbb{R}^m\times\mathbb{R}^s$. For theoretical results on chance constrained problems (like e.g. convexity of the constraint set) we refer to [4],[7],[11].

In the present paper we study the behaviour of (1) with respect to (small) perturbations of the probability distribution μ. Recently such studies have found considerable interest in the literature which is essentially caused by two lines of research in stochastic programming. The first one is the stability analysis for stochastic programs with incomplete information (see e.g.[5],[6]) and the second one is the use of approximation techniques ([17],[20],[21]) and of computational methods (e.g. those introduced in [18] for the calculation of probabilities etc.) for solving chance constrained programs.
Our approach to the study of perturbations of (1) relies on (qualitative and quantitative) stability results for parametric optimization problems with parameters varying in metric spaces obtained by D. Klatte in [9]. As the space of parameters we consider $\mathcal{P}(\mathbb{R}^s)$ - the space of all Borel probability measures on $\mathbb{R}^s$ equipped with a suitable metric. Of course, at first sight it seems especially desirable to cover the case of weak convergence of probability measures (see e.g.[2]) as it

* Humboldt-Universität Berlin, Sektion Mathematik, DDR-1086 Berlin, PSF 1297, German Democratic Republic.

has been done in the stability analysis carried out by Kall [8] and Wang [19]. In [8] , the author stresses the difficulties that arise when looking for comprehensible sufficient conditions for the lower semicontinuity of the constraint-set mapping understood as a multifunction defined on $\mathcal{P}(\mathbb{R}^s)$ endowed with the topology of weak convergence. In [19] additional smoothness assumptions on the measure μ (describing the unperturbed problem) are imposed to establish the desired lower semi-continuity of the constraint set. The present paper continues the line of research begun in [15] and [16]. We present a suitable topology on $\mathcal{P}(\mathbb{R}^s)$ to gain stability results for the (locally) optimal value and the (locally) optimal solutions. To this end, let us consider the distance

$$\alpha_{\mathcal{B}_o}(\mu,\nu) := \sup_{B\in\mathcal{B}_o} |\mu(B)-\nu(B)| \qquad (\mu,\nu\in\mathcal{P}(\mathbb{R}^s))$$

where $\mathcal{B}_o$ is a proper subclass of Borel sets. In our situation $\mathcal{B}_o$ will be chosen adapted to the multifunction X, i.e. in such a way that it contains all the pre-images $X^-(x) := \{z\in\mathbb{R}^s: x\in X(z)\}$ $(x\in\mathbb{R}^m)$. Simultaneously, $\mathcal{B}_o$ is taken rich enough in the sense that $\alpha_{\mathcal{B}_o}$ forms a metric. With $\alpha_{\mathcal{B}_o}$ selected according to these objectives we state a fairly general qualitative stability result (Proposition 1) and a quantitative continuity property of the optimal value (Theorem 7). In Remark 2 we reveal additional hypotheses to identify those measures μ at which stability with respect to the topology of weak convergence holds. We present an example (Example 3) to show that under the hypotheses of Proposition 1 stability with respect to weak convergence may collapse.

Next we introduce some basic concepts and notations which are used throughout. For $\nu\in\mathcal{P}(\mathbb{R}^s)$ and $p\in[0,1]$ the set $C_p(\nu)$ is given as $C_p(\nu) := \{x\in\mathbb{R}^m: \nu(X^-(x)) \geq p\}$, hence problem (1) becomes $\min\{f(x): x\in C_{p_o}(\mu)\}$. Given $V\subseteq\mathbb{R}^m$ and $\nu\in\mathcal{P}(\mathbb{R}^s)$ we denote

$\varphi_V(\nu) := \inf\{f(x): x\in C_{p_o}(\nu)\cap \mathrm{cl}\, V\}$ and

$\psi_V(\nu) := \{x\in C_{p_o}(\nu)\cap \mathrm{cl}\, V: f(x) = \varphi_V(\nu)\}$.

By cl V and bd V we understand the closure resp. the boundary of V. Following [13],[9] we call a nonempty subset M of $\mathbb{R}^m$ a complete local minimizing set (CLM set) for f on $C_{p_o}(\mu)$ if there is an open set $Q\supset M$ such that $M = \psi_Q(\mu)$. Examples for CLM sets are the set of global minimizers or strict local minimizing points.

We recall that a multifunction Γ from a metric space T to $\mathbb{R}^m$ is said to be closed at $t_o\in T$ if $t_k\to t_o$, $x_k\to x_o$, $x_k\in\Gamma(t_k)$ $(k\in\mathbb{N})$ imply $x_o\in\Gamma(t_o)$, Γ is said to be upper semicontinuous (usc) at t_o if for any open set $\Omega\supset\Gamma(t_o)$ there exists a neighbourhood $U(t_o)$ such that $\Gamma(t)\subset\Omega$ whenever $t\in U(t_o)$, Γ is said to be locally lower semicontinuous (locally lsc) at $(x_o,t_o)\in\Gamma(t_o)\times T$ if for any open set $\Omega\supset\{x_o\}$

there exists a neighbourhood $U(t_o)$ such that $\Omega \cap \Gamma(t) \neq \emptyset$ whenever $t \in U(t_o)$ and finally Γ is said to be pseudo-Lipschitzian at $(x_o,t_o) \in \Gamma(t_o) \times T$ (cf. [14]) if there are neighbourhoods $U = U(t_o)$, $V = V(x_o)$ and a positive constant L such that

$$\Gamma(t') \cap V \subset \Gamma(t'') + L\, d(t',t'')B_m \quad \text{whenever } t',t'' \in U,$$

here d denotes the metric in T and B_m is the closed unit ball in $\mathbb{R}^m$. Note that the above property especially implies that Γ is locally lsc at (x_o,t_o).

Our first proposition is an adaptation of a general parametric programming result ([9], Prop. 1, [13] Prop. 3.2 and Prop. 3.3) to the situation of distribution sensitivity in stochastic programming.

<u>Proposition 1:</u>

Let $\mu \in \mathcal{P}(\mathbb{R}^s)$, $p_o \in (0,1)$ and $\{X^-(x): x \in \mathbb{R}^m\} \subseteq \mathcal{B}_o$. Let X be a closed multifunction and f be continuous. Assume that there exists a bounded open set $V \subset \mathbb{R}^m$ such that $\psi_V(\mu)$ is a CLM set for f on $C_{p_o}(\mu)$. Let the multifunction $p \mapsto C_p(\mu)$ be locally lsc at (x_o,p_o) for some $x_o \in \psi_V(\mu)$.

Then (i) φ_V is continuous at μ and ψ_V is usc at μ with respect to the metric $\alpha_{\mathcal{B}_o}$ on $\mathcal{P}(\mathbb{R}^s)$, and

(ii) there exists $\delta > 0$ such that $\psi_V(\nu)$ is a CLM set for f on $C_{p_o}(\nu)$ whenever $\nu \in \mathcal{P}(\mathbb{R}^s)$ and $\alpha_{\mathcal{B}_o}(\mu,\nu) < \delta$.

<u>Proof:</u>

Let us consider the metric space $(\mathcal{P}(\mathbb{R}^s), \alpha_{\mathcal{B}_o})$. It is known from Berge's classical stability theory for parametric optimization problems (applied to $\min\{f(x): x \in C_{p_o}(\nu) \cap \mathrm{cl}\, V\}$, $\nu \in \mathcal{P}(\mathbb{R}^s)$) that assertion (i) holds if the multifunction (from $\mathcal{P}(\mathbb{R}^s)$ into $\mathbb{R}^m$) $\nu \mapsto C_{p_o}(\nu) \cap \mathrm{cl}\, V$ is closed at μ and locally lsc at (x_o,μ). (cf. e.g. [1] , Theorem 4.2.2). In the proof of Theorem 5.4 in [15] it was shown that $\nu \mapsto C_{p_o}(\nu)$ is closed at μ since X is closed. Since $x_o \in V$, it remains to show that $\nu \mapsto C_{p_o}(\nu)$ is locally lsc at (x_o,μ). Let $\varepsilon > 0$ be arbitrary, but fixed. Since $p \mapsto C_p(\mu)$ is locally lsc at (x_o,p_o), there exists $\delta_o > 0$ such that $C_p(\mu) \cap B(x_o,\varepsilon) \neq \emptyset$ for every $p \in \mathbb{R}$ with $|p-p_o| < \delta_o$. Let $\nu \in \mathcal{P}(\mathbb{R}^s)$ be such that $\tilde{\delta} := \alpha_{\mathcal{B}_o}(\mu,\nu) < \delta_o$. Put $p := p_o + \tilde{\delta}$. Then we have for every $y \in C_p(\mu)$ that

$$\nu(X^-(y)) \geq \mu(X^-(y)) - |\mu(X^-(y)) - \nu(X^-(y))| \geq p - \alpha_{\mathcal{B}_o}(\mu,\nu) = p_o.$$

Hence we obtain $y \in C_{p_o}(\nu)$ and $C_{p_o}(\nu) \cap B(x_o,\varepsilon) \supseteq C_p(\mu) \cap B(x_o,\varepsilon) \neq \emptyset$. Thus the multifunction $\nu \mapsto C_{p_o}(\nu)$ is locally lsc at (x_o,μ). Since ψ_V is usc at μ (with respect to $\alpha_{\mathcal{B}_o}$) and $\psi_V(\mu)$ is contained in the open set V, there exists $\delta > 0$ such that $\psi_V(\nu) \subset V$ for every $\nu \in \mathcal{P}(\mathbb{R}^s)$ such that $\alpha_{\mathfrak{B}_o}(\mu,\nu) < \delta$. This completes the proof. □

Remark 2:

Proposition 1 may also be viewed as a qualitative distribution stability result with respect to perturbations of μ in the space $(\mathcal{P}(\mathbb{R}^s), \tau_w)$ (τ_w denoting the topology of weak convergence, cf. [2]) if $\mathfrak{B}_o$ is a μ-uniformity class of Borel subsets of $\mathbb{R}^s$. Recall that $\mathfrak{B}_o$ is a μ-uniformity class if $\alpha_{\mathfrak{B}_o}(\mu_n,\mu) \to 0$ holds for every sequence (μ_n) converging weakly to μ ([3]). Let us consider two particular classes $\mathfrak{B}_o$ of Borel sets:

(i) $\mathfrak{B}_R := \{\emptyset, (-\infty,z] : z \in \mathbb{R}^s\}$ is a μ-uniformity class if the distribution function of $\mu \in \mathcal{P}(\mathbb{R}^s)$ is continuous.

(ii) $\mathfrak{B}_C := \{B \subseteq \mathbb{R}^s : B$ is convex and Borel$\}$ is a μ-uniformity class if $\mu \in \mathcal{P}(\mathbb{R}^s)$ is absolutely continuous w.r. to Lebesgue measure on $\mathbb{R}^s$.

Hence, at least, if $\mu \in \mathcal{P}(\mathbb{R}^s)$ has a density and all pre-images $X^-(x)$ ($x \in \mathbb{R}^m$) are convex, Prop. 5.1 appears to be a useful stability result w.r. to perturbations of μ in $(\mathcal{P}(\mathbb{R}^s), \tau_w)$.
Furthermore, the above proof shows that it is just the shape of the metric $\alpha_{\mathfrak{B}_o}$ which allows to carry over the local lower semicontinuity of the mapping $p \mapsto C_p(\mu)$ at some (x_o,p_o) to the mapping $\nu \mapsto C_{p_o}(\nu)$ at (x_o,μ). In the general situation, local lower semicontinuity of $p \mapsto C_p(\mu)$ at (x_o,p_o) implies $x_o \in \mathrm{cl}(\{x \in \mathbb{R}^m : \mu(X^-(x)) > p_o\})$; if now a metric d on $\mathcal{P}(\mathbb{R}^s)$ is chosen such that the functions $t_{\mu_n}(x) := \mu_n(X^-(x))$ converge pointwise to $t_\mu(x) := \mu(X^-(x))$ as $d(\mu_n,\mu) \to 0$ one obtains that $\nu \mapsto C_{p_o}(\nu)$ is locally lsc at (x_o,μ) with respect to $(\mathcal{P}(\mathbb{R}^s), d)$ (cf. [17], Corr. 3.2.1). For the topology of weak convergence, however, the desired pointwise convergence of t_{μ_n} as $\mu_n \xrightarrow{\tau_w} \mu$ holds, if $\mu(\mathrm{bd}\, X^-(x)) = 0$ for all $x \in \mathbb{R}^m$, i.e. all sets $X^-(x)$ are μ-continuity sets ([2], Portmonteau Theorem). We refer to a corresponding discussion in [8], p. 404.

By the following example we illustrate what has been said in the above remark. We present a multifunction $\nu \mapsto C_{p_o}(\nu)$ (from $(\mathcal{P}(\mathbb{R}^s), \tau_w)$ to $\mathbb{R}^m$) not locally lsc at some (x_o,μ) although the multifunction $p \mapsto C_p(\mu)$ is locally lsc at (x_o,p_o).

Example 3:

Let $X(z) := \{x \in (-\infty,0] : x \geq z\}$, $\mu_o \in \mathcal{P}(\mathbb{R})$ be the uniform distribution on $[-1,0]$, $\delta_x \in \mathcal{P}(\mathbb{R})$ the distribution with unit mass at $x \in \mathbb{R}$.

Define $\mu := \frac{1}{2}\delta_0 + \frac{1}{2}\mu_0$

$$\mu_n := (\tfrac{1}{2} + \tfrac{1}{n})\delta_{\frac{1}{n}} + (\tfrac{1}{2} - \tfrac{1}{n})\mu_0 \quad (n \in \mathbb{N}).$$

One confirms $\mu_n \xrightarrow{\tau_w} \mu$ as $n \to \infty$, but $\alpha_{\mathcal{B}_R}(\mu_n,\mu) \not\to 0$. Furthermore, $p \mapsto C_p(\mu)$ is locally lsc at $(x_0,p_0) := (0,\frac{1}{2})$ since $C_p(\mu) = \{0\}$ for $p \in [\frac{1}{2}, 1]$ and $C_p(\mu) = [2p-1,0]$ for $p \in (0,\frac{1}{2})$. On the other hand, $C_{p_0}(\mu_n) = \emptyset$ for all $n \in \mathbb{N}$. Thus, the mapping $\nu \mapsto C_{p_0}(\nu)$ is not locally lsc at (x_0,μ).

To see that assertion (i) in Proposition 1 does no longer hold when the multifunction $p \mapsto C_p(\mu)$ is not locally lsc at (x_0,p_0) at any $x_0 \in \psi_V(\mu)$ consider the next example.

<u>Example 4:</u>

Let $\mu \in \mathcal{P}(\mathbb{R})$ be given by the distribution function

$$F_\mu(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \le x < -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \le x < \frac{1}{2} \\ x & \frac{1}{2} \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Let further $p_0 := \frac{1}{2}$, $X(z) := \{x \in \mathbb{R}: x \ge z\}$ and $f(x) := (x + \frac{1}{8})^2$. then with $V := (-\frac{1}{4},\frac{1}{4})$ the set $\psi_V(\mu) = \{-\frac{1}{8}\}$ is a CLM set for f on $C_{p_0}(\mu)$. However, the mapping $p \mapsto C_p(\mu)$ is not locally lsc at (x_0,p_0) with $x_0 = -\frac{1}{8}$.
Let

$$F_{\mu_n}(x) = \begin{cases} 0 & x < -1 \\ 1-\frac{1}{n} + (1-\frac{1}{n})x & -1 \le x < -\frac{1}{2} \\ \frac{1}{2} + \frac{1}{n}x & -\frac{1}{2} \le x < \frac{1}{2} \\ \frac{1}{n} + (1-\frac{1}{n})x & \frac{1}{2} \le x < 1 \\ 1 & 1 \le x \end{cases} \qquad (n \in \mathbb{N})$$

With $\mathcal{B}_0 = \{(-\infty,x] : x \in \mathbb{R}\}$ we have $\{X^-(x) : x \in \mathbb{R}\} \subseteq \mathcal{B}_0$ and $\alpha_{\mathcal{B}_0}(\mu,\mu_n) \to 0$ $(n \to \infty)$. On the other hand $\psi_V(\mu_n) = \{0\}$ and $\varphi_V(\mu_n) = \frac{1}{64}$ for all n. Hence ψ_V and φ_V are not usc resp. continuous at μ.

In view of the consequences for the stability of local optimal values resp. locally optimal solutions it is interesting to ask for conditions

under which multifunctions (especially the mapping $p \mapsto C_p(\mu)$) behave locally lsc. In the literature this question is related to certain constraint qualifications ([1],[12],[14]).
The next Lemma, which is due to Rockafellar [14] , reflects this approach and gives also quantitative information about the continuity of certain constraint-set-mappings.

Lemma 5:

Let $g: \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^r$ be continuously differentiable and $(x_o,p_o) \in \mathbb{R}^m \times \mathbb{R}^d$ with $g(x_o,p_o) \geqq 0$. Assume that the "Mangasarian-Fromowitz constraint qualification" holds, i.e.
there exists a $y \in \mathbb{R}^m$ such that

$$y^T \nabla_x g_i(x_o,p_o) > 0 \quad \text{for all } i \in \{1,\dots,r\} \text{ with } g_i(x_o,p_o) = 0.$$

Then the multifunction $p \mapsto \{x \in \mathbb{R}^m : g(x,p) \geqq 0\}$ (from $\mathbb{R}^d$ to $\mathbb{R}^m$) is pseudo-Lipschitzian at (x_o,p_o).

Proof: follows immediately from Corollary 3.5. and Remark 3.6. in [14].

As we will see lateron it is possible to derive a local Lipschitz property for φ_V at μ if the mapping $p \mapsto C_p(\mu)$ is even pseudo-Lipschitzian at suitable (x_o,p_o). In this context let us recall that without Mangasarian-Fromowitz condition the constraint-set-mapping may fail to be pseudo-Lipschitzian even when it is already locally lsc. This will be illustrated by the next example which is a mapping of the type $p \mapsto C_p(\mu)$.

Example 6:

Let $\mu \in \mathcal{P}(\mathbb{R})$ be given by a distribution function $F_\mu(\cdot)$ which is continuously differentiable and for which $F_\mu(x) = x^3 + \frac{1}{2}$ on some neighbourhood around 0. Let further $p_o = \frac{1}{2}$, $X(z) := \{x \in \mathbb{R} : x \geqq z\}$. Then $C_p(\mu) = \{x \in \mathbb{R} : F_\mu(x) \geqq p\}$. Consider $x_o = 0$. Then $p \mapsto C_p(\mu)$ is locally lsc at (x_o,p_o) but not pseudo-Lipschitzian at (x_o,p_o).

Theorem 7:

Assume the hypotheses of Proposition 1 and let additionally f be locally Lipschitzian and the multifunction $p \mapsto C_p(\mu)$ be pseudo-Lipschitzian at each (x_o,p_o) belonging to $\psi_V(\mu) \times \{p_o\}$. Then there exist constants $L > 0$ and $\delta > 0$ such that

$$|\varphi_V(\mu) - \varphi_V(\nu)| \leq L \alpha_{\mathfrak{B}_o}(\mu,\nu)$$

whenever $\alpha_{\mathfrak{B}_o}(\mu,\nu) < \delta$.

The above theorem has been proven in [15] (Theorem 5.4) relying on quantitative stability results obtained by D. Klatte in [9] (Theorem 1).

Now, let us consider a particular chance constrained model with random right-hand side. The multifunction X (from $\mathbb{R}^s$ into $\mathbb{R}^m$) is given as

$$X(z) := \{x \in \mathbb{R}^m : g_i(x) \geq 0,\ i=1,\dots,r,\ Ax \geq z\}$$

where $A \in L(\mathbb{R}^m, \mathbb{R}^s)$ and $g_i : \mathbb{R}^m \to \mathbb{R}$ continuously differentiable $(i=1,\dots,r)$. Note that $X^-(x) \in \mathfrak{B}_R$ for each $x \in \mathbb{R}^m$ and

$$C_p(\mu) = \{x \in \mathbb{R}^m : g_i(x) \geq 0,\ i=1,\dots,r,\ F_\mu(Ax) \geq p\}$$

where F_μ denotes the distribution function of $\mu \in \mathcal{P}(\mathbb{R}^s)$ and $p \in (0,1)$.

Corollary 8:

Let $\mu \in \mathcal{P}(\mathbb{R}^s)$ with continuously differentiable F_μ, $p_o \in (0,1)$ and f be locally Lipschitzian. Assume that there exists a bounded open set $V \subset \mathbb{R}^m$ such that $\psi_V(\mu)$ is a CLM set for f on $C_{p_o}(\mu)$. Suppose further that for each $x_o \in \psi_V(\mu)$ there exists $y \in \mathbb{R}^m$ such that

$$y^T \nabla g_i(x_o) > 0, \quad \text{for all } i \in \{1,\dots,r\} \text{ with } g_i(x_o) = 0, \text{ and}$$

$$y^T A^T \nabla F_\mu(Ax_o) > 0 \quad \text{if} \quad F_\mu(Ax_o) = p_o.$$

Then ψ_V is usc at μ and there exist constants $L > 0$ and $\delta > 0$ such that $\psi_V(\nu)$ is a CLM set for f on $C_{p_o}(\nu)$ and

$$|\varphi_V(\mu) - \varphi_V(\nu)| \leq L \sup_{z \in \mathbb{R}^s} |F_\mu(z) - F_\nu(z)|$$

whenever $\sup_{z \in \mathbb{R}^s} |F_\mu(z) - F_\nu(z)| < \delta$.

Proof:

According to Lemma 5 the mapping $p \mapsto C_p(\mu)$ is pseudo-Lipschitzian at any $(x_o, p_o) \in \psi_V(\mu) \times \{p_o\}$. The assertion now follows from Proposition 1 and Theorem 7 with $\mathfrak{B}_o := \mathfrak{B}_R$ (noting that $\alpha_{\mathfrak{B}_R}(\mu,\nu) = \sup_{z \in \mathbb{R}^s} |F_\mu(z) - F_\nu(z)|$). □

For another chance constrained model which was introduced in [10] a quantitative stability result relying on Theorem 7 (with $\mathfrak{B}_o := \mathfrak{B}_C$) has been developed in [15].

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