OBTAINING CONVERGENCE RATES FOR APPROXIMATIONS IN STOCHASTIC PROGRAMMING *

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Abstract:

Stochastic programming problems are viewed as parametric programs with respect to the underlying probability distribution. General results about continuity properties of expectation functionals and of certain set-valued mappings (defined by probabilistic constraints) with respect to the weak topology on the set of probability distributions are proved. They are applied to obtain quantitative distribution sensitivity results for the optimal values of recourse and chance-constrained problems, respectively, of stochastic (linear) programming. These results lead to convergence rates for approximations of such problems (e.g. via certain discrete probability measures or via empirical measures).

1. Introduction

In the (theoretical and numerical) treatment of stochastic programming problems, their stability with respect to a perturbation of the distribution of the underlying random variables plays an essential role.

If, e.g., this distribution (or some of its parameters) is not known, the stability w.r. to the distribution is important when estimates for the distributions resp. its parameters are used. Questions of this kind were the starting point in [10], [11], [18], [20], [27], [28] and [30]. There, "stability" means continuity of optimal values, optimal solutions and (or) Kuhn-Tucker points with respect to the distribution resp. its parameters. (For a survey on stability results in parametric optimization, see, e.g., [2].)

Another motivation for the interest in such stability results comes from the application of numerical methods for the solution of stochastic programming problems. One of the main lines in the development of numerical methods consists in approximating the (known) probability distribution by "simpler" (e.g. discrete) distributions ([5], [14], [16], [17], [20], [22], [26], see also the survey [31])

The present paper gives a contribution to the quantitative distribution sensitivity analysis of stochastic programming problems, which, though suitable for both of the directions mentioned above, stresses, as an application, the investigation of convergence rates of approximative methods. We explain our aim by the following rather general class of stochastic programming problems:

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$$\min\{\int_Z f(z,x)\mu(dz) \mid x \in \mathbb{R}^m,\ \mu(\{z \in Z \mid x \in X(z)\}) \geq \alpha\} \tag{1.1}$$

where Z is a Borel subset of $\mathbb{R}^s$, f a mapping from $Z \times \mathbb{R}^m$ into $\mathbb{R}$, X is a set-valued mapping from Z into $\mathbb{R}^m$, $\alpha \in [0,1]$ and μ is a probability distribution on Z.

Note that a number of stochastic programs with recourse or with chance (i.e. probabilistic) constraints fit into this class (see [15], [31] and Sections 3,4).

We consider (1.1) as a parametric optimization problem with parameter μ varying in $\mathcal{P}(Z)$, the set of all Borel probability measures on Z, equipped with the topology of weak convergence ([4], [3, Ch. 1], [9]). It is well known that (even if Z is a seperable metric space) this topology is metrizable, e.g. by the Prokhorov metric ρ and the bounded Lipschitz (or Dudley) metric β defined by ([7], [9], [32])

$$\begin{aligned} \rho(\mu,\nu) &:= \inf\{\varepsilon > 0 \mid \mu(B) \leq \nu(B^\varepsilon) + \varepsilon, \text{ for all Borel sets } B \subseteq Z\} \\ \beta(\mu,\nu) &:= \sup\{|\int g(z)\mu(dz) - \int g(z)\nu(dz)| / g \in BL(Z,d), \|g\|_{BL} \leq 1\} \end{aligned} \tag{1.2}$$

where $B^\varepsilon := \{z \in Z \mid \inf\{d(z,\tilde{z}) \mid \tilde{z} \in B\} < \varepsilon\}$ $\quad (B \subseteq Z,\ \varepsilon > 0)$,

$$BL(Z,d) := \{g : Z \to \mathbb{R} \mid \|g\|_{BL} :=$$

$$:= \sup_{z \in Z} |g(z)| + \sup_{z \neq \tilde{z}} \frac{|g(z) - g(\tilde{z})|}{d(z,\tilde{z})} < \infty\},$$

d denotes the metric on Z.

In this paper, we will impose only mild regularity assumptions on f and μ, but will restrict ourselves to the stability of the optimal value $\varphi(\mu)$ of (1.1) with respect to μ.

More precisely, we will prove results on the Hölder- and local Lipschitz continuity of $\varphi : \mathcal{P}(Z) \to \mathbb{R}$ with respect to the metrics β and ρ. (Let us recall here that the metrics β and ρ are equivalent in view of $\frac{1}{2}\beta(\mu,\nu) \leq \rho(\mu,\nu) \leq (\frac{3}{2}\beta(\mu,\nu))^{1/2}$; $\mu,\nu \in \mathcal{P}(Z)$ ([7])).

From [1, Theorem 7, p. 53] resp. [19] it is rather obvious that Hölder- resp. local Lipschitz properties of the mappings

$$(x,\mu) \mapsto \int_Z f(z,x)\mu(dz)$$

$$\mu \mapsto C_\alpha(\mu) = \{x \in \mathbb{R}^m \mid \mu(\{z \in Z \mid x \in X(z)\}) \geq \alpha\}$$

are essential for the desired continuity properties of $\varphi : \mathcal{P}(Z) \to \mathbb{R}$. This will be discussed in more detail in Sections 2 and 4. In Section 2, we obtain a result about the Hölder continuity (with respect to the metric β) of the mapping

$$\mu \mapsto \int_Z g(z)\mu(dz)$$

on certain subsets of $\mathcal{P}(Z)$ (defined by moment conditions which are related to a quantitative local Lipschitz property ([25], [12]) imposed on g).

In Section 3, we apply this result to stochastic linear programming problems with complete fixed recourse, obtaining a convergence rate

$$|\varphi(\mu) - \varphi(\mu_n)| = O(\beta(\mu,\mu_n)^{1-1/p})$$

(where p is linked with moment conditions on μ and μ_n) if (μ_n) converges weakly towards μ. Especially, we investigate in Section 3 discrete approximations of μ by means of conditional expectations and the approximation of μ by empirical measures.

In Section 4, we prove a result on the local Lipschitz continuity of the mapping

$$\mu \longmapsto C_\alpha(\mu)$$

with respect to the Hausdorff-distance and the Prokhorov-metric ρ and apply it to a stochastic linear programming problem with chance constraints (Corollary 4.7).

2. Continuity properties of expectation functionals of probability measures with respect to the weak topology

Let (Z,d) be a separable metric space and $\mathcal{P}(Z)$ be the set of all Borel probability measures on Z equipped with the topology of weak convergence (which will be denoted by $\xrightarrow{w}$). Let g be a mapping from Z into IR such that

$$|g(z)-g(\tilde{z})| \le L(\max\{d(z,0),d(\tilde{z},0)\})d(z,\tilde{z}) \quad (z,\tilde{z} \in Z) \tag{2.1}$$

where $L: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and monotonically increasing, and $0 \in Z$ is some distinguished element.
For the following we put

$$L_1(t) := L(t).t \qquad (t \in \mathbb{R}_+),$$

$$M_p(\mu) := (\int_Z L_1(d(z,0))^p \,\mu(dz))^{1/p} \quad (\mu \in \mathcal{P}(Z),\ 1 \le p < \infty).$$

Note that $\int_Z |g(z)|\mu(dz)$ is finite if $M_1(\mu) < \infty$.

We are aiming at "quantitative continuity results" of the type

$$|\int_Z g(z)\mu(dz) - \int_Z g(z)\nu(dz)| \le \psi(\mu,\nu) \quad (\mu,\nu \in \mathcal{P}(Z)) \tag{2.2}$$

If $g: Z \to \mathbb{R}$ is a bounded Lipschitz function, then (2.2) holds with $\psi(\mu,\nu) := \|g\|_{BL}\ \beta(\mu,\nu)$.
The main result of this section is an estimate of the type (2.2), where ψ depends on g, on the generalized moments $M_p(\mu)$ and $M_p(\nu)$, and on $\beta(\mu,\nu)$:

2.1 Theorem: For all $p \in]1,\infty[$, there holds (with notations and general assumptions as above)

$$\left| \int_Z g(z)\mu(dz) - \int_Z g(z)\nu(dz) \right| \leq C(1 + M_p(\mu) + M_p(\nu))\ \beta(\mu,\nu)^{1-1/p} \tag{2.3}$$

for all $\mu,\nu \in \mathcal{P}(Z)$ such that $M_p(\mu) + M_p(\nu) < \infty$ (where $C := \max\{4, 2L(1) + 10\ |g(0)|\}$).

Proof: Let $\mu,\nu \in \mathcal{P}(Z)$ be such that $M_p(\mu) + M_p(\nu) < \infty$. For arbitrary $r > 0$, g is Lipschitz continuous on $K_r := \{z \in Z \mid d(z,0) \leq r\}$ with Lipschitz constant $L(r)$, and there holds $\sup_{z \in K_r} |g(z)| \leq |g(0)| + L_1(r)$.

Any bounded Lipschitz function from a subset of Z into $\mathbb{R}$ has a bounded Lipschitz extension to the whole of Z with the same BL-norm ([9, Corollary 7.4]).

Let g_r be such an extension of $g|_{K_r}$, i.e.

$$\|g_r\|_{BL} = \|g|_{K_r}\|_{BL} \leq |g(0)| + L_1(r) + L(r).$$

Hence we obtain

$$\begin{aligned}
&\left| \int_Z g\ d(\mu-\nu) \right| \leq \\
&\leq \left| \int_Z (g-g_r)d\mu \right| + \left| \int_Z g_r d(\mu-\nu) \right| + \left| \int_Z (g-g_r)d\nu \right| \leq \\
&\leq \int_{Z\setminus K_r} (|g| + |g_r|)\, d(\mu+\nu) + \|g_r\|_{BL}\ \beta(\mu,\nu) \leq \qquad (2.4)\\
&\leq \int_{Z\setminus K_r} (L_1(d(z,0)) + L_1(r) + 2|g(0)|)(\mu(dz) + \nu(dz)) + (|g(0)| + L_1(r) + L(r))\beta(\mu,\nu) = \\
&= \int_{Z\setminus K_r} L_1(d(z,0))(\mu(dz)+\nu(dz)) + (L_1(r) + 2|g(0)|)(\mu+\nu)(Z\setminus K_r) + \\
&\qquad + (|g(0)| + L_1(r) + L(r))\beta(\mu,\nu)
\end{aligned}$$

From Chebyshev's inequality we get

$$\mu(Z\setminus K_r) \leq L_1(r)^{-p} \int_Z L_1(d(z,0))^p \mu(dz) = \tag{2.5}$$

$$= \left(\frac{M_p(\mu)}{L_1(r)}\right)^p$$

this estimate together with Hölder's inequality yields

$$\int_{Z\setminus K_r} L_1(d(z,0))\mu(dz) \leq \left(\int_{Z\setminus K_r} L_1(d(z,0))^p \mu(dz)\right)^{1/p} \mu(Z\setminus K_r)^{1-1/p} \leq$$

$$\leq M_p(\mu)^p L_1(r)^{1-p} \tag{2.6}$$

Using (2.5), (2.6) and the same estimates for ν instead of μ, we continue (2.4) by

$$
\begin{aligned}
&\Big|\int_Z g\,d(\mu-\nu)\Big| \le \\
&\le (M_p(\mu)^p + M_p(\nu)^p)L_1(r)^{1-p} + \\
&+ (M_p(\mu)^p + M_p(\nu)^p)(L_1(r) + 2|g(0)|)L_1(r)^{-p} + \\
&+ (L_1(r) + L(r) + |g(0)|)\beta(\mu,\nu) \le \\
&\le (M_p(\mu) + M_p(\nu)).2A_p(L_1(r)^{1-p} + |g(0)|L_1(r)^{-p}) + \\
&+ (2L_1(r) + L(1) + |g(0)|)\beta(\mu,\nu)
\end{aligned}
\tag{2.7}
$$

where we put $A_p := \max\{M_p(\mu)^{p-1}, M_p(\nu)^{p-1}\}$.

Now we choose $r > 0$ such that

$$L_1(r) = \max\{M_p(\mu), M_p(\nu)\}.\beta(\mu,\nu)^{-1/p} = A_p^{1/(p-1)}\,\beta(\mu,\nu)^{-1/p}$$

(note that $L_1(0) = 0$, $\lim_{t\to+\infty} L_1(t) = +\infty$ and L_1 is continuous, hence L_1 is surjective).

We then have $A_pL_1(r)^{1-p} = \beta(\mu,\nu)^{1-1/p}$ and
$A_pL_1(r)^{-p} = A_p^{-1/(1-p)}\,\beta(\mu,\nu) = (\max\{M_p(\mu),M_p(\nu)\})^{-1}\,\beta(\mu,\nu)$
and continue (2.7) by

$$
\begin{aligned}
&\Big|\int_Z g\,d(\mu-\nu)\Big| \le \\
&\le 4(M_p(\mu)+M_p(\nu))\beta(\mu,\nu)^{1-1/p} + (L(1)+5|g(0)|)\beta(\mu,\nu) \le \\
&\le \max\{4,2L(1)+10|g(0)|\}(1+M_p(\mu)+M_p(\nu))\beta(\mu,\nu)^{1-1/p}
\end{aligned}
$$

(note that $\beta(\mu,\nu)^{1/p} \le 2^{1/p} \le 2$) □

<u>2.2 Remark:</u> In view of $\beta(\mu,\nu) \le 2\rho(\mu,\nu)$ $(\mu,\nu\in \mathcal{P}(Z))$, Theorem 2.1 improves the estimate in [24, Theorem 1] considerably.
Theorem 2.1 states that the mapping $\mu \mapsto \int_Z g\,d\mu$ is, for each $p>1$, Hölder continuous on the subset $\{\mu\in \mathcal{P}(Z)\,|\,M_p(\mu)<\infty\}$ of $\mathcal{P}(Z)$ with exponent $1-1/p$.

The next proposition (which, in some sense, is a converse to Theorem 2.1), states that this exponent is "optimal"

<u>2.3 Proposition:</u> Let $Z = \mathbb{R}$, $g(z) := z^k$ $(k\in\mathbb{N}$ fixed), $L(t) := k\,t^{k-1}$ Let further $\mu\in \mathcal{P}(Z)$

and $p \geq 1$ be such that $M_p(\mu) < \infty$. Then there exists a sequence (μ_n) in $\mathcal{P}(Z)$ converging weakly towards μ such that $(M_p(\mu_n))$ is bounded and

$$\left| \int_Z g(z)\mu(dz) - \int_Z g(z)\mu_n(dz) \right| \geq C \cdot \beta(\mu,\mu_n)^{1-1/p} \tag{2.8}$$

for all $n \in \mathbb{N}$ and some constant $C > 0$.

Proof: Put $\mu_n := (1-\frac{1}{n})\mu + \frac{1}{n}\delta_{n^{1/(kp)}}$ $(n \in \mathbb{N})$.

Then we observe that $M_p(\mu_n) = ((1-\frac{1}{n}) \int_Z k^p\, |z|^{kp}\, \mu(dz) + \frac{1}{n}\, k^p\, n)^{1/p}$

is bounded, and that

$$\left| \int_Z gd(\mu-\mu_n) \right| = \left| \frac{1}{n} \int_Z z^k \mu(dz) - \frac{1}{n}\; n^{1/p} \right| \geq C_1 n^{1/p-1} \tag{2.9}$$

for a suitable $C_1 > 0$.

On the other hand there holds $\beta(\mu,\mu_n) \leq 2\rho(\mu,\mu_n) \leq \frac{2}{n}$, which implies that

$$\beta(\mu,\mu_n)^{1-1/p} \leq C_2 n^{1/p-1} \tag{2.10}$$

for a suitable $C_2 > 0$.

Combining (2.9) and (2.10) we arrive at (2.8). □

2.4 Remark: In a similar way as in Proposition 2.3 it can be shown that for general Z, unbounded $g: Z \to \mathbb{R}$ and $\mu \in \mathcal{P}(Z)$ such that $\int |g(z)|^p \mu(dz) < \infty$, there exists a sequence (μ_n) in $\mathcal{P}(Z)$ converging weakly towards μ such that (2.8) holds.

3. Convergence rates for approximations in stochastic linear programming with complete fixed recourse

We consider the following class of problems, which has been studied in detail in [29] and [15]:

$$\min\{c^T x + \int_Z Q(z,x)\mu(dz) \,|\, x \in X_0\} \tag{3.1}$$

where $X_0 \subseteq \mathbb{R}^m$ is a convex, compact polyhedron, and $c \in \mathbb{R}^m$,

$$Q(z,x) := \eta(a,b-Ax), \quad \forall z = (a,b,A) \in Z,\ \forall x \in X_0, \tag{3.2}$$

$$Z := \{(a,b,A) \,|\, b \in \mathbb{R}^r,\ A \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^r),\ a \in \mathbb{R}^d,\ \{u \in \mathbb{R}^r \,|\, W^T u \leq a\} \neq \emptyset\}, \tag{3.3}$$

$$\eta(a,v) := \inf\{a^T y \mid y \in \mathbb{R}^d,\ Wy = v,\ y \geq 0\}, \quad \forall v \in \mathbb{R}^r \tag{3.4}$$

$W \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^r)$ is the "fixed recourse" matrix, and $\mu \in \mathcal{P}(Z)$.

We generally assume that for all $v \in \mathbb{R}^r$ the set $\{y \in \mathbb{R}^d \mid Wy = v,\ y \geq 0\}$ is non-void ("complete fixed recourse"). This implies that $\eta(a,v) < +\infty$ for all $a \in \mathbb{R}^d$ and $v \in \mathbb{R}^r$. Besides, due to the duality theorem of linear optimization, for all $a \in \mathbb{R}^d$ such that $\{u \in \mathbb{R}^r \mid W^T u \leq a\} \neq \emptyset$ the dual problem

$$\max\{v^T u \mid u \in \mathbb{R}^r,\ W^T u \leq a\}$$

has a solution u_* with $\eta(a,v) = v^T u^*$. Hence $\eta(a,v)$ is finite for all $(a,v) \in D$ where

$$D := \{(a,v) \in \mathbb{R}^d \times \mathbb{R}^r \mid \{u \in \mathbb{R}^r \mid W^T u \leq a\} \neq \emptyset\}. \tag{3.5}$$

In the sequel we will use the following result from linear parametric optimization:

<u>3.1 Lemma:</u> [21, Theorem 8.8, p. 219]
η (as defined in (3.4)) is continuous on D (defined in (3.5)). Moreover, there exists a finite number of convex cones K_j $(j = 1,\dots,N)$ such that $\bigcup_j K_j = D$, and there exist matrices $C_j \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^r)$ $(j = 1,\dots,N)$ such that

$$\eta(a,v) = (C_j a)^T v \quad \text{for all } (a,v) \in K_j. \tag{3.6}$$

In addition to Lemma 3.1, we need a quantitative continuity result on η:

<u>3.2 Lemma:</u> η obeys the following Lipschitz condition on D:

$$|\eta(a_1,v_1) - \eta(a_2,v_2)| \leq MR(\|a_1-a_2\| + \|v_1-v_2\|),$$

if $\max\{\|a_1\| + \|v_1\|,\ \|a_2\| + \|v_2\|\} \leq R$,

where $M := \max_{j=1,\dots,N} \|C_j\|$ and C_j is as in Lemma 3.1.

<u>Proof:</u> Obviously there holds in view of (3.6) for all $(a_1,v_1), (a_2,v_2) \in K_j$ $(j = 1,\dots,N)$:

$$|\eta(a_1,v_1) - \eta(a_2,v_2)| \leq \|C_j\|(\|a_1\|.\|v_1-v_2\| + \|v_2\|.\|a_1-a_2\|) \leq$$

$$\leq M.\max\{\|a_1\| + \|v_1\|, \|a_2\| + \|v_2\|\}\ (\|a_1-a_2\| + \|v_1-v_2\|).$$

Now put, for $R > 0$,

$$D_R := D \cap \{(a,v) \in \mathbb{R}^d \times \mathbb{R}^r \mid\ \|a\| + \|v\| \leq R\}$$

Then η is continuous on D_R (Lemma 3.1), and, for all $a,v \in D_R$ there holds

$$\eta(a,v) \in \bigcup_{j=1}^{N} \{(C_j a)^T v\},$$

i.e. $\eta(a,v)$ takes one of the values of N mappings, all of which are Lipschitz continuous with constant MR. Using [13, Theorem 2.1], one shows now, analogously to [6, Theorem 2.2], that η itself is Lipschitz continuous on D_R with constant MR. □

We now consider the mapping $f: Z \times X_0 \longrightarrow \mathbb{R}$,

$$f(z,x) := c^T x + Q(z,x) = c^T x + \eta(a,b-Ax) \quad \text{for all } z = (a,b,A) \in Z,\ x \in X_0,$$

where Z is the subset of $\mathbb{R}^d \times \mathbb{R}^r \times \mathcal{L}(\mathbb{R}^m,\mathbb{R}^r)$ defined in (3.3) (equipped with the norm $\|z\| := \|a\| + \|b\| + \|A\|$).
f is real valued, as $(a,b-Ax) \in D$ for all $x \in X_0$.

For $\mu \in \mathcal{P}(Z)$, we denote by

$$\varphi(\mu) := \inf\{ \int_Z f(z,\mu)\mu(dz) | x \in X_0\}$$

the optimal value of problem (3.1).

The following result deals with the quantitative continuity properties of φ with respect to the bounded Lipschitz metric on $\mathcal{P}(Z)$:

<u>3.3 Theorem:</u> Let (μ_n) be a sequence which converges weakly towards $\mu \in \mathcal{P}(Z)$, and assume that $\int_Z \|z\|^{2p}\mu(dz) < \infty$ and $\int_Z \|z\|^{2p}\mu_n(dz) < \infty$
$(n \in \mathbb{N})$. Then there holds

$$|\varphi(\mu)-\varphi(\mu_n)| = O((1 + (\int_Z \|z\|^{2p}\mu_n(dz))^{1/p}\ \beta(\mu,\mu_n)^{1-1/p}))$$

<u>Proof:</u> We are going to apply Theorem 2.1; hence we need an estimate of the type (2.1):
For arbitrary $x \in X_0$ and $z = (a,b,A)$, $\tilde{z} = (\tilde{a},\tilde{b},\tilde{A}) \in Z$ there holds according to Lemma 3.2:

$$|f(z,x) - f(\tilde{z},x)| = |\eta(a,b-Ax) - \eta(\tilde{a},\tilde{b}-\tilde{A}x)| \le$$

$$\le M \max\{\|a\| + \|b-Ax\|, \|\tilde{a}\| + \|\tilde{b}-\tilde{A}x\|\}(\|a-\tilde{a}\| + \|b-\tilde{b}+(A-\tilde{A})x\|) \le$$

$$\le M . K^2 \max\{\|z\|, \|\tilde{z}\|\} \|z-\tilde{z}\|$$

where $K := \max\{1, \max\{\|x\| | x \in X_0\}\}$.

Hence, for each $x \in X_0$, $g := f(.,x): Z \to \mathbb{R}$ fulfills the Lipschitz condition (2.1) with $L(t) := L_0 t$, $t \in \mathbb{R}_+$, and $L_0 := MK^2$, which yields, for each $x \in X_0$, the estimate

$$|f(z,x)| \le L_0\|z\|^2 + |f(0,x)| \le L_0\|z\|^2 + K\|c\|$$

From this we get, due to our assumption, that

$$\int_Z |f(z,x)|\mu(dz) < \infty, \int_Z |f(z,x)|\mu_n(dz) < \infty \quad (n \in \mathbb{N}, x \in X_0)$$

and $\varphi(\mu) > -\infty$, $\varphi(\mu_n) > -\infty$.

Now take an arbitrary $x \in X_0$. Putting $g := f(.,x)$, we conclude from Theorem 2.1 that there exists a constant $C > 0$ (independent of x) such that

$$\left|\int_Z f(z,x)\mu(dz) - \int_Z f(z,x)\mu_n(dz)\right| \leq$$
$$\leq C(1 + M_p(\mu_n))\beta(\mu,\mu_n)^{1-1/p} \quad (n \in \mathbb{N}).$$

From this estimate our assertion results in view of

$$M_p(\mu_n) = L_0(\int_Z \|z\|^{2p} \mu_n(dz))^{1/p} \quad (n \in \mathbb{N}) \quad \text{and}$$

$$|\varphi(\mu) - \varphi(\mu_n)| \leq \sup_{x \in X_0} \left|\int_Z f(z,x)\mu(dz) - \int_Z f(z,x)\mu_n(dz)\right|$$ □

Theorem 3.3 improves the result of [24, Theorem 5] and can be used to investigate speed of convergence of approximations of different type for the problem (3.1). We demonstrate this first for the case of discrete approximations of μ by conditional expectations. This type of approximation has gained particular attention in the literature ([5], [16], [17], [31]).

3.4 Corollary: Let $\mu \in \mathcal{P}(Z)$ be such that $\int_Z \|z\|^{2p} \mu(dz)$ is finite for some $p > 1$. Let, for every $n \in \mathbb{N}$, $S_n := \{Z_{nk} | k = 1,\ldots,m(n)\}$ be a partition of Z into Borel sets such that S_{n+1} is a refinement of S_n $(n \in \mathbb{N})$ and that $\bigcup_{n \in \mathbb{N}} S_n$ generates the σ-algebra of Borel subsets of Z.

Let $\mu_n := \sum_{k=1}^{m(n)} \mu(Z_{nk})\delta_{e_{nk}}$, where

$$e_{nk} := \mu(Z_{nk})^{-1} \int_{Z_{nk}} z\mu(dz) \quad \text{for } k = 1,\ldots,m(n) \text{ and } n \in \mathbb{N}.$$

Then $|\varphi(\mu) - \varphi(\mu_n)| = O(\beta(\mu,\mu_n)^{1-1/p})$.

Proof: By martingale convergence we have $\mu_n \xrightarrow{w} \mu$, and from Jensen's inequality we get

$$\int_Z \|z\|^{2p}\mu_n(dz) \leq \int_Z \|z\|^{2p}\mu(dz) \quad (n \in \mathbb{N})$$

Now Theorem 3.3 yields the assertion. □

As a further application we mention the approximation of $\mu \in \mathcal{P}(Z)$ by empirical measures. Let $(z_i)_{i \in \mathbb{N}}$ be a sequence of independent Z-valued random variables with distribution μ, and consider the empirical measures

$$\mu_n(\omega) := \frac{1}{n} \sum_{i=1}^{n} \delta_{z_i(\omega)}, \; n \in \mathbb{N}, \; \omega \in \Omega.$$

<u>3.5 Corollary:</u>

a) If $\mu \in \mathcal{P}(Z)$ is such that $\int_Z \|z\|^{2p} \mu(dz)$ is finite for some $p > 1$, we have

$$E[|\varphi(\mu) - \varphi(\mu_n(\omega))|] = O(E[\beta(\mu,\mu_n(\omega))]^{1-1/p})$$

b) If $\int_Z \|z\|^{\alpha} d\mu$ is finite for all $\alpha > 1$, and if the support of μ is a subset of a k-dimensional space, then we have

$$E[|\varphi(\mu) - \varphi(\mu_n(\omega))|] = O(n^{-1/\Delta}) \quad \text{for every } \Delta > \max(2,k).$$

<u>Proof:</u> From the strong law of large numbers there follows (cf. [7,9]) that $\mu_n(\omega) \xrightarrow{w} \mu$ almost surely.

Besides, there holds

$$\int_Z \|z\|^{2p} \mu_n(\omega)(dz) = \frac{1}{n} \sum_{i=1}^{n} \|z_i(\omega)\|^{2p} < \infty \qquad (\omega \in \Omega, \; n \in \mathbb{N}).$$

Hence Theorem 3.3 is applicable, and there exists a constant $C > 0$ such that for all $\omega \in \Omega$ there holds

$$|\varphi(\mu) - \varphi(\mu_n(\omega))| \le C(1 + (\int_Z \|z\|^{2p} \mu_n(\omega)(dz)))^{1/p}) \beta(\mu,\mu_n(\omega))^{1-1/p} =$$

$$= C(1 + (\frac{1}{n} \sum_{i=1}^{n} \|z_i(\omega)\|^{2p})^{1/p}) \beta(\mu,\mu_n(\omega))^{1/q} \qquad (\frac{1}{p} + \frac{1}{q} = 1)$$

Applying Hölder's inequality, we get the estimate

$$E[|\varphi(\mu) - \varphi(\mu_n(\omega))|] \le C(1 + E[(\frac{1}{n} \sum_{i=1}^{n} \|z_i(\omega)\|^{2p})]^{1/p}) E[\beta(\mu,\mu_n(\omega))]^{1/q}$$

$$= C(1 + (\int_Z \|z\|^{2p} \mu(dz))^{1/p}) E[\beta(\mu,\mu_n(\omega))]^{1-1/p}$$

This proves assertion a)

b) From part a) there follows for any $p > 1$.

$$E[|\varphi(\mu) - \varphi(\mu_n(\omega))|] = O(E[\beta(\mu,\mu_n(\omega))]^{1-1/p})$$

From [8, Prop. 3.4] we obtain that

$$E[\beta(\mu,\mu_n(\omega))] = O(n^{-1/\Delta}) \quad \text{for all } \Delta > \max\{2,k\}.$$

Hence the assertion results. □

The just stated results on the mean-convergence of the optimal values may, on one hand, be interpreted as a theoretical basis for methods of Monte-Carlo-type for the numerical treatment of recourse problems in stochastic linear programming. On the other hand they provide asymptotic properties of the optimal values, if estimates for the (unknown) distribution μ, based on empirical distributions, are used. For such a situation, similar asymptotic properties of the optimal values (and also of optimal solutions under stronger regularity assumptions on (3.1) - (3.4) and μ) are derived in [10], [11], [27], [30] and in [18], where a rate-of-convergence-estimate for the optimal values is obtained w.r. to convergence in probability ([18, Theorem 2], but for the case that the support of μ is bounded (since the integrand $f(z,x)$ is not bounded on $Z \times X_0$ as required in [18, p. 350]).

4. On distribution sensitivity of the optimal value of chance-constrained programming problems

We are going to investigate (quantitative) continuity results of the following set-valued mapping (induced by a probabilitstic constraint) from $\mathcal{P}(Z)$ into $\mathbb{R}^m$

$$\mu \mapsto C_\alpha(\mu) := \{x \in \mathbb{R}^m \,|\, \mu(X^{-1}(\{x\})) \geq \alpha\} \tag{4.1}$$

with respect to the weak topology in $\mathcal{P}(Z)$ and the Hausdorff distance (on the class of all nonempty subsets of $\mathbb{R}^m$), respectively.

Thereby, $Z := \mathbb{R}^s$ (equipped with the norm $\|\ \|_\infty$), X is a set-valued mapping from Z into $\mathbb{R}^m$ (where we denote, as usual, $X^{-1}(B) := \{z \in Z \,|\, X(z) \cap B \neq \emptyset\}$ for $B \subseteq \mathbb{R}^m$) and $\alpha \in]0,1[$.

We are aiming at a result on the (local) Lipschitz-property of the mapping (4.1), where (based on the following lemma) the Prokhorov- metric (1.2) offers itself as a metric for the weak topology in $\mathcal{P}(Z)$.

<u>4.1 Lemma:</u> Let $\mathcal{B}$ be a class of Borel subsets of Z and $\mu \in \mathcal{P}(Z)$. Assume that

$$\left.\begin{array}{l}\text{there exists a constant } M>0 \text{ such that}\\ \sup\limits_{B\in\mathcal{B}} \mu((\partial B)^\varepsilon) \leq M\varepsilon \qquad \text{for all } \varepsilon>0\end{array}\right\} \tag{4.2}$$

(where ∂B is the topological boundary of B).

Then we have for all $\nu \in \mathcal{P}(Z)$:

$$\alpha_{\mathcal{B}}(\mu,\nu) := \sup_{B\in\mathcal{B}} |\mu(B) - \nu(B)| \leq (M+1)\rho(\mu,\nu) \tag{4.3}$$

<u>Proof:</u> For $\varepsilon > 0$ and $B \subseteq Z$, put $B^{-\varepsilon} := B\backslash(Z\backslash B)^\varepsilon$ and note that $(\partial B)^\varepsilon = B^\varepsilon \backslash B^{-\varepsilon}$ ([3], p.14). Now take an arbitrary $\varepsilon > \rho(\mu,\nu)$ and $B \in \mathcal{B}$, and observe that

$$\mu(B^{-\varepsilon}) \leq \nu((B^{-\varepsilon})^\varepsilon) + \varepsilon \leq \nu(B) + \varepsilon \qquad \text{and}$$

$$\nu(B) \le \mu(B^{\varepsilon}) + \varepsilon$$

Hence follows

$$\mu(B) - \nu(B) \le \mu(B) - \mu(B^{-\varepsilon}) + \varepsilon \le \mu(B^{\varepsilon} \setminus B^{-\varepsilon}) + \varepsilon \le (M+1)\varepsilon$$

and

$$\nu(B) - \mu(B) \le \mu(B^{\varepsilon}) - \mu(B) + \varepsilon \le \mu(B^{\varepsilon} \setminus B^{-\varepsilon}) + \varepsilon \le (M+1)\varepsilon$$

which implies (4.3). □

4.2 Remark: Condition (4.2) implies that $\mathcal{B}$ is a μ-uniformity class ([3], p.14). We mention two important examples of classes $\mathcal{B}$ that fulfill (4.2) under certain assumptions on μ:

(i) $\mathcal{B} \subseteq \{]-\infty, z] := \mathop{\times}\limits_{i=1}^{s}]-\infty, z_i] \mid z = (z_1, \dots, z_s) \in \mathbb{R}^s\}$.

If the distribution function F_μ belonging to μ is Lipschitz continuous (i.e. if all marginal distributions of μ have bounded densities), then (4.2) is fulfilled for $\mathcal{B}$ and μ. Indeed, for $B =]-\infty, z]$ $(z \in \mathbb{R}^s)$ there holds:

$$\begin{aligned} \mu((\partial B)^{\varepsilon}) &\le F_\mu(z + \bar{\varepsilon}) - F_\mu(z - \bar{\varepsilon}) \le \\ &\le L_\mu \| (z + \bar{\varepsilon}) - (z - \bar{\varepsilon}) \|_\infty = 2L_\mu \|\bar{\varepsilon}\|_\infty = 2L_\mu \varepsilon \end{aligned} \tag{4.4}$$

(where $\bar{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^s$, and L_μ is a Lipschitz constant of F_μ).

(ii) $\mathcal{B} \subseteq \{B \subseteq \mathbb{R}^s \mid B \text{ is convex and Borel}\}$.

For this case, [3, Theorem 3.1] contains conditions on the density of $\mu \in \mathcal{P}(\mathbb{R}^s)$ that imply (4.2).

Now we state a first continuity result on the mapping (4.1):

4.3 Lemma: Let $\mu \in \mathcal{P}(Z)$, and $\mathcal{B} := \{X^{-1}(\{x\}) \mid x \in \mathbb{R}^m\}$ be a class of Borel subsets of Z. Assume that

$$\left.\begin{array}{l} \text{there are constants } L > 0 \text{ and } \delta_0 > 0 \text{ such that} \\ C_{\alpha'}(\mu) \subseteq C_{\alpha'+\delta}(\mu)^{L\delta} \quad \text{for all } \alpha' \in [\alpha - \delta_0, \alpha],\ \delta \in]0, \delta_0]. \end{array}\right\} \tag{4.5}$$

Then there holds

$$\left.\begin{array}{l} D(C_\alpha(\mu), C_\alpha(\nu)) \le L\alpha_{\mathcal{B}}(\mu, \nu) \\ \text{for all } \nu \in \mathcal{P}(Z) \text{ with } \alpha_{\mathcal{B}}(\mu, \nu) \le \delta_0. \end{array}\right\} \tag{4.6}$$

Proof: If $\alpha_{\mathcal{B}}(\mu, \nu) = 0$, we have $C_\alpha(\mu) = C_\alpha(\nu)$.

Hence we assume $0 < \alpha_{\mathcal{B}}(\mu,\nu) =: \delta \leq \delta_0$.

For all $x \in C_{\alpha+\delta}(\mu)$, there holds
$\nu(X^{-1}(\{x\})) \geq \mu(X^{-1}(\{x\})) - \delta \geq \alpha + \delta - \delta = \alpha$,
thus we get the inclusion $C_{\alpha+\delta}(\mu) \subseteq C_\alpha(\nu)$
and analogously $C_\alpha(\nu) \subseteq C_{\alpha-\delta}(\mu)$.
Using (4.5), we obtain $C_\alpha(\nu) \subseteq C_{\alpha-\delta}(\mu) \subseteq C_\alpha(\mu)^{L\delta}$
and $C_\alpha(\mu) \subseteq C_{\alpha+\delta}(\mu)^{L\delta} \subseteq C_\alpha(\nu)^{L\delta}$,
which immediately yields (4.6). □

Combining Lemmata 4.1 and 4.3, we arrive directly at

4.4 Proposition: Let $\mu \in \mathcal{P}(Z)$, and $\mathcal{B} := \{X^{-1}(\{x\}) | x \in \mathbb{R}^m\}$ be a class of Borel sets of Z. Assume the validity of (4.2) and (4.5). Then there holds
$$D(C_\alpha(\mu), C_\alpha(\nu)) \leq L(M+1)\rho(\mu,\nu)$$
for all $\nu \in \mathcal{P}(Z)$ with $\rho(\mu,\nu)$ sufficiently small.

4.5 Remark: Condition (4.2) in Proposition 4.4 was discussed in Remark 4.2. The essential new condition we impose is the local Lipschitzian property (4.5) of the mapping

$$\alpha' \mapsto C_{\alpha'}(\mu) = \{x \in \mathbb{R}^m | \mu(X^{-1}(\{x\})) \geq \alpha'\}$$

at $\alpha \in]0,1[$. [23] contains general results which seem to be useful in this particular case, too (e.g. [23, Corollary 3.5] states conditions on $t(x) = \mu(X^{-1}(\{x\}))$, $x \in \mathbb{R}^m$ (even for nonsmooth t) that imply (4.5).

But, for the particular case we shall discuss now, we prefer to give a self-contained proof.

For the rest of this section, we consider the case

$$X(z) := \{x \in \mathbb{R}^m | Ax \geq z\} \qquad (z \in Z = \mathbb{R}^s)$$

where $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^s)$, and check the applicability of Proposition 4.4 on the set-valued mapping

$$\left.\begin{aligned} \mu \mapsto C_\alpha(\mu) &= \{x \in \mathbb{R}^m | \mu(X^{-1}(\{\alpha\})) \geq \alpha\} \\ &= \{x \in \mathbb{R}^m | \mu(\{z \in \mathbb{R}^s | Ax \geq z\}) \geq \alpha\} \\ &= \{x \in \mathbb{R}^m | F_\mu(Ax) \geq \alpha\} \end{aligned}\right\} \qquad (4.7)$$

where $0 < \alpha < 1$, and F_μ is the distribution function of $\mu \in \mathcal{P}(Z)$.

4.6 Theorem: Let $\alpha \in]0,1[$ and $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^s)$ have rank $s \leq m$. Assume that the distribution function F_μ of $\mu \in \mathcal{P}(\mathbb{R}^s)$ is Lipschitzian and satisfies the following "(local) inverse Lipschitz condition at α":

$$\left.\begin{array}{l} \text{There exist positive constants } \gamma, \delta_1 \text{ and } \delta_2 \text{ such that} \\ F_\mu(z+\bar{\varepsilon}) \geq F_\mu(z) + \gamma\varepsilon \\ \text{for all } \varepsilon \in [0,\delta_2],\ \bar{\varepsilon} := (\varepsilon,\dots,\varepsilon),\ z \in F_\mu^{-1}([\alpha-\delta_1, \alpha+\delta_1]) \end{array}\right\} \quad (4.8)$$

Then there is a constant $L_0 > 0$ such that

$$D(C_\alpha(\mu), C_\alpha(\nu)) \leq L_0 \rho(\mu,\nu)$$

for all $\nu \in \mathcal{P}(\mathbb{R}^s)$ with sufficiently small $\rho(\mu,\nu)$.

Proof: As $\mathcal{B} = \{\chi^{-1}(\{x\}) \mid x \in \mathbb{R}^m\} \subseteq \{]-\infty,z] \mid z \in \mathbb{R}^s\}$, and as F_μ is assumed to be Lipschitzian, condition (4.2) is fulfilled in view of Remark 4.2 (i). We now show that also condition (4.5) is satisfied:
Due to the assumption that A has rank $s \leq m$, there exists an $x_0 \in \mathbb{R}^m$ such that $Ax_0 = (1,\dots,1) \in \mathbb{R}^s$
We now put $\delta_0 := \min\{\delta_1, \delta_2\gamma\}$ and $L := \frac{\|x_0\|}{\gamma} + 1$, and show that

$$C_{\alpha'}(\mu) \subseteq C_{\alpha'+\delta}(\mu)^{L\delta} \quad \text{for all } \alpha' \in [\alpha-\delta_0, \alpha],\ \delta \in [0,\delta_0].$$

Choose such an α' and δ, take an arbitrary $x \in C_{\alpha'}(\mu)$ and put $y := x + \frac{\delta}{\gamma} x_0$.

Then there holds $d(x,y) \leq \frac{\|x_0\|}{\gamma}\delta < L\delta$ and

$$\begin{aligned} F_\mu(Ay) &= F_\mu(Ax + \tfrac{\delta}{\gamma} Ax_0) = F_\mu(Ax + \tfrac{\delta}{\gamma}(1,\dots,1)) \\ &\geq F_\mu(Ax) + \tfrac{\delta}{\gamma}\gamma \geq \alpha' + \delta. \end{aligned}$$

This means that $y \in C_{\alpha'+\delta}(\mu)$, and consequently, that $x \in C_{\alpha'+\delta}(\mu)^{L\delta}$. Hence Proposition 4.4 is applicable, yielding the assertion. □

4.7 Corollary: Assume that the conditions of Theorem 4.6 are satisfied for $\alpha \in]0,1[$, $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^s)$ and $\mu \in \mathcal{P}(\mathbb{R}^s)$. Let $c \in \mathbb{R}^m$ and $\varphi(\mu) := \inf\{c^T x \mid x \in \mathbb{R}^m,\ F_\mu(Ax) \geq \alpha\}$ (and likewise $\varphi(\nu)$ for $\nu \in \mathcal{P}(\mathbb{R}^s)$). Assume that $\varphi(\mu)$ and $\varphi(\nu)$ are finite, and that $\rho(\mu,\nu)$ is sufficiently small. Then there is a constant $L_1 > 0$ (not depending on ν) such that
$$|\varphi(\mu) - \varphi(\nu)| \leq L_1 \rho(\mu,\nu)$$

Proof: For all $\varepsilon > 0$, there exists an $x \in C_\alpha(\mu)$ such that $c^T x \leq \varphi(\mu) + \varepsilon$. Due to Theorem 4.6 there exists a $y \in C_\alpha(\nu)$ such that $d(x,y) \leq L_0\rho(\mu,\nu) + \varepsilon$. This implies

$$\varphi(\nu) - \varphi(\mu) \le c^T y - c^T x + \varepsilon \le |c^T(x-y)| + \varepsilon$$
$$\le \|c\| d(x,y) + \varepsilon$$
$$\le \|c\| L_0 \rho(\mu,\nu) + \varepsilon(\|c\| + 1)$$

Interchanging the roles of μ und ν, we arrive at our assertion. □

4.8 Remark: The proof of Theorem 4.6 (together with Lemma 4.3) shows that the mapping $\mu \mapsto C_\alpha(\mu)$ is (locally) Lipschitz continuous even w.r. to the Levy-metric

$$\lambda(\mu,\nu) := \inf\{\varepsilon > 0 \mid \mu(B) \le \nu(B^\varepsilon) + \varepsilon,\ \nu(B) \le \mu(B^\varepsilon) + \varepsilon,$$
$$\text{for every } B =]-\infty, z],\ z \in \mathbb{R}^s\}$$

(which also metrizes the weak topology on $\mathcal{P}(\mathbb{R}^s)$).

Thereby, the essential condition is the (local) inverse Lipschitz property (4.8) for F_μ in α. For this property we finally give a sufficent condition, which shows that, e.g. the multivariate normal distribution μ obeys all conditions of Theorem 4.6 (see also Remark 4.2 (i)).

4.9 Lemma: Assume that the distribution function F_μ of $\mu \in \mathcal{P}(\mathbb{R}^s)$ has a continuous, strictly positive density f_μ. Then condition (4.8) of Theorem 4.6 is satisfied for all $\alpha \in]0,1[$.

Proof: We choose $\delta_1 > 0$ such that $0 < \alpha - \delta_1 < \alpha + \delta_1 < 1$ and put $M := F_\mu^{-1}([\alpha - \delta_1, \alpha + \delta_1]) \subseteq \mathbb{R}^s$.

As $\lim_{z \to (-\infty,\dots,-\infty)} F_\mu(z) = 0$, there exist a constant $C > 0$ such that

$$\left|\min_{i=1,\dots,s} z_i\right| \le C \quad (z = (z_1,\dots,z_s) \in M).$$

We now put $m := \min_{z \in [-2C,2C]^s} f_\mu(z)$ and $\gamma := \delta_2 = mC^{s-1}$.

Then there holds for arbitrary $z \in M$ and $\varepsilon \in [0,\delta_2]$ (we are assuming without loss of generality that $z_1 = \min_{i=1,\dots,s} z_i$):

$$F_\mu(z+\bar{\varepsilon}) - F_\mu(z) \ge \int_{z_1}^{z_1+\varepsilon} \int_{-\infty}^{z_2} \dots \int_{-\infty}^{z_s} f_\mu(t_1,\dots,t_s)dt_1 \dots dt_s \ge$$

$$\ge \int_{z_1}^{z_1+\varepsilon} \int_{-2C}^{-C} \dots \int_{-2C}^{-C} f_\mu(t_1,\dots,t_s)dt_1 \dots dt_s \ge m \cdot \varepsilon C^{s-1} = \gamma\varepsilon$$

where $\bar{\varepsilon} := (\varepsilon,\dots,\varepsilon)$. Consequently, F_μ satisfies condition (4.8). □

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