Quantitative stability analysis of stochastic generalized equations

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Introduction

We consider stochastic generalized equations (SGE) of the form

$$0 \in \mathbb{E}_P[\Gamma(x,\xi)] + \mathcal{G}(x),$$

where $\Gamma :: \mathcal{X} \times \Xi \to 2^{\mathcal{Y}}$ and $\mathcal{G} : \mathcal{X} \to 2^{\mathcal{Y}}$ are closed set-valued mappings, $\mathcal{X}$ and $\mathcal{Y}$ are subsets of Banach spaces X and Y (with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$) respectively, $\xi : \Omega \to \Xi$ is a random vector defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with support set $\Xi \in \mathbb{R}^d$ and probability distribution P, and $\mathbb{E}_P[\cdot]$ denotes Aumann's set-valued integral with respect to P, i.e.,

$$\mathbb{E}_P[\Gamma(x,\xi)] := \int_\Xi \Gamma(x,\xi)P(d\xi)$$
$$= \left\{ \int_\Xi \gamma_x(\xi)P(d\xi) : \gamma_x \text{ is integrable selection of } \Gamma(x,\cdot) \right\}.$$

If X is separable and P non-atomic, Γ closed-valued and integrably bounded, then $E_P[\Gamma(x,\xi)]$ is convex.

Stochastic generalized equations were first studied by Ralph-Xu in MOR 2011.

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Aim: **Framework for stability analysis of stochastic variational problems**

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Prerequisites about support functions

Lemma:

Let C, D be nonempty compact and convex subsets of a Banach space E with support functions $\sigma(\cdot, C)$ and $\sigma(\cdot, D)$ given on the dual E^*, i.e., $\sigma(u, C) = \sup_{x \in C} \langle u, x \rangle$.

Then it holds for the excess

$$\mathbb{D}(C, D) := \sup_{x \in C} d(x, D) = \max_{\|u\|_* \leq 1} (\sigma(u, C) - \sigma(u, D))$$

and for the Pompeiu-Hausdorff distance

$$\mathbb{H}(C, D) := \max\{\sup_{x \in C} d(x, D), \sup_{x \in D} d(x, C)\} = \max_{\|u\|_* \leq 1} |\sigma(u, C) - \sigma(u, D)|.$$

Lemma:

If $\Gamma : \Xi \longrightarrow E$ is closed convex-valued and integrably bounded, then it holds for all $u \in E^*$

$$\mathbb{E}_P[\sigma(u, \Gamma(\xi))] = \sigma(u, \mathbb{E}_P[\Gamma(\xi)]).$$

Main quantitative stability result

We consider the stochastic generalized equation

$$(SGE(P)) \qquad 0 \in \mathbb{E}_P[\Gamma(x,\xi)] + \mathcal{G}(x)$$

and its perturbation

$$(SGE(Q)) \qquad 0 \in \mathbb{E}_Q[\Gamma(x,\xi)] + \mathcal{G}(x).$$

for probability measures P and Q on Ξ.

We consider the following "distance" of probability measures

$$\mathscr{D}(Q,P) := \sup_{g\in\mathscr{F}} \big(\mathbb{E}_Q[g(\xi)] - \mathbb{E}_P[g(\xi)]\big)$$

where $\mathscr{F} := \{g : g(\xi) := \sigma(u, \Gamma(x,\xi)), \text{ for } x \in \mathcal{X}, \|u\|_* \le 1\}$.

Note that $\mathscr{D}(Q,P)$ may be bounded by the ζ-metric

$$\zeta_{\mathscr{F}}(Q,P) := \sup_{g\in\mathscr{F}} \big|\mathbb{E}_Q[g(\xi)] - \mathbb{E}_P[g(\xi)]\big|.$$

Theorem:

Let $\mathcal{X}$ be a compact subset of X and $S(P)$ and $S(Q)$ denote the solution sets of (SGE(P)) and (SGE(Q)) restricted to $\mathcal{X}$. Assume

(a) Γ is set-valued taking convex and compact values in Y,

(b) Y is finite-dimensional or Γ is single-valued,

(c) $\Gamma(\cdot,\xi)$ is upper semi-continuous for every $\xi \in \Xi$ and integrably bounded, i.e., $\sup_{y\in\Gamma(x,\xi)} \|y\|$ integrable for all $x \in \mathcal{X}$,

(d) $\mathcal{G}$ is upper semi-continuous,

(e) $S(Q)$ is nonempty if $\mathscr{D}(Q,P)$ is small.

For any $\epsilon > 0$, let

$$R(\epsilon) := \inf_{x\in\mathcal{X},\, d(x,S(P))\geq\epsilon} d(0, \mathbb{E}_P[\Gamma(x,\xi)] + \mathcal{G}(x)).$$

Then $R(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ and

$$\mathbb{D}(S(Q), S(P)) \leq R^{-1}(2\mathscr{D}(Q,P)),$$

where $R^{-1}(t) := \min\{\epsilon \in \mathbb{R}_+ : R(\epsilon) = t\}$.

Stability of linear two-stage stochastic programs

We consider

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & c^\top x + \mathbb{E}_P[v(x,\xi)] \\ \text{s.t.} & x \in X, \end{array}$$

where X is convex polyhedral and $v(x,\xi)$ is the second stage optimal value function

$$\begin{array}{ll} \min_{y\in\mathbb{R}^m} & q(\xi)^\top y \\ \text{s.t.} & T(\xi)x + Wy = h(\xi),\ y \geq 0, \end{array}$$

where $W \in \mathbb{R}^{r\times m}$ is a fixed recourse matrix, $T(\xi) \in \mathbb{R}^{r\times n}$ is a random matrix, and $h(\xi) \in \mathbb{R}^r$ and $q(\xi) \in \mathbb{R}^m$ are random vectors. We assume that $T(\cdot)$, $h(\cdot)$ and $q(\cdot)$ are affine functions of ξ and that Ξ is a polyhedral subset of $\mathbb{R}^s$ (for example, $\Xi = \mathbb{R}^s$).

Stochastic generalized equation

$$0 \in \mathbb{E}_P[c - T(\xi)^\top D(x,\xi)] + \mathbb{N}_X(x),$$

where $D(x,\xi)$ is the solution set of the dual second stage problem

$$D(x,\xi) := \arg \max_{W^\top \zeta \leq q(\xi)} \zeta^\top (h(\xi) - T(\xi)x).$$

Theorem: Assume that

(a) $h(\xi) - T(\xi)x \in W(\mathbb{R}^m_+)$ for all $(\xi, x) \in \Xi \times X$,

(b) $\mathcal{M}(q(\xi)) = \{\pi : W^\top \pi \le q(\xi)\} \neq \emptyset$ is bounded for all $\xi \in \Xi$,

(c) P has finite second order moments, i.e., $\mathbb{E}_P[\|\xi\|^2] < +\infty$ and

(d) X is a nonempty and bounded polyhedron.

Then it holds for any probability measure Q such that $\mathscr{D}(Q, P)$ is sufficiently small

$$\mathbb{D}(S(Q), S(P)) \le R^{-1}(2\mathscr{D}(Q, P)),$$

where the function R is defined by

$$R(\epsilon) := \inf_{x \in X, d(x, S(P)) \ge \epsilon} d(0, \mathbb{E}_P[\Gamma(x, \xi)] + \mathbb{N}_X(x)).$$

The class $\mathscr{F}$ for defining $\mathscr{D}$ is contained in

$$\{g : g(\xi) - g(\tilde{\xi}) \le C \max\{1, \|\xi\|, \|\tilde{\xi}\|\}^2 \|\xi - \tilde{\xi}\|, \forall \xi, \tilde{\xi} \in \Xi\}.$$

Stability of two-stage SMPCC

The theory applies to

$$\begin{array}{ll} \min\limits_{x,\, y(\cdot)\in\mathscr{Y}} & \mathbb{E}_P[f(x, y(\omega), \xi(\omega))] \\ \text{subject to} & x \in X \text{ and for almost every } \ \omega \in \Omega : \\ & g(x, y(\omega), \xi(\omega)) \le 0, \\ & h(x, y(\omega), \xi(\omega)) = 0, \\ & 0 \le G(x, y(\omega), \xi(\omega)) \ \perp H(x, y(\omega), \xi(\omega)) \ge 0, \end{array}$$

where X is a nonempty closed convex subset of $\mathbb{R}^n$, f, g, h, G, H are continuously differentiable functions from $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$ to $\mathbb{R}, \mathbb{R}^s, \mathbb{R}^r, \mathbb{R}^m, \mathbb{R}^m$, respectively, $\xi : \Omega \to \Xi$ is a vector of random variables defined on probability $(\Omega, \mathbb{F}, P)$ with compact support set $\Xi \subset \mathbb{R}^q$, and $\mathbb{E}_P[\cdot]$ denotes the expected value with respect to probability measure P, and '$\perp$' denotes the perpendicularity of two vectors, $\mathscr{Y}$ is a space of functions $y(\cdot) : \Omega \to \mathbb{R}^m$ such that $\mathbb{E}_P[f(x, y(\omega), \xi(\omega))]$ is well defined.

The two-stage SPCC may be reformulated as

$$\begin{array}{ll} \min\limits_{x} & \theta(x) = \mathbb{E}_P[v(x,\xi)] \\ \text{s.t.} & x \in X, \end{array}$$

where $v(x, \xi)$ denotes the optimal value function of the following second stage problem:

$$\mathrm{MPCC}(x,\xi): \quad \begin{array}{ll} \min\limits_{y} & f(x,y,\xi) \\ \text{s.t.} & g(x,y,\xi) \le 0, \\ & h(x,y,\xi) = 0, \\ & 0 \le G(x,y,\xi) \perp H(x,y,\xi) \ge 0. \end{array}$$

Under certain assumptions $v(\cdot, \xi)$ is locally Lipschitz continuous (with a P-integrable Lipschitz constant) and one may consider the necessary optimality conditions (using the Clarke subdifferential)

$$0 \in \mathbb{E}_P[\partial_x v(x,\xi)] + \mathbb{N}_X(x).$$

as SGE that hopefully satisfies the assumptions of the stability result.

Let us, in particular, consider the stochastic optimization model

$$\min\{c^\top x + \mathbb{E}_P[q(\xi)^\top y] : 0 \in Wy + T(\xi)x - h(\xi) + N_{R_+^{\bar{m}}}(y), x \in X\},$$

where similar conditions are imposed as in the (standard) two-stage model before. The linear generalized equation is equivalent to the linear complementarity problem

$$Wy + T(\xi)x \geq h(\xi), y \geq 0, y^\top(Wy + Tx - h(\xi)) = 0.$$

Its solution set is a polyhedral multifunction (of $a = h(\xi) - T(\xi)x$) and, hence, is locally upper Lipschitz continuous at each a (with the same modulus $L > 0$). Hence, the reformulation reads

$$\min\{c^\top x + \mathbb{E}[v(x,\xi)] : x \in X\}$$

and the function $v(\cdot,\xi)$ is locally Lipschitz continuous (with constant $L\|q(\xi)\|\|T(\xi)\|$). Then the general theory implies (local) upper Lipschitz continuity of the solution set mapping at P with respect to the ζ-distance $\zeta_{\mathscr{F}}$ and the function class

$$\mathscr{F} = \{v^o(x,\cdot;u) : x \in X, \|u\| \leq 1\},$$

where $v^o(x,\xi;u)$ denotes the Clarke directional derivative of $v(\cdot,\xi)$ at x.

Stability of convex programs with second order dominance constraints

We consider convex programs with second order dominance constraints

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & \mathbb{E}_P[(\eta - G(x,\xi))_+] \le \mathbb{E}_P[(\eta - Y(\xi))_+], \ \forall \eta \in [a,b], \\ & x \in X, \end{aligned}$$

where X is a closed convex subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable and $G : \mathbb{R}^n \times \Xi \to \mathbb{R}$ is concave in the first component and has linear growth in the second, ξ is a random vector with distribution P and support Ξ in $\mathbb{R}^d$.

The constraint satisfies the uniform dominance condition (udc) at P if $\bar{x} \in X$ exists such that

$$\min_{\eta\in[a,b]} \left(\mathbb{E}_P[(\eta - G(\bar{x},\xi))_+] - \mathbb{E}_P[(\eta - Y(\xi))_+]\right) > 0.$$

Optimality condition:

Let udc be satisfied at P. If a feasible $x^* \in X$ is optimal, there exists $u^* \in \mathcal{U}_1$ satisfying

$$0 \in \mathbb{E}_P[\Gamma(x^*, u^*, \xi)] + \mathcal{G}(x^*)$$

i.e.,

$$\begin{aligned} 0 &\in f'(x^*) + \mathbb{E}_P[\partial_x(-u^*(G(x^*,\xi))] + N_X(x^*) \\ 0 &= \mathbb{E}_P[u^*(G(x^*,\xi)) - u^*(Y(\xi)))], \end{aligned}$$

where $\mathcal{U}_1 = \{u \in C^1(\mathbb{R}) : \exists \varphi : I \longrightarrow \mathbb{R}_+$ nonincreasing, left-continuous and bounded such that $u'(t) = \varphi(t), t \in [a, b]$, $u'(t) = \varphi(a), t < a, u(t) = 0, t \geq b\}$.

Theorem:

Let udc be satisfied at P and X be compact. Then it holds

$$\mathbb{D}(S(Q), S(P)) \leq R^{-1}(2\mathscr{D}(Q, P)),$$

where R and R^{-1} are defined in the stability theorem.

Characterization of the class $\mathscr{F}$?

Another stability result for such models (Dentcheva/Römisch 12):

Let $v(\xi, Y)$ denote the optimal value and $S(\xi, Y)$ the solution set and $\mathcal{X}(\xi, Y)$ the feasible set.

We consider the growth function

$$\psi_{(\xi,Y)}(\tau) := \inf\{f(x) - v(\xi, Y) : d(x, S(\xi, Y)) \geq \tau, \, x \in \mathcal{X}(\xi, Y)\}$$

and

$$\Psi_{(\xi,Y)}(\theta) := \theta + \psi^{-1}_{(\xi,Y)}(2\theta) \quad (\theta \in \mathbb{R}_+),$$

where we set $\psi^{-1}_{(\xi,Y)}(t) = \sup\{\tau \in \mathbb{R}_+ : \psi_{(\xi,Y)}(\tau) \leq t\}$.
Note that $\Psi_{(\xi,Y)}$ is inreasing and vanishes at $\theta = 0$.

Theorem:
Let X be compact and assume that the function G satisfies

$$|G(x,u) - G(x,\tilde{u})| \le L_G \|u - \tilde{u}\|$$

for all $x \in D$, $u, \tilde{u} \in \Xi$ and some constant $L_G > 0$. Assume that udc is satisfied at (ξ, Y).
Then there exist positive constants L and δ such that

$$\begin{aligned} |v(\xi,Y) - v(\tilde{\xi},\tilde{Y})| &\le L\, d_2((\xi,Y),(\tilde{\xi},\tilde{Y})) \\ \mathbb{D}(S(\tilde{\xi},\tilde{Y}), S(\xi,Y)) &\le \Psi_{(\xi,Y)}(L\, d_2((\xi,Y),(\tilde{\xi},\tilde{Y}))) \end{aligned}$$

whenever $d_2((\xi,Y),(\tilde{\xi},\tilde{Y})) < \delta$.

The metric d_2 is defined by

$$d_2((\xi,Y),(\tilde{\xi},\tilde{Y})) = \ell_1(\xi,\tilde{\xi}) + \sup_{t\in\mathbb{R}} |F_Y^{(2)}(t) - F_{\tilde{Y}}^{(2)}(t)|$$

with the L_1-minimal metric ℓ_1 and $F_Y^{(2)}(t) = \int_{-\infty}^{t} F_Y(x)dx$ ($t \in \mathbb{R}$).

Conclusions

- The stability analysis of SGEs allows to extend the stability theory to more general stochastic variational problems.
- In particular, quantitative stability results for two-stage SPCCs and programs with stochastic dominance constraints were obtained.
- A characterization of the distances $\mathscr{D}$ and the function classes $\mathscr{F}$ might improve the understanding of scenario generation for such models.

Thank you !

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