O&D Revenue Management: A Multistage Stochastic Programming Approach

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Mathematical Model

O&D Revenue Management has become standard in the airline industry. The entire airline network is considered because costumers often require multiple flights.

Literature: K. Talluri, G. van Ryzin: Revenue Management, Kluwer 2004.

We present an optimization model for O&D RM that

- models the dynamic booking control process consisting of recursive decisions and observations,
- incorporates the stochastic nature of the passenger behaviour,
- determines protection levels for all origin destination itineraries, fare classes, points of sale and data collection points (dcp's),
- represents a multi-stage stochastic program where its stages correspond to the dcp's of the booking horizon,
- reduces to a specially structured large scale MILP if the stochastic demand and cancellations processes are represented by a scenario tree.

Notation:

Input data

π^s - probability of scenario s

stochastic

$d^s_{i,j,k,t}$ - passenger demands

$\gamma^s_{i,j,k,t}$ - cancellation rates

deterministic

$f_{i,j,k,t}$ - fares

$C_{l,m}$ - leg capacities

Variables

$P^s_{i,j,k,t}$ - protection levels

$B^s_{i,j,k,t}$ - cumulative bookings

$b^s_{i,j,k,t}$ - bookings

$C^s_{i,j,k,t}$ - cumulative cancelations

$c^s_{i,j,k,t}$ - cancelations

$z^{b,s}_{i,j,k,t}$, $z^{P,s}_{i,j,k,t}$ - slack variables

$\tilde{z}^s_{i,j,k,t}$ - binary variables

For node variables superscript n is used instead of s.

Indices

$s = 1, \ldots, S$ - scenarios

$t = 0, \ldots, T$ - data collection points (dcp's)

$i = 1, \ldots, I$ - Origin-Destination-Itineraries

$j = 1, \ldots, J$ - fare classes

$k = 1, \ldots, K$ - points of sale

$l = 1, \ldots, L$ - legs

$\mathcal{I}_l \subset \{1, \ldots, I\}$ - index set of itineraries containing leg l

$m = 1, \ldots, M(l)$ - compartments on leg l

$\mathcal{J}_m(l) \subset \{1, \ldots, J\}$ - index set of fare classes of compartment m on leg l

$n = 0, \ldots, N$ - nodes

Stochastic Optimization Model

Objective

$$\max_{(P^s_{i,j,k,t})} \left\{ \sum_{s=1}^{S} \pi^s \sum_{t=1}^{T} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} f_{i,j,k,t} \left[b^s_{i,j,k,t} - c^s_{i,j,k,t} \right] \right\}$$

Constraints

Cumulative bookings

$$B^s_{i,j,k,0} := \bar{B}^0_{i,j,k}; \qquad B^s_{i,j,k,t} := B^s_{i,j,k,t-1} + b^s_{i,j,k,t}$$

Cumulative cancelations ($\vartheta \in (0, 0.5])$

$$\gamma^s_{i,j,k,t} B^s_{i,j,k,t} - \vartheta \le C^s_{i,j,k,t} < \gamma^s_{i,j,k,t} B^s_{i,j,k,t} + \vartheta$$

Cancelations

$$c^s_{i,j,k,t} = C^s_{i,j,k,t} - C^s_{i,j,k,t-1}$$

Passenger demands

$$b^s_{i,j,k,t} + z^{b,s}_{i,j,k,t} = d^s_{i,j,k,t}$$

Protection levels

$$B^s_{i,j,k,t} - C^s_{i,j,k,t} + z^{P,s}_{i,j,k,t} = P^s_{i,j,k,t-1}$$

Number of bookings (disjunctive constraints) ($\omega > 0$, adequately large)

$$0 \le z^{b,s}_{i,j,k,t} \le (1 - \tilde{z}^s_{i,j,k,t}) d^s_{i,j,k,t} \quad 0 \le z^{P,s}_{i,j,k,t} \le \tilde{z}^s_{i,j,k,t} \omega \quad \tilde{z}^s_{i,j,k,t} \in \{0, 1\}$$

Leg capacity limits

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^{K} P^s_{i,j,k,T-1} \le C_{l,m}$$

Integrality and nonnegativity constraints

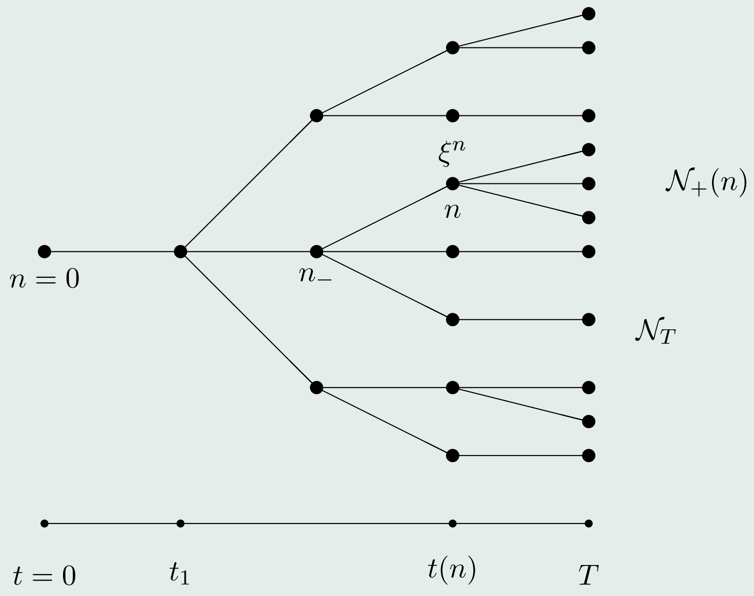
$$B^s_{i,j,k,t}, C^s_{i,j,k,t}, P^s_{i,j,k,t} \in \mathbb{Z}; \qquad b^s_{i,j,k,t} \ge 0; \qquad c^s_{i,j,k,t} \ge 0$$

Nonanticipativity constraints

the decisions at t only depend on the demand until t

Demand process approximation by scenario trees

The demand process $\{d_t\}_{t=0}^T$ is approximated by a process forming a scenario tree which is based on a finite set $\mathcal{N} = \{0, 1, \ldots, N\}$ of nodes.



Scenario tree with $t_1 = 1$, $T = 4$, $N = 22$ and 11 leaves

$n = 0$ root node, n_- unique predecessor of node n,
$\text{path}(n) = \{0, \ldots, n_-, n\}$, $t(n) := |\text{path}(n)| - 1$,
$\mathcal{N}_+(n)$ set of successors to n, $\mathcal{N}_T := \{n \in \mathcal{N} : \mathcal{N}_+(n) = \emptyset\}$ set of leaves,
$\text{path}(n)$, $n \in \mathcal{N}_T$, scenario with (given) probability π^n,
$\pi^n := \sum_{n_+ \in \mathcal{N}_+(n)} \pi^{n_+}$ probability of node n, ξ^n realization of $\xi_{t(n)}$.

Stochastic optimization model in node formulation

Objective

$$\max_{(P^n_{i,j,k})} \left\{ \sum_{n=1}^{N} \pi^n \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[f_{i,j,k,t(n)} b^n_{i,j,k} - f_{i,j,k,t(n)} c^n_{i,j,k} \right] \right\}$$

Constraints

Cumulative bookings

$$B^s_{i,j,k,0} := \bar{B}^0_{i,j,k}; \qquad B^n_{i,j,k} := B^{n-}_{i,j,k} + b^n_{i,j,k}$$

Cumulative cancelations ($\vartheta \in (0.0, 0.5]$)

$$\gamma^n_{i,j,k} B^n_{i,j,k} - \vartheta \leq C^n_{i,j,k} < \gamma^n_{i,j,k} B^n_{i,j,k} + \vartheta$$

Cancelations

$$c^n_{i,j,k} = C^n_{i,j,k} - C^{n-}_{i,j,k}$$

Passenger demands

$$b^n_{i,j,k} + z^{b,n}_{i,j,k} = d^n_{i,j,k}$$

Protection levels

$$B^n_{i,j,k} - C^n_{i,j,k} + z^{P,n}_{i,j,k} = P^{n-}_{i,j,k}$$

Number of bookings (disjunctive constraints) ($\omega > 0$, adequately large)

$$0 \leq z^{b,n}_{i,j,k} \leq (1 - \tilde{z}^n_{i,j,k}) d^n_{i,j,k} \quad 0 \leq z^{P,n}_{i,j,k} \leq \tilde{z}^n_{i,j,k} \omega \quad \tilde{z}^n_{i,j,k} \in \{0, 1\}$$

Leg capacity limits

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^{K} P^n_{i,j,k} \leq C_{l,m} \qquad \forall n \in \mathcal{N}_{T-1}$$

Integrality and nonnegativity constraints

$$B^n_{i,j,k}, C^n_{i,j,k}, P^n_{i,j,k} \in \mathbb{Z}; \qquad b^n_{i,j,k} \geq 0; \qquad c^n_{i,j,k} \geq 0$$

Nonanticipativity constraints are satisfied by construction.

Generation of scenario trees

(i) Development of a stochastic model for the data process ξ (parametric [e.g. time series model], nonparametric [e.g. resampling]) and generation of simulation scenarios;

(ii) Construction of a scenario tree out of the stochastic model or of the simulation scenarios.

(1) Bound-based approximation methods, (Frauendorfer 96, Edirisinghe 99, Casey/Sen 02).

(2) Monte Carlo-based schemes (inside or outside decomposition methods) (e.g. Shapiro 00, 03, Higle/Rayco/Sen 01, Chiralaksanakul/Morton 03).

(3) the use of Quasi Monte Carlo integration quadratures (Koivu/Pennanen 03, Pennanen 03, 04).

(4) EVPI-based sampling schemes (inside decomposition schemes) (Consigli/Dempster 98).

(5) Moment-matching principle (Høyland/Wallace 01, Høyland/Kaut/Wallace 03).

(6) (Nearly) best approximations based on probability metrics (Pflug 01, Hochreiter/Pflug 02, Gröwe-Kuska/Heitsch/Römisch 01, 03).

Survey: Dupačová/Consigli/Wallace 00

A Distance of Probability Distributions

Let $\mathbb{P}$ denote the probability distribution of the stochastic process $d = \{d_t\}_{t=0}^T$ with d_t in $\mathbb{R}^d$, i.e., $\mathbb{P}$ has support in some $\Xi \subseteq \mathbb{R}^{dT} = \mathbb{R}^r$, and let $\tilde{\mathbb{P}}$ be the distribution of an approximation $\tilde{d}$ of d. The Kantorovich or transportation distance is of the form

$$\begin{aligned} \ell_1(\mathbb{P}, \tilde{\mathbb{P}}) &:= \inf \left\{ \int_{\mathbb{R}^r \times \mathbb{R}^r} \|\xi - \tilde{\xi}\| \eta(d\xi, d\tilde{\xi}) \mid \pi_1 \eta = \mathbb{P},\ \pi_2 \eta = \tilde{\mathbb{P}} \right\} \\ &= \mathbb{E}[\|d - \tilde{d}\|] \end{aligned}$$

on some probability space, where $\| \cdot \|$ is a norm in $\mathbb{R}^r$.

(Rachev 91, Rachev/Rüschendorf 98)

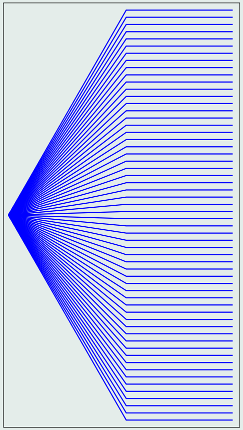
The case of finitely many scenarios:
$\mathbb{P}$: scenarios d^s with probabilities p^s, $s = 1, \ldots, S$,
$\tilde{\mathbb{P}}$: scenarios $\tilde{d}^\sigma$ with probabilities q^σ, $\sigma = 1, \ldots, \tilde{S}$.

$$\ell_1(\mathbb{P}, \mathbb{Q}) = \inf \left\{ \sum_{s,\sigma} \eta_{s\sigma} \|d^s - \tilde{d}^\sigma\| \mid \eta_{s\sigma} \geq 0, \sum_\sigma \eta_{s\sigma} = p^s, \sum_s \eta_{s\sigma} = q^\sigma \right\}$$

(*linear transportation problem*)

Constructing Scenario Trees

Let $\mathbb{P}$ be the probability distribution of a fan of (multivariate) data scenarios $\xi^s = (\xi_1^s, \ldots, \xi_T^s)$ with probabilities π^s, $s = 1, \ldots, S$, i.e., all scenarios coincide at $t = 1$, i.e., $\xi_1^1 = \ldots = \xi_1^S =: \xi_1^*$.



The fan may be regarded as a scenario tree with $1 + S(T - 1)$ nodes. We develop an algorithm that constructs new scenarios such that their t-th component belongs to the set $\{\xi_t^1, \ldots, \xi_t^S\}$. The algorithm is based on recursive scenario reduction on the time horizon $\{1, \ldots, t\}$ starting from $t = T$ and ending at $t = 1$. For the time horizon $\{1, \ldots, t\}$ we consider the norm $\|\xi\|_t := \|(\xi_1, \ldots, \xi_t, 0, \ldots, 0)\|$ and the corresponding Kantorovich distance $\ell_{1,t}$ based on the norm $\|\cdot\|_t$.

Algorithm: (**Recursive Scenario Reduction**)

Let ε, $\varepsilon_t > 0$, $t = 1, \ldots, T$, be such that $\sum_{t=1}^{T} \varepsilon_t \leq \varepsilon$.

Step 0: Determine $\mathbb{Q}_T$ with scenario index set $I_T \subset \{1, \ldots, N\}$ by optimal scenario reduction such that $\ell_{1,T}(\mathbb{Q}_T, \mathbb{P}) < \varepsilon_T$ and $\mathbb{Q}_T = \sum_{s \in I_T} \pi_T^s \delta_{\xi^s}$.

Step t: Determine $\mathbb{Q}_{T-t}$ with scenario index set $I_{T-t} \subset I_{T-t+1}$ by optimal scenario reduction such that $\ell_{1,t}(\mathbb{Q}_{T-t}, \mathbb{Q}_{T-t+1}) < \varepsilon_{T-t}$ and determine $j_{T-t}(i) \in \arg\min_{j \in I_{T-t}} \|\xi^i - \xi^j\|_t$ for $i \in I_{T-t+1} \setminus I_{T-t}$.

Step T: Construction of $\mathbb{P}_\varepsilon$: Determine the following mappings recursively $\alpha_t : I_T \to I_t$ for $t = T, \ldots, 1$ where $\alpha_T := id|_{I_T}$ and

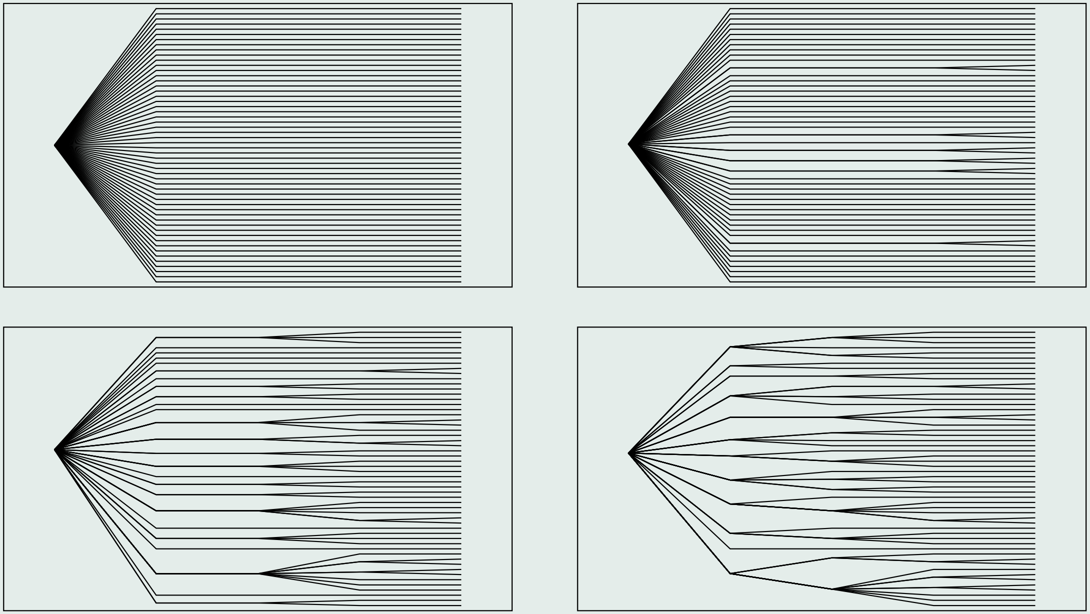
$$\alpha_t(i) := \begin{cases} j_t(\alpha_{t+1}(i)) & , \alpha_{t+1}(i) \in I_{t+1} \setminus I_t, \\ \alpha_{t+1}(i) & , \text{else} \end{cases} \qquad (t = T-1, \ldots, 1).$$

Determine scenarios $\hat{\xi}^s$ with $\hat{\xi}_t^s := \xi_t^{\alpha_t(s)}$ for $s \in I_T$ and set $\mathbb{P}_\varepsilon := \sum_{s \in I_T} \pi_T^s \delta_{\hat{\xi}^s}$.

(Dupačová/Gröwe-Kuska/Römisch 03, Gröwe-Kuska/Heitsch/Römisch 03)

Example:

Recursive construction of a bivariate load-price scenario tree starting with $N = 58$ scenarios (illustration, time period: 1 year)



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Approximation Results

Theorem: If $\mathbb{P}_\varepsilon$ is determined by the Algorithm starting with $\mathbb{P}$, we have

$$\ell_1(\mathbb{P}, \mathbb{P}_\varepsilon) < \varepsilon.$$

(Heitsch/Römisch 05)

Theorem: (Stability of multistage programs)
Let d be the demand process with probability distribution $\mathbb{P}$. Then there exists a constant $L > 0$ such that the estimate

$$|v(d) - v(\tilde{d})| \leq L[\mathbb{E}[\|d - \tilde{d}\|] + \sum_{t=2}^{T-1} D_t(d^t, \tilde{d}^t)]$$

holds for the optimal values of the original and approximate programs, respectively, where $\tilde{d}$ is the approximate demand process with distribution $\tilde{\mathbb{P}}$. Here,

$$D_t(d^t, \tilde{d}^t) := \max\{\mathbb{E}[\|x_t - \mathbb{E}[x_t|\tilde{d}_1, \dots, \tilde{d}_t]\|], \mathbb{E}[\|\tilde{x}_t - \mathbb{E}[\tilde{x}_t|d_1, \dots, d_t]\|]\},$$

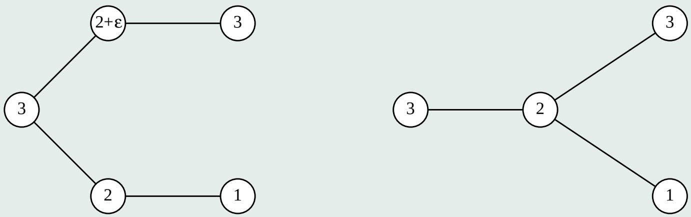
where x and $\tilde{x}$ are solutions of the original and approximate programs, respectively. D_t is called the information distance at t.
(Heitsch/Römisch/Strugarek 05)

Example: (Optimal purchase under uncertainty)
The decisions x_t correspond to the amounts to be purchased at each time period with uncertain prices are ξ_t, $t = 1, \ldots, T$, and such that a prescribed amount a is achieved at the end of a given time horizon. The problem is of the form

$$\min \left\{ \mathbb{E}\left[\sum_{t=1}^{T} \xi_t x_t\right] \;\middle|\; \begin{array}{l} (x_t, s_t) \in X_t = \mathbb{R}^2_+, \\ (x_t, s_t) \text{ is } (\xi_1, \ldots, \xi_t)\text{-measurable}, \\ s_t - s_{t-1} = x_t,\ t = 2, \ldots, T, \\ s_1 = 0,\ s_T = a. \end{array} \right\},$$

where the state variable s_t corresponds to the amount at time t.
Let $T := 3$ and $\mathbb{P}_\varepsilon$ denote the probability distribution of the stochastic price process having the two scenarios $\xi_\varepsilon^1 = (3, 2{+}\varepsilon, 3)$ ($\varepsilon \in (0, 1)$) and $\xi_\varepsilon^2 = (3, 2, 1)$ each endowed with probability $\frac{1}{2}$. Let $\tilde{\mathbb{P}}$ denote the approximation of $\mathbb{P}$ given by the two scenarios $\tilde{\xi}^1 = (3, 2, 3)$ and $\tilde{\xi}^2 = (3, 2, 1)$ with the same probabilities $\frac{1}{2}$.

Let the scenario trees of the processes ξ_ε and $\tilde{\xi}$ be of the form



Scenario trees for $\mathbb{P}_\varepsilon$ (left) and $\tilde{\mathbb{P}}$

We obtain

$$v(\xi_\varepsilon) = \frac{3+\varepsilon}{2}a \quad \text{and} \quad v(\tilde{\xi}) = 2a\,, \text{ but} \quad \hat{\mu}_1(P_\varepsilon, Q) = \frac{\varepsilon}{2}.$$

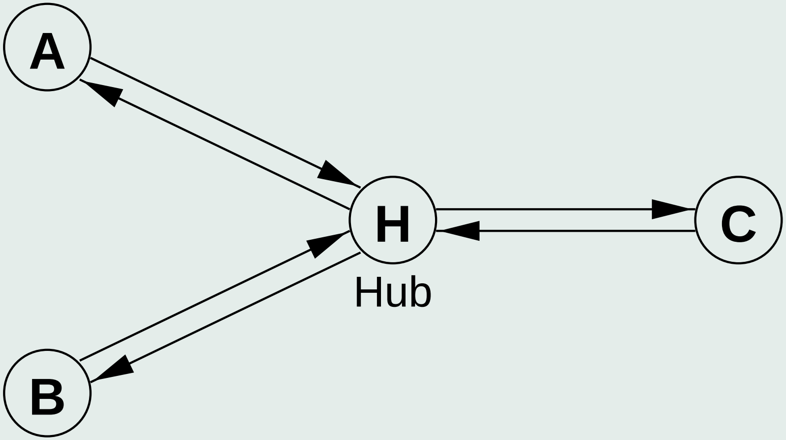
Hence, the multistage stochastic purchasing model is not stable with respect to the Kantorovich distance ℓ_1.

However, the estimate for $|v(\xi) - v(\tilde{\xi})|$ in the previous Theorem is valid with $L = a$ since $D_2(\xi^2, \tilde{\xi}^2) = 1$.

O&D Example and Demand Tree

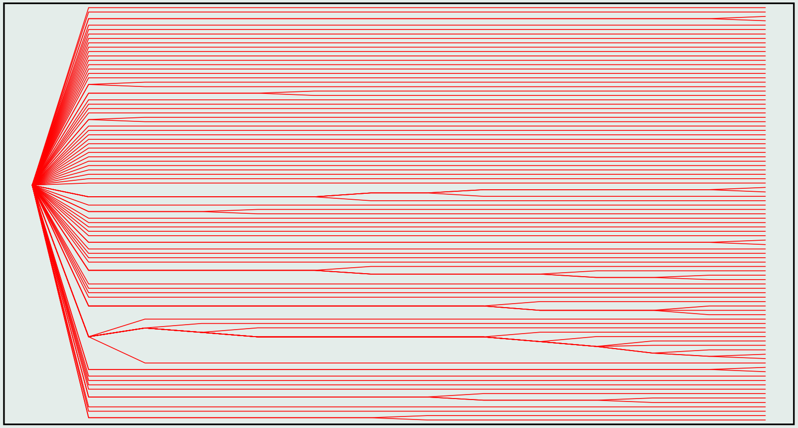
Hub-and-Spokes Network

#Legs	6
#ODIs	12
#Compartments	2
#Fare Classes	6
#POS	1
#DCPs	14



Tree and Size

#Scenarios	95
#Nodes	1022
#cont. Variables	434304
#bin. Variables	73512
#Constraints	442212



Numerical Results

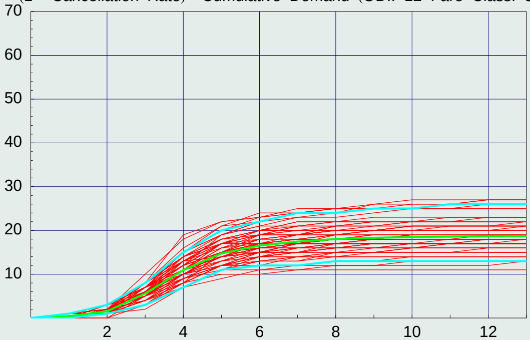
CPLEX-Results

Version	8.1
MIP Gap	0.001
Solution Status	Optimal
#MIP Nodes passed	0 (root)

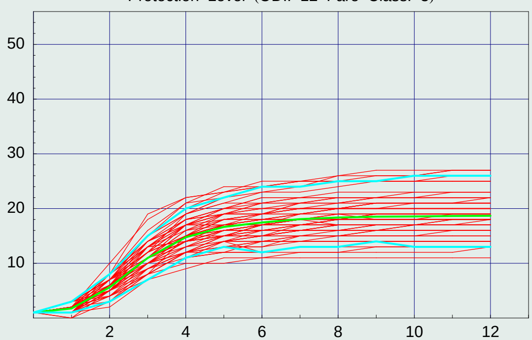
Computing times

Total	41.920 s
→ Solving Problem (CPLEX)	36.81 s
→→ CPLEX Presolve Time	13.32 s
(Intel Celeron, 2.0 GHz, Linux)	

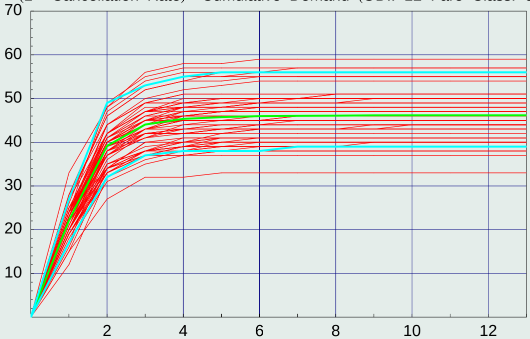
(1 − Cancellation Rate) ∗ Cumulative Demand (ODI: 12 Fare Class: 3)



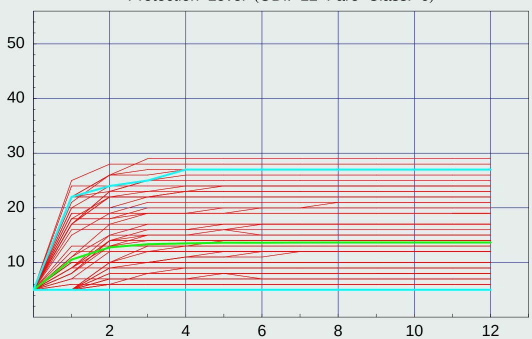
Protection Level (ODI: 12 Fare Class: 3)



(1 − Cancellation Rate) ∗ Cumulative Demand (ODI: 12 Fare Class: 6)



Protection Level (ODI: 12 Fare Class: 6)



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Conclusions and Future Work

We presented an approach to O&D Revenue Management using a scenario tree-based dynamic stochastic optimization model. The approach

- starts from a finite number of demand scenarios and probabilities,
- requires no assumptions on the demand distributions except their decision-independence.

Stochastic programming approaches lead to solutions that are more robust with respect to perturbations of input data. However, the models have higher complexity.

Future work:

- Analysis of O&D data and setting up suitable demand models (essentially done by Lufthansa Systems)
- Generation of large scale scenario trees
- Implementation of an itinerary-based decomposition scheme
- Numerical comparison with other approaches

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