TOPICS IN TOPOLOGY ("TOPOLOGIE III'), SOMMERSEMESTER 2024, HU BERLIN

CHRIS WENDL

At the beginning of the semester, I wrote in this spot that these are not a set of lecture notes, but merely a brief summary of the contents of each lecture, with reading suggestions and a compendium of exercises. But who am I kidding: these are lecture notes! They are not very polished, and the format is probably a bit annoying, but from at least week 3 onward they contain almost all of the material that was covered in the lectures, and some that wasn't.

Prologue: notation

Before getting into the content of the course, here is a glossary of important notation that is used in the lectures, including some comparison with other sources such as [tD08, DK01, Wen23] where different notation is sometimes used. This glossary will be updated during the semester as needed, and it is not in alphabetical order, but there is some kind of ordering principle... maybe you can figure out what it is.[1]

Categories.

- General shorthand: For any category $\mathscr{C}$, I often abuse notation by writing $X \in \mathscr{C}$ to mean "X is an **object** in $\mathscr{C}$"; many other authors denote this by "$X \in \operatorname{Ob}(\mathscr{C})$" or something similar. For two objects $X, Y \in \mathscr{C}$, I write

  $$\operatorname{Hom}_{\mathscr{C}}(X, Y) \qquad \text{or sometimes just} \qquad \operatorname{Hom}(X, Y)$$

  for the set of **morphisms** $X \to Y$. The notation $\operatorname{Mor}(X, Y)$ is also frequently used in many sources, and would make more sense linguistically, but it seems to be less popular. Given two functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$, the notation

  $$T : \mathcal{F} \to \mathcal{G}$$

  means that T is a **natural transformation** from $\mathcal{F}$ to $\mathcal{G}$.
- Top: the category of topological spaces and continuous maps
- Top_*: the category of **pointed spaces** and **pointed maps**, i.e. an object (X, x) is a topological space X equipped with a base point $x \in X$, and morphisms $f : (X, x) \to (Y, y)$ are continuous maps $X \to Y$ that send x to y. This notation is common but not universal, e.g. [tD08] uses a superscript 0 to indicate base points, so Top_* is called TOP^0.
- Set: the category of sets and maps (with no continuity requirement)
- Set_*: the category of **pointed sets** and (not necessarily continuous) **pointed maps**, i.e. an object (X, x) is a set X with a base point $x \in X$, and morphisms $f : (X, x) \to (Y, y)$ are arbitrary maps $X \to Y$ that send x to y.
- $\mathsf{Top}^{\mathrm{rel}}$: the category of **pairs of spaces** (X, A) and **maps of pairs**, i.e. an object (X, A) is a topological space X equipped with a subset $A \subset X$, and morphisms $f : (X, A) \to (Y, B)$ are continuous maps $X \to Y$ that send A into B. Despite the uniquity of this category, there doesn't seem to be any common standard notation for it; [tD08] calls it $\mathsf{TOP}(2)$, and

[1] The notational glossary didn't really get updated much after the first half of the semester, because I lost interest.

similarly writes $\mathsf{TOP}(3)$ for the category of **triples** (X, A, B) with $B \subset A \subset X$, and so forth. In [Wen23] I used a subscript instead of a superscript, but I'm changing it so that I can also define the next item on this list.

- $\mathsf{Top}_*^{\mathrm{rel}}$: the category of **pointed pairs of spaces**, i.e. an object (X, A, x) is a topological space X equipped with a subset $A \subset X$ and a base point $x \in A$, and morphisms $f : (X, A, x) \to (Y, B, y)$ are maps of pairs $(X, A) \to (Y, B)$ that also send x to y. I have no idea what anyone else calls this, but it's a subcategory of what [tD08] calls $\mathsf{TOP}(3)$, and is in any case clearly important since e.g. it is the domain of the relative homotopy functors π_n.
- hTop, hTop_*, $\mathsf{hTop}^{\mathrm{rel}}$, $\mathsf{hTop}_*^{\mathrm{rel}}$: the **homotopy categories** associated to Top, Top_*, $\mathsf{Top}^{\mathrm{rel}}$ and $\mathsf{Top}_*^{\mathrm{rel}}$ respectively, meaning we define categories with the same objects, but instead of taking morphisms to be actual maps, we define them to be *homotopy classes* of maps (respecting subsets and/or base points where appropriate, so e.g. **pointed homotopy** for hTop_*, and homotopy of maps of pairs for $\mathsf{hTop}^{\mathrm{rel}}$). This notation (or similar) for homotopy categories is very common, but different from my Topology I–II notes [Wen23], which wrote e.g. Top_*^h instead of hTop_*.
- Diff: the category of smooth finite-dimensional **manifolds** without boundary, and **smooth maps**
- Grp: the category of **groups** and group homomorphisms
- Ab: the category of **abelian groups** and homomorphisms, which is a subcategory of Grp
- $\mathsf{Ring} \supset \mathsf{CRing} \supset \mathsf{Fld}$: the category of **rings with unit** and its subcategories of **commutative rings** and **fields** respectively, with ring homomorphisms (preserving the unit)
- R-Mod: the category of **modules** over a given commutative ring R and **R-module homomorphisms**. In [Wen23] I called this Mod^R, and other variations such as Mod-R are also common.
- $\mathbb{K}$-Vect: the category of **vector spaces** over a given field $\mathbb{K}$ and **$\mathbb{K}$-linear maps**, i.e. this is R-Mod in the special case where R is a field $\mathbb{K}$. In [Wen23] I called this $\mathsf{Vec}_{\mathbb{K}}$.
- Categories of **(co-)chain complexes**: given any *additive* category $\mathscr{A}$ such as Ab or R-Mod,
$$\mathsf{Ch}(\mathscr{A}) \qquad \text{or sometimes simply} \qquad \mathsf{Ch}$$
denotes the category of chain complexes $\ldots \to A_{n+1} \to A_n \to A_{n-1} \to \ldots$ formed out of objects and morphisms in $\mathscr{A}$, with the morphisms of $\mathsf{Ch}(\mathscr{A})$ defined to be **chain maps**. There is a similar category $\mathsf{CoCh}(\mathscr{A})$ of cochain complexes $\ldots \to A_{n-1} \to A_n \to A_{n+1} \to \ldots$, though I am not really happy with this notation and I doubt that anyone else is either. In [Wen23] I denoted $\mathsf{Ch}(\mathsf{Ab})$, $\mathsf{CoCh}(\mathsf{Ab})$, $\mathsf{Ch}(R\text{-}\mathsf{Mod})$ and $\mathsf{CoCh}(R\text{-}\mathsf{Mod})$ by Chain, $\mathsf{Cochain}$, Chain^R and $\mathsf{Cochain}^R$ respectively. One sometimes sees a meaningless subscript such as $\mathsf{Ch}_\bullet(\mathscr{A})$ added, but there are also meaningful subscripts that define important subcategories such as e.g. $\mathsf{Ch}_{\geqslant 0}(\mathscr{A})$, the chain complexes that are trivial in all negative degrees.
- **Homotopy categories of chain complexes**: analogously to the homotopy categories of spaces, one can take the objects in $\mathsf{Ch}(\mathscr{A})$ and define morphisms to be chain homotopy classes of chain maps instead of actual chain maps. The internet seems quite insistent that I should call the resulting category
$$\mathsf{K}(\mathscr{A}) := \text{the (naive) homotopy category associated to } \mathsf{Ch}(\mathscr{A}),$$
even though I'd rather call it $\mathsf{hCh}(\mathscr{A})$, and in [Wen23] I wrote e.g. Chain^h instead of $\mathsf{K}(\mathsf{Ab})$; on occasion I have even seen $\mathsf{Ho}(\mathscr{A})$ in place of $\mathsf{K}(\mathscr{A})$. I have no idea what notation to use for the homotopy category of cochain complexes. People who like derived categories will tell you that there are other things more deserving of the name "homotopy category

of chain complexes," and I added the word "naive" above in order to avoid getting into conversations about it with those people, which would be completely unnecessary for the purposes of the present course.

- Top_B, $\mathsf{Top}_{B,*}$, hTop_B and $\mathsf{hTop}_{B,*}$: Given a space B, these are the various categories of (unpointed or pointed) **spaces over** B with **maps over** B or homotopy classes thereof, as defined in Week 4, Lecture 6. The notation used in [tD08] for Top_B and hTop_B is not identical but sufficiently similar; I cannot find definitions in [tD08] for the pointed variants $\mathsf{Top}_{B,*}$ and $\mathsf{hTop}_{B,*}$.

Topological constructions.

- $X \amalg Y$: This is how I write the **disjoint union** of two topological spaces (and similarly for pairs of spaces), and most sensible people use either this notation or $X \sqcup Y$, but [tD08] instead writes $X + Y$ and calls it the **topological sum** of X and Y, presumably because—like the direct sum of abelian groups and many other constructions that use the word "sum"—it is a coproduct. The book by tom Tieck becomes significantly easier to read once you realize this.
- $X \coprod Y$: the **coproduct** of X and Y, whatever that means in whichever category X and Y happen to live in, so e.g. in Top, it means the same thing as $X \amalg Y$, though in Top_* it means $X \vee Y$.
- $[X, Y]$: If X and Y are just topological spaces (i.e. objects in Top), then this denotes the **set of homotopy classes** of maps $X \to Y$, i.e.

$$[X, Y] := \mathrm{Hom}_{\mathsf{hTop}}(X, Y).$$

If X and Y are equipped with additional data (which may be suppressed in the notation) and are thus objects in Top_*, $\mathsf{Top}^{\mathrm{rel}}$ or $\mathsf{Top}_*^{\mathrm{rel}}$, then I use the same notation $[X, Y]$ to mean the corresponding notion of homotopy classes in each category, so e.g. in the context of pointed spaces, I would write

$$[X, Y] := \mathrm{Hom}_{\mathsf{hTop}_*}(X, Y),$$

and similarly for (pointed or unpointed) pairs of spaces. This convention is popular but not universal, e.g. [tD08] writes $[X, Y]^0$ for the set of pointed homotopy classes and uses $[X, Y]$ only to mean unpointed homotopy classes; [DK01] does the same but writes $[X, Y]_0$ instead of $[X, Y]^0$.

- $X \vee Y$ and $X \wedge Y$: these are the **wedge sum** and **smash product** respectively of pointed spaces, and mercifully, everyone seems to agree on what they mean and how to write them.
- **Implied base points**: for a pair of spaces (X, A), the **quotient space** X/A is often interpreted as a pointed space, with the collapsed subset A as base point. Similarly, for two pointed spaces X, Y, the set of **pointed homotopy classes** $[X, Y]$ is viewed as a pointed set (i.e. an object in Set_*) whose base point is the homotopy class of the constant map to the base point of Y.
- **One-point spaces**: the symbol $*$ is often used to mean either a one-point space, the unique point in that space, or sometimes a previously unnamed base point of a given pointed space. It should usually be clear from context which is meant.
- I: this usually denotes the **unit interval**

$$I := [0, 1],$$

as appears in domains of paths, homotopies etc.

- **Homotopy relations**: Given maps $f, g : X \to Y$, I write

$$f \underset{h}{\sim} g$$

to means that f and g are homotopic ([tD08] writes "$f \simeq g$"), and

$$f \overset{H}{\rightsquigarrow} g$$

to mean that H is a homotopy from f to g, thought of as a *path* in the space of maps, hence $H : I \times X \to Y$ with $H(0, \cdot) = f$ and $H(1, \cdot) = g$. This can also mean e.g. pointed homotopy or homotopy of maps of pairs if working in Top_* or $\mathsf{Top}^{\mathrm{rel}}$ respectively. Where I write $f \overset{H}{\rightsquigarrow} g$, [tD08] writes $H : f \simeq g$.

- **Homotopy commutative diagrams**: I use a diagram of the form

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\sim"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & Q \end{tikzcd}$$

  to mean that $\varphi \circ f$ and $\psi \circ g$ need not be identical but are homotopic, whatever that means in whichever category the objects of the diagram live in, e.g. if they are pointed spaces it means pointed homotopic, for spaces without base points it just means homotopic—it may also mean *chain* homotopic if the objects are chain complexes. If I write the variant

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\underset{\alpha}{\sim}"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & Q \end{tikzcd},$$

  then it means that α is a homotopy (or chain homotopy as the case may be) from $\varphi \circ f$ to $\psi \circ g$. It wasn't easy to figure out how to render this in LaTeX, so maybe that's why most textbooks don't do it.
- $Z(f)$, $Z(f, g)$, $\mathrm{cone}(f)$: **mapping cylinders**, **double mapping cylinders** and **mapping cones** (see Week 2, Lecture 3)
- CX, ΣX: the **cone** and **suspension** respectively of a space X. In the context of pointed spaces the same notation may instead mean the *reduced* cone/suspension.
- $P(f)$, $P(f, g)$, $F(f)$: the **mapping path space** of a map, **double mapping path space** of two maps, and **homotopy fiber** of a map respectively, as defined in Week 3, Lecture 5. These constructions are dual to $Z(f)$, $Z(f, g)$ and $\mathrm{cone}(f)$ respectively, in the sense that they fit into analogous diagrams with all arrows reversed.
- $[X, Y]_B$: For two spaces X, Y over another space B, this is the set of (unpointed or pointed) **homotopy classes of maps over** B, i.e. morphisms in the category hTop_B or $\mathsf{hTop}_{B,*}$.

1. Week 1

Lecture 1 (15.04.2024): Motivation and colimits.

- Motivational theorem on exotic spheres (Milnor 1956): There exists a smooth manifold Y that is homeomorphic but not diffeomorphic to S^7. (In fact, Kervaire and Milnor proved shortly afterwards that there are exactly 28 such manifolds up to diffeomorphism.)
- Outline of a proof (slightly ahistorical), with notions that will be major topics in this course written in red:
  1. Pontryagin classes: Associate topological invariants $p_k(E) \in H^{4k}(X; \mathbb{Z})$ for each $k \in \mathbb{N}$ to every isomorphism class of vector bundles E over a given space X. Since every smooth manifold M has a tangent bundle TM, we can define $p_k(M) := p_k(TM) \in H^{4k}(M; \mathbb{Z})$ as an invariant of smooth (but not topological) manifolds.

(2) Intersection form and signature: For a compact oriented $4k$-manifold M (possibly with boundary), the intersection form is the quadratic form Q_M on $H^{2k}(M, \partial M; \mathbb{Z})$ defined by

$$Q_M(\alpha) := \langle \alpha \cup \alpha, [M] \rangle \in \mathbb{Z},$$

and it's called the "intersection form" because it can be interpreted as a signed count of intersections between two generic closed oriented submanifolds representing the class in $H_{2k}(M; \mathbb{Z})$ Poincaré dual to α. The signature $\sigma(M) \in \mathbb{Z}$ is essentially the number of positive eigenvalues minus the number of negative eigenvalues[2] of this quadratic form.

(3) Hirzebruch signature theorem (8-dimensional case): For M a closed oriented 8-manifold,

$$\sigma(M) = \frac{1}{45} \langle 7p_2(M) - p_1(M) \cup p_1(M), [M] \rangle.$$

(4) (the clever bit) Construct a compact oriented smooth 8-manifold X with simply connected boundary $Y := \partial X$ such that $\sigma(X) = 8$, $H_2(Y)$ and $H_3(Y)$ both vanish, and the tangent bundle TX is stably trivial, which implies its Pontryagin classes vanish. The construction can be described (key words: "plumbing of spheres"), and the computations carried out, using only methods from Topology 2.

(5) Deduce via Poincaré duality, the Hurewicz theorem and Whitehead's theorem[3] that Y is homotopy equivalent to S^7. By Smale's solution to the higher-dimensional Poincaré conjecture,[4] it follows that Y is homeomorphic to S^7.

(6) Argue by contradiction: If Y is diffeomorphic to S^7, then one can construct a closed smooth 8-manifold M by gluing X to an 8-disk along a diffeomorphism $\partial X = Y \cong S^7 = \partial \mathbb{D}^8$,

$$M := X \cup_{S^7} \mathbb{D}^8.$$

Methods from Topology 2 (e.g. Mayer-Vietoris) now imply $p_1(M) = 0$ and $\sigma(M) = 8$, so Hirzebruch says

$$45\sigma(M) = 45 \cdot 8 = 7\langle p_2(M), [M] \rangle.$$

But the right hand side of this relation is a multiple of 7, and the left hand side is not.

- Interpretation of a functor $\mathcal{F} : \mathscr{J} \to \mathscr{C}$ as a **diagram** in $\mathscr{C}$ over $\mathscr{J}$, constant functors $\mathcal{X} : \mathscr{J} \to \mathscr{C}$ as **targets**, the **universal property** and definition of the **colimit** $\operatorname{colim}(\mathcal{F})$
- Interpreting direct systems as diagrams and direct limits as colimits
- Defining the quotient space X/A as colimit of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & * \\ X & \end{tikzcd}$$

understood as a functor $\mathscr{J} \to \mathsf{Top}$, where $\mathscr{J}$ is a category with three objects and only two nontrivial morphisms.

[2] What I really mean is: first rewrite Q_M as a quadratic form on $H^{2k}(M, \partial M; \mathbb{Q})$ or $H^{2k}(M, \partial M; \mathbb{R})$, which is a vector space, so that by standard linear algebra, you can present it in terms of a symmetric linear transformation and look at the eigenvalues of that transformation. One can define this in a more obviously invariant way by talking about maximal subspaces on which Q_M is positive/negative definite.

[3] A 3-dimensional version of this same argument is described in [Wen23, Lecture 57], using the theorems of Hurewicz and Whitehead as black boxes.

[4] This is the one major black box in this proof that I do not intend to fill in, because that would be a whole course in itself.

Lecture 2 (18.04.2024): From coproducts to pullbacks and pushouts.

- The **limit** $\lim(\mathcal{F})$ of a diagram $\mathcal{F} : \mathscr{I} \to \mathscr{C}$
- Inverse limits as limits of diagrams
- Important special cases of limits and colimits:
  - **Coproducts** $\coprod$, and examples in the categories Top (disjoint union), Top_* (wedge sum), Ab (direct sum) and Grp (free product)
  - **Products** $\times$ (or $\prod$), and examples in Top
  - **Equalizers** and **co-equalizers**, realization in Top as subspaces or quotient spaces respectively
- Word of caution: limits and colimits are not guaranteed to exist, e.g. in the category Diff of smooth finite-dimensional manifolds without boundary, finite or countable coproducts exist (and are the same thing as in Top), but uncountable disjoint unions are not second countable and are thus not objects in Diff. Similarly, finite products exist in Diff but infinite products typically do not.
- Theorem: In any category $\mathscr{C}$, all (co-)limits can be presented in terms of (co-)products and (co-)equalizers, if they exist.
- Proof sketch (co-limit case): Given $\mathcal{F} : \mathscr{I} \to \mathscr{C} : \alpha \mapsto X_\alpha$, construct $\operatorname{colim}(\mathcal{F})$ as the equalizer of two morphisms $Y \xrightarrow{f,g} Z$ defined as follows. Write the set of all morphisms in $\mathscr{I}$ as $\operatorname{Hom}(\mathscr{I}, \mathscr{I})$; we then take Y to be the coproduct

  $$Y := \coprod_{\phi \in \operatorname{Hom}(\mathscr{I},\mathscr{I})} X_\phi, \qquad \text{where} \qquad \text{for } \phi \in \operatorname{Hom}(\alpha,\beta),\ X_\phi := X_\alpha,$$

  while Z is the slightly simpler coproduct

  $$Z := \coprod_{\beta \in \mathscr{I}} X_\beta.$$

  For each $\alpha, \beta \in \mathscr{I}$ and $\phi \in \operatorname{Hom}(\alpha,\beta)$, let $f_\phi : X_\phi \to Z$ denote the composition of the morphism $\phi_* : X_\phi = X_\alpha \to X_\beta$ with the canonical morphism $X_\beta \to \coprod_{\gamma \in \mathscr{I}} X_\gamma$ of the coproduct; the universal property of the coproduct then dictates that the collection of morphisms $f_\phi : X_\phi \to Z$ determines a morphism $f : Y \to Z$. Similarly, $g : Y \to Z$ is determined by the collection of morphisms $g_\phi : X_\phi \to Z$ defined for each $\phi \in \operatorname{Hom}(\alpha,\beta)$ as the compositions of $\operatorname{Id}_{X_\alpha} : X_\phi = X_\alpha \to X_\alpha$ with the canonical morphism $X_\alpha \to \coprod_{\gamma \in \mathscr{I}} X_\gamma$. Now check that the universal property is satisfied (exercise).
- Upshot: In Top, colimits are quotients of disjoint unions, limits are subspaces of products.
- **Fiber products**: presenting the fiber product of two maps $f : X \to Z$ and $g : Y \to Z$ in Top as the "intersection locus"

  $$X\ {}_f\!\times_g Y := \left\{ (x,y) \in X \times Y \;\middle|\; f(x) = g(y) \right\}$$

  with the obvious projections to X and Y.
- Interpreting fiber products as **pullbacks**
- **Pushouts**: presenting the pushout of two maps $f : Z \to X$ and $g : Z \to Y$ in Top as "gluing spaces together" along a map:

  $$X\ {}_f\!\cup_g Y := (X \amalg Y) \Big/ f(z) \sim g(z) \text{ for all } z \in Z.$$

- Question for thought: How many of these constructions of limits or colimits work in the homotopy categories hTop or hTop_*? (Hint: Do not try too hard to make sense of equalizers and co-equalizers.)

Suggested reading. The main definitions involving direct systems and direct limits can all be found in [Wen23, Lecture 39], with the generalization to colimits explained in Exercise 39.24. If you're really serious about this stuff, you can also try reading [Mac71].

If you want to read more about exotic spheres, there's a nice collection of relevant literature assembled at https://www.maths.ed.ac.uk/~v1ranick/exotic.htm.

Exercises (for the Übung on 25.04.2024). Since the Übung on 25.04 was cancelled due to illness, most of the exercises for Week 1 have now been supplemented with written answers and/or some discussion.

Exercise 1.1. In what sense precisely are the limit and colimit of a diagram $\mathcal{F} : \mathscr{J} \to \mathscr{C}$ unique, if they exist?

Answer: If the limit or colimit exists (of which there is no guarantee, cf. Exercise 1.7), then it is unique up to canonical isomorphisms. Precisely: Suppose $X, Y \in \mathscr{C}$ are two objects, together with collections of morphisms $\mathcal{F}(\alpha) \xrightarrow{\varphi_\alpha} X$ and $\mathcal{F}(\alpha) \xrightarrow{\psi_\alpha} Y$ for all $\alpha \in \mathscr{J}$, such that both satisfy the universal property for $\operatorname{colim}(\mathcal{F})$. Then there is a uniquely determined isomorphism

$$f : X \xrightarrow{\cong} Y \qquad \text{such that} \qquad \psi_\alpha = f \circ \varphi_\alpha \text{ for all } \alpha \in \mathscr{J}.$$

The existence and uniqueness of a morphism f satisfying this condition follows from the universal property of X, and the fact that it is an isomorphism follows by reversing the roles of X and Y, since Y also satisfies the universal property. For $\lim(\mathcal{F})$ there is a similar uniqueness statement, proved in a similar way.

Note that in most categories, uniqueness "up to canonical isomorphisms" is the best that one could hope to get from universal properties, as one will always have the freedom to replace a given object playing the role of $\operatorname{colim}(\mathcal{F})$ or $\lim(\mathcal{F})$ with a different object that is isomorphic to it. In practice, our favorite categories often come with canonical constructions that lead to specific objects, e.g. the disjoint union (also known as the coproduct) of a given collection of topological spaces is a specific space, not just an equivalence class of spaces up to homeomorphism. But in various situations, limits or colimits can also arise from something other than the canonical construction, and finding an isomorphism with that canonical construction may be harder than explicitly verifying the universal property.

Exercise 1.2 (morphisms between (co-)products)**.** Assume J is a set, and $\{X_\alpha\}_{\alpha\in J}$ and $\{Y_\alpha\}_{\alpha\in J}$ are collections of objects in some category $\mathscr{C}$ such that the products

$$\left\{\prod_{\alpha\in J} X_\alpha \xrightarrow{\pi^X_\beta} X_\beta\right\}_{\beta\in J}, \qquad \left\{\prod_{\alpha\in J} Y_\alpha \xrightarrow{\pi^Y_\beta} Y_\beta\right\}_{\beta\in J},$$

and coproducts

$$\left\{X_\beta \xrightarrow{i^X_\beta} \coprod_{\alpha\in J} X_\alpha\right\}_{\beta\in J}, \qquad \left\{Y_\beta \xrightarrow{i^Y_\beta} \coprod_{\alpha\in J} Y_\alpha\right\}_{\beta\in J}$$

exist. In what sense does an arbitrary collection of morphisms $\{f_\alpha : X_\alpha \to Y_\alpha\}_{\alpha\in J}$ uniquely determine morphisms

$$\coprod_{\alpha\in J} f_\alpha : \coprod_{\alpha\in J} X_\alpha \to \coprod_{\alpha\in J} Y_\alpha, \qquad \text{and} \qquad \prod_{\alpha\in J} f_\alpha : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha?$$

Argue in terms of universal properties, without using your knowledge of how to represent products and coproducts in any specific categories.

Answer: The morphisms $\coprod_\alpha f_\alpha$ and $\prod_\alpha f_\alpha$ are uniquely determined by the condition that the diagrams

$$\begin{tikzcd} X_\beta \arrow[r, "f_\beta"] \arrow[d, "i^X_\beta"] & Y_\beta \arrow[d, "i^Y_\beta"] \\ \coprod_\alpha X_\alpha \arrow[r, "\coprod_\alpha f_\alpha"] & \coprod_\alpha Y_\alpha \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \prod_\alpha X_\alpha \arrow[r, "\prod_\alpha f_\alpha"] \arrow[d, "\pi^X_\beta"] & \prod_\alpha Y_\alpha \arrow[d, "\pi^Y_\beta"] \\ X_\beta \arrow[r, "f_\beta"] & Y_\beta \end{tikzcd}$$

commute for every $\beta \in J$. One gets the existence and uniqueness of $\coprod_\alpha f_\alpha$ from the universal property of the coproduct $\coprod_\alpha X_\alpha$, because the morphisms $i^Y_\beta \circ f_\beta : X_\beta \to \coprod_\alpha Y_\alpha$ make $\coprod_\alpha Y_\alpha$ a target of the diagram whose colimit is $\coprod_\alpha X_\alpha$. Similarly, the existence and uniqueness of $\prod_\alpha f_\alpha$ follows from the universal property of the product $\prod_\alpha Y_\alpha$, using the collection morphisms $f_\beta \circ \pi^X_\beta : \prod_\alpha X_\alpha \to Y_\beta$.

Exercise 1.3 (finite limits and colimits)**.** Show that in any category $\mathscr{C}$, finite colimits always exist if and only if all pushouts exist and $\mathscr{C}$ has an initial object (see Exercise 1.5). Dually, finite limits always exist if and only if all pullbacks (also known as fiber products) exist and $\mathscr{C}$ has a terminal object.[5]
Hint: By a theorem stated in the lecture, it suffices if you can express arbitrary (co-)equalizers and finite (co-)products in terms of pushouts or pullbacks.

Solution: Note that the statement of this exercise has been revised; the original version had two errors, one being its failure to mention initial and terminal objects, and the other an oversimplification of what it means for a limit or colimit to be *finite*—we need the category $\mathscr{J}$ underlying the diagram to have finitely-many morphisms, not just finitely-many objects.

With that understood, let's assume all pushouts exist and that $\mathscr{C}$ also has an initial object $0 \in \mathscr{C}$. If we can show that all finite coproducts and all coqualizers exist, then the theorem from lecture uses these to construct a colimit for any diagram $\mathcal{F} : \mathscr{J} \to \mathscr{C}$ such that $\mathscr{J}$ has only finitely many objects and morphisms. (Regarding the errors in the original version: note that if $\mathscr{J}$ has finitely-many objects but infinitely-many morphisms, then one of the coproducts needed for the theorem from lecture is not finite.)

You should be able to convince yourself via an inductive argument that if the coproduct of two objects $X, Y \in \mathscr{C}$ always exists, then all finite coproducts exist. So let's show first that $X \coprod Y$ exists for arbitrary $X, Y \in \mathscr{C}$. At this point I find it helpful to think about how coproducts and pushouts are constructed concretely in the example $\mathscr{C} = \mathsf{Top}$: the coproduct of X and Y is their disjoint union, and the pushout of a pair of maps $f : Z \to X$ and $g : Z \to Y$ is a quotient of that disjoint union by the equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$. If we want to make that equivalence relation trivial so that the pushout turns out to be the same thing as the coproduct, the solution is to choose the empty set for Z; the maps f, g are uniquely determined by this choice, because the empty set is an initial object in Top (see Exercise 1.5). This suggests that in our given category $\mathscr{C}$ with initial object $0 \in \mathscr{C}$, the pushout of the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

should be the coproduct of X and Y; note that only one diagram of this form is possible since 0 being initial means that the morphisms $0 \to X$ and $0 \to Y$ are unique. Now suppose P is the

[5]The word "finite" in this context refers to limits or colimits of diagrams $\mathcal{F} : \mathscr{J} \to \mathscr{C}$ such that $\mathscr{J}$ has only finitely many objects and morphisms.

pushout of this diagram, equipped with morphisms $\varphi : X \to P$ and $\psi : Y \to P$, and suppose we are given another object Z with morphisms $f : X \to Z$ and $g : Y \to Z$. The diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

then trivially commutes, since there is only one morphism $0 \to Z$, and the universal property of the pushout gives rise to a unique morphism $u : P \to Z$ such that $f = u \circ \varphi$ and $g = u \circ \psi$, which amounts to the statement that P with its morphisms φ and ψ also satisfies the universal property of the coproduct $X \coprod Y$.

We show next that the coequalizer of an arbitrary pair of morphisms

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

in $\mathscr{C}$ can also be constructed as a pushout. Think again about how it works in the case $\mathscr{C} = \mathsf{Top}$: the coequalizer here is the quotient of Y by the equivalence relation such that $f(x) \sim g(x)$ for all $x \in X$. If we instead take the pushout of f and g, the resulting space is too large: it is a quotient of $Y \amalg Y$ instead of Y, meaning that we glue together *two* copies of Y by identifying $f(x)$ in one copy with $g(x)$ in the *other* copy for each $x \in X$. But the correct space can be obtained from this by making the equivalence relation larger, so that for every $y \in Y$, y in the first copy gets identified with y in the second copy. The way to realize this is by enlarging the domain of the pair of maps used in defining the pushout: instead of the two maps $f, g : X \to Y$, we consider the pushout of the two maps $f \amalg \mathrm{Id}, g \amalg \mathrm{Id} : X \amalg Y \to Y$.

Let's say that again without assuming $\mathscr{C} = \mathsf{Top}$. We've already shown that the coproduct $X \coprod Y$ of two objects in $\mathscr{C}$ can be constructed, and if we write $i_X : X \to X \coprod Y$ and $i_Y : Y \to X \coprod Y$ for the canonical morphisms that coproducts come equipped with, then by the universal property of the coproduct, every morphism $\varphi : X \to Y$ determines a unique morphism $\varphi \coprod \mathrm{Id} : X \coprod Y \to Y$ for which the diagram

$$\begin{tikzcd} X \arrow[d, "i_X"] \arrow[dr, bend left, "\varphi"] & \\ X \coprod Y \arrow[r, "\varphi \coprod \mathrm{Id}"] & Y \\ Y \arrow[u, "i_Y"] \arrow[ur, bend right, "\mathrm{Id}"'] & \end{tikzcd}$$

commutes. Claim: Given two morhisms $f, g : X \to Y$, a diagram of the form

$$\begin{tikzcd} X \coprod Y \arrow[r, "f \coprod \mathrm{Id}"] \arrow[d, "g \coprod \mathrm{Id}"] & Y \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & Z \end{tikzcd}$$

commutes if and only if $\varphi = \psi$ and $\varphi \circ f = \varphi \circ g$. To see this, we can enhance the diagram in two ways using the universal property of the coproduct: first,

$$\begin{tikzcd}
Y \arrow[dr, "i_Y"] \arrow[drr, bend left, "\mathrm{Id}"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\
& X \coprod Y \arrow[r, "f \coprod \mathrm{Id}"] \arrow[d, "g \coprod \mathrm{Id}"] & Y \arrow[d, "\varphi"] \\
& Y \arrow[r, "\psi"] & Z
\end{tikzcd}$$

shows that if the given diagram commutes, then $\varphi = \varphi \circ \mathrm{Id} = \psi \circ \mathrm{Id} = \psi$. Assuming this, the second enhanced diagram

$$\begin{tikzcd}
X \arrow[dr, "i_X"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \coprod Y \arrow[r, "f \coprod \mathrm{Id}"] \arrow[d, "g \coprod \mathrm{Id}"] & Y \arrow[d, "\varphi"] \\
& Y \arrow[r, "\psi"] & Z
\end{tikzcd}$$

then proves $\varphi \circ f = \psi \circ g = \varphi \circ g$. Conversely, if one assumes $\varphi = \psi$ and $\varphi \circ f = \varphi \circ g$, then $\varphi \circ (f \coprod \mathrm{Id})$ and $\psi \circ (g \coprod \mathrm{Id})$ are two morphisms $X \coprod Y \to Z$ whose compositions with i_X and i_Y are identical, so the uniqueness in the universal property of the coproduct requires them to be the same.

The result of the claim is that pushout diagrams for the two morphisms $f \coprod \mathrm{Id} : X \coprod Y \to Y$ and $g \coprod \mathrm{Id} : X \coprod Y \to Y$ are equivalent to coequalizer diagrams for $f, g : X \to Y$. It is a short step from there to the conclusion that an object Z with morphism $Y \to Z$ satisfies the universal property of the coequalizer if and only if Z with two copies of that same morphism $Y \to Z$ satisfies the universal property of the pushout.

For the dual case of this whole story, I will just say this: if $1 \in \mathscr{C}$ is a terminal object, then the uniqueness of morphisms to 1 implies that the pullback of the diagram

$$\begin{tikzcd}
& X \arrow[d] \\
Y \arrow[r] & 1
\end{tikzcd}$$

satisfies the universal property of the product $X \times Y$. Having shown that finite products exist, one then obtains the equalizer of any pair of morphisms $f, g : X \to Y$ as the pullback of the diagram

$$\begin{tikzcd}
& X \arrow[d, "\mathrm{Id} \times f"] \\
X \arrow[r, "\mathrm{Id} \times g"] & X \times Y
\end{tikzcd}.$$

If finite products and equalizers always exist, then all finite limits can be constructed out of them.

Exercise 1.4. Let's talk about some coproducts and products in algebraic settings.

(a) What is a coproduct of two objects in the category Ring of rings with unit? Try to describe it explicitly.
(b) Same question about products in Ring. (This one is perhaps easier.)
(c) Show that two fields of different characteristic can have neither a product nor a coproduct in the category Fld of fields.

Answers: The coproduct of two rings A, B is their tensor product $A \otimes B$, equipped with the ring homomorphisms

$$A \xrightarrow{i_A} A \otimes B : a \mapsto a \otimes 1, \qquad B \xrightarrow{i_B} A \otimes B : b \mapsto 1 \otimes b.$$

As a set, $A \otimes B$ is the same thing as the tensor product of A and B as abelian groups; one then gives it a ring structure by defining

$$(a \otimes b)(a' \otimes b') := (aa') \otimes (bb').$$

It is easy to check that the required universal property is satisfied. Perhaps more interesting is to observe that in the more familiar categories Ab and R-Mod in which we are used to talking about tensor products, they do *not* arise as colimits, and there is an obvious reason why they shouldn't: the only obviously canonical homomorphisms I can think of from a pair of abelian groups A and B to their tensor product $A \otimes B$ are the trivial ones. The big difference in Ring as that rings have multiplicative units, and these give rise to canonical *nontrivial* morphisms from A and B to $A \otimes B$ as described above. (For similar reasons, you also should not try to think of tensor products as categorical products—for a more useful categorical perspective on tensor products, see Exercise 1.9.)

The product in Ring is exactly what you'd expect: the product of rings.

For fields, the problem is that there are in fact no field homomorphisms *at all* between a pair of fields with different characteristics. So for any fields A and B, the need to have morphisms $A, B \to A \coprod B$ and $A \times B \to A, B$ means that neither the coproduct nor the product can exist unless A and B have the same characteristic (which their product and coproduct must then also have). For example, $\mathbb{Z}_2$ and $\mathbb{Q}$ have no coproduct in Fld, though they do have a coproduct in Ring, namely $\mathbb{Z}_2 \otimes \mathbb{Q}$, which is an extremely indirect way of writing the trivial ring. (Amusing exercise: show that $1 = 0$ in $\mathbb{Z}_2 \otimes \mathbb{Q}$. The elements 1 and 0 are never equal in a field.)

Exercise 1.5 (initial and terminal objects)**.** In defining limits and colimits of diagrams $\mathcal{F} : \mathscr{I} \to \mathscr{C}$, the set of objects in $\mathscr{I}$ is not required to be nonempty. When it is empty, we can think of $\operatorname{colim}(\mathcal{F})$ is a coproduct of an empty collection of objects in $\mathscr{C}$, and $\operatorname{colim}(\mathcal{F})$ is then called an **initial object** in $\mathscr{C}$. Similarly, the product $\lim(\mathcal{F})$ of an empty collection of objects is called a **terminal** (or **final**) object in $\mathscr{C}$.

(a) Reformulate the definitions given above for the terms "initial object" and "terminal object" in a way that makes no reference to limits or colimits, and using this reformulation, give a short proof that both are unique up to canonical isomorphisms, if they exist.
(b) Show that for any initial object $0 \in \mathscr{C}$, the coproducts $0 \coprod X$ and $X \coprod 0$ exist and the canonical morphisms of X to each are isomorphisms. Similarly, for any terminal object $1 \in \mathscr{C}$, the products $1 \times X$ and $X \times 1$ exist and their canonical morphisms to X are isomorphisms.
(c) Describe what initial and terminal objects are in each of the following categories, if they exist: Top, Top_*, Ab, Ring, and Fld.
Hint: You might guess the last two from Exercise 1.4.

Answers: If $\mathscr{I}$ is the empty category, then there is a unique diagram $\mathcal{F} : \mathscr{I} \to \mathscr{C}$, but it carries no information. If we want to define a colimit of this diagram, then any object $X \in \mathscr{C}$ can be considered a target; there is no need to specify any morphisms since $\mathscr{I}$ has no objects. The condition of X being a *universal* target is, however, nontrivial: it means that for any other target Y, there is a unique morphism $u : X \to Y$ such that... well, at this point we would normally say that certain morphisms admit factorizations through the morphism u, but since $\mathscr{I}$ has no objects, there are no morphisms to be factored and thus no further conditions to impose. We are left only with this: *$X \in \mathscr{C}$ is an initial object if and only if for every object $Y \in \mathscr{C}$, there is a* *unique morphism* $X \to Y$. That's the usual definition—we stated it in a much more roundabout way by talking about coproducts over the empty category.

Here's the dual version: $X \in \mathscr{C}$ *is a terminal object if and only if for every object* $Y \in \mathscr{C}$*, there is a unique morphism* $Y \to X$.

With these definitions understood: if $0, 0' \in \mathscr{C}$ are two initial objects, then there is a unique morphism $0 \to 0'$, and there is also a unique morphism $0' \to 0$. Moreover, there are unique morphisms $0 \to 0$ and $0' \to 0'$, and both of those have to be identity morphisms, since identity morphisms must always exist. It follows that the unique morphisms $0 \to 0'$ and $0' \to 0$ are inverse to each other, and are thus isomorphisms. The uniqueness of terminal objects up to unique isomorphisms is proved similarly; there is only a slightly different reason for the uniqueness of the morphisms $1 \to 1'$ and so forth.

Let's consider the coproduct of an initial object $0 \in \mathscr{C}$ with an arbitrary $X \in \mathscr{C}$. We claim that X itself plays the role of the coproduct, together with the two morphisms

$$\begin{tikzcd} 0 \arrow[dr] & \\ & X , \\ X \arrow[ur, "\mathrm{Id}"] & \end{tikzcd}$$

the first of which is determined by the condition that 0 is an initial object. Indeed, suppose Y is given, along with a morphism $f : X \to Y$ and the unique morphism $0 \to Y$ (for which there is no freedom of choice). The dashed arrow in the following diagram is then uniquely determined,

$$\begin{tikzcd} 0 \arrow[dr] \arrow[drr, bend left] & & \\ & X \arrow[r, dashed] & Y , \\ X \arrow[ur, "\mathrm{Id}"] \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

and this establishes the universal property of the coproduct. In this way of representing $0 \coprod X$, the canonical morphism $X \to 0 \coprod X$ is imply the identity morphism $X \to X$, and thus an isomorphism. Similar arguments prove the analogous statements about $X \coprod 0$, $1 \times X$ and $X \times 1$.

Here is an inventory of initial and terminal objects in specific categories:

- Top: the empty set $\varnothing$ is initial, and every one-point space $*$ is terminal. Note that the initial object in this case is not just unique *up to isomorphism*, but is actually unique, i.e. there really is only one object in Top called $\varnothing$. By contrast, the unique point in a one-point space can be anything, and the collection of all possible one-point spaces is therefore too large to qualify as a set; it is a proper class. Nonetheless, there is indeed a unique homeomorphism between any two of them.
- Top_*: every one-point space is both an initial and a terminal object.
- Ab: every trivial group is both initial and terminal. The answer in R-Mod is the same, in case you'd wondered.
- Ring: this one's more interesting. According to Exercise 1.4, tensor products are coproducts in Ring, so an initial object $R \in \mathsf{Ring}$ should be a ring with the property that $R \otimes A \cong A \cong A \otimes R$ for all rings $A \in \mathsf{Ring}$; plugging in $A := \mathbb{Z}$ as a special case, one deduces $R \cong \mathbb{Z}$. And indeed, for any other ring B, a ring homomorphism $\mathbb{Z} \to B$ is uniquely determined

by the condition that it preserve the 0 and 1 elements. Terminal objects are trivial rings, i.e. those in which $1 = 0$.

- Fld: there are no initial or terminal objects in Fld, because as discussed in the answer to Exercise 1.4(c), there do not exist any fields that admit homomorphisms either to or from every other field (of arbitrary characteristic).

Exercise 1.6 (biproducts)**.** Assume $\mathscr{A}$ is a category in which the sets $\operatorname{Hom}(A, B)$ of morphisms $A \to B$ for each $A, B \in \mathscr{A}$ are equipped with the structure of abelian groups such that composition $\operatorname{Hom}(A, B) \times \operatorname{Hom}(B, C) : (f, g) \mapsto g \circ f$ is always a bilinear map. (Popular examples are the categories Ab of abelian groups and R-Mod of modules over a commutative ring R.) A **biproduct** of two objects $A, B \in \mathscr{A}$ is an object $C \in \mathscr{A}$ equipped with four morphisms

$$\begin{tikzcd} A \arrow[dr, "i_A"] & & A \\ & C \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"] & \\ B \arrow[ur, "i_B"] & & B \end{tikzcd} \tag{1.1}$$

that satisfy the five relations

$$\pi_A i_A = \mathbb{1}_A, \quad \pi_B i_B = \mathbb{1}_B, \quad \pi_A i_B = 0, \quad \pi_B i_A = 0, \quad i_A \pi_A + i_B \pi_B = \mathbb{1}_C. \tag{1.2}$$

In the categories Ab or R-Mod, an example of a biproduct of A and B is the direct sum $A \oplus B$ with its canonical inclusion and projection maps. The category $\mathscr{A}$ is called **additive** if every pair of objects has a biproduct.

(a) Show that for any biproduct as in the diagram (1.1), C with the morphisms i_A, i_B is a coproduct of A and B, and with the morphisms π_A, π_B it is also a product of A and B.

(b) Show that in the categories Ab and R-Mod, every biproduct of two objects A, B admits an isomorphism to $A \oplus B$ that identifies the four maps in (1.1) with the obvious inclusions and projections.

(c) A (covariant or contravariant) functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ between two additive categories is called an **additive functor** if the map defined by $\mathcal{F}$ from $\operatorname{Hom}(A, B)$ to $\operatorname{Hom}(\mathcal{F}(A), \mathcal{F}(B))$ or (in the contravariant case) $\operatorname{Hom}(\mathcal{F}(B), \mathcal{F}(A))$ is a group homomorphism for all $A, B \in \mathscr{A}$. Show that additive functors send all biproducts in $\mathscr{A}$ to biproducts in $\mathscr{B}$.

Remark: Popular examples of additive functors $\mathsf{Ab} \to \mathsf{Ab}$ *or* R-$\mathsf{Mod} \to R$-Mod *are* $\otimes G$, $G\otimes$, $\operatorname{Hom}(\cdot, G)$ *and* $\operatorname{Hom}(G, \cdot)$ *for any fixed module* G*, as these arise in the universal coefficient theorems for homology and cohomology.*

Answers: Let's show first that (1.1) and (1.2) make C with the morphisms $i_A : A \to C$ and $i_B : A \to B$ into a coproduct of A and B. We need to show that the dashed morphism u in the diagram

$$\begin{tikzcd} A \arrow[dr, "i_A"] \arrow[drr, bend left, "f_A"] & & \\ & C \arrow[r, dashed, "u"] & X \\ B \arrow[ur, "i_B"] \arrow[urr, bend right, "f_B"'] & & \end{tikzcd}$$

exists and is unique for any given object $X \in \mathscr{A}$ with morphisms f_A, f_B from A and B respectively. Start with uniqueness: if u is a morphism for which this diagram commutes, then using (1.2) and the assumption that composition is bilinear, we have

$$u = u(i_A \pi_A + i_B \pi_B) = (u i_A)\pi_A + (u i_B)\pi_B = f_A \pi_A + f_B \pi_B.$$

For existence, we then just need to define u by this formula and show that it satisfies $u i_A = f_A$ and $u i_B = f_B$, which also follows easily from the relations (1.2). The proof that C with the morphisms π_A, π_B is a product of A and B is similar.

For part (b), we already know that $A \oplus B$ defines a biproduct of R-modules A and B, so what we really need is a general result about uniqueness of biproducts up to isomorphism. We already have such results for products and coproducts separately, but we cannot directly apply them here, even though we know that biproducts are both; the trouble is that doing so will produce *two* isomorphisms between any two biproducts of A and B, one that arises by viewing them as products, and another by viewing them as coproducts. We want to see that those two isomorphisms are *the same one*.

Concretely, let's suppose that (1.1) and (1.2) are given, and that we also have a second object C' and set of morphisms $i'_A, i'_B, \pi'_A, \pi'_B$ satisfying the same set of relations. We do not need to assume $\mathscr{A}$ is Ab or R-Mod for this discussion, as it will make sense in any category for which biproducts can be defined, but some intuition about direct sums may nonetheless be helpful for writing down suitable morphisms between C and C'. Explicitly, define

$$f := i'_A \pi_A + i'_B \pi_B : C \to C', \qquad \text{and} \qquad g := i_A \pi'_A + i_B \pi'_B : C' \to C.$$

Using (1.2), we then have

$$\begin{aligned} gf &= (i_A \pi'_A + i_B \pi'_B)(i'_A \pi_A + i'_B \pi_B) = i_A(\pi'_A i'_A)\pi_A + i_A(\pi'_A i'_B)\pi_B + i_B(\pi'_B i'_A)\pi_A + i_B(\pi'_B i'_B)\pi_B \\ &= i_A \pi_A + i_B \pi_B = \mathbb{1}_C, \end{aligned}$$

and by a similar calculation, $fg = \mathbb{1}_{C'}$, so f is an isomorphism with $g = f^{-1}$. Using f to identify C with C' now transforms the morphism $i_A : A \to C$ into

$$f i_A = (i'_A \pi_A + i'_B \pi_B) i_A = i'_A(\pi_A i_A) + i'_B(\pi_B i_A) = i'_A : A \to C',$$

and it transforms the morphism $\pi_A : C \to A$ into

$$\pi_A f^{-1} = \pi_A(i_A \pi'_A + i_B \pi'_B) = (\pi_A i_A)\pi'_A + (\pi_A i_B)\pi'_B = \pi'_A : C' \to A,$$

and by similar calculations,

$$f i_B = i'_B, \qquad \pi_B f^{-1} = \pi'_B.$$

One can now appeal to abstract principles (i.e. the universal properties of products and coproducts) to deduce that f is indeed the *only* isomorphism $C \to C'$ that relates the morphisms i_A, i'_A and so forth in this way.

For a covariant additive functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, it is easy to check that $\mathcal{F}$ sends the four morphisms of (1.1) to morphisms

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[dr, "\mathcal{F}(i_A)"] & & \mathcal{F}(A) \\
& \mathcal{F}(C) \arrow[ur, "\mathcal{F}(\pi_A)"] \arrow[dr, "\mathcal{F}(\pi_B)"] & \\
\mathcal{F}(B) \arrow[ur, "\mathcal{F}(i_B)"] & & \mathcal{F}(B)
\end{tikzcd}$$

in $\mathscr{B}$ that satisfy the five relations (1.2), making $\mathcal{F}(C)$ a biproduct of $\mathcal{F}(A)$ and $\mathcal{F}(B)$. The amusing detail is what happens if $\mathcal{F}$ is *contra*variant: it still works, but the reversal of arrows means that some roles need to be switched, e.g. the diagram in $\mathscr{B}$ arising from (1.1) must be written as

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[dr, "\mathcal{F}(\pi_A)"] & & \mathcal{F}(A) \\
& \mathcal{F}(C) \arrow[ur, "\mathcal{F}(i_A)"] \arrow[dr, "\mathcal{F}(i_B)"] & \\
\mathcal{F}(B) \arrow[ur, "\mathcal{F}(\pi_B)"] & & \mathcal{F}(B)
\end{tikzcd} .$$

With $\mathcal{F}(\pi_A), \mathcal{F}(\pi_B)$ now playing the roles formerly played by i_A, i_B and $\mathcal{F}(i_A), \mathcal{F}(i_B)$ playing the roles of π_A, π_B, one easily checks that the five relations (1.2) are satisfied, so $\mathcal{F}(C)$ is again a biproduct of $\mathcal{F}(A)$ and $\mathcal{F}(B)$, with contravariance having transformed inclusions into projections and vice versa.

Exercise 1.7 (fiber products in Diff)**.** As mentioned in lecture, the category Diff of smooth manifolds is one in which many limits and colimits do not exist. An important example is the fiber product of two smooth maps $f : M \to Q$ and $g : N \to Q$, which matches the usual topological fiber product

$$M \,_f\!\times_g N := \{(x, y) \in M \times N \mid f(x) = g(y)\} \subset M \times N$$

if the maps f and g are **transverse** to each other (written $f \pitchfork g$), because the implicit function theorem then gives $M \,_f\!\times_g N$ a natural smooth manifold structure for which the obvious projections to M and N are smooth.[6] If, on the other hand, f and g are not transverse, then the examples below show that all bets are off.

(a) Suppose $F : P \to M$ and $G : F \to N$ are smooth maps that define a target in Diff for the fiber product diagram defined by f and g; in other words, the diagram

$$\begin{tikzcd}
P \arrow[r, "F"] \arrow[d, "G"] & M \arrow[d, "f"] \\
N \arrow[r, "g"] & Q
\end{tikzcd}$$

commutes and consists entirely of smooth manifolds and smooth maps. Interpret this diagram as defining a smooth map

$$u : P \to M \times N$$

whose image lies in the *topological* fiber product $M \,_f\!\times_g N \subset M \times N$, and show that if F and G satisfy the universal property for a fiber product in Diff, then u is a continuous bijection of P onto $M \,_f\!\times_g N \subset M \times N$.

(b) Deduce that if $M \,_f\!\times_g N \subset M \times N$ is a smooth submanifold of $M \times N$, then $M \,_f\!\times_g N$ with its projection maps to M and N does in fact define a fiber product in Diff. (Note that this may sometimes hold even if f and g are not transverse.)

(c) Consider the example $M = N = Q := \mathbb{R}$ with $f(x) := x^2$ and $g(y) := y^2$, thus

$$M \,_f\!\times_g N = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\} .$$

[6] Transversality is a condition on the derivatives of f and g at all points $x \in M$ and $y \in N$ such that $f(x) = g(y) =: p$; writing the derivatives at these points as linear maps $df(x) : T_x M \to T_p Q$ and $dg(y) : T_y M \to T_p Q$ between the appropriate tangent spaces, it means that the subspaces $\operatorname{im} df(x)$ and $\operatorname{im} dg(y)$ span all of $T_p Q$. Choosing suitable local coordinates near each point $(x, y) \in M \,_f\!\times_g N$, one can identify $M \,_f\!\times_g N$ locally with the zero-set of a smooth map whose derivative at (x, y) is surjective if and only if the transversality condition holds, so that the implicit function theorem makes $M \,_f\!\times_g N$ a smooth submanifold of $M \times N$.

You will easily convince yourself that this topological fiber product is not a manifold. Show that the pair of maps f, g does not admit any fiber product in Diff. Note that this is a stronger statement than just the observation that $\{x^2 = y^2\} \subset \mathbb{R}^2$ is not an object of Diff.
Hint: You can use parts (a) and (b) to show that if P is a smooth fiber product, then it contains a special point $p \in P$ such that $P \backslash \{p\}$ is diffeomorphic to $\{x^2 = y^2\} \backslash \{(0,0)\}$.

(d) Here's a weirder example: Let $M = Q := \mathbb{R}$, define $N := *$ as a manifold of one point with $g : N \to Q = \mathbb{R}$ mapping to 0, and choose $f : M = \mathbb{R} \to \mathbb{R} = Q$ to be any smooth function with

$$f^{-1}(0) = \{-1, -1/2, -1/3, \dots\} \cup \{0\} \cup \{\dots, 1/3, 1/2, 1\}.$$

(If you have doubts about the existence of such a function, try making minor modifications to the function e^{-1/x^2}, or something similar.) Show that in this case, a fiber product in Diff does exist, but is not homeomorphic to the topological fiber product.
Hint: What can you say about continuous maps from locally path-connected spaces to $f^{-1}(0) \subset \mathbb{R}$?

Answers: For part (a), note first that a fiber product diagram in Diff can always also be interpreted as a fiber product diagram in Top, so applying the universal property of the topological fiber product $M \,{}_f\!\times_g N$ immediately gives us a unique *continuous* map $u : P \to M \,{}_f\!\times_g N$ such that the diagram

$$\begin{tikzcd} & M \\ P \arrow[ur, bend left, "F"] \arrow[r, "u"] \arrow[dr, bend right, "G"'] & M \,{}_f\!\times_g N \arrow[u] \arrow[d] \\ & N \end{tikzcd}$$

commutes, where the vertical arrows are the obvious projections. This diagram also gives us an explicit formula for u: its composition with the inclusion $M \,{}_f\!\times_g N \hookrightarrow M \times N$ is just

$$(F, G) : P \to M \times N,$$

which is a smooth map since F and G are smooth, though we cannot sensibly call it a smooth map to $M \,{}_f\!\times_g N$ unless the latter is known to be a smooth submanifold of $M \times N$.

We want to show that if P with the maps F and G satisfies the universal property for a fiber product in Diff, then the map $u : P \to M \,{}_f\!\times_g N$ described above is a bijection. Indeed, pick any point $(x, y) \in M \,{}_f\!\times_g N$ and consider the pullback diagram

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "y"] & M \arrow[d, "f"] \\ N \arrow[r, "g"] & Q \end{tikzcd},$$

where the labels "x" and "y" on arrows are used to indicate the images of maps from a one-point space labelled $*$. The latter is (trivially) a smooth 0-manifold, and the maps defined on it are (trivially) smooth, so this diagram lives in Diff, and the universal property of the fiber product P

therefore produces a unique map $u : * \to P$ for which the diagram

$$\begin{tikzcd} & M \\ * \arrow[ur, bend left, "x"] \arrow[r, "u"] \arrow[dr, bend right, "y"'] & P \arrow[u, "F"'] \arrow[d, "G"] \\ & N \end{tikzcd}$$

commutes. The image of $u : * \to P$ is thus the unique point $p \in P$ satisfying $u(p) = (F(p), G(p)) = (x, y)$.

Part (b) follows almost immediately from what was said above: if $M \,{}_f\!\times_g N$ is a smooth submanifold of $M \times N$, then the map $u : P \to M \,{}_f\!\times_g N$ obtained from any smooth fiber product diagram by applying the universal property in Top is automatically also smooth, with the consequence that $M \,{}_f\!\times_g N$ also satisfies the universal property in Diff.

For the example in part (c), $M \,{}_f\!\times_g N \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the union of the two lines $\{y = x\}$ and $\{y = -x\}$, so it is not globally a manifold, though it becomes a smooth 1-manifold if one deletes the singular point $(0, 0)$. Suppose there exists a smooth manifold P and smooth functions $F, G : P \to \mathbb{R}$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "G"] & \mathbb{R} \arrow[d, "f"] \\ \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

defines a fiber product in Diff. By part (a), the smooth map $(F, G) : P \to \mathbb{R}^2$ is then a bijection onto the set $\{y = \pm x\}$, so that there is a unique point $p \in P$ with $F(p) = G(p) = 0$. The manifold P must be path-connected, because any point in $\{y = \pm x\}$ can be joined to $(0, 0)$ by a smooth path lying in one of the smooth submanifolds $\{y = x\}$ or $\{y = -x\}$, and the universal property will then produce a smooth map from this submanifold to P, whose image thus contains a path from any given point to p. Now let $\Sigma := \{y = \pm x\} \backslash \{(0, 0)\} \subset \mathbb{R}^2$, defining a smooth 1-dimensional submanifold of $\mathbb{R}^2$, and observe that the restrictions to Σ of the two projections $\mathbb{R}^2 \to \mathbb{R}$ define a smooth fiber product diagram, and thus (since P satisfies the universal property) give rise to a smooth map $u : \Sigma \to P$, which is inverse to the bijection $P \backslash \{p\} \to \Sigma$ defined by (F, G). This shows that $P \backslash \{p\}$ and Σ are diffeomorphic, thus P is a connected smooth manifold that can be turned into a 1-manifold with four connected components by deleting one point. There is no such manifold, so this is a contradiction.

For the example in part (d), we can identify $M \times N = \mathbb{R} \times *$ with $\mathbb{R}$ and thus identify the topological fiber product with the set

$$M \,{}_f\!\times_g N = f^{-1}(0) \subset \mathbb{R},$$

carrying the subspace topology it inherits as a subset of $\mathbb{R}$. It is not a manifold, because the point $0 \in f^{-1}(0)$ does not have any connected neighborhood. However, for any given smooth fiber product diagram

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d] & \mathbb{R} \arrow[d, "f"] \\ * \arrow[r, "0"] & \mathbb{R} \end{tikzcd},$$

P is a smooth manifold with a smooth function $F : P \to \mathbb{R}$ whose image is contained in $f^{-1}(0)$, and there is very little freedom in finding functions F with this property: since P is locally path-connected, F must be locally constant. It follows that F does factor through a smooth manifold with an obvious smooth bijection onto $f^{-1}(0)$: the manifold in question is $f^{-1}(0)$ itself, but with the *discrete* topology instead of the subspace topology. Conclusion: the fiber product in Diff for our given pair of maps is given by

$$\begin{tikzcd} f^{-1}(0) \arrow[r, hook] \arrow[d] & \mathbb{R} \arrow[d, "f"] \\ * \arrow[r, "0"] & \mathbb{R} \end{tikzcd},$$

where $f^{-1}(0)$ in the corner is understood to carry the discrete topology and is thus a smooth 0-manifold. Its obvious bijection to the topological fiber product ($f^{-1}(0)$ with the subspace topology) is continuous, but not a homeomorphism.

Exercise 1.8. The following bit of abstract nonsense provides a useful tool for proving that objects are isomorphic in various categories, e.g. one can apply it in hTop to establish homotopy equivalences, or (as in Exercise 1.9 below) to deduce properties of tensor products from a universal property.

In any category $\mathscr{C}$, each object $X \in \mathscr{C}$ determines a covariant functor

$$\mathrm{Hom}(X, \cdot) : \mathscr{C} \to \mathsf{Set},$$

which associates to each object $Y \in \mathscr{C}$ the set $\mathrm{Hom}(X, Y)$ of morphisms and to each morphism $f : Y \to Z$ in $\mathscr{C}$ the map

$$\mathrm{Hom}(X, Y) \xrightarrow{f_*} \mathrm{Hom}(X, Z) : g \mapsto f \circ g.$$

There is similarly a contravariant functor $\mathrm{Hom}(\cdot, X) : \mathscr{C} \to \mathsf{Set}$ for which morphisms $f : Y \to Z$ induce maps

$$\mathrm{Hom}(Z, X) \xrightarrow{f^*} \mathrm{Hom}(Y, X) : g \mapsto g \circ f.$$

(a) Show that for any two objects $X, Y \in \mathscr{C}$, each morphism $f : X \to Y$ determines a natural transformation $T_f : \mathrm{Hom}(Y, \cdot) \to \mathrm{Hom}(X, \cdot)$ associating to each object $Z \in \mathscr{C}$ the set map $f^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, and that if f is an isomorphism, then the map f^* is bijective for every $Z \in \mathscr{C}$, i.e. T_f is then a *natural isomorphism*.[7]

(b) Show conversely that every natural transformation $T : \mathrm{Hom}(Y, \cdot) \to \mathrm{Hom}(X, \cdot)$ is T_f for a unique morphism $f : X \to Y$, which is an isomorphism of $\mathscr{C}$ if and only if T_f is a natural isomorphism. It follows that X and Y are isomorphic whenever the sets of morphisms $\mathrm{Hom}(X, Z)$ and $\mathrm{Hom}(Y, Z)$ are in bijective correspondence for every third object Z, in a way that is natural with respect to Z.

(c) Prove contravariant analogues of parts (a) and (b) involving the functors $\mathrm{Hom}(\cdot, X)$ and $\mathrm{Hom}(\cdot, Y)$.

Solution: The interesting step is part (b), so let's just talk about that. (One could give a quick answer to part (a) more or less by mumbling the word "functor".) Suppose a natural transformation $T : \mathrm{Hom}(Y, \cdot) \to \mathrm{Hom}(X, \cdot)$ is given, so for every object $Z \in \mathscr{C}$, T defines a set map

$$T_Z : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

which is required to fit into certain commutative diagrams as dictated by the word "natural". In particular, choosing $Z := Y$, we observe that T determines a distinguished morphism $f : X \to Y$ by

$$f := T_Y(\mathrm{Id}_Y) \in \mathrm{Hom}(X, Y).$$

[7]A **natural isomorphism** $T : \mathcal{F} \to \mathcal{G}$ between two functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$ is a natural transformation such that the morphism $T(\alpha) : \mathcal{F}(\alpha) \to \mathcal{G}(\alpha)$ in $\mathscr{B}$ associated to each object $\alpha \in \mathscr{A}$ is an isomorphism. It follows that T has an inverse, which is also a natural transformation $T^{-1} : \mathcal{G} \to \mathcal{F}$.

We claim now that, in fact, $T = T_f$. Indeed, given any $Z \in \mathscr{C}$ and $g \in \mathrm{Hom}(Y, Z)$, naturality implies that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(Y,Y) & \xrightarrow{T_Y} & \mathrm{Hom}(X,Y) \\ \big\downarrow{\scriptstyle g_*} & & \big\downarrow{\scriptstyle g_*} \\ \mathrm{Hom}(Y,Z) & \xrightarrow{T_Z} & \mathrm{Hom}(X,Z) \end{array}$$

commutes, hence

$$T_Z(g) = T_Z(g \circ \mathrm{Id}_Y) = (T_Z \circ g_*)(\mathrm{Id}_Y) = (g_* \circ T_Y)(\mathrm{Id}_Y) = g_* f = g \circ f = f^* g = T_f(g).$$

Now that we know all natural transformations arise in this way, and after verifying the formula $T_{f \circ g} = T_g \circ T_f$, it follows easily that the morphism $f : X \to Y$ has an inverse if and only if the corresponding natural transformation T_f has an inverse.

One way to apply this result in homotopy theory is as follows. Suppose we are given a map $f : X \to Y$ for which we can verify that for all spaces Z, the induced maps

$$f^* : [Y, Z] \to [X, Z] : g \mapsto g \circ f$$

are bijective. This means that the natural transformation on Hom-functors corresponding to f is a natural isomorphism, therefore implying that f itself is an isomorphism, i.e. the conclusion in this setting is that f is a homotopy equivalence. The variant in part (c) would imply similarly that if the maps

$$f_* : [Z, X] \to [Z, Y] : g \mapsto f \circ g$$

are known to be bijective for all spaces Z, then f is a homotopy equivalence.

Exercise 1.9 (tensor products). On the category R-Mod of modules over a commutative ring R, the tensor product satisfies the following universal property: for any three R-modules A, B, C, the natural map

$$\mathrm{Hom}(A \otimes B, C) \xrightarrow{\alpha} \mathrm{Hom}(A, \mathrm{Hom}(B, C)), \qquad \alpha(\Phi)(a)(b) := \Phi(a \otimes b)$$

is a bijection. Indeed,

$$\mathrm{Hom}_2(A, B; C) := \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

can be interpreted as the set of R-bilinear maps $A \times B \to C$, so the fact that α is bijective means that every such bilinear map factors through the canonical R-bilinear map $A \times B \to A \otimes B$ and a uniquely determined R-module homomorphism $A \otimes B \to C$. In fact, α is not just a bijection; it is also an R-module isomorphism, though we will not make use of this fact in the following. The important observation for now is that α defines a natural isomorphism between the two functors $\mathrm{Hom}(\cdot \otimes \cdot, \cdot)$ and Hom_2 from R-$\mathsf{Mod} \times R$-$\mathsf{Mod} \times R$-Mod to Set, which are contravariant in the first two variables and covariant in the third.

More generally, suppose $\mathscr{C}$ is any category for which the sets $\mathrm{Hom}(X, Y)$ can be regarded as objects in $\mathscr{C}$ for every $X, Y \in \mathscr{C}$, and suppose $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a functor such that the functors $\mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathsf{Set}$ defined by $\mathrm{Hom}(\cdot \otimes \cdot, \cdot)$ and $\mathrm{Hom}_2 := \mathrm{Hom}(\cdot, \mathrm{Hom}(\cdot, \cdot))$ are naturally isomorphic, so in particular, for every triple of objects $X, Y, Z \in \mathscr{C}$, there is a bijection of sets

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

that is natural with respect to all three.

(a) Prove that there is a natural isomorphism relating any two functors $\otimes, \otimes' : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ that satisfy the condition described above. In other words: tensor products are uniquely determined (up to natural isomorphism) by the universal property.

(b) Prove that $\otimes$ is associative in the sense that the functors $\mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ defined by $(X, Y, Z) \mapsto X \otimes (Y \otimes Z)$ and $(X, Y, Z) \mapsto (X \otimes Y) \otimes Z$ are naturally isomorphic. Prove it using only the universal property, i.e. do not use any knowledge of how $\otimes$ is actually defined in any specific categories.

Solutions: Both parts are applications of Exercise 1.8, which is the right tool for the job because the universal property of $\otimes$ does not tell us what $X \otimes Y$ is, but instead tells us what other functor $\mathrm{Hom}(X \otimes Y, \cdot)$ is naturally isomorphic to, namely $\mathrm{Hom}_2(X, Y; \cdot) := \mathrm{Hom}(X, \mathrm{Hom}(Y, \cdot))$. If we are given two versions $\otimes$ and $\otimes'$ that both satisfy the universal property, we obtain from this a natural isomorphism

$$\mathrm{Hom}(X \otimes Y, \cdot) \cong \mathrm{Hom}(X \otimes' Y, \cdot)$$

for every pair of objects $X, Y \in \mathscr{C}$, and therefore (via Exercise 1.8) an isomorphism $X \otimes Y \cong X \otimes' Y$.

Associativity follows similarly because one can follow two chains of natural bijections that both end at the same destination: for any spaces X, Y, Z, V we have:

$$\mathrm{Hom}(X \otimes (Y \otimes Z), V) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y \otimes Z, V)) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, V))),$$

and also

$$\mathrm{Hom}((X \otimes Y) \otimes Z, V) \cong \mathrm{Hom}(X \otimes Y, \mathrm{Hom}(Z, V)) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, V))).$$

Exercise 1.10 (tensor products of pairs)**.** Let $\mathsf{Top}^{\mathrm{rel}}$ denote the category of pairs of spaces and maps of pairs. When defining the cross and cup products on relative homology and cohomology, one often sees the product of two pairs defined as

$$(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B).$$

(a) Why is this definition of $\times$ not actually a *product* (in the sense of category theory) on the category $\mathsf{Top}^{\mathrm{rel}}$? What do categorical products in $\mathsf{Top}^{\mathrm{rel}}$ actually look like?

(b) In the spirit of Exercise 1.9, I would like to argue that $\times$ as defined above should be interpreted as a *tensor product* on $\mathsf{Top}^{\mathrm{rel}}$. Due to some subtle point-set topological issues that I'd rather not get into until next week, it's best for now to dispense with topologies and work instead in the category $\mathsf{Set}^{\mathrm{rel}}$, whose objects are pairs (X, A) of sets with $A \subset X$, and whose morphisms $(X, A) \to (Y, B)$ are arbitrary (not necessarily continuous) maps $X \to Y$ that send A into B. In this setting, how can you regard each of the sets $\mathrm{Hom}((X, A), (Y, B))$ as an object of $\mathsf{Set}^{\mathrm{rel}}$ such that there are natural bijections

$$\mathrm{Hom}((X, A) \times (Y, B), (Z, C)) \cong \mathrm{Hom}\big((X, A), \mathrm{Hom}((Y, B), (Z, C))\big)$$

for all choices of pairs?

Answers: Categorical products require projection morphisms, but e.g. the projection map $X \times Y \to X$ does not generally send $A \times Y \cup X \times B$ into A, and thus does not define a map of pairs $(X, A) \times (Y, B) \to (X, Y)$. For a categorical product on $\mathsf{Top}^{\mathrm{rel}}$, the correct definition would be the obvious one,

$$(X, A) \times (Y, B) := (X \times Y, A \times B).$$

If (X, A) and (Y, B) are objects in $\mathsf{Set}^{\mathrm{rel}}$, then $\mathrm{Hom}((X, A), (Y, B))$ also becomes an object in $\mathsf{Set}^{\mathrm{rel}}$ after singling out the subset

$$\big\{\phi \in \mathrm{Hom}((X, A), (Y, B)) \bigm| \phi(X) \subset B\big\} \subset \mathrm{Hom}((X, A), (Y, B)).$$

It is then straightforward to check that set maps of pairs from (X, A) to $\mathrm{Hom}((Y, B), (Z, C))$ are in natural bijective correspondence with set maps of pairs from $(X, A) \times (Y, B)$ to (Z, C).

The case of this with $A = B = C = \varnothing$ is often written in a more appealing way by using the notation

$$X^Y := \mathrm{Hom}(Y, X) \qquad \text{in } \mathsf{Set},$$

so that $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ becomes the so-called **exponential law**

$$Z^{X \times Y} \cong (Z^Y)^X.$$

Note that this is one of the few situations in which the categorical product can also sensibly be called a tensor product; they are not the same thing in $\mathsf{Set}^{\mathrm{rel}}$, but in Set they are.

The reason we removed topologies from the picture before starting this discussion was that one needs to be very careful about defining the right topology on the set $C(X, Y)$ of continuous maps $X \to Y$ between two spaces if one wants to have a natural bijection

$$C(X \times Y, Z) \cong C(X, C(Y, Z)).$$

In fact, there is *no* right way to define the topology on $C(X, Y)$ so that this works for *all* spaces; one must first restrict the category of spaces under consideration, and then make slight modifications to the definitions of both $C(X, Y)$ and $X \times Y$ as topological spaces. We will go into a little bit of detail about this when it becomes necessary, as without it, one would miss out on some very clever tools coming from stable homotopy theory.

2. Week 2

The lecture on 22.04.2024 was cancelled due to illness, so this week contains only one lecture.

Lecture 3 (25.04.2024): The homotopy category and mapping cylinders.

- The homotopy categories hTop (without base points) and hTop_* (with base points)
- Notation for diagrams that commute up to homotopy (see the notational glossary above)
- The **double mapping cylinder** of two maps $f : Z \to X$ and $g : Z \to Y$,

$$Z(f, g) := \left(X \amalg (I \times Z) \amalg Y\right) \big/ \sim, \qquad \text{where } (0, z) \sim f(z) \text{ and } (1, z) \sim g(z) \text{ for all } z \in Z.$$

- Role of $Z(f, g)$ as a weak form of pushout in hTop (it is called a **homotopy pushout**): the diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\sim"] & X \arrow[d, hook, "i_X"] \\ Y \arrow[r, hook, "i_Y"'] & Z(f,g) \end{tikzcd}$$

commutes up to an obvious homotopy, though not on the nose (the obvious inclusions i_X and i_Y have disjoint images). Diagrams

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\underset{H}{\sim}"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"'] & Q \end{tikzcd}$$

determine maps $Z(f, g) \xrightarrow{u} Q$, constructed in an obvious way out of φ, ψ and the homotopy $\varphi \circ f \overset{H}{\rightsquigarrow} \psi \circ g$, so that the diagram

$$\begin{tikzcd} X \arrow[d, hook, "i_X"'] \arrow[dr, bend left, "\varphi"] & \\ Z(f,g) \arrow[r, "u"] & Q \\ Y \arrow[u, hook, "i_Y"] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$

commutes (on the nose, i.e. not just up to homotopy).

- Special cases:
  (1) **Mapping cylinder** of $f : X \to Y$:
  $$Z(f) := Z(\mathrm{Id}_X, f) = (I \times Z) \cup_f Y,$$
  where the gluing occurs along $\{1\} \times Z$. Convenient feature: $Z(f)$ deformation retracts to Y, so $i_Y : Y \hookrightarrow Z(f)$ is a homotopy equivalence. We can therefore view *every* map $X \to Y$ "up to homotopy equivalence" as inclusion of a subspace, namely $i_X : X \hookrightarrow Z(f)$. (This trick was used once at the end of *Topologie II*, cf. the last two pages of [Wen23].)
  (2) **Mapping cone** of $f : X \to Y$: using the unique map $\epsilon : X \to *$, we define
  $$\mathrm{cone}(f) := Z(\epsilon, f) = CX \cup_f Y,$$
  where $CX := (I \times X)/(\{0\} \times X)$ is the usual cone of X.
  (3) **Suspension** (unreduced): Not the most direct way to define it, but the familiar suspension ΣX of a space X is also the double mapping cylinder of a pair of maps from X to one-point spaces:
  $$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\sim"] & * \arrow[d, hook] \\ * \arrow[r, hook] & \Sigma X \end{tikzcd}.$$
  Here the two maps from $*$ to ΣX have images at the opposite poles, which are points obtained by collapsing $I \times X$ at $\{0\} \times X$ and $\{1\} \times X$ separately.
- Variant for hTop_*: If X, Y, Z are pointed spaces and f, g are pointed maps, defining a base point on $Z(f, g)$ requires modifying its definition by
$$Z(f,g) := \left(X \vee \frac{I \times Z}{I \times *} \vee Y \right) \Big/ \sim, \qquad \text{where } (0,z) \sim f(z) \text{ and } (1,z) \sim g(z) \text{ for all } z \in Z.$$
  Note: Quotienting $I \times Z$ is necessary because $I \times Z$ on its own has no natural base point, but whenever Z, Z' are two pointed spaces,
  $$\text{pointed homotopies } I \times Z \to Z' \qquad \Leftrightarrow \qquad \text{pointed maps } \frac{I \times Z}{I \times *} \to Z'.$$
  Everything discussed above has analogues in which all maps are base-point preserving. The pointed version is sometimes called the **reduced** double mapping cylinder, and one can also derive from it special cases such as the **reduced mapping cone** and **reduced suspension**, which we'll have much more to say about later.
- Why is $Z(f, g)$ not really a pushout in hTop?
  (1) Our construction of the map $u : Z(f, g) \to Q$ uses more information than a diagram in hTop: it uses the actual maps in the diagram (not just their homotopy classes), plus a choice of homotopy. This doesn't mean it cannot work, but is a hint that we may be cheating.

(2) (The real reason): The diagram

$$\begin{tikzcd} X \arrow[d, hook, "i_X"'] \arrow[dr, bend left, "\varphi"] \arrow[d, phantom, shift left=4ex, "\sim"] & \\ Z(f,g) \arrow[r, "u"] & Q \\ Y \arrow[u, hook', "i_Y"] \arrow[ur, bend right, "\psi"'] \arrow[u, phantom, shift left=4ex, "\sim"] & \end{tikzcd}$$

does not always uniquely determine $[u] \in [Z(f,g), Q]$. Example: The mapping cone $\mathrm{cone}(\alpha)$ of a degree 2 map $\alpha : S^1 \to S^1$, say $\alpha(e^{i\theta}) := e^{2i\theta}$ if we think of S^1 as the unit circle in $\mathbb{C}$. Now $\mathrm{cone}(\alpha) \cong \mathbb{RP}^2$ and the natural inclusion $S^1 \hookrightarrow \mathrm{cone}(\alpha)$ defines the nontrivial element of $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. A homotopy pushout diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\sim"] & * \arrow[d] \\ S^1 \arrow[r, "\beta"'] & Q \end{tikzcd}$$

now means a choice of space Q and homotopy class $\beta \in [S^1, Q]$ such that $\beta \cdot \beta$ is homotopic to a constant loop. The latter always holds if Q is simply connected, so take $Q := S^2$, and then observe that the diagram

$$\begin{tikzcd} * \arrow[d, hook] \arrow[dr, bend left] \arrow[d, phantom, shift left=4ex, "\sim"] & \\ \mathrm{cone}(\alpha) \arrow[r, "u"] & S^2 \\ S^1 \arrow[u, hook'] \arrow[ur, bend right] \arrow[u, phantom, shift left=4ex, "\sim"] & \end{tikzcd}$$

always commutes up to homotopy, since $[S^1, S^2] \cong * \cong [*, S^2]$. But $[\mathbb{RP}^2, S^2]$ has more than one element, because there exist maps $\mathbb{RP}^2 \to S^2$ having either possible value of the mod-2 mapping degree (cf. Exercise 2.1).

- Theorem: There exists a category $\mathscr{P}$ whose objects are pushout diagrams (in Top)

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \\ Y & \end{tikzcd}$$

such that
(1) Changing the maps f and g by homotopies produces isomorphic objects of $\mathscr{P}$;
(2) There is a functor $\mathscr{P} \to \mathsf{hTop}$ sending each pushout diagram to its mapping cylinder $Z(f,g)$.

- Proof sketch: Morphisms in $\mathscr{P}$ are diagrams

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, "\gamma"] \arrow[dd, "g"'] & & X \arrow[dr, "\alpha"] \arrow[dl, phantom, "\underset{\phi}{\sim}"] & \\
& Z' \arrow[rr, "f'"] \arrow[dd, "g'"] & & X' \\
Y \arrow[dr, "\beta"'] \arrow[ur, phantom, "\underset{\psi}{\sim}"] & & & \\
& Y' & &
\end{tikzcd},$$

including choices of homotopies ϕ and ψ as part of the data. The notion of composition of such morphisms arises naturally by composing maps and concatenating homotopies.[8] Such a morphism determines a homotopy pushout diagram

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\underset{H}{\sim}"] & X \arrow[d, "i_{X'} \circ \alpha"] \\
Y \arrow[r, "i_{Y'} \circ \beta"'] & Z(f', g')
\end{tikzcd}$$

and therefore also an induced map $Z(f,g) \xrightarrow{u} Z(f',g')$. It is a bit tedious but straightforward to check:
(1) The map induced by a composition of two morphisms in $\mathscr{P}$ is homotopic to the composition of the two induced maps.
(2) If the maps α, β, γ all have homotopy inverses, one can use them to construct an inverse morphism in $\mathscr{P}$.
Both only require the same ideas that are needed for proving e.g. that multiplication in the fundamental group is associative. The second point implies, in particular, that the map $Z(f,g) \to Z(f',g')$ is a homotopy equivalence whenever α, β, γ are.
- Corollary: If $f \underset{h}{\sim} f'$ and $g \underset{h}{\sim} g'$, then $Z(f,g)$ and $Z(f',g')$ are homotopy equivalent.
- Theorem: Pushouts in hTop and hTop_* do not always exist.[9]
- Proof sketch in hTop_*: Fix the obvious base point in S^1 so that our previous degree 2 map $\alpha : S^1 \to S^1$ preserves base points. A pushout diagram in hTop_* of the form

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\sim"] & * \arrow[d] \\
S^1 \arrow[r, "\beta"'] & P
\end{tikzcd}$$

then means a pointed space P together with an element in the 2-torsion subgroup of its fundamental group

$$\beta \in \pi_1(P)_{(2)} := \left\{ \gamma \in \pi_1(P) \;\middle|\; \gamma^2 = 0 \right\}.$$

---

[8] It seems likely that I'm oversimplifying this and ought to talk about "homotopy classes of homotopies" if I really want the composition in $\mathscr{P}$ to be associative, but I do not want to give these details more attention than they deserve. I am attempting to present a slightly more highbrow perspective on a sequence of lemmas in [tD08, §4.1–4.2] that seem rather technical and tedious.

[9] ...which is why we need to use *homotopy* pushouts instead.

Then P and β satisfy the universal property for a pushout in hTop_* if and only if for every space Q and $\gamma \in \pi_1(Q)_{(2)}$, the map

$$[P,Q] \to \pi_1(Q)_{(2)} : u \mapsto u_*\beta$$

is a bijection. Assume this is true, and then consider the surjective map

$$\mathrm{SO}(3) \xrightarrow{p} S^2 : A \mapsto Ae_1,$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $e_1, e_2, e_3 \in \mathbb{R}^3$ denotes the standard basis. Taking e_1 as a base point in S^2, we have

$$p^{-1}(e_1) \cong \mathrm{SO}(2) \cong S^1,$$

giving rise to an exact sequence of pointed spaces

$$S^1 \overset{i}{\hookrightarrow} \mathrm{SO}(3) \overset{p}{\to} S^2.$$

We will see next week that the map $p : \mathrm{SO}(3) \to S^2$ has a special property: it is a *fibration*, with the consequence that for every space P, the induced sequence of pointed sets

$$[P, S^1] \overset{i_*}{\to} [P, \mathrm{SO}(3)] \overset{p_*}{\to} [P, S^2]$$

is also exact, meaning the preimage of the base point under p_* matches the image of i_*. (Here $[X,Y]$ means the set of homotopy classes of pointed maps $X \to Y$, so it is a set with an obvious base point.) Combining this with the bijection that we deduced above from the universal property of the pushout, we obtain an exact sequence

$$\pi_1(S^1)_{(2)} \to \pi_1(\mathrm{SO}(3))_{(2)} \to \pi_1(S^2)_{(2)},$$

in which the first and last terms both vanish. But $\mathrm{SO}(3) \cong \mathbb{RP}^3$ and thus $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$, so the middle term does not vanish, and this is a contradiction.

- To do next week: Define what a fibration is and explain why the sequence of sets of homotopy classes in that proof was exact.

Suggested reading. A more comprehensive treatment of mapping cylinders (including details that I left out of the proof of the theorem about the functor $\mathscr{P} \to \mathsf{hTop}$) can be found in [tD08, §4.1–4.2]. This does not include the proof that pushouts in hTop_* don't exist; I found that in [Cut21, Week 8 exercises].

Exercises (for the Übung on 2.05.2024).

Exercise 2.1. Review the notions of the $\mathbb{Z}_2$-valued and $\mathbb{Z}$-valued mapping degrees for maps between closed and connected topological manifolds of the same dimension, as covered e.g. in [Wen23, Lecture 35]. Then:

(a) Show that for every closed and connected topological manifold M of dimension $n \in \mathbb{N}$, the set $[M, S^n]$ contains at least two elements, and infinitely many if M is orientable.
(b) Does the set $[S^n, M]$ also always have more than one element?

Exercise 2.2. Deduce from the properties of double mapping cylinders the standard fact that there is a functor $\Sigma : \mathsf{Top} \to \mathsf{Top}$ assigning to every space $X \in \mathsf{Top}$ its (unreduced) suspension ΣX. *Note: This is just intended as a sanity check. There is nothing especially nontrivial to be done here, and there are also more direct ways to show that suspensions define a functor.*

Exercise 2.3. Show that the mapping cone $\mathrm{cone}(f)$ of any homotopy equivalence $f : X \to Y$ is a contractible space.
Hint: Find a useful morphism in the category $\mathscr{P}$ of pushout diagrams.

Exercise 2.4. Show that for any two maps $f : Z \to X$ and $g : Z \to Y$, the singular homologies (with arbitrary coefficients) of the spaces X, Y, Z and $Z(f, g)$ are related by a long exact sequence of the form

$$\ldots \to H_{n+1}(Z(f,g)) \to H_n(Z) \to H_n(X) \oplus H_n(Y) \to H_n(Z(f,g)) \to H_{n-1}(Z) \to \ldots,$$

and describe explicitly what the two homomorphisms in the middle of this sequence look like. Show that it also works with all homology groups replaced by their reduced counterparts, then write down the special case of a mapping cone and check that what you have is consistent with Exercise 2.3.
Hint: There is a relatively straightforward way to apply the Mayer-Vietoris sequence here, but you could also deduce this as a special case of the exact sequence of the generalized mapping torus derived in [Wen23, Lecture 34].

Exercise 2.5. Prove that pushouts in hTop do not always exist.
Hint: The proof carried out in lecture for hTop_* requires only minor modifications. Note that even if X and Y are spaces without base points, the set of homotopy classes $[X, Y]$ still has a natural base point whenever Y is path-connected. (Why?)

Exercise 2.6. Give explicit examples of homotopic maps

$$f \underset{h}{\sim} f' : Z \to X \qquad \text{and} \qquad g \underset{h}{\sim} g' : Z \to Y$$

such that the mapping cylinders $Z(f, g)$ and $Z(f', g')$ are not homeomorphic. (They will of course be homotopy equivalent!)

Exercise 2.7. The **join** $X * Y$ of two spaces X and Y is the double mapping cylinder $Z(\pi_X, \pi_Y)$ defined via the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Prove that the join of two spheres is always homeomorphic to a sphere: concretely, for every $m, n \in \mathbb{N}$,

$$S^m * S^n \cong S^{m+n+1}.$$

Hint: Split the double mapping cylinder in half so that you see $S^m * S^n$ as the union of two pieces glued along boundaries that both look like $S^m \times S^n$. Can you think of two compact manifolds that both have $S^m \times S^n$ as boundary? Stare closely at the two pieces, you might recognize them! Now glue them together and ask: what is $S^m * S^n$ the boundary of?

Exercise 2.8. Many constructions in homotopy theory have analogues in homological algebra, and one of these is the mapping cone. For two chain complexes (A_*, ∂_A) and (B_*, ∂_B) with a chain map $f : A_* \to B_*$, the **mapping cone of** f is the chain complex $(\mathrm{cone}(f)_*, \partial)$ with

$$\mathrm{cone}(f)_n := A_{n-1} \oplus B_n \qquad \text{and} \qquad \partial := \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix}.$$

The analogy to the mapping cone in Top goes through cellular homology: if X, Y are two CW-complexes and $f : X \to Y$ is a cellular map, then the cone of f inherits a natural cell decomposition whose augmented cellular chain complex $\widetilde{C}_*^{\mathrm{CW}}(\mathrm{cone}(f))$ is the cone of the chain map $f_* : \widetilde{C}_*^{\mathrm{CW}}(X) \to \widetilde{C}_*^{\mathrm{CW}}(Y)$.[10]

Show that the mapping cone $\mathrm{cone}(f)_*$ of a chain map $f : A_* \to B_*$ similarly plays the role of a *homotopy pushout* in the category Ch of chain complexes and chain maps, with the role of a one-point space played by the trivial chain complex $0_* \in \mathsf{Ch}$. Specifically:

[10] This was Problem 2(b) on the take-home midterm for last semester's *Topologie II* course, but for Exercise 2.8, you do not need to know about it.

(a) There is a natural chain map $i_B : B_* \to \mathrm{cone}(f)_*$ such that the diagram

$$\begin{tikzcd} A_* \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\sim"] & 0_* \arrow[d] \\ B_* \arrow[r, "i_B"'] & \mathrm{cone}(f)_* \end{tikzcd}$$

commutes up to chain homotopy.

(b) Any homotopy-commutative diagram in Ch of the form

$$\begin{tikzcd} A_* \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\underset{H}{\sim}"] & 0_* \arrow[d] \\ B_* \arrow[r, "\psi"'] & D_* \end{tikzcd}$$

naturally determines a chain map $u : \mathrm{cone}(f)_* \to D_*$ such that $u \circ i_B$ is chain homotopic to ψ.

(c) If we were being strict about the analogy via cellular homology, then the trivial complex 0_* in the diagrams above ought to be replaced by $\widetilde{C}^{\mathrm{CW}}_*(*)$, the augmented cellular chain complex of a one-point space, which is not trivial: it has nontrivial entries in degrees 0 and -1, with the boundary operator giving an isomorphism between them. Explain why this discrepancy does not matter, and nothing in the discussion above would change if we used $\widetilde{C}^{\mathrm{CW}}_*(*)$ in place of 0_*.

Hint: None of this is hard. . . the quickest approach may be by guessing.

3. Week 3

Lecture 4 (29.04.2024): Introduction to fibrations.

- The set of (free or pointed) homotopy classes $[X, Y]$ as a pointed set (assuming Y is path-connected in the unpointed case)
- What it means for a sequence of three pointed sets to be exact
- Motivational question: Given a map $p : E \to B$ and the inclusion $i : F := p^{-1}(*) \hookrightarrow E$, what condition makes the sequence

$$[X, F] \xrightarrow{i_*} [X, E] \xrightarrow{p_*} [X, B]$$

  exact for all other spaces X?
- Definition of the homotopy lifting property (free case) and (free, i.e. unpointed) fibrations $p : E \to B$. (See next lecture for a precise roundup of the crucial definitions.)
- Terminology: the **base** B and **fibers** $E_b := p^{-1}(b) \subset E$ of a fibration $p : E \to B$
- Example 1: covering spaces (discrete fibers, lifts of homotopies are *unique*, which does not hold for more general fibrations)
- Example 2: fiber bundles (to be studied later in this course): $\{E_b\}_{b \in B}$ is a continuous family of *homeomorphic* spaces (assuming B is path-connected)
  - Example 2a: For M any smooth n-manifold, its **tangent bundle** $TM = \bigcup_{x \in M} T_x M$ is a fiber bundle whose fibers (the **tangent spaces**) $T_x M$ are all homeomorphic to $\mathbb{R}^n$. (One can cook up examples with more interesting fibers e.g. by equipping each tangent space with an inner product and taking the unit sphere in each—this produces a fiber bundle with fibers homeomorphic to S^{n-1}, a so-called **sphere bundle**.)
  - Example 2b: The map $p : \mathrm{SO}(3) \to S^2$ that we used in Lecture 3 for showing that pushouts in hTop_* do not always exist. Observation 1: For base point $e_1 \in S^2$, the fiber $F := p^{-1}(e_1)$ is a subgroup isomorphic to $\mathrm{SO}(2)$, thus homeomorphic to S^1.

Observation 2: That subgroup acts continuously (in fact smoothly), *freely* and *transitively* from the right on every other fiber, implying that all fibers are homeomorphic (in fact diffeomorphic) to S^1.

- Remark: When we study fiber bundles in earnest, we will prove that they all have the homotopy lifting property, and are thus fibrations. If you are already familiar with *smooth* fiber bundles and connections, then you should believe this easily for the following reason: any choice of connection on $p : E \to B$ defines parallel transport maps which uniquely determine a lift of any *smooth* homotopy $X \times I \to B$. (One has to work harder to also get lifts of all *continuous* homotopies... for this, differential geometry is not enough.)
- Theorem (already proved): For any fibration $p : E \to B$ with B path-connected, and any space X, the induced sequence of free homotopy classes $[X, F] \to [X, E] \to [X, B]$ is exact. (Here the map $F \to E$ is the inclusion of the fiber $F := p^{-1}(b_0) \subset E$ over any chosen point $b_0 \in B$.)[11]
- Idea: If we can show that *every* map $f : X \to Y$ becomes a fibration after replacing X with some space $X' \underset{h.e.}{\simeq} X$, then we can do this with the inclusion $F \hookrightarrow E$ and thus extend the exact sequence $[X, F] \to [X, E] \to [X, B]$ one more term to the left. Then we can do it again, and again, and extend the sequence as far as we want...
- Example (**path space fibrations**): for $(X, x_0) \in \mathsf{Top}_*$, we define
  - the **free path space**: $C(I, X) := \{\text{continuous maps } I \to X\}$ with the compact-open topology
  - the **based path space**: $PX := P_{x_0}X := \{\gamma \in C(I, X) \mid \gamma(0) = x_0\}$
  - the **based loop space**: $\Omega X := \Omega_{x_0}X := \{\gamma \in PX \mid \gamma(1) = x_0\}$.

  Notice: $C(I, X)$ does not depend on a base point, and it has no natural base point of its own. The spaces PX and ΩX do have natural base points defined by constant paths. Define maps $C(X, I) \xrightarrow{p} X$ and $PX \xrightarrow{p} X$ by $p(\gamma) := \gamma(1)$; for the latter, we notice $p^{-1}(x_0) = \Omega X \subset PX$, making

  $$\Omega X \hookrightarrow PX \xrightarrow{p} X$$

  an exact sequence of pointed spaces.
- Theorem: (1) $C(I, X) \xrightarrow{p} X$ and $PX \xrightarrow{p} X$ are fibrations. (2) The map $C(I, X) \xrightarrow{p} X$ is also a homotopy equivalence. (3) The space PX is contractible.

The following is a digression, subtitled "The revenge of *Topologie I*":

- Why is $p : C(I, X) \to X$ continuous? More generally, is the map

  $$\mathrm{ev} : C(X, Y) \times X \to Y : (f, x) \mapsto f(x)$$

  continuous for all spaces X and Y? (One can show that it is always *sequentially* continuous.)
- Counterexample: $\mathrm{ev} : C(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q} \to \mathbb{R}$ is not continuous for the obvious (subspace) topology on $\mathbb{Q} \subset \mathbb{R}$. Quick proof: If ev is continuous, then for every continuous $f_0 : \mathbb{Q} \to \mathbb{R}$, every $x_0 \in \mathbb{Q}$ and every neighborhood $\mathcal{U} \subset \mathbb{R}$ of $y_0 := f_0(x_0)$, there are open neighborhoods $f_0 \in \mathcal{O} \subset C(\mathbb{Q}, \mathbb{R})$ and $x_0 \in \mathcal{W} \subset \mathbb{Q}$ such that $(f, x) \in \mathcal{O} \times \mathcal{W}$ implies $f(x) \in \mathcal{U}$. Without loss of generality, the set $\mathcal{O} \subset C(\mathbb{Q}, \mathbb{R})$ has the form

  $$\mathcal{O} = \{f \mid f(K_i) \subset \mathcal{V}_i \text{ for all } i = 1, \dots, N\} \subset C(\mathbb{Q}, \mathbb{R})$$

[11] In the lecture I somewhat sloppily asserted that this statement was equally valid in the unpointed and pointed cases, but in fact the pointed case involves some subtleties that I brushed under the rug. These gaps got filled in in Lecture 5.

for some finite collection of compact subsets $K_i \subset \mathbb{Q}$ and open subsets $\mathcal{V}_i \subset \mathbb{R}$, $i = 1, \ldots, N$. But since compact subsets of $\mathbb{Q}$ *cannot contain any open subsets*, one can then find two irrational numbers $a < b$ such that

$$\mathcal{W}_0 := (a, b) \cap \mathbb{Q} \subset \mathcal{W}$$

is a nonempty open subset of $\mathcal{W}$ disjoint from $K_1 \cup \ldots \cup K_N$. Now define a continuous function $f : \mathbb{Q} \to \mathbb{R}$ that matches f_0 outside of $\mathcal{W}_0$ but takes a value $f(x) \notin \mathcal{U}$ for some $x \in \mathcal{W}_0$; this is easy since $a, b \notin \mathbb{Q}$. Then $(f, x) \in \mathcal{O} \times \mathcal{W}$ but $f(x) \notin \mathcal{U}$, a contradiction.

- Message: $\mathbb{Q}$ is a terrible topological space. The main problem: It is not locally compact.
- Lemma 1: If X is locally compact and Hausdorff,[12] then $\mathrm{ev} : C(X, Y) \times X \to Y$ is continuous. (For the proof, see Exercise 3.3.)
- The **exponential law**: For two sets X, Y (not necessarily with topologies), let X^Y denote the set of all (not necessarily continuous) maps $Y \to X$. Then there is a natural bijection

$$Z^{X \times Y} \cong (Z^Y)^X,$$

identifying each map $f : X \times Y \to Z$ with the map $\widehat{f} : X \to Z^Y$ defined by $\widehat{f}(x)(y) := f(x, y)$.
- Lemma 2: For all topological spaces X, Y, Z, if $f : X \times Y \to Z$ is continuous, then the corresponding map $\widehat{f} : X \to Z^Y$ is a continuous map into $C(Y, Z)$. The converse also holds if Y is locally compact and Hausdorff. (Proof: see Exercise 3.3.)
- Corollary (since I is locally compact and Hausdorff): Homotopies $X \times I \to Y$ are naturally equivalent to continuous maps on X with values in the path space $C(I, Y)$.[13]

End of *Topologie I* digression.

- Proof of the theorem on path space fibrations: see [DK01, Theorem 6.15], supplemented by the following remark. In this proof, there are several maps and homotopies to be written down, most of which are pretty straightforward, one just needs to think a little about why they are continuous. Thanks to the digression above, the fact that I is locally compact and Hausdorff ensures this.

Lecture 5 (2.05.2024): Replacing maps with fibrations. This lecture began with some minor extensions and clarifications to the main definition from Lecture 4.

- Definition: A map $p : E \to B$ has the (free) **homotopy lifting property** (**HLP**) with respect to some class of spaces $\mathscr{C} \subset \mathsf{Top}$ if the lifting problem

$$\begin{tikzcd} X \arrow[r, "\widetilde{H}_0"] \arrow[d, hook, "i_0"] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

[12]Whether the Hausdorff condition here is truly necessary depends on what definition one takes for the term *locally compact*. I typically define locally compact to mean simply that every point has a compact neighborhood, but many authors (such as tom Dieck [tD08]) prefer a stricter definition in which the compact neighborhood can always be assumed arbitrarily small: concretely, for every point $x \in X$, every neighborhood of x contains a neighborhood of x that is compact. The latter is the condition that one really needs for proving $\mathrm{ev} : C(X, Y) \times X \to Y$ is continuous, but it is equivalent to the simpler definition whenever X is Hausdorff. I have no plans to consider any examples in which X is not Hausdorff.

[13]I'm not certain, but in the lecture I may have stated this wrongly and said homotopies $X \times I \to Y$ are equivalent to paths in the space $C(X, Y)$, i.e. maps $I \to C(X, Y)$. The latter is not true in general unless X is also locally compact and Hausdorff.

is solvable for all $X \in \mathscr{C}$, i.e. given a homotopy H and an initial condition $\widetilde{H}_0$ for a lift, the lifted homotopy $\widetilde{H}$ exists. Here i_0 denotes the inclusion $X = X \times \{0\} \hookrightarrow X \times I$.

- Notation convention: For a homotopy $H : X \times I \to Y$, we will often write
$$H_t := H(\cdot, t) : X \to Y \qquad \text{for each } t \in I.$$
- Definition: $p : E \to B$ is a **free** (Hurewicz) **fibration** if it satisfies the HLP with respect to all spaces $X \in \mathsf{Top}$. The word "free" (or the synonyms "unpointed" or "unbased") is included in order to distinguish this from the pointed variant below, but will be omitted whenever possible. The word "Hurewicz" will almost always be omitted, but is meant to distinguish this from certain useful weaker conditions, such as:
- Definition: $p : E \to B$ is a **Serre fibration** if it satisfies the HLP with respect to all CW-complexes X. Note that E and B do not need to be CW-complexes. This condition is often easier to verify, and has some very nice applications to higher homotopy groups (we'll get there in a few lectures).
- Definition: A pointed map $p : E \to B$ has the (pointed) **homotopy lifting property** with respect to some class of pointed spaces $\mathscr{C} \subset \mathsf{Top}_*$ if the lifting problem
$$\begin{tikzcd} (X, *) \arrow[r, "\widetilde{H}_0"] \arrow[d, hook, "i_0"] & (E, *) \arrow[d, "p"] \\ (X \times I, \{*\} \times I) \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & (B, *) \end{tikzcd}$$
is solvable for all $X \in \mathscr{C}$; in other words, we require the HLP but with maps and homotopies replaced by *pointed maps* and *pointed homotopies*.
- Definition: A pointed map $p : E \to B$ is a **pointed** (Hurewicz) **fibration** if it satisfies the pointed HLP with respect to all $X \in \mathsf{Top}_*$.
- Theorem ("the main property of fibrations"): Assume $p : E \to B$ satisfies the (free or pointed) HLP with respect to some class $\mathscr{C}$ in Top or Top_* respectively; in the free case, assume also that B is path-connected, so that sets of (free or pointed) homotopy classes $[X, B]$ have natural base points in either case. Denote the inclusion $i : F := p^{-1}(b_0) \hookrightarrow E$, where $b_0 \in B$ is the base point in the pointed case, or any chosen point in the free case. Then for every $X \in \mathscr{C}$, the sequence
$$[X, F] \xrightarrow{i_*} [X, E] \xrightarrow{p_*} [X, B]$$
is exact.
- Convenient fact (see Exercise 3.2): Pointed fibrations are also free fibrations after forgetting their base points.
- Inconvenient fact: If $p : E \to B$ is a free fibration, choosing base points $* \in B$ and $* \in p^{-1}(*) \subset E$ to make p into a pointed map does *not* automatically make it into a pointed fibration! On the other hand, actual counterexamples are not easy to find, mainly because. . .
- Sufficiently convenient fact: The aforementioned pointed map $p : E \to B$ does however satisfy the pointed HLP with respect to all "reasonable" pointed spaces. This means that in practice, one rarely actually needs to worry about the distinction between free and pointed fibrations. (Giving more details on this will require some discussion of cofibrations, which is coming next week.)
- Definition: A sequence of maps $Z \xrightarrow{j} X \xrightarrow{f} Y$ has the **homotopy type of a fibration** if there exists a fibration $p : E \to B$ with fiber inclusion $i : F := p^{-1}(*) \hookrightarrow E$ and a

homotopy commutative diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\sim"] & X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\sim"] & Y \arrow[d] \\ F \arrow[r, "i"] & E \arrow[r, "p"] & B \end{tikzcd}$$

in which the vertical maps are all homotopy equivalences. (Note: This definition is sensible in either the free or the pointed case—for the latter, one takes all maps and homotopies to be pointed.) It follows that sequences of the form

$$[Q, Z] \xrightarrow{j_*} [Q, X] \xrightarrow{f_*} [Q, Y]$$

are exact for all Q (assuming as usual in the unpointed case that Y is path-connected). Remark: There are obvious generalizations of this conclusion for cases where $p : E \to B$ only satisfies the HLP with respect to some smaller class of spaces $\mathscr{C}$; then one must also assume $Q \in \mathscr{C}$ in writing down such exact sequences.

- Convention: Unless the words "free" or "pointed" are included explicitly, every statement in the rest of this lecture is meant to be valid for both cases, with closely analogous proofs in either context.
- Theorem 1: For every map $f : X \to Y$, there exists a space Z (the "**homotopy fiber**" of f) and a map j such that $Z \xrightarrow{j} X \xrightarrow{f} Y$ has the homotopy type of a fibration. In other words, "every map is a fibration up to homotopy equivalence". Proof at the end of the lecture.
- The dual perspective on the HLP: For topological spaces X, Y, abbreviate

$$Y^X := C(X, Y)$$

with the compact-open topology.[14] This makes X^I the space of paths in X, and since I is locally compact and Hausdorff, the evaluation map

$$\mathrm{ev} : X^I \times I \to X : (\gamma, t) \mapsto \gamma(t)$$

is a homotopy between $\mathrm{ev}_0 := \mathrm{ev}(\cdot, 0)$ and $\mathrm{ev}_1 := \mathrm{ev}(\cdot, 1)$; one can deduce from this (see Exercise 3.3) that for every continuous map $f : X \to Y$, the induced map

$$f^I : X^I \to Y^I : \gamma \mapsto f \circ \gamma$$

is continuous, thus defining a functor $(\cdot)^I = C(I, \cdot) : \mathsf{Top} \to \mathsf{Top}$. Moreover, the natural bijection $Y^{X \times I} \cong (Y^I)^X$ identifies homotopies $H : X \times I \to Y$ with maps $H : X \to Y^I$ into path space, and this translates the HLP into the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\widetilde{H}"] \arrow[drr, bend left, "\widetilde{H}_0"] \arrow[ddr, bend right, "H"'] & & \\ & E^I \arrow[r, "\mathrm{ev}_0"] \arrow[d, "p^I"] & E \arrow[d, "p"] \\ & B^I \arrow[r, "\mathrm{ev}_0"] & B. \end{tikzcd}$$

Interpretation: the HLP is satisfied if and only if E^I with its maps to E and B^I defines a *"weak fiber product"* of the maps $p : E \to B$ and $\mathrm{ev}_0 : B^I \to B$, i.e. the map $X \to E^I$ is required to exist, but need not be unique (as an actual universal property would require).

[14] We had previously used the notation Y^X to mean *all* (not necessarily continuous) maps $X \to Y$, but we are now altering the definition of this notation in the context of topological spaces, because it's a convenient shorthand.

- Constructions of fibrations (proofs are straightforward and mostly consist of drawing some diagrams and adding some dotted arrows):
  (1) **Projection** maps $B \times F \to B$ are always fibrations. (Note that here one can clearly see the non-uniqueness of the lifted homotopy, outside of special cases such as when F is a discrete space, which would make the projection a covering map.)
  (2) **Path space**: By a very slight extension of what we proved last time, the map
  $$X^I \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} X \times X$$
  is always a fibration.
  (3) **Compositions**: If $p : E \to B$ and $f : B \to A$ are fibrations, then so is $f \circ p : E \to A$. (Remark: If we didn't already know this, we could now deduce from the first three items on this list that the maps $\mathrm{ev}_0, \mathrm{ev}_1 : X^I \to X$ individually are also fibrations.)
  (4) **Products**: Given two fibrations $p_i : E_i \to B_i$ for $i = 1, 2$, the product map $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is also a fibration.
  (5) **Pullbacks**: Assume E' is a fiber product of $p : E \to B$ and another map $f : B' \to B$, so we have a diagram
  $$\begin{array}{ccc} E' & \xrightarrow{f'} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array},$$
  and E' can be identified with $B' {}_f\!\times_p E \subset B' \times E$ so that f' and p' become the obvious projections. For any $b \in B'$, writing $E'_b := (p')^{-1}(b) \subset E'$, it follows that
  $$E'_b \xrightarrow{f'} E_{f(b)}$$
  is a homeomorphism, thus we think of E' as a union of the same collection of fibers as E, but parametrized over B' instead of B. Proposition: If p is a fibration, then so is p'. (We then call $p' : E' \to B'$ the **pullback** of $p : E \to B$ via the map $B' \to B$, and sometimes emphasize this by writing $f^*E := E'$. It is also often called an **induced fibration**.) Sketch of proof: Given a homotopy $X \times I \to B'$, composing it with f gives a homotopy to B, which can be lifted to E. The universal property of the pullback determines from this a unique map $X \times I \to E'$, which turns out to be the lift we need.
  (6) **Path/loop** spaces: Analogously to the free path space functor $\mathsf{Top} \to \mathsf{Top} : X \mapsto X^I$, the based path and loop spaces define functors $\mathsf{Top}_* \to \mathsf{Top}_*$ sending X to PX or ΩX. Proposition: For any free fibration $p : E \to B$, the map $p^I : E^I \to B^I$ is also a (free) fibration; similarly for any pointed fibration $p : E \to B$, the maps $Pp : PE \to PB$ and $\Omega p : \Omega E \to \Omega B$ are pointed fibrations. Proof in the free case: The correspondence $Y^{X \times I} \cong (Y^I)^X$ translates the HLP for $p^I : E^I \to B^I$ with respect to a space X into a lifting problem of the form
  $$\begin{array}{ccc} X \times I & \longrightarrow & E \\ \downarrow{\scriptstyle i_0 \times \mathrm{Id}} & & \downarrow{\scriptstyle p} \\ X \times I \times I & \longrightarrow & B \end{array},$$
  which is solvable because $p : E \to B$ has the HLP with respect to $X \times I$. The proofs for the based path and loop spaces are Exercise 3.4.

- Proposition: Every pointed fibration $F \overset{i}{\hookrightarrow} E \xrightarrow{p} B$ (we will often write the inclusion of the fiber $F = p^{-1}(*) \subset E$ as part of the data) determines a canonical pointed homotopy class of maps
$$\Omega B \xrightarrow{\delta} F.$$
Part 1 of the proof: The idea is the same as in covering space theory, where each based loop $\gamma : I \to B$ gets interpreted as a path and then has a (in this case non-unique) lift $\widetilde{\gamma} : I \to E$ that starts at the base point but may end in some other point of $F = p^{-1}(*)$. Since $p : E \to B$ has the HLP with respect to the space ΩB, we can do this for all loops at once by interpreting $\mathrm{ev} : \Omega B \times I \to B$ as a homotopy and lifting it:
$$\begin{tikzcd} \Omega B \arrow[r, "\mathrm{const}"] \arrow[d, hook] & E \arrow[d, "p"] \\ \Omega B \times I \arrow[r, "\mathrm{ev}"] \arrow[ur, dashed, "\widetilde{\mathrm{ev}}"] & B \end{tikzcd}.$$
On ΩB, ev_0 and ev_1 are both constant maps to the base point of B, thus $\delta := \widetilde{\mathrm{ev}}_1 : \Omega B \to E$ takes values in F. (We will need some machinery developed next week in order to show that the homotopy class of δ is independent of the choice of lift.)
- Theorem 2 (just a preview of our goal for next week, with the caveat that the statement may need minor modifications before it is strictly correct): For any pointed fibration $F \xrightarrow{i} E \xrightarrow{p} B$, every triple of consecutive terms in the sequence of pointed (homotopy classes of) maps
$$\ldots \longrightarrow \Omega^2 E \xrightarrow{\Omega^2 p} \Omega^2 B \xrightarrow{\delta} \Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{\delta} F \xrightarrow{i} E \xrightarrow{p} B$$
has the homotopy type of a pointed fibration. (Note: Implicit in this sequence is the observation that ΩF has an obvious identification with the fiber of $\Omega E \xrightarrow{\Omega p} \Omega B$ over the base point of ΩB, such that $\Omega i : \Omega F \to \Omega E$ becomes its inclusion.)
- Remark: Theorem 2 gives us long exact sequences of sets of pointed homotopy classes
$$\ldots \to [X, \Omega^2 B] \to [X, \Omega F] \to [X, \Omega E] \to [X, \Omega B] \to [X, F] \to [X, E] \to [X, B],$$
and since pointed fibrations are also free fibrations, the corresponding sequence of sets of *free* homotopy classes is also exact wherever exactness makes sense (i.e. when the relevant space is known to be path-connected).
- Definition: The (double) **mapping path space** of two maps $f : X \to Z$ and $g : Y \to Z$ is
$$P(f, g) := X \ {}_f{\times}_{\mathrm{ev}_0} Z^I \ {}_{\mathrm{ev}_1}{\times}_g Y = \left\{ (x, \gamma, y) \in X \times Z^I \times Y \;\middle|\; \gamma \text{ is a path in } Z \text{ from } f(x) \text{ to } g(y) \right\}.$$
This construction is “dual” to the double mapping cylinder, in the sense that it fits into all the same diagrams but with the arrows reversed, e.g. with the obvious projections to X and Y, the diagram
$$\begin{tikzcd} P(f,g) \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\sim"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
commutes up to an obvious homotopy, and any homotopy commutative diagram of the form
$$\begin{tikzcd} Q \arrow[r, "\varphi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\underset{H}{\sim}"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

naturally determines a map $u : Q \to P(f,g)$ for which the diagram

$$\begin{tikzcd} & X \\ Q \arrow[ur, "\varphi"] \arrow[r, "u"] \arrow[dr, "\psi"'] & P(f,g) \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd}$$

commutes. In summary: $P(f,g)$ is a **homotopy pullback** of the maps f and g. (As with homotopy pushouts: We are *not* claiming that $P(f,g)$ defines an actual pullback in the category hTop, which would require the homotopy class of $u : Q \to P(f,g)$ above to be determined uniquely by the commutativity (up to homotopy) of the diagram. Exercise for those who are so inclined: show that pullbacks in hTop do not always exist.)

- Remark: If f and g are pointed maps, then there is an obvious choice of base point for $P(f,g)$ that makes everything in the above discussion pointed. In contrast to the case of mapping cylinders, this does not require any modification to the definition of the space $P(f,g)$ itself.
- Proposition: The map

$$(\pi_X, \pi_Y) : P(f,g) \to X \times Y$$

is a fibration (and by composition, so therefore are the individual projections π_X and π_Y). Proof: It's a pullback of the path space fibration $Z^I \to Z \times Z$:

$$\begin{tikzcd} P(f,g) \arrow[r] \arrow[d, "(\pi_X,\pi_Y)"'] & Z^I \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ X \times Y \arrow[r, "f\times g"'] & Z \times Z \end{tikzcd},$$

where the map $P(f,g) \to Z^I$ is $(x,\gamma,y) \mapsto \gamma$.

- Proof of Theorem 1: Define the **mapping path space** of $f : X \to Y$ as

$$\begin{aligned} P(f) := P(f, \mathrm{Id}_Y) &= \left\{ (x,\gamma,y) \in X \times Y^I \times Y \;\middle|\; \gamma(0) = f(x) \text{ and } \gamma(1) = y \right\} \\ &= \left\{ (x,\gamma) \in X \times Y^I \;\middle|\; \gamma(0) = f(x) \right\}. \end{aligned}$$

By contracting every path back to its starting point, we find a deformation retraction of $P(f)$ to an embedded copy of X, i.e. the map $h : X \to P(f) : x \mapsto (x, \mathrm{const}_{f(x)})$ is a homotopy inverse of the projection $\pi_X : P(f) \to X : (x,\gamma) \mapsto x$. Moreover, π_X is a fibration, and more importantly, so is the other projection

$$p := \pi_Y : P(f) \to Y : (x,\gamma) \mapsto \gamma(1),$$

which now fits into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] & P(f) \arrow[dl, "p"] \\ Y \end{tikzcd},$$

in which h is a homotopy equivalence. One can now take the fiber $Z := p^{-1}(*) \subset P(f)$ with inclusion $i : Z \hookrightarrow P(f)$ and define $j := \pi_X \circ i : Z \to X$, producing the diagram

$$\begin{tikzcd} Z \arrow[r, "j"] & X \arrow[r, "f"] \arrow[dr, phantom, "\sim"] & Y \\ Z \arrow[u, "\mathrm{Id}"] \arrow[r, hook, "i"] & P(f) \arrow[u, "\pi_X"] \arrow[r, "p"'] & Y \arrow[u, "\mathrm{Id}"'] \end{tikzcd},$$

in which the vertical maps are all homotopy equivalences.

- To do list for next week: Clarify in what sense the fibration $P(f) \xrightarrow{p} Y$ and homotopy fiber Z associated to $f : X \to Y$ are unique, why the homotopy class of $\delta : \Omega B \to F$ is well defined, where the long exact sequence in Theorem 2 comes from, why free fibrations with added base points are almost as good in practice as pointed fibrations, and along the way, what a *cofibration* is and what this whole story looks like with all the arrows reversed. That will keep us busy enough.

Suggested reading. The main nontrivial things we did this week can be found in [DK01, §6.2, §6.4 and §6.9]. An unfortunate omission in both [DK01] and [tD08] is the pointed variant of the homotopy lifting property, but there's a fuller discussion of this and the associated subtleties in [Cut21, Week 6: Fibrations IV].

Exercises (for the Übung on 16.05.2024). Thursday the 9th is a holiday, so we'll talk about these exercises (and probably some others) in the Übung for the following week.

Exercise 3.1. The following are two examples of maps $p : E \to B$ with the property that all fibers $E_b := p^{-1}(b)$ are homotopy equivalent—we will see next week that this is a property that fibrations must have, though in these examples, the fibers are *not* all homeomorphic, so they cannot be fiber bundles. Determine whether each is actually a fibration.

(a) The projection $E \to \mathbb{R} : (x, y) \mapsto x$ of the subset $E := \{(x, y) \in \mathbb{R}^2 \mid |y| \leqslant |x|\}$.
(b) The projection $E \to I : (x, y) \mapsto x$ of the subset $E := (I \times \{0\}) \cup (\{0\} \times I)$.

Exercise 3.2. Prove that every pointed fibration becomes a free fibration after forgetting the base points.
Hint: For any $X \in \mathsf{Top}$ and $Y \in \mathsf{Top}_*$, unpointed maps $X \to Y$ are equivalent to pointed maps $X_+ \to Y$, for a pointed space X_+ defined as the disjoint union of X with a one point space.

Exercise 3.3. For this exercise, let's agree to call a space X **locally compact** if every neighborhood of every point $x \in X$ contains a compact neighborhood of x.[15] If you prefer the convention that "locally compact" just means every point has a compact neighborhood, then feel free to add the assumption that X is Hausdorff, which makes the simpler definition of locally compact equivalent to the stricter one stated above. We assume as usual that the space $C(X, Y)$ of continuous maps $X \to Y$ carries the compact-open topology. The first three parts below add up to the proofs of two lemmas that were stated without proof in lecture.

(a) Prove that if X is locally compact, then the evaluation map $\mathrm{ev} : C(X, Y) \times X \to Y : (f, x) \mapsto f(x)$ is continuous.
(b) Prove that for any spaces X, Y, Z and any continuous map $f : X \times Y \to Z$, the map $\hat{f} : X \to C(Y, Z)$ defined by $\hat{f}(x)(y) := f(x, y)$ is also continuous, thus defining an injective map

$$C(X \times Y, Z) \to C(X, C(Y, Z)) : f \mapsto \hat{f}. \tag{3.1}$$

[15] This definition presumes the term **neighborhood of** x to mean any set that contains an open set containing x, i.e. the neighborhood itself need not be open.

Remark: One would ideally also like to know that the map (3.1) is continuous, but let's not worry about that for now.

(c) Prove that for two given spaces Y and Z, the evaluation map $\mathrm{ev} : C(Y,Z) \times Y \to Z$ is continuous if and only if the map (3.1) is surjective for all spaces X.
Hint: The identity map is continuous on all spaces.
Comment: It follows in particular that (3.1) is a bijection whenever Y is locally compact; we have already made ample use of the special case $Y := I$ in the lectures.

(d) Give a concrete example of three spaces for which the map (3.1) is not surjective.

(e) Writing $X^I := C(I, X)$ for the space of paths in X, show that for any continuous map $f : X \to Y$, the induced map $f^I : X^I \to Y^I : \gamma \mapsto f \circ \gamma$.
Remark: There is a quite easy way to deduce this from the previous parts of this exercise, exploiting the fact that I is locally compact, but in fact, much more than this is true, and it can be proved without using local compactness at all (see Exercise 5.9).

Before we continue, here is a definition: A continuous map $q : \widetilde{X} \to X$ is called a **quotient map** if it is surjective and the open sets $\mathcal{U} \subset X$ are precisely the sets for which $q^{-1}(\mathcal{U}) \subset \widetilde{X}$ is open. Equivalently, q is a quotient map if and only if it descends to a homeomorphism $\widetilde{X}/\sim \to X$, for the equivalence relation $\sim$ on $\widetilde{X}$ such that $x \sim y$ means $q(x) = q(y)$. Most crucially, being a quotient map means that in any diagram of the form

$$\begin{tikzcd} \widetilde{X} \arrow[d, "q"] \arrow[dr, "\widetilde{f}"] & \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

continuity of the map $\widetilde{f}$ implies that f is also continuous. (The converse is of course obvious, since q is continuous.)

(f) Given two quotient maps $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$, can you show that the product map $p \times q : \widetilde{X} \times \widetilde{Y} \to X \times Y$ is also a quotient map? Give it a try, but do not try too hard... Once you've gotten stuck and realized that it isn't obvious, take a look at [Mun75, pp. 143–144].

(g) Prove that if Y is a space with the property that $\mathrm{ev} : C(Y,Z) \times Y \to Z$ is continuous for every space Z, then for every quotient map $q : \widetilde{X} \to X$, the product

$$q \times \mathrm{Id}_Y : \widetilde{X} \times Y \to X \times Y$$

is also a quotient map. In particular, this is true whenever Y is locally compact.

(h) In last week's Übung, I sketched an approach to proving $S^m * S^n \cong S^{m+n+1}$ (Exercise 2.7) that led to the more general formula

$$X * Y \cong \big(CX \times Y\big) \cup_{X \times Y} \big(X \times CY\big),$$

obtained by splitting the double mapping cylinder in the middle and reinterpreting the quotients that one sees in the two halves. I also mentioned however that it is not so obvious how generally this formula holds, because e.g. $CX \times Y$ is a product of a quotient, which is not always homeomorphic to the corresponding quotient of a product. Can you name some conditions on X and Y that will guarantee that the formula holds? (Your conditions should preferably include the special case with $X = S^m$ and $Y = S^n$!)

Exercise 3.4. In lecture, we exploited the natural bijective correspondence between maps $X \to Y^I$ and maps $X \times I \to Y$ to prove that for any fibration $p : E \to B$, the map $p^I : E^I \to B^I$ is also a fibration, give or take some minor details (e.g. the continuity of p^I is provided by Exercise 3.3(e) above).

(a) Describe a pointed space $P'X$ associated to every pointed space X with the property that there is a natural bijective correspondence between pointed maps $X \to PY$ to the based path space and pointed maps $P'X \to Y$. Moreover, there should also be a bijective correspondence between pointed homotopies $X \times I \to PY$ and pointed homotopies $P'X \times I \to Y$.
Achtung: The detail about homotopies will require you to think about products of quotients, so Exercise 3.3(g) may be useful.
(b) Do the same thing as in part (a) for pointed maps/homotopies to the based loop space ΩY.
(c) Prove the result stated in lecture that for any pointed fibration $p : E \to B$, the induced maps $Pp : PE \to PB$ and $\Omega p : \Omega E \to \Omega B$ are also pointed fibrations.

Exercise 3.5. Prove that if X is path-connected, then the homotopy type of the based loop space ΩX is independent of the choice of base point.

Exercise 3.6. Formulate an analogue for mapping path spaces $P(f,g)$ of the theorem we previously proved about mapping cylinders $Z(f,g)$ defining a functor from a category of pushout diagrams to hTop. Convince yourself in this way that the homotopy type of $P(f,g)$ only depends on the homotopy classes of the two maps $f : X \to Z$ and $g : Y \to Z$.

Exercise 3.7. The mapping path space $P(f) = \{(x,\gamma) \in X \times Y^I \mid \gamma(0) = f(x)\}$ of a map $f : X \to Y$ can be described as the fiber product of the maps $f : X \to Y$ and $\mathrm{ev}_0 : Y^I \to Y$, so by the universal property of the fiber product, the diagram

$$\begin{tikzcd} X^I \arrow[r, "f^I"] \arrow[d, "\mathrm{ev}_0"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

determines a map $u : X^I \to P(f)$. Show that $f : X \to Y$ is a fibration if and only if the map $u : X^I \to P(f)$ admits a right-inverse $\lambda : P(f) \to X^I$; in this situation, λ is sometimes called a **lifting function** for the fibration $f : X \to Y$.

4. Week 4

Thursday this week is a holiday, so there is only one lecture and no Übung.

Lecture 6 (6.05.2024): The transport functor.

- Recall: We constructed for every (unpointed or pointed) map $f : X \to Y$ a diagram

$$\begin{tikzcd} F(f) \arrow[r, "j"] & X \arrow[r, "f"] & Y \\ F(f) \arrow[u, "\mathrm{Id}"] \arrow[r, hook, "i"] & P(f) \arrow[u, "\pi_X"] \arrow[r, "p"'] \arrow[ur, phantom, "\sim"] & Y \arrow[u, "\mathrm{Id}"'] \end{tikzcd},$$

  where the bottom row is a (free or pointed) fibration with fiber $F(f) := p^{-1}(*) \subset P(f)$ (preimage of the base point $* \in Y$ if pointed, an arbitrary point if not), and all the vertical maps are homotopy equivalences. We call $F(f)$ the **homotopy fiber** of $f : X \to Y$. We also had a homotopy inverse $h : X \to P(f)$ of π_X fitting into the diagram

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, "h"] & X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "\mathrm{Id}"] \\ F(f) \arrow[r, hook] & P(f) \arrow[r, "p"] & Y \end{tikzcd},$$

which commutes on the nose (not just up to homotopy), where $F := f^{-1}(*) \subset X$, thus defining a **comparison map**

$$F \xrightarrow{h} F(f)$$

from the "actual" fiber of f to its homotopy fiber.

- Question: Does $f : X \to Y$ uniquely determine (up to what notion of equivalence?) the fibration $p : E \to Y$ in any diagram of the form

$$\begin{tikzcd} X \arrow[rr, "\simeq"', "h.e."] \arrow[dr, "f"'] & \sim & E \arrow[dl, "p"] \\ & Y & \end{tikzcd} \quad ?$$

For instance, if $f : X \to Y$ is already a fibration, are the two fibrations (and thus their fibers) equivalent in some sense?[16]

- Inspiration from differential geometry: For a smooth fiber bundle $p : E \to B$, any choice of *connection* associates to each smooth path $x \overset{\gamma}{\rightsquigarrow} y$ in B a *parallel transport* diffeomorphism

$$E_x \xrightarrow{P_\gamma} E_y,$$

and it is compatible with smooth concatenation of paths: $P_{\alpha\cdot\beta} = P_\beta \circ P_\alpha$. Connections live in a contractible space of choices, so up to homotopy, P_γ is independent of this choice and depends only on the (smooth) homotopy class of the path γ. Given any smooth homotopy $H : X \times I \to B$ between maps $H_0, H_1 : X \to B$, parallel transport determines a correspondence

$$\text{lifts} \begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{H}_0"] \arrow[r, "H_0"'] & B \end{tikzcd} \quad \mapsto \quad \text{lifts} \begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{H}_1"] \arrow[r, "H_1"'] & B \end{tikzcd}$$

defined by $\widetilde{H}_1(x) := P_{H(x,\cdot)} \circ \widetilde{H}_0(x)$. At the level of homotopy classes of lifts, this correspondence is independent of choices, and depends on H only up to (smooth) homotopy of homotopies. In homotopy theory, we have no smooth structures and cannot talk about connections. . . but we probably *can* prove that things are unique up to homotopy!

- Definition: Given $B \in \mathsf{Top}$, the category Top_B of **spaces over** B has objects that are pairs (X, f) with X a space and $f : X \to B$ a map, and the set of morphisms $\mathrm{Hom}((X, f), (Y, g))$ consists of **maps over** B, meaning maps $\varphi : X \to Y$ that fit into the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd} .$$

Two such morphisms φ, ψ are **homotopic over** B if there exists a homotopy $\varphi \overset{H}{\rightsquigarrow} \psi$ such that H_t is a morphism $(X, f) \to (Y, g)$ for every $t \in I$. This notion defines the corresponding homotopy category hTop_B, and isomorphisms in this category are called **homotopy equivalences over** B. There are similar definitions for categories $\mathsf{Top}_{B,*}$ and $\mathsf{hTop}_{B,*}$ in which all maps and homotopies are required to be pointed.

[16]In the lecture I stated this question a bit differently, involving a more complicated diagram, but I later realized that that version was not exactly the question we are going to answer, nor is it the one that we really *need* to answer.

- Notation: Given two objects X, Y in Top_B or $\mathsf{Top}_{B,*}$, we denote by
$$[X,Y]_B := \operatorname{Hom}(X,Y) \text{ in } \mathsf{hTop}_B \text{ or } \mathsf{hTop}_{B,*} \text{ resp.}$$
the set of homotopy classes of (unpointed or pointed) maps $X \to Y$ over B. We can also write $[(X,f),(Y,g)]_B$ whenever the maps $f : X \to B$ and $g : Y \to B$ defining these objects need to be specified.
- Definition: Given $X, Y \in \mathsf{Top}$, the **homotopy groupoid** $\Pi(X,Y)$ is a category whose objects are maps $f : X \to Y$, with morphisms
$$\operatorname{Hom}(f,g) := \left\{\text{homotopies } f \overset{H}{\rightsquigarrow} g\right\} \Big/ \sim,$$
where the equivalence relation is "homotopy of homotopies": $H \sim H'$ means there is a homotopy $H \overset{\Phi}{\rightsquigarrow} H'$ of maps $X \times I \to Y$ such that $\Phi_s := \Phi(\cdot,\cdot,s) : X \times I \to Y$ for each $s \in I$ is also a homotopy $f \overset{\Phi_s}{\rightsquigarrow} g$. Composition of morphisms is defined by concatenation of homotopies. (Easy exercise: The equivalence relation makes this notion of composition associative. The proof is essentially the same as the proof that multiplication in the fundamental group is associative.)
- Remark: $\Pi(X,Y)$ is called a *groupoid* (and not just a category) because all of its morphisms are invertible; one can always reverse homotopies.
- Special case: $\Pi(Y) := \Pi(*,Y)$ is the **fundamental groupoid** of Y, and for each $y \in Y$, $\operatorname{Hom}(y,y)$ is then the (opposite of the) fundamental group $\pi_1(Y,y)$.[17]
- For $X, Y \in \mathsf{Top}_*$, there is a pointed variant of $\Pi(X,Y)$ whose objects are pointed maps and morphisms are homotopy classes of pointed homotopies. Amusing exercise: Is $\Pi(*,Y)$ interesting in the pointed case?
- Theorem: For every (free or pointed) fibration $p : E \to B$ and every space X (unpointed or pointed), there is a well-defined **transport functor**
$$\Pi(X,B) \to \mathsf{Set}$$
which associates to each map $f : X \to B$ the set $[(X,f),(E,p)]_B$ of homotopy classes of maps over B; we can interpret these as homotopy classes of *lifts* $\tilde{f} : X \to E$ of $f : X \to B$. To each homotopy class of homotopies $f \overset{H}{\rightsquigarrow} g$ of maps $f, g : X \to B$, it associates the map
$$[(X,f),(E,p)]_B \xrightarrow{H_\#} [(X,g),(E,p)]_p$$
which sends the homotopy class of the lift $\tilde{f}$ to the homotopy class of a lift $\tilde{g}$ obtained by lifting $H : X \times I \to B$ to a homotopy $\tilde{H} : X \times I \to E$ from $\tilde{f}$ to $\tilde{g}$.
- Remark: It is educational to try using the HLP to prove that $H_\#$ is independent of choices, but you will get stuck at some point and notice that the lifting problem you need to solve is more complicated than the one addressed by the HLP. We will deal with this next week, after talking a bit about the homotopy *extension* property and cofibrations. For the rest of this lecture, we take the existence of the transport functor as a black box and explore some of its applications.

[17] A slightly annoying detail here is that while $\operatorname{Hom}(y,y)$ has a natural group structure defined by composition of morphisms—which in this case means homotopy classes of concatenation of paths—the conventions of category theory then force multiplication in $\operatorname{Hom}(y,y)$ to be defined by $[\alpha][\beta] := [\beta \cdot \alpha]$. This is why, strictly speaking $\operatorname{Hom}(y,y)$ is the *opposite* group of $\pi_1(Y,y)$, rather than $\pi_1(Y,y)$ itself. For any group G with multiplication of elements $g,h \in G$ denoted by $gh \in G$, the **opposite group** G^{op} can be defined as the same set but with a new multiplication law "$\cdot$" defined by $g \cdot h := hg$, so there is no difference if G happens to be abelian, but in general G and G^{op} are different (though isomorphic!) groups. One occasionally sees claims in the literature that the "correct" definition of $\pi_1(Y,y)$ really should be what we normally call $\pi_1(Y,y)^{\mathrm{op}}$, so that it matches $\operatorname{Hom}(y,y)$ rather than its opposite group. But this idea does not seem to have caught on.

- Theorem: For any fibration $p : E \to B$ and any two homotopic maps $f_0, f_1 : B' \to B$, the pullback fibrations $f_0^* E \to B'$ and $f_1^* E \to B'$ are homotopy equivalent over B'. It follows in particular that for every $b \in B'$, there is a homotopy equivalence $(f_0^* E)_b \to (f_1^* E)_b$ between corresponding fibers.
- Corollary (the case $B' := *$): For a fibration $p : E \to B$, any two fibers over the same path-component of B are homotopy equivalent.
- Proof of the theorem: For each $i = 0, 1$ we have pullback diagrams

$$\begin{tikzcd} f_i^* E \arrow[r, "f_i'"] \arrow[d, "p_i"] & E \arrow[d, "p"] \\ B' \arrow[r, "f_i"] & B \end{tikzcd},$$

Let $B' \xrightarrow{f_t} B$ for $t \in I$ denote the family of maps defined by a given homotopy

$$f_0 \overset{F}{\rightsquigarrow} f_1.$$

The family $f_0^* E \xrightarrow{f_t \circ p_0} B$ then defines a homotopy

$$f_0 \circ p_0 \overset{H}{\rightsquigarrow} f_1 \circ p_0$$

of maps $f_0^* E \to B$, and using the transport functor, we obtain a bijection $H_\#$ that associates to each homotopy class of lifts of $f_0 \circ p_0$ a homotopy class of lifts of $f_1 \circ p_0$. Since $f_0' : f_0^* E \to E$ is a lift of $f_0 \circ p_0 : f_0^* E \to B$, we can feed this into $H_\#$ and thus obtain a lift $g : f_0^* E \to E$ of $f_1 \circ p_0$, and by the universal property of the pullback $f_1^* E$, this uniquely determines the map $\Phi_F : f_0^* E \to f_1^* E$ in the following diagram

$$\begin{tikzcd} f_0^* E \arrow[dr, dashed, "\Phi_F"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "p_0"'] & & \\ & f_1^* E \arrow[r, "f_1'"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\ & B' \arrow[r, "f_1"] & B \end{tikzcd}.$$

By Exercise 4.1 below, this construction defines a functor

$$\Pi(B', B) \to \mathsf{hTop}_{B'}$$

which associates to each map $f : B' \to B$ the induced fibration $f^* E \to B'$ and to each homotopy class of homotopies $f \overset{F}{\rightsquigarrow} g$ the homotopy class of maps over B' represented by $\Phi_F : f^* E \to g^* E$ as constructed above via the transport functor. Since morphisms in $\Pi(B', B)$ are all invertible, the maps Φ_F obtained in this way are all isomorphisms in $\mathsf{hTop}_{B'}$, meaning homotopy equivalences over B'.
- Theorem: If $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B$ are two fibrations and $f : E \to E'$ is a homotopy equivalence of spaces that is also a map over B (with respect to p, p'), then f is also a homotopy equivalence over B.

- Remark: Using Exercise 4.2 below, it follows that whenever we have two ways of replacing a map $f : X \to Y$ by fibrations $p_i : E_i \to Y$ as in the diagram

$$\begin{tikzcd} E_0 \arrow[dr, bend right, "p_0"'] & X \arrow[l, "\simeq"', "h.e."] \arrow[r, "\simeq", "h.e."'] \arrow[d, "f"] & E_1 \arrow[dl, bend left, "p_1"] \\ & Y & \end{tikzcd} ,$$

the two fibrations must be homotopy equivalent over Y, and their corresponding fibers therefore homotopy equivalent. In particular, all reasonable definitions of the term "homotopy fiber" give the same thing up to homotopy equivalence.
- Preparation for the proof: Given a map $X \xrightarrow{f} Y$ and a space E, f induces a map
$$[Y, E] \xrightarrow{f^*} [X, E] : \varphi \mapsto f^*\varphi := \varphi \circ f,$$
which is obviously bijective if f is a homotopy equivalence. If we are also given maps $q : Y \to B$ and $p : E \to B$, then for any map $\varphi : Y \to E$ over B, the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "f^*\varphi := \varphi\circ f"] \arrow[dr, "f^*q := q\circ f"'] & Y \arrow[r, "\varphi"] \arrow[d, "q"] & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$
means that f also induces a map
$$[(Y, q), (E, p)]_B \xrightarrow{f^*} [(X, f^*q), (E, p)]_B.$$
- Lemma: In the situation above, if $p : E \to B$ is a fibration and $f : X \to Y$ is a homotopy equivalence, then the map $[(Y, q), (E, p)]_B \xrightarrow{f^*} [(X, f^*q), (E, p)]_B$ is also bijective.
- Proof of the lemma: Given a homotopy inverse $g : Y \to X$ of X, choose a homotopy $\mathrm{Id}_Y \overset{H}{\rightsquigarrow} f \circ g$, so that $q \circ H$ is then a homotopy of maps $Y \to B$ from q to $q \circ f \circ g = g^*f^*q$. We claim that the diagram

$$\begin{tikzcd} {[(Y, q), (E, p)]_B} \arrow[r, "f^*"] \arrow[rr, bend left, "(q\circ H)_\#"] & {[(X, f^*q), (E, p)]_B} \arrow[r, "g^*"] & {[(Y, g^*f^*q), (E, p)]_B} \end{tikzcd} \tag{4.1}$$

commutes. The reason is that for any given map $(Y, q) \xrightarrow{\varphi} (E, p)$ over B, the following diagram reveals that there is an obvious choice of lift for the homotopy $q \circ H : Y \times I \to B$ with initial condition $\varphi : Y \to E$:

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{Id}"] \arrow[dd, hook] & & E \arrow[dd, "p"] \\ & Y \arrow[ur, "\varphi"] \arrow[dr, "q"] & \\ Y \times I \arrow[ur, "H"] \arrow[rr, "q\circ H"] & & B \end{tikzcd}$$

Choosing $\varphi \circ H : Y \times I \to E$ as the lifted homotopy, it defines a homotopy from φ to $\varphi \circ f \circ g = g^*f^*\varphi$ and thus proves the claim. Since $(q \circ H)_\#$ is a bijection, it follows that

f^* is injective and g^* is surjective. Using a homotopy of $g \circ f$ to Id_X, one can apply the same trick again to show that the composition

$$\begin{tikzcd} {[(X, f^*q), (E, p)]_B} \arrow[r, "g^*"] \arrow[rr, bend left] & {[(Y, g^*f^*q), (E, p)]_B} \arrow[r, "f^*"] & {[(X, f^*g^*f^*q), (E, p)]_B} \end{tikzcd}$$

is also bijective, implying that the same map g^* is also injective, and thus bijective. Since the composition g^*f^* in (4.1) is bijective, it now follows that f^* is bijective.

- Remark: The proof of the lemma should remind you of the proof that homotopy equivalences induce isomorphisms of fundamental groups (in spite of the annoying detail that the homotopy inverse need not respect base points). In fact, there is a dual version of this lemma for cofibrations, a special case of which involves homotopy classes of maps $S^1 \to X$ over a one point space, and the result in that case is precisely the isomorphism of fundamental groups.
- The proof of the theorem about homotopy equivalence of fibrations now follows from abstract nonsense; see Exercise 4.3 below.

Suggested reading. The notions of "spaces/maps over B" and the homotopy groupoid are introduced in [tD08, §2.2 and §2.9], with the special case of the fundamental groupoid treated at length in §2.5. My presentation of the transport functor is based essentially on [tD08, §5.6], though tom Dieck only gives very brief sketches of proofs in that section, since it appears after the corresponding discussion about cofibrations (which is formally similar).

In [DK01, §6.6], you will also find a fairly down-to-earth proof of the fact that for the fibration $P(f) \to Y$ constructed out of the mapping path space of any map $f : X \to Y$, the associated homotopy equivalence $h : X \to P(f)$ is also a homotopy equivalence over Y whenever $f : X \to Y$ itself is a fibration. This is less general than what we proved, because it applies only to a specific fibration $P(f) \to Y$ rather than an arbitrary fibration over Y that fits into a suitable diagram with $f : X \to Y$. Unfortunately, the proof of the main theorem about the long exact fibration sequence in [DK01, §6.11] sneakily uses the more general version of this uniqueness result, so as far as I can tell, this is a logical gap in the book.

Exercises (also for the Übung on 16.05.2024).

Exercise 4.1. In lecture we used the transport functor to associate to any fibration $p : E \to B$ and any homotopy class of homotopies F between two maps $f_0, f_1 : B' \to B$ a homotopy class of maps over B' in the form

$$\begin{tikzcd} f_0^*E \arrow[rr, "\Phi_F"] \arrow[dr, "p_0"'] & & f_1^*E \arrow[dl, "p_1"] \\ & B' & \end{tikzcd},$$

relating the two pullback fibrations $p_i : f_i^*E \to B'$ induced by $f_i : B' \to B$ for $i = 0, 1$. Complete the proof that this construction defines a functor

$$\Pi(B', B) \to \mathsf{hTop}_{B'},$$

which associates to each map $f : B' \to B$ the pullback fibration $f^*E \to B'$, with the important consequence that the map Φ_F determined by a homotopy is always a homotopy equivalence over B'. *Hint: Consider a family of maps $B' \xrightarrow{f_t} B$ parameterized by $t \in [0, 2]$, which you can think of as a concatenation of a homotopy from f_0 to f_1 with a homotopy from f_1 to f_2. Defining the induced maps $f_0^*E \to f_1^*E$ and $f_1^*E \to f_2^*E$ requires choosing lifts of certain homotopies of maps*

$f_0^*E \to B$ (for $0 \leqslant t \leqslant 1$) and $f_1^*E \to B$ (for $1 \leqslant t \leqslant 2$) respectively. Let these choices determine how you can continue the lift of the homotopy of maps $f_0^*E \to B$ over the interval $1 \leqslant t \leqslant 2$, thus defining the induced map $f_0^*E \to f_2^*E$.

Exercise 4.2. Assume $p : E \to Y$ is a fibration and $f : X \to Y$ is a map.

(a) Show that if $\varphi : X \to E$ is a map for which the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & \arrow[d, phantom, "\sim"] & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

commutes up to homotopy, then φ can be replaced with a homotopic map $X \to E$ that makes the diagram commute on the nose.

(b) Deduce the basic uniqueness result about fibrations associated to a map $f : X \to Y$, namely that for any diagram of the form

$$\begin{tikzcd} E_0 \arrow[dr, bend right, "p_0"'] & X \arrow[l, "\simeq"', "h.e."] \arrow[r, "\simeq", "h.e."'] \arrow[d, "f"] \arrow[dl, phantom, "\sim"] \arrow[dr, phantom, "\sim"] & E_1 \arrow[dl, bend left, "p_1"] \\ & Y & \end{tikzcd}$$

in which $p_0 : E_0 \to Y$ and $p_1 : E_1 \to Y$ are both fibrations, the two fibrations are homotopy equivalent over Y.

Exercise 4.3. Suppose $\mathscr{C}$ is a category and $X \xrightarrow{f} Y$ is a morphism in $\mathscr{C}$ with the property that the maps

$$\operatorname{Hom}(Y, X) \xrightarrow{f^*} \operatorname{Hom}(X, X) \qquad \text{and} \qquad \operatorname{Hom}(Y, Y) \xrightarrow{f^*} \operatorname{Hom}(X, Y)$$

defined via $f^*\varphi := \varphi \circ f$ are both bijections. Prove that f is an isomorphism of $\mathscr{C}$. Then use this to finish the proof of the theorem stated in lecture that every homotopy equivalence $E \to E'$ that is also a map over B for two fibrations $E, E' \to B$ is also a homotopy equivalence over B.

5. Week 5

Lecture 7 (13.05.2024): Cofibrations.

- Tricky lifting problem 1: If $p : E \to B$ is a free fibration and we choose base points $* \in B$ and $* \in p^{-1}(*) \subset E$ to make it a pointed map, then it satisfies the pointed HLP with respect to a pointed space X if and only if the lifting problem

$$\begin{tikzcd} X \times \{0\} \cup \{*\} \times I \arrow[r, "\widetilde{H}_0 \cup \mathrm{const}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

is solvable. Having $\widetilde{H}$ prescribed on $\{*\} \times I$ and not just on $X \times \{0\}$ means that the usual (free) HLP does not guarantee a solution to this problem.
- Tricky lifting problem 2: Showing that the transport functor for a free fibration $p : E \to B$ is well defined requires solving the lifting problem

$$\begin{tikzcd} X \times (\partial I \times I \cup I \times \{0\}) \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I^2 \arrow[r, "G"] \arrow[ur, dashed, "\tilde{G}"] & B \end{tikzcd}.$$

Here, $G : X \times I^2 \to B : (x,s,t) \mapsto H^{(s)}(x,t)$ is a 1-parameter family of homotopies $\{H_0 \overset{H^{(s)}}{\rightsquigarrow} H_1\}_{s\in I}$ between two fixed maps $H_0, H_1 : X \to B$, the lift $\widetilde{G}$ is prescribed on $X \times \partial I \times I$ because lifts of the two specific homotopies $H^{(0)}$ and $H^{(1)}$ have already been chosen, and it is prescribed on $X \times I \times \{0\}$ because we are also given a homotopy $\{\widetilde{H}_0^{(s)}\}_{s\in I}$ of lifts of H_0. The existence of $\widetilde{G}$ then implies a corresponding homotopy $\{\widetilde{H}_1^{(s)}\}_{s\in I}$ of lifts of H_1.

- More general question: Given a free fibration $p : E \to B$ and a map $j : A \to X$, under what conditions is the problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

solvable? We refer to this in the following as problem (FLP), for "fundamental lifting problem".
- Theorem FLP (the "fundamental lifting property"): Problem (FLP) is solvable whenever $j : A \to X$ is a free cofibration (see definition below) and either j or p is a homotopy equivalence.
Remark: We will only need a special case of Theorem FLP and thus will not prove it in full generality. Notice that the case where $j : A \to X$ is the inclusion $Y \times \{0\} \hookrightarrow Y \times I$ for some space Y is simply the HLP with respect to Y. Since the map $\mathrm{ev}_0 : Y^I \to Y$ is always both a fibration and a homotopy equivalence, the HEP defined below is another special case.
- Definition: A map $j : A \to X$ satisfies the (free) **homotopy extension property** (HEP) with respect to a space Y if the lifting problem

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "j"] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "H_0"] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

is solvable for all given maps H_0 and h. Interpretation: Since maps $A \to Y^I$ are equivalent to homotopies $A \times I \to Y$, the diagram asks that for any given homotopy $h : A \times I \to Y$ and map $H_0 : X \to Y$ satisfying $H_0 \circ j = h_0$, there should exist a homotopy $H : X \times I \to Y$ satisfying $H_t \circ j = h_t$ for all t. In other words, the problem

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "j"] & A \times I \arrow[d, "j\times \mathrm{Id}"] \arrow[ddr, bend left, "h"] & \\ X \arrow[r, hook, "i_0"] \arrow[drr, bend right, "H_0"] & X \times I \arrow[dr, dashed, "H"] & \\ & & Y \end{tikzcd}$$

is solvable, allowing us to interpret $X \times I$ as a *weak pushout* of the maps $i_0 : A \hookrightarrow A \times I$ and $j : A \to X$. (The word "weak" is included because the map H is not required to be unique, and in typical examples it is not.) We will see in Exercise 5.2 that without loss of generality, $j : A \to X$ is always the inclusion of a subspace $A \subset X$, in which case H is literally an extension of $h : A \times I \to Y$ to the larger domain $X \times I$.
- Definition: $j : A \to X$ is a (free) **cofibration** if it has the HEP with respect to all spaces Y.

- Main application: Assume for simplicity that $j : A \to X$ is the inclusion of a subspace $A \subset X$, and let $q : X \to X/A$ denote the quotient projection. Given a path-connected space Y, we can plug the maps $A \xrightarrow{j} X \xrightarrow{q} X/A$ into the contravariant functor $[\cdot, Y] : \mathsf{hTop} \to \mathsf{Set}_*$ and obtain a sequence of homotopy sets
$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{j^*} [A, Y].$$
Theorem: This sequence is exact whenever $j : A \to X$ has the free HEP with respect to Y, so in particular whenever it is a free cofibration. (The proof is an easy exercise.)
- Terminology: for a cofibration $j : A \to X$, we call A the **cobase** and $X/j(A)$ the **cofiber**.
- There is an analogous **pointed homotopy extension property** and thus a notion of **pointed cofibrations** in which all maps and homotopies are required to be pointed. For these, the theorem above is true for sets of pointed homotopy classes of maps to Y, and the presence of a base point removes the necessity of assuming Y is path-connected. (Note that we never need any path-connectedness assumption on A, X or X/A, in contrast to the case of fibrations.)
- Convention: As with fibrations, any statement we make about cofibrations without specifying the words *free/unpointed* or *pointed/based* should be understood to be valid in two parallel versions, one in the category Top or hTop, the other in Top_* or hTop_*. This is, however, possible less often with cofibrations than with fibrations, due to the more-than-cosmetic differences between spaces such as $X \times I$ and $(X \times I)/(\{*\} \times I)$.
- Constructions of cofibrations (analogous to the list in Lecture 5 for fibrations; for proofs, see Exercise 5.3):
  1. **Inclusions in coproducts**: For all spaces $A, Q \in \mathsf{Top}$, the inclusion $A \hookrightarrow A \amalg Q$ is a free cofibration, and for pointed spaces $A, Q \in \mathsf{Top}_*$, the inclusion $A \hookrightarrow A \vee Q$ is a pointed cofibration.
  2. **Cylinders**: Inclusions of the form
  $$X \amalg X \xrightarrow{i_0 \amalg i_1} X \times I \qquad \text{or} \qquad X \vee X \xrightarrow{i_0 \vee i_1} \frac{X \times I}{\{*\} \times I}$$
  are free or pointed cofibrations respectively, where $i_t(x) := (x, t)$.
  3. **Compositions**: The composition of two cofibrations is a cofibration.
  4. **Coproducts**: Given two (free or pointed) cofibrations $j_i : A_i \to X_i$ for $i = 1, 2$, the map
  $$j_1 \amalg j_2 : A_1 \amalg A_2 \to X_1 \amalg X_2 \qquad \text{or} \qquad j_1 \vee j_2 : A_1 \vee A_2 \to X_1 \vee X_2$$
  is a (free or pointed) cofibration respectively.
  5. **Pushouts**: Assume X' is the pushout of two maps $j : A \to X$ and $f : A \to A'$, giving rise to the diagram
  $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & A' \arrow[d, "j'"] \\ X \arrow[r, "f'"] & X' \end{tikzcd}.$$
  If $j : A \to X$ is a cofibration, then so is $j' : A' \to X'$. In this case we call $j' : A' \to X'$ the cofibration **induced** from $j : A \to X$ by the map $f : A \to A'$; this construction is sometimes called **change of cobase**.
- Proposition: For any two maps $f : Z \to X$ and $g : Z \to Y$ in Top or Top_*, the natural inclusion of $X \amalg Y$ or $X \vee Y$ respectively into the (unreduced or reduced) double mapping cylinder $Z(f, g)$ is a (free or pointed) cofibration.

Proof: Present it as the pushout of the maps $Z \amalg Z \hookrightarrow Z \times I$ and $f \amalg g : Z \amalg Z \to X \amalg Y$ in the unpointed case, or $Z \vee Z \hookrightarrow (Z \times I)\big/(\{*\} \times I)$ and $f \vee g : Z \vee Z \to X \vee Y$ in the pointed case.

- Corollary: Every map $f : X \to Y$ has the homotopy type of a cofibration whose cofiber (known as the **homotopy cofiber** of f) is the mapping cone of f. Proof:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
X \arrow[u, "\mathrm{Id}"] \arrow[r, hook, "i_X"] & Z(f) \arrow[u, "h"'] \arrow[r] & Z(f)\big/X \cong CX \cup_X Y = \operatorname{cone}(f),
\end{tikzcd}$$

where $h : Z(f) \to Y$ is the homotopy equivalence defined on $Z(f) = \big((X \times I) \amalg Y\big)\Big/ \sim$ by $h([(x,t)]) := f(x)$ for $(x,t) \in X \times I$ and $h(y) := y$ for $y \in Y$.
Remark: Following our usual convention, this result is equally valid in the unpointed and pointed cases. In the latter version, $Z(f)$ and $\operatorname{cone}(f)$ are the *reduced* mapping cylinder and cone respectively.

- In the following, we redefine the unreduced mapping cylinder of a map $j : A \to X$ by

$$Z(j) := Z(j, \mathrm{Id}) = X \cup_j (A \times I),$$

where $A \times I$ is glued to X along $A \times \{0\}$ instead of $A \times \{1\}$.
Theorem: There is a natural map $\Psi : Z(j) \to X \times I$ such that the following conditions are equivalent:
(1) $j : A \to X$ is a free cofibration;
(2) $\Psi : Z(j) \to X \times I$ admits a right-inverse $r : X \times I \to Z(j)$;
(3) $j : A \to X$ has the HEP with respect to the space $Z(j)$.
(For the pointed version of this theorem and its consequences, see Exercise 5.1.)
Proof: Look at the diagram

(5.1)
$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "j"] & A \times I \arrow[d, "\varphi_A"'] \arrow[dr, bend left, "j \times \mathrm{Id}"] \arrow[ddrr, bend left, "\varphi_A"] \arrow[dddrrr, bend left, "h"] & & & \\
X \arrow[r, "\varphi_X"] \arrow[drr, bend right, "i_0"] \arrow[ddrrr, bend right, "\varphi_X"] \arrow[dddrrrr, bend right, "H_0"'] & Z(j) \arrow[dr, "\Psi"] & & & \\
& & X \times I \arrow[dr, dashed, "r"] & & \\
& & & Z(j) \arrow[dr, "u"] & \\
& & & & Y
\end{tikzcd}$$

The top left square is a pushout square, with φ_A and φ_X denoting the maps canonically associated with the pushout. The universal property of the pushout implies that the maps Ψ and u exist and are unique; in light of uniqueness, it also implies that $r \circ \Psi = \mathrm{Id}$ if r exists. The map r does exist (but need not be unique) if j has the free HEP with respect to $Z(j)$, and in that case, $u \circ r$ solves the homotopy extension problem with respect to an arbitrary given space Y.

- Remark: If $j : A \to X$ is the inclusion of a subspace $A \subset X$ (which is not a loss of generality according to Exercise 5.2), then (5.1) shows that Ψ is the canonical bijection
$$Z(j) \to X \times \{0\} \cup A \times I,$$
which need not be a homeomorphism in general because the subspace topology on $X \times \{0\} \cup A \times I \subset X \times I$ may be different from the topology of $Z(j) = \big(X \amalg (A \times I)\big)/\sim$. But if $A \hookrightarrow X$ is a cofibration, then r restricts to $X \times \{0\} \cup A \times I$ as a continuous inverse of this bijection, meaning we have a homeomorphism $Z(j) \cong X \times \{0\} \cup A \times I$, and r can then be interpreted as a *retraction*
$$X \times I \xrightarrow{r} X \times \{0\} \cup A \times I.$$
Corollary: The inclusion $A \hookrightarrow X$ of a subspace $A \subset X$ is a cofibration if and only if there exists a retraction $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I$.
- Definition: For a closed subset $A \subset X$, we call (X, A) an **NDR-pair** (stands for "neighborhood deformation retract") if there exists a continuous function $u : X \to I$ and a homotopy $\rho : X \times I \to X$ such that
$$\rho_1 = \mathrm{Id}_X, \qquad \rho_t|_A = \mathrm{Id}_A \text{ for all } t \in I, \qquad \text{and} \qquad \rho_0(\{u < 1\}) \subset A.$$
Further, we call it a **DR-pair** if additionally $u < 1$ everywhere on X, in which case the open subset $\{u < 1\}$ is all of X and ρ is therefore a deformation retraction of X to A.
- Lemma (see Exercise 5.4):
  (1) If (X, A) and (Y, B) are NDR-pairs then so is $(X \times Y, A \times Y \cup X \times B)$, and it is a DR-pair whenever either of (X, A) or (Y, B) is a DR-pair.
  (2) If $A \subset X$ is a closed subset such that there exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$, then (X, A) is an NDR-pair.
- Corollary: For a closed subset $A \subset X$, the inclusion $A \hookrightarrow X$ is a free cofibration if and only if (X, A) is an NDR-pair.
Proof: Cofibration $\Rightarrow$ retraction $\Rightarrow$ NDR-pair according to the lemma and the previous corollary. Conversely, one easily checks that $(I, \{0\})$ is a DR-pair, so if (X, A) is an NDR-pair, then $(X \times I, X \times \{0\} \cup A \times I)$ is a DR-pair, implying the existence of the required retraction.
- Theorem (a useful special case of Theorem FLP): The lifting problem (FLP) is solvable whenever $j : A \to X$ is the inclusion of a subspace $A \subset X$ and (X, A) is a DR-pair. (For applications to tricky lifting problems 1 and 2, see Exercise 5.5.)
- Proof: Assume $u : X \to I$ and $\rho : X \times I \to X$ make (X, A) a DR-pair, so in particular, $u^{-1}(0) = A$, $\rho_1 = \mathrm{Id}_X$, $\rho_t|_A = \mathrm{Id}_A$ for all $t \in I$ and $\rho_0(X) = A$. The problem to be solved is
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "j"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}.$$
As an ansatz, we try to define $h : X \to E$ in the form
$$h(x) = \widetilde{H}(x, u(x)),$$
where $\widetilde{H} : X \times I \to E$ is a lift of the homotopy
$$H : X \times I \to B, \qquad H(x, t) := g \circ \rho(x, t).$$
The condition $h(a) = f(a)$ for $a \in A$ is then satisfied if and only if $\widetilde{H}_0|_A = f$, which we can arrange by requiring the initial lift of the homotopy to be $\widetilde{H}_0 := f \circ \rho_0$. The condition

$p \circ h(x) = H(x, u(x)) = g \circ \rho(x, u(x)) = g(x)$ is then satisfied if $\rho(x, u(x)) = x$ for all $x \in X$, which is not necessarily true in general, but can be arranged without loss of generality. Indeed, for each $x \in X$, $\rho(x, \cdot) \in X^I$ is a path starting in A and ending at x, and is a constant path for every $x \in A$. It therefore suffices to reparametrize ρ by speeding up each of these paths so that for each $x \notin A$, $\rho(x, \cdot)$ reaches the end already by time $t = u(x) > 0$. (Exercise: Write this down explicitly and reassure yourself that the modified version of ρ can be made continuous—in $X \backslash A$ and in the interior of A this is obvious, but one needs to think more carefully about the boundary of A.)

Lecture 8 (16.05.2024): The Puppe sequence of a fibration.

- The **Puppe exact sequence** of a pointed map $(X, x_0) \xrightarrow{f} (Y, y_0)$ is the sequence of pointed maps
$$\ldots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega i_f} \Omega F(f) \xrightarrow{-\Omega \pi_f} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{i_f} F(f) \xrightarrow{\pi_f} X \xrightarrow{f} Y.$$
All notation used here is clarified below.
Theorem 1: The Puppe sequence is natural (in the category-theoretic sense), and any three consecutive terms in the sequence have the homotopy type of a pointed fibration.
- Remark 1: The Puppe sequence induces long exact sequences of pointed homotopy sets
$$\ldots \to [Z, \Omega^2 Y] \to [Z, \Omega F(f)] \to [Z, \Omega X] \to [Z, \Omega Y] \to [Z, F(f)] \to [Z, X] \to [Z, Y]$$
for any pointed space Z. We will later derive exact sequences of higher homotopy groups as special cases of this. Since pointed fibrations are also free fibrations, the corresponding sequences of unpointed homotopy sets are also exact wherever exactness makes sense, i.e. whenever the relevant spaces in the Puppe sequence are known to be path-connected. If an unpointed map $f : X \to Y$ is given, one can apply the Puppe sequence after choosing $y_0 \in Y$ and $x_0 \in f^{-1}(y_0) \subset X$ arbitrarily.
- Remark 2: Naturality means that any diagram of pointed maps
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$
will induce maps between the corresponding terms of the Puppe sequences of f and f', and thus morphisms of exact sequences of homotopy sets. The proof is straightforward.
- Ingredient 1: *Mapping path spaces.*
Redefine the **based path space** of X by[18]
$$PX := \left\{ \gamma \in X^I \;\middle|\; \gamma(1) = x_0 \right\},$$
Recall from Lecture 6 the mapping path space and homotopy fiber of f,
$$P(f) := X \,{}_f\!\times_{\mathrm{ev}_0} Y^I, \qquad F(f) := X \,{}_f\!\times_{\mathrm{ev}_0} PY,$$
which come with a natural base point $(x_0, \mathrm{const}_{y_0}) \in F(f) \subset P(f)$ and projection maps
$$F(f) \subset P(f) \xrightarrow{\pi_f := \pi_X} X : (x, \gamma) \mapsto x, \qquad P(f) \xrightarrow{\pi_Y} Y : (x, \gamma) \mapsto \gamma(1).$$
Since mapping path spaces are all pullbacks of fibrations of the form $(\mathrm{ev}_0, \mathrm{ev}_1) : Z^I \to Z \times Z$, these projections are all pointed fibrations, and $F(f)$ is the fiber of $\pi_Y : P(f) \to Y$. The map $\pi_X : P(f) \to X$ has homotopy inverse $h : X \to P(f) : x \mapsto (x, \mathrm{const}_{f(x)})$, and this is why $F(f) \xrightarrow{\pi_f} X \xrightarrow{f} Y$ has the homotopy type of a fibration.

[18] We had previously defined it with $\gamma(0) = x_0$ instead of $\gamma(1) = x_0$.

- Lemma 1 (obvious): The natural embedding
$$\Omega Y \xrightarrow{i_f} F(f) : \gamma \mapsto (x_0, \gamma)$$
is a homeomorphism onto the fiber of the fibration $\pi_f : F(f) \to X$.
Theorem 1 is now already proved for the last four terms in the sequence.
- Lemma 2: If $f : X \to Y$ is a pointed fibration with fiber $F := f^{-1}(y_0) \subset X$, then the map $h : F \to F(f)$ defined by restriction of $h : X \to P(f)$ is also a homotopy equivalence.
Proof: We deduced from the existence of the transport functor in Lecture 6 that h in this situation is a homotopy equivalence over Y, i.e. an isomorphism in the category $\mathsf{hTop}_{Y,*}$, so it has an inverse in this category that restricts to a map $F(f) \to F$ that is a homotopy inverse of h.
- Ingredient 2: *Loop spaces.*
The functor $\Omega : \mathsf{Top}_* \to \mathsf{Top}_*$ sends each space X to its based loop space $\Omega X = \{\gamma \in X^I \mid \gamma(0) = \gamma(1) = x_0\}$ and each pointed map $f : X \to Y$ to the pointed map
$$\Omega f : \Omega X \to \Omega Y : \gamma \mapsto f \circ \gamma.$$
For any $\gamma \in \Omega X$, let $-\gamma \in \Omega X$ denote the inverted loop $t \mapsto \gamma(1-t)$, so that $f : X \to Y$ also determines a map
$$-\Omega f : \Omega X \to \Omega Y : \gamma \mapsto -(f \circ \gamma),$$
satisfying
$$-\Omega(f \circ g) = (-\Omega f) \circ \Omega g = \Omega f \circ (-\Omega g)$$
for any composable maps f, g. For iterated loop spaces, we denote
$$\Omega^2 X := \Omega(\Omega X), \qquad \Omega^3 X := \Omega^2(\Omega X) \quad \text{etc.},$$
and can thus view $\Omega^n : \mathsf{Top}_* \to \mathsf{Top}_*$ as a functor for every $n \in \mathbb{N}$.
- Lemma 3: Ω descends to a functor $\mathsf{hTop}_* \to \mathsf{hTop}_*$, i.e. homotopic maps $X \to Y$ induce homotopic maps $\Omega X \to \Omega Y$.
Proof: Easy, though one should think a bit about why the obvious homotopies $\Omega X \times I \to \Omega Y$ one defines are continuous (cf. Exercise 5.9).
- Lemma 4: For any pointed map $f : X \to Y$, the maps $\Omega^2 f := \Omega(\Omega f)$ and $-\Omega(-\Omega f)$ from $\Omega^2 X$ to $\Omega^2 Y$ are homotopic.

The proof of Lemma 4 involves some subtleties that merit a continuation of the "revenge of *Topologie I*" digression from Lecture 4:

- Let's say that a space $X \in \mathsf{Top}$ is **friendly**[19] if the evaluation map $\mathrm{ev} : Y^X \times X \to Y : (f, x) \mapsto f(x)$ is continuous for every other space $Y \in \mathsf{Top}$. According to Exercise 3.3, we know:
  - For all spaces X, Y, Z, there is a natural injection

(5.2)
$$Z^{X \times Y} \xrightarrow{\alpha} (Z^Y)^X : f \mapsto \hat{f}$$
defined by $\hat{f}(x)(y) := f(x, y)$.
  - The map α is not surjective in general, e.g. one can find counterexamples with $Y = \mathbb{Q} \subset \mathbb{R}$, carrying the subspace topology.
  - The map α is bijective whenever Y is friendly.
  - All locally compact spaces are friendly.

[19] It's not standard terminology, but I personally think it should be.

We have already quite often made use of the fact that α is bijective in particular when $Y = I$. Note that while $Z^{X\times Y}$ and $(Z^Y)^X$ both have natural compact-open topologies, we have not yet considered whether the map α or its inverse (when it exists) is continuous. It will be a lot easier to understand loop spaces and iterated loop spaces if we can answer this question.

- Lemma 5: If X, Y and $X \times Y$ are all friendly, then the map α in (5.2) is a homeomorphism. (Note that there is no condition on Z.)
  Proof sketch: To show that α is continuous, it will suffice if we can show that α lies in the image of the natural map
  $$C(Z^{X\times Y} \times X, Z^Y) \to C(Z^{X\times Y}, (Z^Y)^X),$$
  which is just another case of α with different spaces. A natural map $\beta : Z^{X\times Y} \times X \to Z^Y$ to consider in this situation is defined by
  $$\beta(f, x)(y) := f(x, y),$$
  and one easily checks that $\widehat{\beta} = \alpha$, so α will be continuous if β is. Since the target of β is another space of maps, we can iterate this trick by observing that β must be continuous if it is in the image of the natural map
  $$C(Z^{X\times Y} \times X \times Y, Z) \to C(Z^{X\times Y} \times X, Z^Y),$$
  which is yet another case of α with different spaces. Here we can try $\mathrm{ev} : Z^{X\times Y} \times (X \times Y) \to Z$, which is indeed continuous since $X \times Y$ is friendly, and it satisfies $\widehat{\mathrm{ev}} = \beta$, thus implying that β is continuous, and therefore so is α. Similar tricks will imply that α^{-1} is also continuous, due to the assumption that X and Y are both friendly.
- Lemma 6: If X and Y are both friendly, then the composition map
  $$Z^Y \times Y^X \xrightarrow{c} Z^X : (f, g) \mapsto f \circ g$$
  is continuous.
  Remark: This is not the most general version I am aware of for this result, e.g. it is also a standard exercise in point-set topology to show that c is continuous whenever Y is locally compact, without any assumption on X. But I do not want to do that exercise here, and we will not need the result at that level of generality.
  Proof: Since the image of c is a space of maps, we can again use the trick used in Lemma 5 and conclude that c is continuous if it is in the image of the natural map
  $$C(Z^Y \times Y^X \times X, Z) \to C(Z^Y \times Y^X, Z^X).$$
  Defining $\kappa : Z^Y \times Y^X \times X \to Z$ by $\kappa(f, g, x) := f \circ g(x)$ gives $\widehat{\kappa} = c$, and the problem is thus reduced to showing that κ is continuous, which it is, because it is the composition of the two maps
  $$Z^Y \times Y^X \times X \xrightarrow{\mathrm{Id}\times\mathrm{ev}} Z^Y \times Y \xrightarrow{\mathrm{ev}} Z,$$
  and these are both continuous because X and Y are friendly.

End of digression.

- Proof of Lemma 4: Since I and $I^2 = I \times I$ are both locally compact (and therefore friendly), we have a natural homeomorphism $(X^I)^I \cong X^{I^2}$, which gives rise to a homeomorphism of the double loop space,
  $$\Omega^2 X \cong \left\{\alpha : I^2 \to X \;\middle|\; \alpha(\partial I^2) = \{x_0\}\right\} \subset C(I^2, X) = X^{I^2}.$$
  Under this identification, the maps $\Omega^2 X \to \Omega^2 Y$ defined by $\Omega^2 f$ and $-\Omega(-\Omega f)$ become
  $$(\Omega^2 f)(\alpha) = f \circ \alpha, \qquad \text{and} \qquad -\Omega(-\Omega f)(\alpha) = f \circ \bar{\alpha}, \qquad \text{where } \bar{\alpha}(s, t) := \alpha(1 - s, 1 - t).$$

If we identify I^2 with a round disk and use rotations, we can find a continuous family of homeomorphisms of pairs

$$\left\{\varphi_\rho : (I^2, \partial I^2) \to (I^2, \partial I^2)\right\}_{\rho \in I} \qquad \text{such that} \qquad \varphi_0 = \mathrm{Id} \text{ and } \varphi_1(s,t) = (1-s, 1-t),$$

and then define a homotopy $H : \Omega^2 X \times I \to \Omega^2 Y$ from $\Omega^2 f$ to $-\Omega(-\Omega f)$ by

$$H(\alpha, \rho) := \alpha \circ \varphi_\rho.$$

Since I^2 is friendly, one can use Lemma 6 to show that H is continuous.

- Lemma 7 (the inductive step): If $F \xrightarrow{j} X \xrightarrow{f} Y$ has the homotopy type of a pointed fibration, then so does $\Omega F \xrightarrow{-\Omega j} \Omega X \xrightarrow{-\Omega f} \Omega Y$.
  Proof: Take a homotopy-commutative diagram relating $F \to X \to Y$ via pointed homotopy equivalences to a pointed fibration $F' \overset{i}{\hookrightarrow} E \overset{p}{\to} B$, plug the whole diagram into the functor $\Omega : \mathsf{hTop}_* \to \mathsf{hTop}_*$, carefully insert a few minus signs as needed, and then cite Exercise 3.4, which tells us that $\Omega F' \overset{\Omega i}{\hookrightarrow} \Omega E \overset{\Omega p}{\to} \Omega B$ is also a fibration.
- Remark: In the proof of Theorem 1, one needs to apply Lemmas 7 and 4 together, so that e.g. after passing from

$$\Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega i_f} \Omega F(f) \qquad \text{to} \qquad \Omega^3 X \xrightarrow{-\Omega^3 f} \Omega^3 Y \xrightarrow{-\Omega(-\Omega i_f)} \Omega^2 F(f),$$

  $-\Omega(-\Omega i_f)$ can be replaced by $\Omega^2 i_f$.
- Combining the lemmas established so far, the proof of Theorem 1 is now reduced to showing that the segment $\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{i_f} F(f)$ has the homotopy type of a pointed fibration. This follows from the diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "-\Omega f"] \arrow[dr, "i_{\pi_f}"'] \arrow[dr, phantom, "\sim", shift left=3] & \Omega Y \arrow[r, hook, "i_f"] \arrow[d, "h", "\simeq"'] & F(f) \arrow[r, "\pi_f"] \arrow[d, "h", "\simeq"'] & X \\
& F(\pi_f) \arrow[r, hook] \arrow[ur, "\pi_{\pi_f}"] \arrow[r, phantom, "\sim", shift left=2] & P(\pi_f) \arrow[ur, "\pi_X"'] &
\end{tikzcd} \tag{5.3}$$

  which comes from identifying the fibration $F(f) \xrightarrow{\pi_f} X$ via homotopy equivalence with another fibration $P(\pi_f) \xrightarrow{\pi_X} X$ formed from the mapping path space of π_f. The following details are nontrivial:
  (1) The map $h : \Omega Y \to F(\pi_f)$ defined as a restriction of the natural homotopy equivalence $h : F(f) \to P(\pi_f)$ is in itself a homotopy equivalence, due to Lemma 2 and the fact that π_f is a fibration.
  (2) The maps $\Omega X \to F(\pi_f)$ defined by i_{π_f} and $h \circ (-\Omega f)$ are homotopic. (This is the detail that forces the appearance of all the minus signs in the Puppe sequence.) Proof by calculation: on close examination of all the definitions, one finds

$$F(\pi_f) = \left\{(x, \beta, \alpha) \in X \times PY \times PX \;\middle|\; \beta(0) = f(x) \text{ and } \alpha(0) = x\right\},$$

  with explicit formulas for the two maps $\Omega X \to F(\pi_f)$ given by

$$i_{\pi_f}(\alpha) = (x_0, \mathrm{const}_{y_0}, \alpha), \qquad h \circ (-\Omega f)(\alpha) = (x_0, f \circ \alpha(1 - \cdot), \mathrm{const}_{x_0}).$$

  A homotopy $H : \Omega X \times I \to F(\pi_f)$ between them can be written in the form

$$H_\rho(\alpha) := (\alpha(\rho), \beta_\rho, \alpha_\rho),$$

  where for each $\rho \in I$, we take $\alpha_\rho \in PX$ to be a traversal of α in the forward direction from $\alpha(\rho)$ to its end point, while $\beta_\rho \in PY$ is a traversal of $f \circ \alpha$ backwards from $f(\alpha(\rho))$ to its starting point.

This completes the proof of Theorem 1.

- Theorem 2: For any pointed fibration $p : (E, x_0) \to (B, y_0)$ with fiber inclusion $F := p^{-1}(y_0) \overset{j}{\hookrightarrow} E$, the sequence
$$\ldots \longrightarrow \Omega^2 E \xrightarrow{\Omega^2 p} \Omega^2 B \xrightarrow{-\Omega\delta} \Omega F \xrightarrow{-\Omega j} \Omega E \xrightarrow{-\Omega p} \Omega B \xrightarrow{\delta} F \xrightarrow{j} E \xrightarrow{p} B,$$
has the same properties as the Puppe sequence in Theorem 1, where $\delta : \Omega B \to F$ is the map that was defined (up to pointed homotopy) in Lecture 5 by lifting the homotopy $\mathrm{ev} : \Omega B \times I \to B$.
- Remarks on the proof of Theorem 2: The point is that since $p : E \to B$ is a fibration, its fiber F is homotopy equivalent to its homotopy fiber $F(p)$, allowing $F(p)$ to be replaced with F in the Puppe sequence. One just needs to check what the map $\Omega B \xrightarrow{i_p} F(p)$ turns into after $F(p)$ is replaced with F; see Exercise 5.8.

Coming next week: We do all this again with cofibrations, and then discuss the additional algebraic structures that we have not yet observed in these sequences.

Suggested reading. The bulk of what we've done so far with cofibrations can be found in [tD08, §5.1 and §5.3], or [May99, Chapter 6]. Cutler's notes [Cut21, Week 2 and 4 exercises] contain a more detailed treatment that carefully avoids assuming X is Hausdorff or $A \subset X$ is closed, at the cost of replacing NDR-pairs with something a bit more complicated.

My treatment of the Puppe fibration sequence was mostly based on [May99, Chapter 8], with some input from [tD08, §2.4] for the point-set topological subtleties.

Exercises (for the Übung on 23.05.2024).

Exercise 5.1. Write down an analogue of the diagram (5.1) for pointed cofibrations, in which $Z(j)$ becomes the reduced mapping cylinder and $X \times I$ is replaced by the quotient $(X \times I)/(\{*\} \times I)$. Deduce from this a theorem characterizing pointed cofibrations $A \hookrightarrow X$ in terms of the existence of a retraction of pointed spaces.

Exercise 5.2. A continuous map $f : X \to Y$ is called an **embedding** if it is injective and is a homeomorphism onto its image $f(X) \subset Y$ with the subspace topology. Embeddings can also be characterized via the following universal property: an injective continuous map $f : X \to Y$ is an embedding if and only if for every space Z and every (not necessarily continuous) map $g : Z \to X$, g is continuous whenever the composition $f \circ g$ is continuous. Before proceeding, take a moment to make sure you understand why these two versions of the definition are equivalent.

(a) Show that for any two maps $f : Z \to X$ and $g : Z \to Y$, the natural inclusions of X and Y into the (unreduced or reduced) double mapping cylinder $Z(f, g)$ are embeddings.
(b) Prove the following dual version of the statement in Exercise 4.2(a): For any cofibration $j : A \to X$ and maps $f : A \to Y$ and $\varphi : X \to Y$ such that the diagram
$$\begin{tikzcd} & A \arrow[dl, "j"'] \arrow[dr, "f"] \arrow[d, phantom, "\sim"] & \\ X \arrow[rr, "\varphi"'] & {} & Y \end{tikzcd}$$
commutes up to homotopy, φ can be replaced with a homotopic map $X \to Y$ that makes the diagram commute on the nose.
(c) Use the universal property of embeddings to deduce from parts (a) and (b) that all cofibrations are embeddings.
Hint: What can you conclude from an embedding that is the composition of two other continuous maps?

(d) Recall that a continuous map $f : X \to Y$ is a **closed map** if it sends all closed subsets of X to closed subsets of Y; in particular, if f is a closed map, then its image $f(X) \subset Y$ is necessarily a closed set in Y. Show that if f is also an embedding, then the converse also holds, i.e. the closed embeddings are precisely those embeddings $f : X \to Y$ whose images in Y are closed.

(e) Show that if $j : A \to X$ is a free cofibration and X is Hausdorff, then j is a closed embedding.
Hint: You can now assume without loss of generality that $A \subset X$ is a subspace with inclusion j. We showed in lecture that whenever such an inclusion is a free cofibration, there exists a retraction

$$X \times I \xrightarrow{r} (X \times \{0\}) \cup (A \times I),$$

so for the inclusion $X = X \times \{1\} \overset{i_1}{\hookrightarrow} X \times I$, it follows that $x \in A$ if and only if $r \circ i_1(x) = i_1(x)$. Use this to present A as the preimage of the diagonal subset for some map from X to a Hausdorff space.
Remark: This has the convenient consequence that the cofiber X/A of a cofibration $j : A \hookrightarrow X$ will be Hausdorff in all examples we ever want to consider (cf. [Wen23, Exercise 6.20]*).*

(f) Under what assumptions on a pointed space X can you also conclude for a pointed cofibration $A \hookrightarrow X$ that A is closed?
Addendum: I wrote this exercise with the idea in my head that one would need a stronger assumption in order to ensure that the reduced cylinder $(X \times I)/(\{*\} \times I)$ is Hausdorff and thus has a closed diagonal, but I was convinced in the Übung that this is already true if X is Hausdorff, so the answer is the same as in the unpointed case.

(g) The natural statement dual to the result of part (c) would be that all fibrations $p : E \to B$ are quotient maps (cf. Exercise 3.3), but this is unfortunately not quite true. Show that if $p : E \to B$ is a fibration and the base B is locally path-connected, then p is an open map, and is therefore a quotient map if and only if it is surjective. Can you find counterexamples in which p is not a quotient map, either because it is not surjective or because B is not locally path-connected?
Hint: Thanks to Exercise 4.2(a), you should have the freedom to replace $p : E \to B$ with the natural fibration $P(p) \to B$ built out of its mapping path space.

Exercise 5.3. Prove the claims stated in lecture about constructions of cofibrations via inclusions into coproducts or cylinders, compositions, coproducts of maps, and pushouts. If you don't have time for all of these, focus on pushouts.

Exercise 5.4. For an NDR-pair (X, A) with associated function $u : X \to I$ and homotopy $\rho : X \times I \to X$, the data (u, ρ) are sometimes called an **NDR-presentation** of (X, A). Parts (a) and (b) below give the proof of a lemma that was quoted in lecture; the precise formulas are adapted from [May99, §6.4].

(a) Prove that if (X, A) and (Y, B) have NDR-presentations (u, ρ) and (v, σ) respectively, then we obtain an NDR-presentation (w, φ) of $(X \times Y, A \times Y \cup X \times B)$ by setting

$$w(x, y) := \min\{u(x), v(y)\}$$

and

$$\varphi(x, y, t) := \left(\rho\left(x, t \cdot \min\left\{1, \frac{v(y)}{u(x)}\right\}\right), \sigma\left(y, t \cdot \min\left\{1, \frac{u(x)}{v(y)}\right\}\right)\right),$$

and in particular, $(X \times Y, A \times Y \cup X \times B)$ is a DR-pair whenever either (X, A) or (Y, B) is a DR-pair.

Remark: We are following a convention that $\min\{1, p/q\} := 1$ *whenever* $q = 0$. *Nonetheless, it is not entirely obvious from the formula that* $\varphi : X \times Y \times I \to X \times Y$ *is continuous, especially near points where* x *or* y *lies on the boundary of* A *or* B *respectively.*

(b) Prove that if $A \subset X$ is a closed subset and $r = (\rho, \tau) : X \times I \to X \times I$ is a retraction onto the subset $X \times \{0\} \cup A \times I$, then (u, ρ) is an NDR-presentation of (X, A), where $u : X \to I$ is defined by
$$u(x) := \sup_{t \in I} |t - \tau(x, t)| .$$
What goes wrong here if you do not assume that $A \subset X$ is closed?

(c) If (like most of us) you learned the basics of homology from [Hat02], then you may have noticed some similarity between NDR-pairs and Hatcher's notion of "good pairs".[20] They are not quite equivalent notions, however; reread both definitions to make sure that you understand why neither implies the other.[21]

(d) Suppose $A \subset X$ is closed and is a deformation retract of an open neighborhood $\mathcal{U} \subset X$ of A, and that X admits a metric (compatible with its topology) for which the distance between A and $X \backslash \mathcal{U}$ is positive. Show that (X, A) is then an NDR-pair.

(e) Show that all CW-pairs (X, A) are NDR-pairs.
Hint: Construct $u : X \to I$ *so that it equals* 1 *on every cell closure does not touch* A*, and also on a neighborhood of the center of every cell that is not contained in* A*. Start with* $A \cup X^0$*, then extend inductively from* $A \cup X^{n-1}$ *to* $A \cup X^n$ *for each* $n \in \mathbb{N}$*.*

(f) Let $\mathbb{R}^J$ denote the vector space $\prod_J \mathbb{R}$, equipped with the product topology; equivalently, you can think of $\mathbb{R}^J$ as the set of all (not necessarily continuous) maps $J \to \mathbb{R}$, with the topology of pointwise convergence. Show that if the set J is uncountable, then $\{0\} \subset \mathbb{R}^J$ is closed but is not the zero set of any continuous function $u : \mathbb{R}^J \to I$, and deduce that the inclusion $\{0\} \hookrightarrow \mathbb{R}^J$ is not a cofibration.
Hint: If such a function $u : \mathbb{R}^J \to I$ *exists, how can you characterize neighborhoods of the form* $u^{-1}([0, 1/n))$ *for* $n \in \mathbb{N}$*? Use this to construct a sequence of functions* $f_n : J \to \mathbb{R}$ *that satisfies* $u(f_n) \to 0$ *but converges pointwise to a nonzero function* $f : J \to \mathbb{R}$*. The latter will be possible specifically because* J *is uncountable.*

Exercise 5.5. Let's start with something easy:

(a) Show that for every free cofibration $j : A \to X$, any choice of base points that makes j into a pointed map makes it also into a pointed cofibration.

Going from free to pointed fibrations is more complicated, and requires the following notion: A pointed space X is called **well-pointed** if the inclusion of its base point $\{*\} \hookrightarrow X$ is a closed free cofibration.

(b) Show that if $p : E \to B$ is a free fibration, then for any choice of base points that makes p into a pointed map, it satisfies the pointed HLP with respect to all well-pointed spaces.
Hint: If $(X, *)$ *is well pointed, then* $(X \times I, X \times \{0\} \cup \{*\} \times I)$ *is a DR-pair. (Why?)*

The result in part (b) is the reason why, in practice, one rarely needs to worry about the distinction between free and pointed fibrations. It suffices for most purposes to restrict attention exclusively to well-pointed spaces, and many books on homotopy theory impose this condition across the board,

[20] Hatcher calls (X, A) a **good pair** if $A \subset X$ is closed and is a deformation retract of some open neighborhood $\mathcal{U} \subset X$ of A.

[21] I will not suggest searching for examples that satisfy one of the definitions but not the other—in practice, almost all of the examples of interest satisfy both. We will see when we study the homotopy-theoretic perspective on homology that the role of good pairs is played in that setting by inclusions that are cofibrations.

simply for convenience, even though it is often not really necessary.[22] Pointed spaces that are *not* well-pointed are typically quite peculiar, cf. Exercise 5.4(f). One can improve part (b) to the statement that for any closed free fibration $p : E \to B$ with a choice of base points such that p is a pointed map and B is well-pointed, $p : E \to B$ is also a pointed fibration.[23] There is also a dual result, stating that any pointed cofibration $j : A \to X$ becomes a free cofibration after forgetting the base points if both A and X are well-pointed. This result is apparently trickier to prove, but we will not make any use of it in this course.[24]

(c) Use DR-pairs to complete the proof that the transport functor for a free fibration is well defined. But if you don't like doing it that way, skip this and proceed to Exercise 5.6.

Exercise 5.6. As mentioned in lecture, the lifting problem

$$\begin{tikzcd} X \times (\partial I \times I \cup I \times \{0\}) \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I^2 \arrow[r, "G"] \arrow[ur, dashed, "\tilde{G}"] & B \end{tikzcd}$$

can indeed be solved by showing that $(X \times I^2, X \times (\partial I \times I \cup I \times \{0\}))$ is a DR-pair, but a few of you ganged up on me after that lecture and convinced me (with some difficulty) that there is an easier way, based on choosing a homeomorphism of pairs

$$(I^2, \partial I \times I \cup I \times \{0\}) \xrightarrow{\Phi} (I^2, I \times \{0\}).$$

Draw enough pictures to convince yourself that such a map exists.

(a) Use the homeomorphism Φ to reduce the lifting problem in the diagram above to an application of the standard homotopy lifting property. This completes the proof that the transport functor is well defined for every free fibration.

(b) What about the transport functor for *pointed* fibrations? Determine what lifting problem needs to be solved in order for the transport functor in the pointed setting to be well-defined, and use the homeomorphism Φ to solve it.
Hint: The most useful way to view pointed homotopies $X \times I \to Y$ in this context is as

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[22] I have been noticing a tendency in the homotopy theory literature that strikes me as unhealthy. It seems to be widely assumed that "most" of the important results in homotopy theory will not reliably work unless one restricts to some "convenient" category of spaces that have better "formal" properties than Top or Top_*. One of the common restrictions is to consider only the well-pointed spaces within Top_*, the standard intuition (so far as I understand it) being that this is what is required in order to make every result about fibrations or cofibrations equally valid in the free and pointed cases. I find that intuition to be a dreadful oversimplification of reality. For example, one cannot simply prove that the transport functor for a free fibration is well defined, and then immediately claim that it is therefore also well-defined in the pointed case as long as everything is well-pointed; that summary does not bear a close resemblence to the correct proof in the pointed case (see Exercise 5.6), in which well-pointedness is not actually relevant at all. I have noticed several places in textbooks where well-pointedness is assumed without being necessary, and this even seems to cause some confusion among experts (see e.g. `https://math.stackexchange.com/questions/175590/importance-of-well-pointedness-in-particular-for-the-pointed-mapping-cylinder-c`). I am therefore making a big effort to avoid imposing such assumptions when they are not truly relevant. In the case of well-pointedness, the price we pay is that we must always keep in mind two parallel definitions of the HLP and HEP—one for the free case and another for the pointed case—but this strikes me as the natural thing to do.

[23] For a proof of this statement, see [Cut21, Week 6: Fibrations IV, Prop. 1.8]. The proof uses the characterization of fibrations in terms of *lifting functions* (Exercise 3.7), which is the dual variant of the characterization of cofibrations in terms of retractions. It also uses a weaker assumption than $\{*\} \hookrightarrow B$ being a closed cofibration; it is sufficient in fact to assume that the base point in B is the zero set of a continuous function $B \to I$.

[24] The full details take about three pages in [MP12, Lemma 1.3.4], where they appear together with a de facto apology for having stated the result casually in [May99, §8.3] as if it were a self-evident fact with no need for justification.

pointed maps $\frac{X \times I}{\{*\} \times I} \to Y$. This also applies to homotopies of pointed homotopies, which you can view as pointed maps $\frac{X \times I^2}{\{*\} \times I^2} \to Y$. Now just check whether what you wrote down in part (a) descends to the relevant quotients.

(c) Without looking up the definition, what do you think the transport functor of a cofibration $A \hookrightarrow X$ should be, and what extension problem needs to be solved in order to prove that it is well defined? Solve it in the unpointed case by combining a well-chosen homeomorphism with the knowledge that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. Then adapt your solution to the pointed case by letting things descend to quotients.

Exercise 5.7. Without using any knowledge of fibrations, give a direct proof that for every pointed map $f : X \to Y$ and every pointed space Z, the induced sequence of pointed homotopy sets

$$[Z, F(f)] \xrightarrow{(\pi_f)_*} [Z, X] \xrightarrow{f_*} [Z, Y]$$

is exact.
Comment: It is also possible—though tedious—to prove the exactness of the rest of the Puppe sequence without using results about fibrations, and this is what [tD08, §4.7] *does.*

Exercise 5.8. Assume $p : E \to B$ is a pointed fibration with fiber inclusion $F = p^{-1}(*) \overset{j}{\hookrightarrow} E$.

(a) Using the transport functor, complete the proof that the map $\delta : \Omega B \to F$ described in Lecture 5 is independent of choices up to pointed homotopy.

(b) Theorem 2 in Lecture 8 (the Puppe sequence for a pointed fibration) follows from the Puppe sequence of Theorem 1 in light of the following variation on the diagram (5.3):

$$\begin{tikzcd}
\Omega B \arrow[r, "\delta"] \arrow[dr, hook, "i_p"'] & F \arrow[r, hook, "j"] \arrow[d, "h", "\simeq"'] & E \arrow[r, "p"] \arrow[d, "h", "\simeq"'] & B \\
& F(p) \arrow[r, hook] \arrow[ur, "\pi_p"] & P(p) \arrow[ur, "\pi_B"'] &
\end{tikzcd}.$$

Here the nontrivial details are again that the map $h : F \to F(p)$ is a homotopy equivalence (because $p : E \to B$ is a fibration), and that the leftmost triangle in the diagram commutes up to homotopy, i.e.

$$h \circ \delta \underset{h}{\sim} i_p.$$

Prove the latter.
Hint: Try first to convince yourself that $h \circ \delta$ is actually just another version of the map δ, namely the one associated to the fibration $\pi_B : P(p) \to B$. The goal is then to establish that $i_p : \Omega B \to F(p)$ is a valid explicit formula for δ in the case of this particular fibration. This will be obvious if you make an intelligent choice of lift of $\mathrm{ev} : \Omega B \times I \to B$ *to $P(p)$.*

(c) Suppose $p : E \to B$ is a *free* fibration, base points $x_0 \in E$ and $y_0 \in B$ have been chosen so as to make p a pointed map and to define the loop spaces ΩB and ΩE, but p is not necessarily a pointed fibration. Is the map $\delta : \Omega B \to F$ still well defined (at least up to *free* homotopy), and do consecutive terms in the sequence of Theorem 2 then have the (free) homotopy type of a free fibration?
Comment: I currently believe the answers to these questions to be yes, but I have not thought about it very much and could be missing some details; I would be interested if someone comes up with a different opinion. The part that had me most concerned was the inductive step, where in the pointed case, we used the fact (from Exercise 3.4) that pointed fibrations remain pointed fibrations after applying the loop space functor. Can you prove something similar about free fibrations?

Exercise 5.9. The fact that $\Omega : \mathsf{Top}_* \to \mathsf{Top}_*$ descends to a functor $\mathsf{hTop}_* \to \mathsf{hTop}_*$ can be deduced from the following much more general phenomenon. For any fixed space $Z \in \mathsf{Top}$, one can define a covariant functor

$$(\cdot)^Z = C(Z, \cdot) : \mathsf{Top} \to \mathsf{Top} : X \mapsto X^Z := C(Z, X),$$

as well as a contravariant functor

$$Z^{(\cdot)} = C(\cdot, Z) : \mathsf{Top} \to \mathsf{Top} : X \mapsto Z^X := C(X, Z).$$

The fact that both are functors depends on part (a) below, which is a straightforward exercise in point-set topology:

(a) Show that for any continuous map $f : X \to Y$, the induced maps

$$X^Z \xrightarrow{f^Z} Y^Z, \qquad f^Z(\varphi) = f_*\varphi := f \circ \varphi,$$
$$Z^Y \xrightarrow{Z^f} Z^X, \qquad Z^f(\varphi) = f^*\varphi := \varphi \circ f$$

are continuous with respect to the compact-open topology.

Let us now consider whether $(\cdot)^Z$ and $Z^{(\cdot)}$ each descend to functors $\mathsf{hTop} \to \mathsf{hTop}$, i.e. does a homotopy between two maps $f, g : X \to Y$ induce homotopies $f^Z \underset{h}{\sim} g^Z$ and $Z^f \underset{h}{\sim} Z^g$? In each case, there is an obvious way to write down the homotopies that one wants, but proving that they are continuous maps takes some effort. It will turn out that they always are, but the amount of effort required to prove it may depend on what assmption you are willing to impose upon Z.

(b) Show that the functor $Z^{(\cdot)}$ descends to $\mathsf{hTop} \to \mathsf{hTop}$.
Hint: You know from part (a) that for any homotopy $H : X \times I \to Y$, the map $Z^H : Z^Y \to Z^{X \times I}$ is continuous. Exploit the fact that I is friendly, and also the existence of a natural continuous map $Z^{X \times I} \to (Z^X)^I$; note that the latter might not be surjective if you don't assume X is friendly, but you will not need it to be surjective.

(c) Give a quick and easy proof that if Z is friendly, then the functor $(\cdot)^Z$ also descends to $\mathsf{hTop} \to \mathsf{hTop}$.

Part (c) covers the cases that typically arise in this course, such as proving that homotopies between maps $f, g : X \to Y$ induce homotopies between the corresponding maps $f^I, g^I : X^I \to Y^I$ on path spaces, and the obvious restriction of that statement to loop spaces. If you want to know why such things would still be true even if you didn't know that I is locally compact, read on.

(d) Prove that for any three spaces X, Y, Z, the map

$$X^Z \times Y \to (X \times Y)^Z : (\varphi, y) \mapsto \varphi^y, \qquad \text{where} \qquad \varphi^y(z) := (\varphi(z), y)$$

is continuous.
Hint/caution: Do not attempt to derive this from any more general statement involving the obvious bijection $X^Z \times Y^Z \cong (X \times Y)^Z$, which is subtler than it looks (see part (f) below). You should instead try a direct proof in the style of Topology I, using the definitions of the compact-open and product topologies.

(e) Combine the results of parts (a) and (d) to give a general proof, without any assumption on the space Z, that the functor $(\cdot)^Z$ descends to $\mathsf{hTop} \to \mathsf{hTop}$.

(f) Here is why I told you what *not* to try in part (d). For any collection of spaces $\{X_\alpha\}_{\alpha \in J}$ and another space Z, there is an obvious bijective map

$$\left(\prod_{\alpha \in J} X_\alpha \right)^Z \to \prod_{\alpha \in J} X_\alpha^Z,$$

i.e. maps to a product space are in one-to-one correspondence with tuples of maps to its factors. Prove that this map is continuous, and that its inverse is also continuous if Z is friendly. Then consider the case where $\{X_\alpha\}_{\alpha\in J}$ is a collection of just two spaces X and Y, and try to give a direct proof that the inverse of the obvious bijection $(X\times Y)^Z \to X^Z \times Y^Z$ is continuous, without assuming that Z is friendly or locally compact. Stop trying once you realize that it isn't so obvious.
Comment: According to [tD08, §2.4, Problem 8], *the inverse* $X^Z \times Y^Z \to (X\times Y)^Z$ *is indeed continuous whenever* Z *is Hausdorff.*[25]

(g) Deduce from part (f) that for any collection of pointed spaces $\{X_\alpha\}_{\alpha\in J}$, there is a natural pointed homeomorphism

$$\Omega\left(\prod_{\alpha\in J} X_\alpha\right) \cong \prod_{\alpha\in J} \Omega X_\alpha.$$

6. Week 6

Again only one lecture this week, because Monday was a holiday.

Lecture 9 (23.05.2024): Adjunction and the cofiber sequence.

- Definition: The functor $\mathcal{R} : \mathscr{A} \to \mathscr{B}$ is a **right-adjoint** of the functor $\mathcal{L} : \mathscr{B} \to \mathscr{A}$ (thus making $\mathcal{L}$ a **left-adjoint** of $\mathcal{R}$) if there exists a natural isomorphism between the two functors $\mathscr{A} \times \mathscr{B} \to \mathsf{Set}$ defined by

$$(A,B) \mapsto \operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(B), A) \qquad \text{and} \qquad (A,B) \mapsto \operatorname{Hom}_{\mathscr{B}}(B, \mathcal{R}(A)).$$

  In other words, there is a bijection $\operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(B), A) \xrightarrow{\alpha} \operatorname{Hom}_{\mathscr{B}}(B, \mathcal{R}(A))$ associated to each $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that for every pair of morphisms $\varphi \in \operatorname{Hom}_{\mathscr{A}}(A, A')$ and $\psi \in \operatorname{Hom}_{\mathscr{B}}(B', B)$, the diagram[26]

$$\begin{tikzcd} \operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(B), A) \arrow[r, "\alpha"] \arrow[d, "\varphi_*\mathcal{L}(\psi)^*"'] & \operatorname{Hom}_{\mathscr{B}}(B, \mathcal{R}(A)) \arrow[d, "\mathcal{R}(\varphi)_*\psi^*"] \\ \operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(B'), A') \arrow[r, "\alpha"] & \operatorname{Hom}_{\mathscr{B}}(B', \mathcal{R}(A')) \end{tikzcd}$$

  commutes. (Note that $\varphi_*\mathcal{L}(\psi)^* = \mathcal{L}(\psi)^*\varphi_*$ and $\mathcal{R}(\varphi)_*\psi^* = \psi^*\mathcal{R}(\varphi)_*$, so the order in which these compositions are written on the vertical maps in the diagram does not matter. One sometimes sees the diagram written as two separate diagrams in which either φ or ψ is taken to be the identity morphism.)
- Remark: A given functor need not have either a left-adjoint or a right-adjoint, and functors that do have one or the other have special properties, e.g. one can show that every functor that is a right- or left-adjoint of another one preserves all limits or colimits respectively (see e.g. [Mac71, §V.5]). One can deduce from Exercise 1.8 (a version of the Yoneda lemma) that for any functor that has a left- or right-adjoint, that adjoint is unique up to natural isomorphism. This fact will not concern us here; we mainly just need to make use of certain examples.
- Examples of adjoints:

[25] I am just passing this along as hearsay—I have not attempted to do the exercise in [tD08, §2.4] myself. I also do not happen to know a counterexample in which the inverse really fails to be continuous. Evidently Z would in this case have to be neither Hausdorff nor locally compact, which sounds pretty terrifying.

[26] This version of the diagram assumes that $\mathcal{L}$ and $\mathcal{R}$ are both covariant; there is an analogous version for the case where both are contravariant.

(1) For any $G \in \mathsf{Ab}$, the functors $\otimes G$ and $\mathrm{Hom}(G, \cdot)$ from Ab to itself are adjoints (left and right respectively) due to the natural bijection

$$\mathrm{Hom}(A \otimes G, B) \cong \mathrm{Hom}(A, \mathrm{Hom}(G, B)).$$

This adjunction is one way of expressing the universal property of the tensor product on Ab, or similarly on R-Mod (cf. Exercise 1.9).

(2) The functor $\mathsf{Top} \to \mathsf{Top}_* : X \mapsto X_+ := X \amalg \{*\}$ is a left-adjoint of the **forgetful functor** $\mathsf{Top}_* \to \mathsf{Top}$, i.e. the one that takes any pointed space and forgets its base point. This follows from the natural bijection

$$\mathcal{F}_+(X_+, Y) := \mathrm{Hom}_{\mathsf{Top}_*}(X_+, Y) \to \mathrm{Hom}_{\mathsf{Top}}(X, Y) =: \mathcal{F}(X, Y)$$

that sends each pointed map $f : X_+ \to Y$ to its restriction $f|_X$. (See below for remarks on the notation $\mathcal{F}$ and $\mathcal{F}_+$.)

(3) If $Y \in \mathsf{Top}$ is *friendly* in the sense defined in the previous lecture, then the natural injection

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Top}}(X \times Y, Z) &=: \mathcal{F}(X \times Y, Z) = Z^{X \times Y} \\ &\to (Z^Y)^X = \mathcal{F}(X, \mathcal{F}(Y, Z)) := \mathrm{Hom}_{\mathsf{Top}}(X, \mathcal{F}(Y, Z)) \end{aligned} \tag{6.1}$$

becomes a bijection and thus makes the functors $\mathsf{Top} \to \mathsf{Top}$ defined by $(\cdot) \times Y$ and $\mathcal{F}(Y, \cdot)$ adjoint to each other. The desire for this adjunction to hold without imposing strong restrictions such as local compactness on Y is often cited as the main motivation for working in the category of *compactly generated* spaces instead of in Top, which is a standard convention in modern homotopy theory.[27]

- Notation: Since our previous notation $C(X, Y) = Y^X$ for the space of continuous maps $X \to Y$ can be confused too easily with the notation for cones or chain complexes, I am replacing it henceforth with

$$\mathcal{F}(X, Y) := \{\text{continuous maps } X \to Y\} = Y^X = \mathrm{Hom}_{\mathsf{Top}}(X, Y)$$

for $X, Y \in \mathsf{Top}$, where the $\mathcal{F}$ stands for "function," and when $\mathcal{F}(X, Y)$ is regarded as a topological space, it should still be assumed to carry the compact-open topology unless stated otherwise. When working in the context Top_*, the same notation can also be used to mean the space of pointed maps

$$\mathcal{F}(X, Y) := \{\text{pointed maps } X \to Y\} = \mathrm{Hom}_{\mathsf{Top}_*}(X, Y)$$

for $X, Y \in \mathsf{Top}_*$. For situations in which we need to discuss the pointed and unpointed categories in the same context, we will distinguish them by writing the subscript $+$ to mean "pointed" and the subscript $\circ$ (this is a small circle, not a zero) to mean "unpointed," thus

$$\begin{aligned} \mathcal{F}_\circ(X, Y) &:= \mathrm{Hom}_{\mathsf{Top}}(X, Y), & [X, Y]_\circ &:= \mathrm{Hom}_{\mathsf{hTop}}(X, Y), \\ \mathcal{F}_+(X, Y) &:= \mathrm{Hom}_{\mathsf{Top}_*}(X, Y), & [X, Y]_+ &:= \mathrm{Hom}_{\mathsf{hTop}_*}(X, Y), \end{aligned}$$

[27] Working in the compactly generated category does two things in order to make (6.1) hold: first, it excludes a pathological subclass of spaces that are not locally compact, and second, it slightly strengthens the topologies assigned to $X \times Y$ and $\mathcal{F}(X, Y)$, without which they would not necessarily be compactly generated even when X and Y are. The more naive idea of working only with locally compact spaces would be inadequate because it would exclude $\mathcal{F}(X, Y)$ in most interesting cases, so e.g. we would no longer be allowed to talk about loop spaces. Having (6.1) without restrictions on Y would be comforting, but I have not observed it to be truly *necessary* so far in this course, and am thus stubbornly refusing to abandon Top until it is.

and similar notation will be used below to distinguish between unreduced and reduced suspensions/cones/cylinders. Needless to say, you will find many other notational conventions different from this in various books.

- The **smash product** of two pointed spaces (X, x_0) and (Y, y_0) is the quotient
$$X \wedge Y := (X \times Y) \Big/ (\{x_0\} \times Y \cup X \times \{y_0\}),$$
regarded as a pointed space with the collapsed subset as the base point. The definition is sometimes abbreviated by identifying the subset in the denominator with a wedge sum: $X \wedge Y = \frac{X \times Y}{X \vee Y}$. The smash product gives rise to the pointed version of example (3) above:
  (4) For $X, Y, Z \in \mathsf{Top}_*$, letting maps $X \times Y \to Z$ descend to the quotient $X \wedge Y$ turns (6.1) into a natural injection

$$\mathrm{Hom}_{\mathsf{Top}_*}(X \wedge Y, Z) = \mathcal{F}_+(X \wedge Y, Z) \to \mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) = \mathrm{Hom}_{\mathsf{Top}_*}(X, \mathcal{F}_+(Y, Z)). \tag{6.2}$$

  If Y is friendly, then this is a bijection, making the functors $\mathsf{Top}_* \to \mathsf{Top}_*$ defined by $(\cdot) \wedge Y$ and $\mathcal{F}_+(Y, \cdot)$ adjoint to each other.

I will pause the lecture summary here and add some discussion of a subtle detail that I got slightly wrong in the lecture. We showed in Lecture 8 that whenever not only Y but also X and $X \times Y$ are friendly, the natural bijection $\mathcal{F}(X \times Y, Z) \cong \mathcal{F}(X, \mathcal{F}(Y, Z))$ is a homeomorphism. I claimed in this lecture—and [tD08, Theorem 2.4.11] makes a similar claim—that trivial modifications of the same proof make the natural bijection
$$\mathcal{F}_+(X \wedge Y, Z) \cong \mathcal{F}_+(X, \mathcal{F}_+(Y, Z))$$
into a homeomorphism whenever X, Y and $X \wedge Y$ are all friendly. On closer inspection, I am no longer sure if this is true, and some googling reveals that I am not the only one with this uncertainty.[28] I can still establish the homeomorphism under stronger hypotheses that are sufficient for our purposes, and I will explain this below.

The trouble arises from the fact (mentioned slightly later in the lecture) that the smash product—in contrast to the ordinary product $\times$ of unpointed spaces—is not associative without further assumptions about the spaces involved. In fact, I also claimed in this lecture that associativity holds under precisely the same hypotheses, namely that X, Y and $X \wedge Y$ are all friendly, but unfortunately, my proof of that claim relied on $\mathcal{F}_+(X \wedge Y, Z) \cong \mathcal{F}_+(X, \mathcal{F}_+(Y, Z))$ being a homeomorphism, so using it here would be circular reasoning. The version of that claim that appears in the summary below is weaker than what I stated in lecture, but still covers the cases of interest.

Here is a statement that I am currently willing to stand behind, though I will not claim that it is the most general valid statement of its kind. In the course of the proof, I will clarify why I am uncertain about the more general version claimed in lecture.

Proposition 6.1. *For any pointed spaces $X, Y, Z \in \mathsf{Top}_*$ such that X and Y are compact and Hausdorff, the natural injection*
$$\mathcal{F}_+(X \wedge Y, Z) \xrightarrow{\alpha} \mathcal{F}_+(X, \mathcal{F}_+(Y, Z))$$
is a pointed homeomorphism.

Proof, part 1. We prove first that α is continuous, which does follow by an adaptation of the same argument as in the unpointed case, though one particular step requires additional care. The actual assumptions needed for this part are that Y and $X \wedge Y$ are friendly, which certainly follows if X

[28]See https://math.stackexchange.com/questions/3934265/adjunction-of-pointed-maps-is-a-homeomorphism

and Y are both compact and Hausdorff, because both spaces and their products are then locally compact.[29] In the following, we denote elements of a smash product $X \wedge Y$ by

$$[x, y] \in X \wedge Y$$

for representatives $(x, y) \in X \times Y$.

The first crucial observation is easily verified: if Y and Z are pointed spaces for which the unpointed evaluation map $\mathrm{ev} : \mathcal{F}_\circ(Y, Z) \times Y \to Z$ is continuous, then its restriction to $\mathcal{F}_+(Y, Z) \times Y \to Z$ is of course also continuous, and this restriction then descends to a continuous map

$$\mathcal{F}_+(Y, Z) \wedge Y \xrightarrow{\mathrm{ev}} Z.$$

This works in particular whenever Y is friendly; so far so good.

Analogously to the unpointed case, the proof that $\alpha : \mathcal{F}_+(X \wedge Y, Z) \to \mathcal{F}_+(X, \mathcal{F}_+(Y, Z))$ is continuous can be reduced to the fact—following from the observation that $X \wedge Y$ is friendly—that

$$\mathcal{F}_+(X \wedge Y, Z) \wedge (X \wedge Y) \xrightarrow{\mathrm{ev}} Z$$

is continuous. In order to make use of the latter, one needs to apply twice in succession the fact that continuous maps $X \wedge Y \to Z$ always have continuous (and in this case pointed) adjoints $X \to \mathcal{F}_+(Y, Z)$, but here a tricky issue arises that is not relevant in the unpointed case: the smash product is not generally associative, so we do not know whether $\mathcal{F}_+(X \wedge Y, Z) \wedge (X \wedge Y)$ can freely be replaced by $(\mathcal{F}_+(X \wedge Y, Z) \wedge X) \wedge Y$ without changing its topology. What saves the situation is Exercise 6.1(d) below, which implies in light of the friendliness of Y that the canonical bijection from the latter product to the former is at least continuous. Composing it with ev above thus gives a continuous map

$$(\mathcal{F}_+(X \wedge Y, Z) \wedge X) \wedge Y \to Z : \big[[f, x], y\big] \mapsto f([x, y]).$$

The adjoint of this map,

$$\mathcal{F}_+(X \wedge Y, Z) \wedge X \to \mathcal{F}_+(Y, Z)$$

is therefore also continuous, and it assigns to each $[f, x]$ the function $y \mapsto f([x, y])$ in $\mathcal{F}_+(Y, Z)$. Performing the adjoint trick one more time then gives a continuous map

$$\mathcal{F}_+(X \wedge Y, Z) \to \mathcal{F}_+(X, \mathcal{F}_+(Y, Z)),$$

and that one is precisely α. $\square$

Before proving also that α^{-1} is continuous, let us clarify why it does *not* follow by a minor adaptation of the argument for the unpointed case. The idea would have been to first prove the continuity of the map

$$\mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \wedge (X \wedge Y) \to Z : \big[f, [x, y]\big] \mapsto f(x)(y),$$

whose adjoint is α^{-1}. Since X and Y are friendly, it certainly is true that the map

$$\big(\mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \wedge X\big) \wedge Y \to Z : \big[[f, x], y\big] \mapsto f(x)(y)$$

is continuous, as this can be written as a composition involving two continuous evaluation maps. We would thus be done if we knew that the canonical bijection

$$\mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \wedge (X \wedge Y) \to \big(\mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \wedge X\big) \wedge Y$$

[29]Recall that we are using the strict definition of local compactness, in which every point is required to have a compact neighborhood that fits inside any other given neighborhood, thus local compactness does not follow automatically from compactness, but it is a straightforward exercise to show that it does if the space is Hausdorff.

is continuous, but unfortunately, Exercise 6.1 does not give us that unless $\mathcal{F}_+(X, \mathcal{F}_+(Y, Z))$ is friendly, which seems highly unlikely without imposing drastically restrictive additional conditions on all three spaces.

But if X and Y are compact and Hausdorff, then we can take a different approach and deduce the continuity of α^{-1} from the knowledge that the corresponding map $(Z^Y)^X \to Z^{X\times Y}$ in the unpointed case is (since X and Y are also friendly) continuous. This requires a few technical lemmas whose proofs are routine exercises in point-set topology.

Lemma 6.2. *For any spaces X, Y, Z and an embedding $j : Y \to Z$, the induced map $j_* : \mathcal{F}(X, Y) \to \mathcal{F}(X, Z) : f \mapsto j \circ f$ is also an embedding.* □

In order to state a result dual to Lemma 6.2, recall that a continuous map $f : X \to Y$ is called **proper** if it is a closed map and $f^{-1}(y) \subset X$ is compact for every point $y \in Y$. It is common in differential geometry to use a weaker variant of this condition that follows from it and is equivalent for reasonable spaces (e.g. Hausdorff and second countable), namely the conclusion of the following statement:

Lemma 6.3. *For any proper map $f : X \to Y$ and any compact subset $K \subset Y$, the set $f^{-1}(K) \subset X$ is also compact.* □

The dual version of Lemma 6.2 is then:

Lemma 6.4. *For any spaces X, Y, Q and a surjective proper map $q : X \to Q$, the induced map $q^* : \mathcal{F}(Q, Y) \to \mathcal{F}(X, Q) : f \mapsto f \circ q$ is an embedding.*

Quick proof. Surjectivity of q implies that q^* is injective, and one then needs to show that q^* is an open map to its image (with the subspace topology), for which it suffices to consider any set $\mathcal{O} \subset \mathcal{F}(Q, Y)$ of the form $\{f : Q \to Y \mid f(K) \subset \mathcal{V}\}$ for some $K \subset Q$ compact and $\mathcal{V} \subset Y$ open, and find a similar open set $\mathcal{O}' \subset \mathcal{F}(X, Q)$ with the property that $f \in \mathcal{O}$ implies $f \circ q \in \mathcal{O}'$. This is easy since $q^{-1}(K) \subset X$ is compact. □

Finally, there happens to be a relevant class of quotient projections that are always proper:

Lemma 6.5. *For any space X and closed subset $A \subset X$, the quotient projection $X \to X/A$ is a closed map, and is thus proper if A is also compact.* □

Proof of Proposition 6.1, part 2. Let us distinguish $\alpha^{-1} : \mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \to \mathcal{F}_+(X \wedge Y, Z)$ from its unpointed counterpart

$$\mathcal{F}_\circ(X, \mathcal{F}_\circ(Y, Z)) = (Z^Y)^X \xrightarrow{\widehat{\alpha}^{-1}} Z^{X\times Y} = \mathcal{F}_\circ(X \times Y, Z),$$

which we already know is continuous since X, Y and $X \times Y$ are friendly. By Lemma 6.2, the obvious inclusion of $\mathcal{F}_+(X, \mathcal{F}_+(Y, Z))$ into $\mathcal{F}_\circ(X, \mathcal{F}_\circ(Y, Z))$ is an embedding, so $\widehat{\alpha}^{-1}$ restricts to this subset as a continuous map

$$\mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \to \mathcal{F}_\circ(X \times Y, Z),$$

and this restricted map takes its values in the image of the injective map

$$\mathcal{F}_\circ(X \wedge Y, Z) \supset \mathcal{F}_+(X \wedge Y, Z) \xrightarrow{q^*} \mathcal{F}_\circ(X \times Y, Z)$$

induced by the quotient map $X \times Y \xrightarrow{q} X \wedge Y$. Since X and Y are compact and Hausdorff, $\{*\} \times Y \cup X \times \{*\} \subset X \times Y$ is a compact and closed subset, implying via Lemma 6.5 that q is a proper map, and by Lemma 6.4, $q^* : \mathcal{F}_+(X \wedge Y, Z) \to \mathcal{F}_\circ(X \times Y, Z)$ is therefore a homeomorphism onto its image. Composing the map $\mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \to \mathcal{F}_\circ(X \times Y, Z)$ above with the continuous inverse of that homeomorphism now gives $\alpha^{-1} : \mathcal{F}_+(X, \mathcal{F}_+(Y, Z)) \to \mathcal{F}_+(X \wedge Y, Z)$, which is therefore continuous. □

With that painful digression out of the way, we now return to the lecture summary.

- Lemma: If $(X, A), (Y, B) \in \mathsf{Top}^{\mathrm{rel}}$ are pairs of spaces such that X and Y/B are friendly, then the canonical continuous bijection
$$\frac{X \times Y}{A \times Y \cup X \times B} \to (X/A) \wedge (Y/B)$$
is a homeomorphism.
Proof: Thanks to Exercise 3.3, friendliness makes the first two maps in the sequence
$$X \times Y \to X \times Y/B \to (X/A) \times (Y/B) \to (X/A) \wedge (Y/B)$$
quotient maps, and the third one is a quotient map by definition, thus so is the composition.
- Example: The lemma tells us that smash products of spheres are spheres, since
$$S^m \wedge S^n \cong \frac{\mathbb{D}^m}{\partial \mathbb{D}^m} \wedge \frac{\mathbb{D}^n}{\partial \mathbb{D}^n} \cong \frac{\mathbb{D}^m \times \mathbb{D}^n}{\partial \mathbb{D}^m \times \mathbb{D}^n \cup \mathbb{D}^m \times \partial \mathbb{D}^n} = \frac{\mathbb{D}^m \times \mathbb{D}^n}{\partial (\mathbb{D}^m \times \mathbb{D}^n)} \cong \frac{\mathbb{D}^{m+n}}{\partial \mathbb{D}^{m+n}} \cong S^{m+n}.$$
- Proposition (further properties of the smash product):
  (1) The canonical bijection $X \wedge Y \cong Y \wedge X$ is a homeomorphism. (Easy.)
  (2) There is a natural homeomorphism $X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$ induced by the canonical bijection $X \times (Y \amalg Z) \cong (X \times Y) \amalg (X \times Z)$. (See Exercise 6.1.)
  (3) The canonical bijection $X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$ is a homeomorphism whenever either of the following is true: (a) X and Z are friendly (see Exercise 6.1); (b) Y and Z are compact and Hausdorff.
- Achtung: The smash product really is *not* associative in general on unfriendly spaces, e.g. the spaces $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N})$ and $(\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}$ are not homeomorphic. If you really want to read a proof of this, see e.g. https://math.uchicago.edu/~may/REU2022/REUPapers/Horowitz.pdf.
- Proof of associativity when Y and Z are compact and Hausdorff:
Knowing that adjunction gives a homeomorphism under these assumptions (Proposition 6.1) makes it possible to deduce associativity from purely formal considerations. For an arbitrary pointed space Q, we obtain a chain of natural bijections
$$\begin{aligned} \mathrm{Hom}_{\mathsf{Top}_*}\big(X \wedge (Y \wedge Z), Q\big) &\cong \mathrm{Hom}_{\mathsf{Top}_*}\big(X, \mathcal{F}_+(Y \wedge Z, Q)\big) \cong \mathrm{Hom}_{\mathsf{Top}_*}\big(X, \mathcal{F}_+(Y, \mathcal{F}_+(Z, Q))\big) \\ &\cong \mathrm{Hom}_{\mathsf{Top}_*}\big(X \wedge Y, \mathcal{F}_+(Z, Q)\big) \cong \mathrm{Hom}_{\mathsf{Top}_*}\big((X \wedge Y) \wedge Z, Q\big), \end{aligned}$$
where we should note that the second bijection depends on the correspondence $\mathcal{F}_+(Y \wedge Z, Q) \cong \mathcal{F}_+(Y, \mathcal{F}_+(Z, Q))$ being not just a bijection but also a pointed homeomorphism, and each depends in any case on a friendliness condition that follows from the assumption that Y and Z are compact and Hausdorff. Exercise 1.8 now tells us that the natural isomorphism in Set determined by these bijections corresponds to an isomorphism between $X \wedge (Y \wedge Z)$ and $(X \wedge Y) \wedge Z$ in Top_*, and one only needs to unpack the definitions to check that this isomorphism is the obvious map.
- Definition: The **reduced suspension** ΣX and **reduced cone** CX of a pointed space X were mentioned in Lecture 3 as special cases of double mapping cylinders in the pointed category, but we can now give more succinct equivalent definitions via the smash product,

along with the **reduced cylinder** ZX:

$$CX := X \wedge I = \frac{X \times I}{X \times \{0\} \cup \{*\} \times I},$$
$$ZX := X \wedge I_+ \cong \frac{X \times I}{\{*\} \times I},$$
$$\Sigma X := X \wedge S^1 \cong \frac{X \times I}{X \times \partial I \cup \{*\} \times I}.$$

We are taking the base point of I to be 0 when viewing it as a pointed space, and defining the base point of S^1 by identifying it with $I/\partial I$. In writing ZX as a quotient, we have omitted the part involving the base point of $I_+ = I \amalg \{*\}$, and can do so without losing any information since it gets collapsed to the same point as $\{*\} \times I$.

- Notation: The cone, cylinder and suspension all have familiar **unreduced** variants, which are the corresponding double mapping cylinders in the unpointed category (so e.g. the unreduced cylinder is just $X \times I$). In keeping with the previously described convention, we will use the notation CX, ZX and ΣX to denote either the unreduced or reduced version of these spaces in situations where we are working purely in the context of Top or Top_* respectively. When it is necessary to specify which one is meant, we will write

$$\Sigma_+ X := \text{reduced suspension}, \qquad \Sigma_\circ X := \text{unreduced suspension},$$

and similarly for cones and cylinders.[30] It will be useful to notice that the reduced versions are all naturally quotients of the unreduced versions, e.g. if X has base point $* \in X$, then

$$\Sigma_+ X = \frac{\Sigma_\circ X}{\Sigma_\circ \{*\}}, \qquad C_+ X = \frac{C_\circ X}{C_\circ \{*\}}.$$

The fact that a one-point space has a contractible (unreduced) suspension/cone/cylinder will play a useful role below.

- Example: The formula $S^m \wedge S^n \cong S^{m+n}$ produces a nice formula for the reduced suspension of spheres that also happens to hold for the unreduced suspension, though this is more of a coincidence than a general phenomenon:

$$\Sigma S^n \cong S^{n+1}, \qquad \text{thus} \qquad S^n \cong \Sigma^n S^0.$$

Since S^1 is compact and Hausdorff, one can apply associativity of the smash product and write the n-fold reduced suspension of any pointed space X as

$$\Sigma^n X \cong X \wedge (S^1 \wedge \ldots \wedge S^1) \cong X \wedge S^n.$$

- Since S^1, I and I_+ are all friendly spaces, we now observe three important examples of adjunction relations that are special cases of the fourth one above:
  (5) $\mathcal{F}_+(C_+ X, Y) \cong \mathcal{F}_+(X, \mathcal{F}_+(I, Y)) = \mathcal{F}_+(X, PY)$, where PY is the based path space of Y with paths starting at the base point;
  (6) $\mathcal{F}_+(Z_+ X, Y) \cong \mathcal{F}_+(X, \mathcal{F}_+(I_+, Y)) \cong \mathcal{F}_+(X, \mathcal{F}_\circ(I, Y)) = \mathcal{F}_+(X, Y^I)$, where we view the free path space Y^I as a pointed space with base point the constant path at $* \in Y$;
  (7) $\mathcal{F}_+(\Sigma_+ X, Y) \cong \mathcal{F}_+(X, \mathcal{F}_+(S^1, Y)) = \mathcal{F}_+(X, \Omega Y)$.
- Remark: These adjunction relations also descend to hTop_*, producing useful natural bijections such as

$$[\Sigma_+ X, Y]_+ \cong [X, \Omega Y]_+.$$

[30]In earlier lectures I also occasionally wrote SX for the unreduced suspension, as I have usually done in Topology 1 and 2 in the past. I am switching to ΣX now because that notation is more prevalent for the reduced suspension in homotopy theory books, and I don't want to use different notation for the unreduced and reduced versions when it isn't necessary.

- Lemma (valid in free and pointed versions): If $A \hookrightarrow X$ is a closed cofibration, then so are $\Sigma A \hookrightarrow \Sigma X$, $CA \hookrightarrow CX$ and $ZA \hookrightarrow ZX$.
  Proof for suspensions (the most important case): We give different arguments for the pointed and free cases. The pointed case is essentially dual to Exercise 3.4(c), and follows as an easy consequence of adjunction: $\Sigma A \hookrightarrow \Sigma X$ satisfies the HEP with respect to an arbitrary pointed space Y due to the fact that $A \hookrightarrow X$ satisfies it with respect to ΩY. This simple argument is not available in the free case due to the lack of a right-adjoint of $\Sigma_\circ$, but since we are assuming $A \subset X$ is closed, we can argue in terms of NDR-pairs: if (X, A) is one, then since $(I, \partial I)$ clearly also is, so is $(X \times I, X \times \partial I \cup A \times I)$. Now one checks that the functions defining an NDR-presentation of the latter descend to the quotient in which $X \times \{0\}$ and $X \times \{1\}$ are separately collapsed to points, which turns $X \times \partial I \cup A \times I$ into ΣA and thus determines an NDR-presentation of $(\Sigma X, \Sigma A)$.
- Recall an important fact about fibrations $p : E \to B$: for any space B', there is a functor $\Pi(B', B) \to \mathsf{Top}_B$ that sends each map $f : B' \to B$ to the induced fibration $f^*E \to B'$ and sends each homotopy class of homotopies $f_0 \overset{F}{\rightsquigarrow} f_1$ of maps $B' \to B$ to a homotopy equivalence over B':
$$\begin{tikzcd} f_0^*E \arrow[rr, "\Phi_F", "\simeq"'] \arrow[dr] & & f_1^*E \arrow[dl] \\ & B' & \end{tikzcd}$$
  This was constructed in Lecture 6 using the transport functor. A useful detail that follows from the construction (and was discussed in the Übung on 23.05.2024) is that if we write the pullback squares for both induced fibrations as
$$\begin{tikzcd} f_j^*E \arrow[r, "f_j'"] \arrow[d, "p_j"] & E \arrow[d, "p"] \\ B' \arrow[r, "f_j"] & B \end{tikzcd}, \qquad j = 0, 1,$$
  then $f_1' \circ \Phi_F$ is homotopic to f_0'.
- Corollary: For any pullback square
$$\begin{tikzcd} f^*E \arrow[r, "f'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$
  in which $p : E \to B$ is a fibration, if $f : B' \to B$ is a homotopy equivalence, then so is $f' : f^*E \to E$.
  Proof: Use a homotopy inverse $g : B \to B'$ of f and a homotopy $\mathrm{Id} \overset{F}{\rightsquigarrow} f \circ g$ to produce the diagram
$$\begin{tikzcd} \mathrm{Id}^*E = E \arrow[r, "\Phi_F", "\simeq"'] \arrow[dr, "p"] & g^*f^*E \arrow[r, "g'"] \arrow[d, "p''"] & f^*E \arrow[r, "f'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ & B \arrow[r, "g"] & B' \arrow[r, "f"] & B \end{tikzcd},$$
  in which $f' \circ g' \circ \Phi_F \underset{h}{\sim} \mathrm{Id}$. Since Φ_F is an isomorphism in hTop, it follows that f' has a right-inverse and g' has a left-inverse in hTop. Reversing the roles and using a homotopy of $g \circ f$ to the identity then shows that both are isomorphisms. (I have written all this

in the unpointed category for simplicity of notation, but it works equally well for pointed spaces.)

- Corollary of the corollary: If $p : E \to B$ is a fibration and B is contractible, then the inclusion of its fiber $F \hookrightarrow E$ is a homotopy equivalence.
  Proof: Apply the previous corollary to the pullback of $p : E \to B$ via the inclusion $\{*\} \hookrightarrow B$.
- Dualizing the above discussion (see Exercise 6.8) gives a similarly important theorem about cofibrations: if $A \hookrightarrow X$ is a cofibration and A is contractible, then the quotient projection $X \to X/A$ is a homotopy equivalence.
- Corollary: For any well-pointed space X, the natural quotient projections

$$\Sigma_\circ X \to \Sigma_+ X = \frac{\Sigma_\circ X}{\Sigma_\circ\{*\}}, \qquad C_\circ X \to C_+ X = \frac{C_\circ X}{C_\circ\{*\}}, \qquad Z_\circ X \to Z_+ X = \frac{Z_\circ X}{Z_\circ\{*\}}$$

  are all homotopy equivalences.
  Proof: Well-pointed means that $\{*\} \hookrightarrow X$ is a closed free cofibration, thus so is $\Sigma_\circ\{*\} \hookrightarrow \Sigma_\circ X$, with a contractible cobase (and similarly for the cone and cylinder).
- Working in either Top or Top_*, the **Puppe cofiber sequence** of a map $f : X \to Y$ is the sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{i_f} \operatorname{cone}(f) \xrightarrow{q_f} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i_f} \Sigma \operatorname{cone}(f) \xrightarrow{-\Sigma q_f} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \ldots,$$

  where $i_f := i_Y$ denotes the natural inclusion of Y into the mapping cone of $f : X \to Y$, q_f is the quotient projection that arises from the obvious identification of $\operatorname{cone}(f)/Y$ with ΣX, and the minus signs in front of maps such as $\Sigma f : \Sigma X \to \Sigma Y$ mean they are composed with an inversion map such as

$$\Sigma Y \to \Sigma Y : [(y,t)] \mapsto [(y,-t)].$$

  Theorem: The cofiber sequence is natural, and any three consecutive terms in the sequence have the homotopy type of a cofibration. In particular, choosing any space Q (which should be assumed path-connected if we are working without base points) and plugging the cofiber sequence into the contravariant functor $[\cdot, Q]$ gives an exact sequence of pointed sets

$$\ldots \to [\Sigma^2 X, Q] \to [\Sigma \operatorname{cone}(f), Q] \to [\Sigma Y, Q] \to [\Sigma X, Q] \to [\operatorname{cone}(f), Q] \to [Y, Q] \to [X, Q].$$

  Just a few remarks on the proof: The inductive step follows from the lemma that suspensions of cofibrations are cofibrations. It should be clear already that the first three terms have the homotopy type of a cofibration: one only needs to replace Y with the mapping cylinder of f, and the cofiber of the cofibration $X \hookrightarrow Z(f)$ is then the homotopy cofiber $\operatorname{cone}(f)$ that appears as the third term. Terms two through four are also not a problem, because i_f already is a cofibration. The main step is thus to understand terms three through five; see Exercise 6.9 below. In the background of that step is an important theorem about cofibrations that is dual to a theorem we derived from the transport functor for fibrations: given a diagram

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "\varphi", "\simeq"'] & & Y \end{tikzcd}$$

  in which i and j are cofibrations and φ is a homotopy equivalence, φ is also a **homotopy equivalence under** A, meaning it has a homotopy inverse with homotopies to the identity that also fit into similar diagrams. It follows in particular that φ descends to a homotopy equivalence $X/A \to Y/A$; this is why the notion of the *homotopy cofiber* of a map is well-defined up to homotopy equivalence.

Suggested reading. Honestly I'm not sure what to suggest: I have found myself deeply unsatisfied with most textbook accounts I've read of the material we covered this week, which is why I ended up writing more detail than usual in my lecture summary. For a similar but slightly flawed account of the topological subtleties behind the smash product, you can look at [tD08, §2.4]. The treatment of the cofiber sequence in [May99, §8.4] is decent, though concise (as befits the title of the book), and I see one or two details on which May's approach does not seem to adapt in a straightforward way for the unpointed case (which is, to be fair, less important).

Exercises (for the Übung on 30.05.2024).

Exercise 6.1. The first three parts of this exercise are aimed at proving the distributive law for the smash product with respect to finite wedge sums. After that, we address the tricky question of associativity.

(a) Show that for any (finite or infinite) collection of pointed quotient maps $\{q_\alpha : X_\alpha \to Y\}_{\alpha \in J}$, the map
$$\bigvee_{\alpha \in J} q_\alpha : \bigvee_{\alpha \in J} X_\alpha \to Y$$
determined by the universal property of the coproduct is also a quotient map.
Hint: You can prove this without knowing the definitions of "wedge sum" and "quotient map," so long as you know what universal properties characterize them.

(b) Show that for any three pointed spaces X, Y, Z and the quotient projection $q : Y \amalg Z \to Y \vee Z$, the map
$$\mathrm{Id} \times q : X \times (Y \amalg Z) \to X \times (Y \vee Z)$$
is also a quotient map. Do you think the statement will remain true if you replace $Y \amalg Z$ with an infinite disjoint union?
Comment: I'd love to know how to give a purely "formal" proof of this, but I don't—the proof that I know requires getting your hands dirty with open sets.

(c) Deduce that for any three pointed spaces, the canonical bijection
$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$$
is a homeomorphism.
Hint: Identify both with quotients of $X \times (Y \amalg Z)$, which is naturally homeomorphic to $(X \times Y) \amalg (X \times Z)$.

(d) Show that the canonical bijection
$$(X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$$
is continuous whenever Z is friendly, and its inverse is continuous whenever X is friendly; in particular, it is a homeomorphism if (but not only if) both X and Z are locally compact.
Hint: There are canonical continuous bijections to both of these spaces from a certain quotient of $X \times Y \times Z$.

(e) Show that for any two pointed spaces X, Y, there is a canonical continuous bijection $\Sigma(X \wedge Y) \to X \wedge \Sigma Y$, where Σ denotes the reduced suspension. Show moreover that this bijection is a homeomorphism whenever either X or Y is compact and Hausdorff.

Exercise 6.2. Recall that a one-point space $\{*\}$ is both an initial object and a terminal object in Top_* (cf. Exercise 1.5).

(a) Compute $X \wedge \{*\}$ for an arbitrary $X \in \mathsf{Top}_*$ without using the definition of the smash product; use only its adjunction property.

(b) Is there a pointed space $E \in \mathsf{Top}_*$ with the property that $X \wedge E$ and $E \wedge X$ are naturally homeomorphic to X for every $X \in \mathsf{Top}_*$?

Exercise 6.3. The *join* $X * Y$ of two spaces was defined in Exercise 2.7 as a special case of a double mapping cylinder, and like all mapping cylinders, it has both an unreduced and a reduced variant.

(a) To what familiar construction is the unreduced join $X * S^0$ for $X \in \mathsf{Top}$ homeomorphic?
(b) Same question about the reduced join $X * S^0$ for $X \in \mathsf{Top}_*$.

Exercise 6.4. Recall from Exercise 5.5 that a pointed space is called *well pointed* if its base point is a closed subset whose inclusion is a free cofibration.

(a) Show that for any closed free cofibration $A \hookrightarrow X$, the quotient X/A with its obvious base point is a well-pointed space.
(b) Is the reduced suspension of a well-pointed space always also a well-pointed space?
(c) Is $X_+ := X \amalg \{*\} \in \mathsf{Top}_*$ a well-pointed space for every $X \in \mathsf{Top}$?

Exercise 6.5. The **forgetful functor** $\mathsf{Top} \to \mathsf{Set}$ sends each topological space X to its underlying set, forgetting the topology. Show that this functor has both a left-adjoint and a right-adjoint $\mathsf{Set} \to \mathsf{Top}$. What are they?

Exercise 6.6. Suppose $\mathcal{R} : \mathscr{A} \to \mathscr{B}$ is a covariant functor with left-adjoint $\mathcal{L} : \mathscr{B} \to \mathscr{A}$, and let

$$\operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(X), Y) \xrightarrow{\alpha} \operatorname{Hom}_{\mathscr{B}}(X, \mathcal{R}(Y)) : f \mapsto \widehat{f}$$

denote the resulting natural bijection of Hom-sets for each $X \in \mathscr{B}$ and $Y \in \mathscr{A}$.

(a) Show that there are natural morphisms

$$Y \xrightarrow{\Phi} \mathcal{R}\mathcal{L}(Y), \qquad \text{and} \qquad \mathcal{L}\mathcal{R}(X) \xrightarrow{\Psi} X$$

for $X \in \mathscr{A}$ and $Y \in \mathscr{B}$ such that the diagrams

$$\begin{tikzcd} \operatorname{Hom}_{\mathscr{B}}(Z, Y) \arrow[r, "\mathcal{L}"] \arrow[dr, "\Phi_*"'] & \operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(Z), \mathcal{L}(Y)) \arrow[d, "\alpha"] \\ & \operatorname{Hom}_{\mathscr{B}}(Z, \mathcal{R}\mathcal{L}(Y)) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \operatorname{Hom}_{\mathscr{A}}(X, Q) \arrow[r, "\mathcal{R}"] \arrow[dr, "\Psi^*"'] & \operatorname{Hom}_{\mathscr{B}}(\mathcal{R}(X), \mathcal{R}(Q)) \arrow[d, "\alpha^{-1}"] \\ & \operatorname{Hom}_{\mathscr{A}}(\mathcal{L}\mathcal{R}(X), Q) \end{tikzcd}$$

commute for every $X, Q \in \mathscr{A}$ and $Y, Z \in \mathscr{B}$. Here, the word "natural" means that Φ and Ψ should define natural transformations relating the identity functors on $\mathscr{B}$ and $\mathscr{A}$ to the functors $\mathcal{R}\mathcal{L} : \mathscr{B} \to \mathscr{B}$ and $\mathcal{L}\mathcal{R} : \mathscr{A} \to \mathscr{A}$ respectively.
Hint: It would be possible to deduce the existence of Φ and Ψ from abstract nonsense in the spirit of Exercise 1.8, but you might find it easier to just guess what Φ and Ψ are and then check if you are right. What morphisms can you imagine they might be adjoint to? You will need to use the naturality of α.

(b) Applying the result of part (a) to the specific functors $\mathcal{L} := \Sigma$ and $\mathcal{R} := \Omega$ from Top_* to itself gives rise to pointed maps

$$X \xrightarrow{\Phi} \Omega\Sigma X \qquad \text{and} \qquad \Sigma\Omega X \xrightarrow{\Psi} X$$

that are canonically associated to every pointed space X. Write down explicit formulas for these maps.

(c) Is it true in general that $f \in \operatorname{Hom}_{\mathscr{A}}(\mathcal{L}(X), Y)$ is an isomorphism if and only if $\widehat{f} \in \operatorname{Hom}_{\mathscr{B}}(X, \mathcal{R}(Y))$ is an isomorphism?

Exercise 6.7. In this exercise, $\Sigma : \mathsf{Top}_* \to \mathsf{Top}_*$ denotes the reduced suspension functor, and $\operatorname{cone}(f)$ is the reduced mapping cone of a pointed map f.

(a) Show that Σ preserves pushouts, i.e. for any two diagrams of the form

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Sigma Z \arrow[r, "\Sigma f"] \arrow[d, "\Sigma g"] & \Sigma X \arrow[d, "\Sigma\varphi"] \\ \Sigma Y \arrow[r, "\Sigma\psi"] & \Sigma P \end{tikzcd},$$

if the diagram on the left is a pushout square, then so is the diagram on the right.
Hint: The only thing you actually need to know about Σ here is that it is a left-adjoint; you don't even really need to know what its right-adjoint is. This is a special case of the general phenomenon that left-adjoint functors preserve colimits, and similarly, right-adjoint functors preserve limits, so e.g. Ω preserves pullbacks. (We saw already in Exercise 5.9 that Ω preserves products.)

(b) Show that for any pointed map $f : X \to Y$, there is a natural pointed homeomorphism $\Sigma\,\mathrm{cone}(f) \cong \mathrm{cone}(\Sigma f)$.

Exercise 6.8. Fill in the details of the proof of the theorem (used in Lecture 9) that for any cofibration $A \hookrightarrow X$ such that A is contractible, the quotient projection $X \to X/A$ is a homotopy equivalence.

Exercise 6.9. The proof of exactness for the Puppe cofiber sequence of a map $f : X \to Y$ hinges on the claim that the rightmost triangle in the following diagram commutes up to homotopy:

$$\begin{tikzcd} Y \arrow[r, hook, "i_f"] \arrow[dr, hook, "i_Y"'] & \mathrm{cone}(f) \arrow[r, "q_f"] \arrow[dr, hook, "i_{i_f}"] & \Sigma X \arrow[r, "-\Sigma f"] \arrow[dr, phantom, "\sim"] & \Sigma Y \\ & Z(i_f) \arrow[u, "h", "\simeq"'] \arrow[r, "q_Y"] & \mathrm{cone}(i_f) \arrow[u, "h", "\simeq"'] \arrow[ur, "q_{i_f}"'] & \end{tikzcd}$$

This diagram makes sense in two parallel versions, with the mapping cones and cylinders understood as the unreduced variants if we are working in Top, and the reduced varisnts if we are working in Top_*. The maps labelled i are the obvious inclusions into mapping cones or cylinders, maps labelled q are the quotient projections that collapse those included subspaces to a point, $h : Z(i_f) \to \mathrm{cone}(f)$ is the obvious homotopy inverse of the inclusion $\mathrm{cone}(f) \hookrightarrow Z(i_f)$, and $-\Sigma f$ is the composition of $\Sigma f : \Sigma X \to \Sigma Y$ with the inversion map $\Sigma Y \to \Sigma Y : [(y,t)] \mapsto [(y,-t)]$. Prove the claim about the rightmost triangle.
Hint: If you can draw a useful picture of the space $\mathrm{cone}(i_f)$, it may become almost obvious how the desired homotopy of maps $\mathrm{cone}(i_f) \to \Sigma Y$ should be defined, and why the minus sign appears in the diagram.

7. Week 7

Lecture 10 (27.05.2024): Group and cogroup objects.

- For a pointed map $f : X \to Y$ and any other pointed space Q, we now have two long exact sequences of pointed homotopy sets, the fiber and cofiber sequence respectively:

$$\ldots \longrightarrow [Q,\Omega^2 Y] \xrightarrow{(\Omega i_f)_*} [Q,\Omega F(f)] \xrightarrow{(\Omega\pi_f)_*} [Q,\Omega X] \xrightarrow{(\Omega f)_*} [Q,\Omega Y] \xrightarrow{(i_f)_*} [Q,F(f)] \xrightarrow{(\pi_f)_*} [Q,X] \xrightarrow{f_*} [Q,Y]$$

$$\ldots \longrightarrow [\Sigma^2 X,Q] \xrightarrow{(\Sigma q_f)^*} [\Sigma\,\mathrm{cone}(f),Q] \xrightarrow{(\Sigma i_f)^*} [\Sigma Y,Q] \xrightarrow{(\Sigma f)^*} [\Sigma X,Q] \xrightarrow{q_f^*} [\mathrm{cone}(f),Q] \xrightarrow{i_f^*} [Y,Q] \xrightarrow{f^*} [X,Q]$$

Note: We have gotten rid of the minus signs, because this does not affect the exactness of the sequences.

- Theorem: For all pointed spaces Z, Q and $n \geqslant 1$, the sets $[Q, \Omega^n Z]$ and $[\Sigma^n Z, Q]$ have natural group structures, which are abelian for $n \geqslant 2$, such that their base points are identity elements and all maps between such terms appearing in the fiber and cofiber sequences are group homomorphisms.
- Remark: An exact sequence in Grp carries a lot more information than an exact sequence in Set_*. You can't conclude e.g. from the exactness of $\{*\} \to A \xrightarrow{f} B$ that f is injective if the maps are only pointed set maps, but you can if they are group homomorphisms!
- Example 1: Taking $Q = S^0$ in the fiber sequence produces the homotopy groups
$$\pi_n(Z) := [S^0, \Omega^n Z] = [\Sigma^n S^0, Z] \cong [S^n, Z],$$
where we've used adjunction to exchange Ω^n for Σ^n and recalled the observation that $\Sigma^n S^0 \cong S^n$. The theorem gives $\pi_n(Z)$ a group structure for every $n \geqslant 1$ (though $\pi_0(Z)$ is only a pointed set), and makes it abelian for $n \geqslant 2$. We will discuss applications of the fiber sequence to the higher homotopy groups in the next lecture.
- Example 2: We will later see that for every $n \geqslant 0$ and abelian group G, there exists a CW-complex $K(G, n)$ such that
$$\Omega K(G,n) \underset{h.e.}{\simeq} K(G, n-1) \qquad \text{and} \qquad [Z, K(G,n)] \cong H^n(Z; G)$$
for all CW-complexes Z. If one took this as a *definition* of cohomology, one could derive its abelian group structure from the theorem above since
$$[Z, K(G,n)] \cong [Z, \Omega^2 K(G, n+2)].$$
Notice, by the way, that if you insert $Q := K(G,n)$ into the cofiber sequence, you obtain terms such as
$$[\Sigma^k Z, K(G,n)] = [Z, \Omega^k K(G,n)] \cong [Z, K(G, n-k)] \cong H^{n-k}(Z; G),$$
which gives a hint that the cofiber sequence reproduces the long exact sequence of a pair in cohomology. (One can also derive homology long exact sequences from the cofiber sequence, but it requires more cleverness.)
- Idea for a "group like" structure on ΩX: Use **concatenation** of loops to define a multiplication map
$$\Omega X \times \Omega X \xrightarrow{m} \Omega X,$$
along with the inversion map
$$\Omega X \xrightarrow{i} \Omega X : \alpha \mapsto -\alpha := \alpha(1 - \cdot),$$
and interpret the base point of ΩX as an identity element, i.e. the image of the unique pointed map
$$\{*\} \xrightarrow{e} \Omega X.$$
- Proposition 1 (vague version): The maps m, i and e described above satisfy the axioms of a group structure "up to homotopy".
- Definition: Assume $\mathscr{A}$ is a category in which all finite products exist (including the "empty" product $1 := \prod_{\varnothing} X$, which is a terminal object; cf. Exercise 1.5). A **group object** in $\mathscr{A}$ is a tuple (X, m, e, i) consisting of an object $X \in \mathscr{A}$ equipped with morphisms $m : X \times X \to X$, $e : 1 \to X$ and $i : X \to X$ that satisfy the following axioms:
  (1) Identity: $m \circ (e \times \mathrm{Id}_X) \circ \pi_X^{-1} = \mathrm{Id}_X$ and $m \circ (\mathrm{Id}_X \times e) \circ \pi_X^{-1} = \mathrm{Id}_X$, where π_X denotes the canonical projection from $1 \times X$ or $X \times 1$ to X (which is an isomorphism since 1 is terminal).
  (2) Associativity: $m \circ (m \times \mathrm{Id}_X) = m \circ (\mathrm{Id}_X \times m)$

(3) Inverse: $m \circ (\mathrm{Id}_X, i) = m \circ (i, \mathrm{Id}_X) = e \circ \epsilon$, where $\epsilon : X \to 1$ denotes the unique morphism to the terminal object.

We call X **abelian** if it additionally satisfies $m \circ \sigma = m$, where for any two objects $Y, Z \in \mathscr{A}$, we denote $\sigma := (\pi_Z, \pi_Y) : Y \times Z \to Z \times Y$ in terms of the projection morphisms π_Y, π_Z from the product $Y \times Z$ to Y and Z respectively.

- There is a category $\mathsf{Grp}(\mathscr{A})$ whose objects are group objects in $\mathscr{A}$, such that morphisms from (X, m_X, e_X, i_X) to (Y, m_Y, e_Y, i_Y) are morphisms $f : X \to Y$ that commute with the structure morphisms, i.e.
$$f \circ m_X = m_Y \circ (f \times f), \qquad f \circ e_X = e_Y, \qquad f \circ i_X = i_Y \circ f.$$
- Examples:
  (1) A group object (G, m, e, i) in Set is just a group, with $gh := m(g, h)$, $g^{-1} := i(g)$, and $e(*)$ defining the identity element. (Note that a terminal object in Set is a set $\{*\}$ of one point.) Thus $\mathsf{Grp}(\mathsf{Set}) = \mathsf{Grp}$.
  (2) Similarly, $\mathsf{Grp}(\mathsf{Top})$ is the category of topological groups, and
  (3) $\mathsf{Grp}(\mathrm{Diff})$ is the category of smooth Lie groups.
- Brief nonessential digression: It can also be useful to define other algebraic structures within the context of categories other than Set: for example, an **H-space** (named after Heinz Hopf) is an object of hTop_* equipped with two morphisms m and e satisfying the identity axiom listed above, but without assuming associativity or the existence of inverses. The argument behind the fact that $[\Sigma^n Z, Q]$ and $[Z, \Omega^n Z]$ are abelian for $n \geqslant 2$ will also imply restrictions on the topologies of H-spaces, notably that their fundamental groups must be abelian. Interesting factoid: the classic algebraic theorem that there are only four real finite-dimensional division algebras ($\mathbb{R}$, $\mathbb{C}$, the quaternions and the octonions) was deduced from a topological result, proving via K-theory that S^n can only be an H-space if $n \in \{0, 1, 3, 7\}$.
- Proposition 1 (precise version): For every $X \in \mathsf{Top}_*$, the loop space ΩX with the maps m, e, i described above is a group object in hTop_*, and this defines a lift of the functor $\Omega : \mathsf{hTop}_* \to \mathsf{hTop}_*$,
$$\begin{tikzcd} & \mathsf{Grp}(\mathsf{hTop}_*) \arrow[d] \\ \mathsf{hTop}_* \arrow[ur, "\tilde{\Omega}"] \arrow[r, "\Omega"] & \mathsf{hTop}_* \end{tikzcd},$$
where the downward arrow is the "forgetful" functor that forgets the group object structure. In particular: The continuous pointed map $\Omega f : \Omega X \to \Omega Y$ induced by any pointed map $f : X \to Y$ is automatically also a morphism of group objects in hTop_*. (This is nearly immediate from the definitions of the group structure morphisms.)
- Proposition 2: For any covariant functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ that preserves finite products, each group object (X, m, e, i) in $\mathscr{A}$ determines a group object $(\mathcal{F}(X), \mathcal{F}(m), \mathcal{F}(e), \mathcal{F}(i))$ in $\mathscr{B}$ and thus defines a functor $\mathcal{F} : \mathsf{Grp}(\mathscr{A}) \to \mathsf{Grp}(\mathscr{B})$.
Remark: The condition "preserves finite products" means in part that for any two objects $X, Y \in \mathscr{A}$ with product $X \times Y \in \mathscr{A}$, the object $\mathcal{F}(X \times Y) \in \mathscr{B}$ is a product of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ in the category-theoretic sense. This condition is needed in order to interpret the morphism $\mathcal{F}(m)$ as
$$\mathcal{F}(X) \times \mathcal{F}(X) = \mathcal{F}(X \times X) \xrightarrow{\mathcal{F}(m)} \mathcal{F}(X).$$
The condition also means that for a terminal object $1 \in \mathscr{A}$, $\mathcal{F}(1)$ is likewise a terminal object of $\mathscr{B}$, making $\mathcal{F}(e)$ a morphism from a terminal object to $\mathcal{F}(X)$.

- Corollary (applied to the functor $[Q,\cdot] : \mathsf{hTop}_* \to \mathsf{Set}_*$): For every $Q, X \in \mathsf{Top}_*$, the set $[Q, \Omega X]$ has a natural group structure such that for all homotopy classes of pointed maps $f : X \to Y$, the induced map $(\Omega f)_* : [Q, \Omega X] \to [Q, \Omega Y]$ is a group homomorphism.
- Remark: In any category, one can apply Proposition 2 to the functor $\mathcal{F} := \mathrm{Hom}_{\mathscr{A}}(Q, \cdot) : \mathscr{A} \to \mathsf{Set}$ for any chosen object $Q \in \mathscr{A}$ since there are always natural bijections
$$\mathrm{Hom}_{\mathscr{A}}(Q, Y \times Z) = \mathrm{Hom}_{\mathscr{A}}(Q, Y) \times \mathrm{Hom}_{\mathscr{A}}(Q, Z)$$
and $\mathrm{Hom}_{\mathscr{A}}(Q, 1) = \{*\}$ is a terminal object in Set. One checks from the definitions that each morphism $f : Y \to Z$ in $\mathscr{A}$ then determines a group homomorphism $f^* : \mathrm{Hom}_{\mathscr{A}}(Z, X) \to \mathrm{Hom}_{\mathscr{A}}(Y, X)$.
- Proposition 3 (abstract nonsense): Assuming $\mathscr{A}$ admits all finite products, for each object $X \in \mathscr{A}$, there is a canonical one-to-one correspondence between group object structures (m, e, i) on X and lifts of the contravariant functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, X) : \mathscr{A} \to \mathsf{Set}$ to a functor $\mathscr{A} \to \mathsf{Grp}$ whose composition with the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is $\mathrm{Hom}_{\mathscr{A}}(\cdot, X)$.
Proof sketch: Assuming $\mathrm{Hom}_{\mathscr{A}}(Q, X)$ is a group for each $Q \in \mathscr{A}$ with multiplication denoted by $\boxtimes$, and that $f^* : \mathrm{Hom}_{\mathscr{A}}(Z, X) \to \mathrm{Hom}_{\mathscr{A}}(Y, X)$ is a group homomorphism for each morphism $f : Y \to Z$ in $\mathscr{A}$, we use the group structure on $\mathrm{Hom}_{\mathscr{A}}(X \times X, X)$ to define the morphism
$$m := \pi_1 \boxtimes \pi_2 \in \mathrm{Hom}_{\mathscr{A}}(X \times X, X)$$
in terms of the projections $\pi_1, \pi_2 : X \times X \to X$ to the first and second factor of the product. Similarly, define $i \in \mathrm{Hom}_{\mathscr{A}}(X, X)$ to be the inverse of Id_X with respect to the group structure on $\mathrm{Hom}_{\mathscr{A}}(X, X)$, and define e to be the identity element in the group $\mathrm{Hom}_{\mathscr{A}}(1, X)$. Now just check that the axioms are satisfied.
- Question: For a *contravariant* functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ such as $\mathrm{Hom}_{\mathscr{A}}(\cdot, Q) : \mathscr{A} \to \mathsf{Set}$ (or more specifically $[\cdot, Q] : \mathsf{hTop}_* \to \mathsf{Set}$ for a pointed space Q), what kind of structure on an object $X \in \mathscr{A}$ makes $\mathcal{F}(X)$ into a group?
- Predefinition: Given a category $\mathscr{A}$, the **opposite category** $\mathscr{A}^{\mathrm{op}}$ is the category that has the same objects as $\mathscr{A}$ but with
$$\mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathscr{A}}(Y, X)$$
for every $X, Y \in \mathscr{A}^{\mathrm{op}} = \mathscr{A}$, i.e. the sets of morphisms are the same but all arrows are reversed. This is the definition one needs if one wants to talk about functors without ever saying the words "covariant" and "contravariant": one can then define a contravariant functor from $\mathscr{A}$ to some other category $\mathscr{B}$ to be a covariant functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}$.
Exercise: An object is terminal in $\mathscr{A}^{\mathrm{op}}$ if and only if it is initial in $\mathscr{A}$, and a product of two objects in $\mathscr{A}^{\mathrm{op}}$ is canonically equivalent to a coproduct of the same objects in $\mathscr{A}$.
- Definition: Assume $\mathscr{A}$ is a category in which all finite coproducts exist (including an initial object $0 \in \mathscr{A}$). A **cogroup object** in $\mathscr{A}$ is then a group object in $\mathscr{A}^{\mathrm{op}}$. More concretely, reversing the arrows of the structure morphisms of a group object in $\mathscr{A}^{\mathrm{op}}$ realizes a cogroup object in $\mathscr{A}$ as a tuple $(X, \mu, \epsilon, \iota)$ with structure morphisms
$$X \xrightarrow{\mu} X \coprod X, \qquad X \xrightarrow{\epsilon} 0, \qquad X \xrightarrow{\iota} X$$
that satisfy analogues of the axioms satisfied by a group object, but with arrows reversed, products replaced by coproducts, and the terminal object 1 replaced by an initial object 0. Let $\mathsf{CoGrp}(\mathscr{A})$ denote the category of cogroup objects of $\mathscr{A}$, where morphisms are morphisms in $\mathscr{A}$ that commute with the cogroup structure morphisms.
- Remark (why you've perhaps never heard of a *cogroup* before): The initial object in Set is $\varnothing$, thus a cogroup object $(X, \mu, \epsilon, \iota)$ in Set must be equipped with a continuous map $\epsilon : X \to \varnothing$, which is only possible if $X = \varnothing$. The situation in Set_* is slightly better, but

not much (exercise). Interesting examples of cogroup objects are only possible in categories that are quite different from Set, e.g. homotopy categories, in which the freedom to have diagrams commute only up to homotopy allows for more possibilites.

- Dual of Proposition 2:[31] Assume $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is a contravariant functor that takes finite coproducts to products (and thus takes any initial object to a terminal object in particular). Then for any cogroup object $(X, \mu, \epsilon, \iota)$ in $\mathscr{A}$, the morphisms

$$\mathcal{F}(X) \times \mathcal{F}(X) \cong \mathcal{F}\Big(X \coprod X\Big) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(X), \qquad 1 \cong \mathcal{F}(0) \xrightarrow{\mathcal{F}(\epsilon)} \mathcal{F}(X), \qquad \mathcal{F}(X) \xrightarrow{\mathcal{F}(\iota)} \mathcal{F}(X)$$

  make $(\mathcal{F}(X), \mathcal{F}(\mu), \mathcal{F}(\epsilon), \mathcal{F}(\iota))$ into a group object in $\mathscr{B}$.
  Remark: The hypothesis about finite products and coproducts is satisfied in particular by $\mathrm{Hom}_{\mathscr{A}}(\cdot, Q) : \mathscr{A} \to \mathsf{Set}$ for any fixed object $Q \in \mathscr{A}$, which includes the special case $[\cdot, Q] : \mathsf{hTop}_* \to \mathsf{Set}$ that appears in the cofiber sequence.
- Remark: There is similarly a dual version of Proposition 3, characterizing cogroup object structures on $X \in \mathscr{A}$ in terms of lifts of the covariant functor $\mathrm{Hom}_{\mathscr{A}}(X, \cdot) : \mathscr{A} \to \mathsf{Set}$ to a functor valued in Grp.
- Dual of Proposition 1: The reduced suspension functor $\Sigma : \mathsf{hTop}_* \to \mathsf{hTop}_*$ admits a unique lift to a functor

$$\begin{tikzcd} & \mathsf{CoGrp}(\mathsf{hTop}_*) \arrow[d] \\ \mathsf{hTop}_* \arrow[ur, "\tilde{\Sigma}"] \arrow[r, "\Sigma"] & \mathsf{hTop}_* \end{tikzcd}$$

  such that for every $X, Y \in \mathsf{Top}_*$, the induced group structure on $[\Sigma X, Y]$ matches the one it inherits from the adjunction relation $[\Sigma X, Y] \cong [X, \Omega Y]$ and the group object structure of ΩY.
  Proof: For any given $X \in \mathsf{Top}_*$, the existence and uniqueness of a suitable cogroup object structure on ΣX can be deduced from a dual version of Proposition 3 after *defining* the group structure of each $[\Sigma X, Y]$ for each $Y \in \mathsf{Top}_*$ to match that of $[X, \Omega Y]$, and then observing that for each pointed map $f : Y \to Z$, the induced map

$$[\Sigma X, Y] \xrightarrow{f_*} [\Sigma X, Z]$$

  is a group homomorphism, which follows via adjunction from the fact that $\Omega f : \Omega Y \to \Omega Z$ is a morphism of group objects in hTop_*. It is not hard to write down explicit structure morphisms $\mu : \Sigma X \to \Sigma X \vee \Sigma X$, $\epsilon : X \to \{*\}$ and $\iota : \Sigma X \to \Sigma X$ that do the trick; for ϵ there is no choice to be made since there is only one pointed map to $\{*\}$, ι will be the inversion map $[(x,t)] \mapsto [(x, 1-t)]$ that appeared in our original version of the Puppe cofiber sequence, and μ is best described with a picture that can be found e.g. in [tD08, p. 91]. One then checks easily that for every pointed map $f : X \to Y$, the induced map $\Sigma f : \Sigma X \to \Sigma Y$ commutes with the cogroup structure morphisms, thus producing a lift of $\Sigma : \mathsf{hTop}_* \to \mathsf{hTop}_*$ to $\mathsf{CoGrp}(\mathsf{hTop}_*)$.
- To clarify next time: Why do all these groups become abelian as soon as Ω or Σ is applied more than once?

[31] I didn't articulate this statement precisely in the lecture because I was running out of time, but the idea was lurking in the background of everything I said about cogroup objects. I'm including the explicit statement here in case it doesn't seem obvious, though the proof is more-or-less immediate from the definitions.

Lecture 11 (30.05.2024): Higher homotopy groups. The beginning of this lecture wraps up the previous lecture's discussion of group and cogroup objects before talking about the higher homotopy groups in earnest.

- Example: We have two equivalent constructions of the group structure on $\pi_1(X) = [S^0, \Omega X] = [\Sigma S^0, X] = [S^1, X]$:
  (i) Using the group object structure $\Omega X \times \Omega X \xrightarrow{m} \Omega X$ defines the product of two homotopy classes of pointed maps $\alpha, \beta : S^0 \to \Omega X$ as
  $$[\alpha][\beta] := [m \circ (\alpha, \beta)] .$$
  (ii) Using the cogroup object structure $\Sigma S^0 \xrightarrow{\mu} \Sigma S^0 \vee \Sigma S^0$ defines the product of two homotopy classes of pointed maps $\alpha, \beta : \Sigma S^0 \to X$ as
  $$[\alpha][\beta] := [(\alpha \vee \beta) \circ \mu] .$$
  We intentionally defined $\mu : \Sigma S^0 \to \Sigma S^0 \vee \Sigma S^0$ (and more generally $\mu : \Sigma Y \to \Sigma Y \vee \Sigma Y$ for every $Y \in \mathsf{Top}_*$) so as to make these two products identical.
- Example: Things are more complicated for $\pi_2(X) = [S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X]$, because we can think of two ways of defining products that are not obviously equivalent. Since $\pi_2(X) = [S^0, \Omega^2 X]$ has an obvious identification with the set of path-components of $\Omega^2 X$, we can use elements of $\Omega^2 X$ as representatives, and think of these as maps
  $$\alpha : (I^2, \partial I^2) \to (X, *),$$
  where for each $s \in I$, $\alpha(s) \in \Omega X$ denotes the loop $t \mapsto \alpha(s,t)$.
  (a) Using the usual recipe to make $\Omega^2 X = \Omega(\Omega X)$ into a group object defines a product $*$ on $\pi_2(X)$ by concatenation of loops in ΩX, which means concatenation of maps $I^2 \to X$ with respect to the s variable, treating the t variable as an extra parameter. Equivalently, this is the product on $[\Sigma S^0, \Omega X]$ arising from the fact that ΣS^0 is a cogroup object.
  (b) We could instead use the group object structure of ΩX to define one on $\Omega^2 X$ by
  $$\Omega^2 X \times \Omega^2 X \cong \Omega(\Omega X \times \Omega X) \xrightarrow{\Omega m} \Omega(\Omega X) = \Omega^2 X.$$
  Elements of ΩX are functions of the t variable, so this defines a product $\boxtimes$ on $\pi_2(X)$ that concatenates maps $I^2 \to X$ with respect to t while treating s as an extra parameter. Equivalently, this product on $[\Sigma S^0, \Omega X]$ arises from the fact that ΩX is a group object, ignoring the fact that ΣS^0 is a cogroup object.
- Lemma (follows immediately from a picture of I^2 partitioned into four squares): The products $*$ and $\boxtimes$ on $\pi_2(X)$ satisfy the relation
  $$(\alpha * \beta) \boxtimes (\gamma * \delta) = (\alpha \boxtimes \gamma) * (\beta \boxtimes \delta). \tag{7.1}$$
  Corollary (easy algebraic exercise, see [tD08, Prop. 4.3.1]): The products $*$ and $\boxtimes$ on $\pi_2(X)$ are identical, and both are commutative.
  The next result reveals that what we are describing here is a much more general phenomenon.
- Proposition 4: Suppose X is a cogroup object and Y is a group object in some category $\mathscr{A}$. Then the group products $*$ and $\boxtimes$ defined on $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ via the cogroup structure of X or the group structure of Y respectively are the same, and they are commutative.
  Proof: Given four morphisms $\alpha, \beta, \gamma, \delta \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, we can construct a morphism
  $$X \coprod X \xrightarrow{F} Y \times Y$$

as $F := (\alpha \coprod \beta, \gamma \coprod \delta)$, which can equivalently be written as $(\alpha, \gamma) \coprod (\beta, \delta)$. Composing it with the cogroup morphism $\mu : X \to X \coprod X$ and group morphism $m : Y \times Y \to Y$ and writing $(m \circ F) \circ \mu = m \circ (F \circ \mu)$ then produces the two sides of the identity (7.1).

- Corollary (via Proposition 3 from last time): All the "natural" definitions of a group object structure (in hTop_*) on $\Omega^n X$ for $n \geqslant 2$ are equivalent, and they are abelian.
- Corollary of the corollary: The n group products defined on $\pi_n(X)$ via concatenation with respect to each of the n coordinates on I^n are all identical, and for $n \geqslant 2$, the group is abelian.
- Commentary: I have reproduced the corollary above with the same vague wording that I used in the lecture—the word "natural" in this case was not intended in its precise category-theoretical sense, and making the statement more precise is tricky without getting too verbose, but I'll make an attempt here, inspired in part by questions I received after the lecture. The point is this: for an arbitrary pointed space X, we have used concatenation to define a recipe for making ΩX into a group object in hTop_*, and we are regarding that recipe as the "natural" one; there might be other ways to make ΩX into a group object based on completely different ideas, but if so, then they are not relevant to this discussion, and the word "natural" was thus meant to exclude them. For $\Omega^2 X = \Omega(\Omega X)$—and more generally for $\Omega^n X$ for any $n \geqslant 2$—one then has the choice of whether to apply the established recipe as it stands, thus ignoring the fact that ΩX already is a group object, or alternatively, to ignore the recipe but reuse the existing group object structure of ΩX in order to define one on $\Omega(\Omega X)$. I would also consider the latter a "natural" thing to do, and the first message of the corollary is that it doesn't matter which of these approaches we choose. The second message is that the fact of having *multiple* choices in this construction for $n \geqslant 2$ has a nontrivial algebraic consequence, forcing the resulting group object structure to be abelian. In the general setting that we are considering here, the key feature beyond Propostion 4 that leads to these conclusions is adjunction, i.e. the fact that we can always choose freely whether to define the product on a set $[X, \Omega Y] = [\Sigma X, Y]$ in terms of the group object structure of ΩY or the cogroup object structure of ΣX, without the result depending on this choice. It's worth noting, however, that this is not the only useful and interesting way to apply Proposition 4, and it can also have consequences involving group-like structures that have nothing to do with concatenation, such as:
- Theorem: For any pointed space X that is a group object in hTop_*, $\pi_1(X)$ is abelian. This applies in particular whenever X is a topological group with the identity as its base point, which means it is a group object in Top_* (and therefore automatically also in hTop_*).
  Quick proof: The standard definition of the product on $\pi_1(X) = [\Sigma S^0, X]$ can be expressed via the cogroup object structure of ΣS^0, but according to Proposition 4, one would get the same product (which is therefore commutative) by ignoring the suspension and using the native group object structure of X instead.
  Exercise: Find a more elementary proof of this result in the case where X is a topological group. (You have probably seen one in a previous course.)
- Consider a pointed pair of spaces (X, A), meaning $* \in A \subset X$, with inclusion map $j : A \hookrightarrow X$. Feeding the Puppe fiber sequence into $[S^0, \cdot]$ and using adjunction to write $[S^0, \Omega^n X] = [\Sigma^n S^0, X] = [S^n, X] = \pi_n(X)$ and similarly for A and the homotopy fiber $F(j)$, we get a long exact sequence

$$\begin{aligned} \ldots \longrightarrow \pi_3(X) \longrightarrow [\Sigma^2 S^0, F(j)] \longrightarrow \pi_2(A) \xrightarrow{j_*} \pi_2(X) \longrightarrow [\Sigma S^0, F(j)] \\ \longrightarrow \pi_1(A) \xrightarrow{j_*} \pi_1(X) \longrightarrow [S^0, F(j)] \longrightarrow \pi_0(A) \xrightarrow{j_*} \pi_0(X) \end{aligned} \tag{7.2}$$

The entire sequence is exact in the category Set_*; removing the last three terms makes it also an exact sequence in Grp, and all of the groups from $\pi_2(X)$ backwards are abelian. The maps to and from the terms $[\Sigma^n S^0, F(j)]$ are induced by the natural inclusion $i_j : \Omega X \hookrightarrow F(j)$ and projection[32] $p_j : F(j) \to A$ respectively, where making sense of the former requires rewriting $\pi_{n+1}(X) = [\Sigma^{n+1} S^0, X]$ as $[\Sigma^n S^0, \Omega X]$.

- Interpretation of $[\Sigma^n S^0, F(j)]$: Since $j : A \hookrightarrow X$ is an inclusion, we can identify the mapping fiber $F(j)$ with the space
$$F(j) = \left\{ \gamma \in X^I \;\middle|\; \gamma(0) \in A \text{ and } \gamma(1) = * \right\} \subset X^I,$$
which makes $i_j : \Omega X \to F(j) \subset X^I$ the obvious inclusion and $p_j : F(j) \to A$ the restriction to $F(j)$ of the map $\mathrm{ev}_0 : X^I \to X : \gamma \mapsto \gamma(0)$. Identifying S^0 with ∂I gives a natural homeomorphism $\Sigma^n S^0 = I^n / \partial I^n$, so that maps $\Sigma^n S^0 \to F(j)$ can be regarded as maps $I^n \to X^I$ that satisfy certain constraints; by adjunction, these are equivalent to maps $\alpha : I^{n+1} = I^n \times I \to X$, and the constraints they satisfy are firstly
$$\alpha(\partial I^n \times I) = \{*\},$$
which ensures that the corresponding map $I^n \to X^I$ descends to the quotient $I^n/\partial I^n = \Sigma^n S^0$, and secondly
$$\alpha(I^n \times \{0\}) \subset A, \qquad \text{and} \qquad \alpha(I^n \times \{1\}) = \{*\},$$
which make the map $I^n \to X^I$ take values in $F(j) \subset X^I$. We can rewrite this more succinctly after noticing that one of the boundary faces $I^n \times \{0\} \subset \partial I^{n+1}$ of the cube evidently plays a special role, so let us identify
$$I^n := I^n \times \{0\} \subset \partial I^{n+1} \subset I^{n+1},$$
and denote the union of the rest of the boundary faces of I^{n+1} by
$$J^n := \overline{\partial I^{n+1} \backslash I^n},$$
so that $I^n \cap J^n = \partial I^n$. We have now identified $[\Sigma^n S^0, F(j)]$ with the set of homotopy classes of maps $I^{n+1} \to X$ that send ∂I^{n+1} into A and the subset $J^n \subset \partial I^{n+1}$ to the base point, in short,
$$\pi_{n+1}(X, A) := [\Sigma^n S^0, F(j)] = \left\{ \text{maps of triples } (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, *) \right\} \big/ \text{homotopy}.$$
This is our official definition of the **relative homotopy groups**; to be precise, $\pi_n(X, A)$ is a pointed set for every $n \geqslant 1$, a group for $n \geqslant 2$, and is also abelian for $n \geqslant 3$. Note that there is no general definition of $\pi_0(X, A)$. The group structure can be described via concatenation in any of the first n variables on the cube I^{n+1}; variable $n+1$ is not appropriate for this purpose because it does not come from the n-fold suspension $\Sigma^n S^0$, but rather from the homotopy fiber $F(j)$, and this is why $\pi_1(X, A)$ has no natural group structure.
- A nicer geometric picture of $\pi_n(X, A)$: notice that there is a homeomorphism of pointed pairs
$$\left(I^n / J^n, I^{n-1} / \partial I^{n-1} \right) \cong (\mathbb{D}^n, S^{n-1})$$
for any choice of base point $* \in S^{n-1} = \partial \mathbb{D}^n \subset \mathbb{D}^n$, which turns $\pi_n(X, A)$ into
$$\left\{ \text{pointed maps of pairs } (\mathbb{D}^n, S^{n-1}) \to (X, A) \right\} \big/ \text{homotopy}.$$

[32] I would normally call this projection map π_j, and probably did so in the lecture, but I've changed the notation here so that it doesn't get confused with a homotopy group.

This motivates the notation for $\pi_n(X, A)$ as a generalization of $\pi_n(X)$, the latter being identified in this picture with $\pi_n(X, \{*\})$. The only reason not to make this the official definition of $\pi_n(X, A)$ is that from this perspective, we do not clearly see any suspensions or loop spaces, making it far less obvious what the group structure of $\pi_n(X, A)$ should be (for $n \geqslant 2$).

- The conclusion of this discussion is that for any pointed pair (X, A) with inclusion $j : A \hookrightarrow X$, there is a natural long exact sequence

(7.3)
$$\begin{aligned} \ldots \longrightarrow \pi_3(X) \xrightarrow{i_*} \pi_3(X, A) \xrightarrow{\partial_*} \pi_2(A) \xrightarrow{j_*} \pi_2(X) \xrightarrow{i_*} \pi_2(X, A) \\ \xrightarrow{\partial_*} \pi_1(A) \xrightarrow{j_*} \pi_1(X) \xrightarrow{i_*} \pi_1(X, A) \xrightarrow{\partial_*} \pi_0(A) \xrightarrow{j_*} \pi_0(X), \end{aligned}$$

in which the last three terms are in general only pointed sets (don't let the subscript on $\pi_1(X, A)$ fool you!), but the rest are groups, all abelian from $\pi_2(X)$ backwards. The maps $i_* : \pi_n(X) \to \pi_n(X, A)$ in this sequence are what you think they should be (see Exercise 7.3). The maps $\partial_* : \pi_n(X, A) \to \pi_{n-1}(A)$ are $(p_j)_* : \pi_n(X, A) = [\Sigma^{n-1} S^0, F(j)] \to [\Sigma^{n-1} S^0, A] = \pi_{n-1}(A)$, induced by $p_j = \mathrm{ev}_0 : F(j) \to A$, so explicitly, if we identify $I^{n-1} = I^{n-1} \times \{0\} \subset \partial I^n$ in order to represent an element $[\alpha] \in \pi_n(X, A)$ as a homotopy class of maps $\alpha : (I^n, \partial I^n, \partial I^n \backslash I^{n-1}) \to (X, A, *)$, then
$$\partial_*[\alpha] = [\alpha|_{I^{n-1}}] \in [I^{n-1}/\partial I^{n-1}, A] = [\Sigma^{n-1} S^0, A] = \pi_{n-1}(A).$$
Choosing a homeomorphism $\left(I^n / J^n, I^{n-1} / \partial I^{n-1}\right) \cong (\mathbb{D}^n, S^{n-1})$ so as to represent $[\alpha] \in \pi_n(X, A)$ via a pointed map of pairs $\alpha : (\mathbb{D}^n, S^{n-1}) \to (X, A)$ puts this formula in the more appealing form
$$\partial_*[\alpha] = [\alpha|_{\partial \mathbb{D}^n}] \in [S^{n-1}, A] = \pi_{n-1}(A).$$

- Theorem: For any pointed fibration $p : E \to B$ with fiber inclusion $j : F \hookrightarrow E$, there is a natural long exact sequence

(7.4)
$$\begin{aligned} \ldots \longrightarrow \pi_3(B) \xrightarrow{\delta} \pi_2(F) \xrightarrow{j_*} \pi_2(E) \xrightarrow{p_*} \pi_2(B) \xrightarrow{\delta} \pi_1(F) \\ \xrightarrow{j_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B) \xrightarrow{\delta} \pi_0(F) \xrightarrow{j_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B), \end{aligned}$$

where the connecting maps $\delta : \pi_n(B) \to \pi_{n-1}(F)$ can be written as
$$\delta[\alpha] = [\tilde{\alpha}|_{S^{n-1}}] \in [S^{n-1}, F] = \pi_{n-1}(F)$$
for any $[\alpha] \in \pi_n(B) = \pi_n(B, \{*\})$ written as the homotopy class of a map of pointed pairs $\alpha : (\mathbb{D}^n, S^{n-1}, *) \to (B, \{*\}, *)$), along with a choice of pointed lift
$$\begin{tikzcd} & E \arrow[d, "p"] \\ \mathbb{D}^n \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"] & B \end{tikzcd}.$$
Proof: Take the Puppe sequence of $p : E \to B$ (Theorem 2 in Lecture 8), remove the minus signs since they do not affect exactness, feed the whole thing into $[S^0, \cdot]$, and unpack the definitions.

- Dependence on base points: Assume $Q \in \mathsf{Top}_*$ is a well-pointed space, and for any $X \in \mathsf{Top}$ and $x \in X$, let
$$[Q, X]_x := [(Q, *), (X, x)]_+$$
denote the set of pointed homotopy classes of maps $Q \to X$ such that x is considered the base point of X. The free cofibration $* \hookrightarrow Q$ determines a transport functor $\Pi(X) \to \mathsf{Set}$, which is defined on the fundamental groupoid $\Pi(X) = \Pi(*, X)$ of X; recall that the objects of the latter are the points of X (equivalently the maps $* \to X$), and morphisms from x to

y are homotopy classes of paths from x to y. The transport functor assigns to each point $x \in X$ the set $[Q, X]_x$, and to each homotopy class of paths $x \overset{\gamma}{\rightsquigarrow} y$ with fixed end points a bijective map

$$\gamma_\# : [Q, X]_x \xrightarrow{\cong} [Q, X]_y,$$

which sends $[H_0] \mapsto [H_1]$ for any choice of (free) homotopy $H : Q \times I \to X$ with $H(*, t) = \gamma(t)$ for all $t \in I$, i.e. an "extension" of the homotopy $\gamma : * \times I \to X$.

- Illuminating exercise (see Exercise 7.4 below): Verify that the bijection $\gamma_\# : [Q, X]_x \to [Q, X]_y$ is automatically a group isomorphism whenever Q is a cogroup object in hTop_*, and that in the special case $Q := S^1$, it is the same isomorphism $\pi_1(X, x) \to \pi_1(X, y)$ that you saw in Topology 1. Applied to $Q := \Sigma^n S^0 = S^n$, this shows that the isomorphism class of each $\pi_n(X)$ for a path-connected space X is independent of the choice of base point.
- Theorem: If $f : X \to Y$ is a (free) homotopy equivalence, then for each $x \in X$ and $y := f(x) \in Y$ and every well-pointed space Q, the induced map

$$[Q, X]_x \xrightarrow{f_*} [Q, Y]_y$$

is a bijection. (Note that if Q is a cogroup object in hTop_*, then f_* is also automatically a group homomorphism, and therefore an isomorphism; this applies in particular to the maps $\pi_n(X, x) \to \pi_n(Y, y)$ for $n \geqslant 1$.)
Proof: Essentially the same as the special case $Q = S^1$, which was probably the first nontrivial theorem of algebraic topology that you saw proved (i.e. that π_1 is a homotopy type invariant). The tricky detail is just that the homotopy inverse $g : Y \to X$ of f need not respect base points, so writing $g(y) := z \in X$, we obtain a sequence of maps

$$[Q, X]_x \xrightarrow{f_*} [Q, Y]_y \xrightarrow{g_*} [Q, X]_z.$$

But a homotopy H of Id_X to $g \circ f$ then determines a path $\gamma(t) := H(*, t)$ from x to z, and composing the maps $H_t : X \to X$ with any given pointed map $\varphi : (Q, *) \to (X, x)$ produces a free homotopy of maps $Q \to X$ that match γ at the base point, so by construction,

$$g_* f_* = \gamma_\#,$$

proving that $g_* f_*$ is a bijection, and therefore that f_* is injective and g_* is surjective. Since $f \circ g$ is also homotopic to the identity, applying the same argument to

$$[Q, Y]_y \xrightarrow{g_*} [Q, X]_z \xrightarrow{f_*} [Q, Y]_{f(z)}$$

then proves that g_* is also injective, and therefore invertible, and it follows that f_* is also invertible.
- Next task: prove that the converse of this theorem also holds when X is a CW-complex (Whitehead's theorem).

Suggested reading. Most of our usual sources discuss group and cogroup objects in the specific context of the category hTop_*, even though it takes almost no extra effort to frame them in the more general context that I opted for in the lectures; on the other hand, most of them also discuss more general algebraic objects in hTop_* such as H-spaces (which do not need to be associative or have inverses). The nicest presentation I've seen was in Cutler's lecture notes [Cut21, H-Spaces I], and despite minor differences in the level of generality, it's fairly close to what we did in the lectures. For the higher homotopy groups, one can look at either [tD08, Chapter 6] or [DK01, §6.13 and §6.15], both of which include more details than are logically necessary, partly in an effort to make their theorems about π_n semi-independent of the general theory of fibrations and cofibrations. (Example: A version of the bijection $\gamma_\# : [Q, X]_x \to [Q, X]_y$ determined by a homotopy class of paths $x \overset{\gamma}{\rightsquigarrow} y$ is constructed in [DK01, Lemma 6.56], with a hands-on proof that would be completely unnecessary if one took for granted that the transport functor of a cofibration exists.)

Exercises (for the Übung on 6.06.2024).

Exercise 7.1. Describe all possible cogroup objects in the category of pointed sets.

Exercise 7.2. Prove the following claims stated in Lecture 10:

(a) If X is a group object in a category $\mathscr{A}$, then the map $f^* : \mathrm{Hom}_{\mathscr{A}}(Z, X) \to \mathrm{Hom}_{\mathscr{A}}(Y, X)$ induced by any morphism $f : Y \to Z$ in $\mathscr{A}$ is a group homomorphism.

(b) The morphisms $m : X \times X \to X$, $i : X \to X$ and $e : 1 \to X$ defined in terms of group structures on the sets $\mathrm{Hom}_{\mathscr{A}}(Q, X)$ for each $Q \in \mathscr{A}$ in the sketched proof of Proposition 3 satisfy the axioms of a group object in $\mathscr{A}$.
Note: The assumption that $f^* : \mathrm{Hom}_{\mathscr{A}}(Z, X) \to \mathrm{Hom}_{\mathscr{A}}(Y, X)$ *is a group homomorphism for each morphism* $f : Y \to Z$ *in* $\mathscr{A}$ *will be crucial here.*

(c) If for some reason you don't have enough to do this week, write down the dual version of Proposition 3 characterizing cogroup object structures on $X \in \mathscr{A}$ in terms of group structures on the sets $\mathrm{Hom}_{\mathscr{A}}(X, Q)$ for each $Q \in \mathscr{A}$, and prove it.

Exercise 7.3. We didn't explictly discuss this in the lectures, but it will not surprise you to learn that the relative homotopy groups $\pi_n(X, A)$ define functors on the category $\mathsf{Top}_*^{\mathrm{rel}}$ of pointed pairs of spaces. In particular, any pointed map of pairs $f : (X, A) \to (Y, B)$ induces a map $f_* : \pi_n(X, A) \to \pi_n(Y, B)$, which is a group homomorphism for $n \geqslant 2$. Stare at the definitions long enough until you feel you understand why this is true. Then:

(a) Show that there is a natural bijection between the relative $\pi_n(X, \{*\})$ and absolute $\pi_n(X)$ for each $X \in \mathsf{Top}_*$ and $n \geqslant 1$, which is a group isomorphism for $n \geqslant 2$, and that under this identification, the maps $\pi_n(X) \to \pi_n(X, A)$ appearing in the long exact sequence of relative homotopy groups become the maps induced by the inclusion of pointed pairs $(X, \{*\}) \hookrightarrow (X, A)$.

(b) Convince yourself that the long exact sequence of relative homotopy groups is natural with respect to morphisms in $\mathsf{Top}_*^{\mathrm{rel}}$.

Exercise 7.4. This exercise concerns the bijection $\gamma_\# : [Q, X]_x \to [Q, X]_y$ defined in Lecture 11 for any well-pointed space $Q \in \mathsf{Top}_*$ and any homotopy class of paths $x \overset{\gamma}{\rightsquigarrow} y$ in X.

(a) Show that if Q is a cogroup object in hTop_*, then $\gamma_\# : [Q, X]_x \to [Q, X]_y$ is automatically a group isomorphism.

(b) Show that in the special case $Q = S^1$, $\gamma_\# : \pi_1(X, x) \to \pi_1(X, y)$ can be expressed in terms of concatenation of paths via the explicit formula
$$\gamma_\#[\alpha] := [\gamma^{-1} \cdot \alpha \cdot \gamma].$$

(c) Can you similarly write down an explicit description of $\gamma_\# : \pi_n(X, x) \to \pi_n(X, y)$ for $n \geqslant 2$? Try to describe it with a picture.

Exercise 7.5. For any path-connected covering[33] map $p : \widetilde{X} \to X$ of a pointed space X, give two proofs that the induced homomorphism
$$\pi_n(\widetilde{X}) \xrightarrow{p_*} \pi_n(X)$$
is an isomorphism for each $n \geqslant 2$ and is injective for $n = 1$. Use covering space theory for the first proof, and the long exact sequence of a fibration for the second.

[33]I originally formulated this exercise with the stronger assumption that $p : \widetilde{X} \to X$ is the universal cover, which seems to be the special case that gets applied most often. But there is actually no need to assume $\widetilde{X}$ is simply connected.

Exercise 7.6. For a pointed fibration $p : E \to B$ with fiber $F := p^{-1}(*) \subset E$, there is a pointed map of pairs $p : (E, F) \to (B, \{*\})$ which (in light of Exercise 7.3) induces a map

$$\pi_n(E, F) \xrightarrow{p_*} \pi_n(B)$$

for each $n \geqslant 1$. Show that this map is in fact a bijection.
Hint: Many books give a direct proof of this result using the homotopy lifting property, and then use it to deduce the exact sequence of homotopy groups for a fibration from the exact sequence of relative homotopy groups. We have done things the other way around, and you should be able to deduce this isomorphism from the exact sequences.

Exercise 7.7. The long exact sequence of homotopy groups for a fibration $p : E \to B$ also works under more general hypotheses: it suffices in fact to assume that $p : E \to B$ is a *Serre* fibration, meaning that it satisfies the homotopy lifting property with respect to all CW-complexes, but not necessarily with respect to all spaces. This is useful, because proving that a given map is a Serre fibration is easier in practice than proving the stronger hypothesis of a Hurewicz fibration. It also should not surprise you, because the homotopy groups $\pi_n(X)$ are defined in terms of $[Q, X]$ for a very restrictive class of spaces Q, all of which are CW-complexes. See how much of the proof of this more general exactness result you can piece together.
Remark: You can find many books that give fairly direct proofs of the result for Serre fibrations, but that would unnecessarily duplicate a lot of effort that we have already made in this course.

Exercise 7.8. Below are some computations that can be carried out with the aid of the exact sequence of homotopy groups for a fibration. You should take it for granted that the fibrations mentioned actually are fibrations; this will become mostly obvious when we get around to studying fiber bundles. The first two parts are preparatory computations that can be carried out using methods from previous semesters, e.g. the mapping degree, or the simplicial approximation theorem.

(a) Prove that $\pi_n(S^n) \neq 0$ for every $n \in \mathbb{N}$.
Comment: We will soon be able to prove $\pi_n(S^n) \cong \mathbb{Z}$, but it does not follow easily from anything we've covered so far.
(b) Prove that $\pi_k(S^n) = 0$ for $0 < k < n$.
(c) Identifying S^{2n+1} with the unit sphere in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ for $n \geqslant 1$, the quotient projection $\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$ restricts to the sphere as a fibration

$$S^{2n+1} \xrightarrow{p} \mathbb{CP}^n,$$

known as the **Hopf fibration**. Prove that the induced map $\pi_{2n+1}(S^{2n+1}) \to \pi_{2n+1}(\mathbb{CP}^n)$ is an isomorphism, implying via part (a) that $\pi_{2n+1}(\mathbb{CP}^n)$ is nontrivial. (The most famous case is $n = 1$, where we have $\mathbb{CP}^1 \cong S^2$, thus $\pi_3(S^2)$ is nontrivial.)
Comment: $\mathbb{CP}^n$ is a manifold of dimension $2n$, so this result stands in marked contrast to what happens in homology, where $H_k(M)$ vanishes for every manifold M of dimension less than k.
(d) For each $n \geqslant 1$, the fibration $\mathrm{SO}(3) \to S^2$ mentioned in Lecture 3 generalizes to a fibration

$$\mathrm{SO}(n) \xrightarrow{p} S^{n-1},$$

defined by letting matrices in $\mathrm{SO}(n)$ act linearly on a chosen base point in the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Deduce from this fibration that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for every $n \geqslant 3$, and describe a specific loop in $\mathrm{SO}(n)$ that is not nullhomotopic.
Hint: You may want to start by recalling how $\mathrm{SO}(3)$ is related to $\mathbb{RP}^3$.
(e) Regarding S^{2n-1} again as the unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$, we can let the group $\mathrm{U}(n)$ act linearly on a chosen base point in S^{2n-1} to define a fibration

$$\mathrm{U}(n) \xrightarrow{p} S^{2n-1}$$

for each $n \geqslant 1$. Deduce from this that $\pi_1(\mathrm{U}(n)) \cong \mathbb{Z}$ for every $n \geqslant 1$, and describe a specific loop that generates the group.

8. Week 8

Lecture 12 (3.06.2024): Weak homotopy equivalences.

- For $Q, X \in \mathsf{Top}_*$ with Q well pointed, the transport functor of the cofibration $* \hookrightarrow Q$ defines an action of the group $\pi_1(X)$ on the set of pointed homotopy classes $[Q, X]_+$:
$$\pi_1(X) \times [Q, X]_+ \to [Q, X]_+ : ([\gamma], [f]) \mapsto [\gamma] \cdot [f] := \gamma_\#[f].$$
Exercise: For $Q := S^1$, this defines the action of $\pi_1(X)$ on itself by conjugation.
Recall that $\gamma_\#[f] = [g]$ if and only if there is a free homotopy $f \overset{H}{\rightsquigarrow} g$ with $H(*, t) = \gamma(t)$ for all t, and the cofibration condition ensures that such a homotopy exists for any given γ and f. It follows that two pointed maps $Q \to X$ are freely homotopic if and only if their pointed homotopy classes are related by the action of $\pi_1(X)$, which proves:
- Theorem: The map $[Q, X]_+ \to [Q, X]_\circ$ defined by forgetting the base point descends to an injective map
$$[Q, X]_+ \big/ \pi_1(X) \to [Q, X]_\circ,$$
which is surjective if X is path-connected.
- Corollary: For Q well-pointed and X simply connected, two maps $Q \to X$ are pointed homotopic if and only if they are freely homotopic, i.e. $[Q, X]_+ = [Q, X]_\circ$.
- Definition: A map $f : X \to Y$ is a **weak homotopy equivalence** if for every $x \in X$ and $y := f(x)$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, y)$ is bijective for all $n \geqslant 0$.
- Proved last time: Homotopy equivalences are also weak homotopy equivalences. In fact, f being a homotopy equivalence implies that $f_* : [Q, X]_x \to [Q, Y]_y$ is bijective for every well-pointed space Q, so in particular when Q is a sphere of any dimension.
- It is obvious (by thinking about morphisms and isomorphisms in hTop) that for any homotopy equivalence $f : X \to Y$ and any space Q, f induces a bijection on *free* homotopy classes
$$[Q, X]_\circ \xrightarrow{f_*} [Q, Y]_\circ.$$
Exercise (sanity check): The maps $[Q, X]_x \xrightarrow{f_*} [Q, Y]_y$ are also equivariant with respect to the π_1-action, i.e. for any $[\gamma] \in \pi_1(X, x)$ and $[\varphi] \in [Q, X]_x$, one has
$$f_* ([\gamma] \cdot [\varphi]) = f_*[\gamma] \cdot f_*[\varphi].$$
(I call this a "sanity check" because if you assume f is a homotopy equivalence between path-connected spaces and combine the bijectivity of $f_* : [Q, X]_x \to [Q, Y]_y$ with the theorem stated above, equivariance implies that f_* descends to the quotient by the π_1-action as a bijection between sets of free homotopy classes. That's a much more complicated proof of a statement that was already obvious, but it reassures us that all the things we're proving are consistent with each other.)
- "Cofibrant" theorem: A map $f : X \to Y$ is a weak homotopy equivalence if and only if the induced map $f_* : [Q, X] \to [Q, Y]$ of free homotopy classes is bijective for every CW-complex Q.
Remark: Below we will prove only the "$\Rightarrow$" direction of this theorem, which is what we need immediately for applications. For the converse, see e.g. [Whi78, Theorem IV.7.17].

- Corollary (Whitehead's theorem):[34] If X and Y are spaces that are homotopy equivalent to CW-complexes, then every weak homotopy equivalence $X \to Y$ is also a homotopy equivalence.
  Remark: Since homotopy equivalences are also weak homotopy equivalences, the theorem in this form follows immediately if one can prove it under the assumption that X and Y are actual CW-complexes. But allowing spaces that only have the homotopy types of CW-complexes widens the range of applicability, and the theorem is often applied in this form: for example, all topological manifolds have this property, but it is unknown (specifically for non-smoothable manifolds of dimension four) whether all manifolds actually admit cell decompositions. For "infinite-dimensional" spaces such as loop spaces, one cannot reasonably expect a cell decomposition, but Milnor [Mil59] showed for instance that ΩX is homotopy equivalent to a CW-complex with countably-many cells whenever X is. Similarly, combining a result of Palais [Pal66, Theorem 14] with some general properties of CW-complexes (e.g. [Hat02, Prop. A.11]), one finds that most naturally occurring infinite-dimensional manifolds—such as spaces of maps from one smooth manifold to another, or of sections of a smooth fiber bundle—have the homotopy types of CW-complexes.
- Deducing Whitehead from the cofibrant theorem: Assuming X and Y are CW-complexes and $f : X \to Y$ is a weak homotopy equivalence, the cofibrant theorem tells us that

$$[X, X] \xrightarrow{f_*} [X, Y] \qquad \text{and} \qquad [Y, X] \xrightarrow{f_*} [Y, Y]$$

  are bijections. One then finds a homotopy inverse of f in the unique homotopy class $[g] \in [Y, X]$ such that $f_*[g] = [\mathrm{Id}_Y]$ (cf. Exercises 1.8 and 4.3).
- Lemma 1 (toward the proof of the cofibrant theorem): If $A \subset X$ and the inclusion $A \hookrightarrow X$ is a weak homotopy equivalence, then $\pi_n(X, A, x) = 0$ for every $n \geqslant 1$ and $x \in A$.
  Proof: Immediate from the long exact sequence of relative homotopy groups. (However, since you've probably only seen this argument before in the context of abelian groups or R-modules, you should take a moment to convince yourself that it still works on the portion of this exact sequence that may contain nonabelian groups and/or pointed sets.)
  Remark: The converse is also almost true, one just needs to be a bit careful about the last two terms in the exact sequence, e.g. it is clearly true if one adds the assumption that X and A are path-connected.
- Lemma 2: For any space X or pair of spaces (X, A) and each $n \geqslant 1$,
  (1) $\pi_n(X, x) = 0$ for every $x \in X$ if and only if every map $S^n = \partial\mathbb{D}^{n+1} \to X$ admits an extension to $\mathbb{D}^{n+1}$.
  (Proof left as an exercise.)
  (2) $\pi_n(X, A, x) = 0$ for every $x \in A$ if and only if every map of pairs $(\mathbb{D}^n, S^{n-1}) \to (X, A)$ is homotopic rel S^{n-1} to a map whose image is contained in A.[35]
  Proof: Given $f : (\mathbb{D}^n, S^{n-1}) \to (X, A)$, choose $x := f(*) \in A$ to make f a pointed map of pairs, the condition $\pi_n(X, A, x) = 0$ then implies that f is homotopic as a pointed map of pairs to the constant map, thus giving a homotopy $H : \mathbb{D}^n \times I \to X$ such that

$$H|_{\mathbb{D}^n \times \{0\}} = f, \qquad H(S^{n-1} \times I) \subset A, \qquad \text{and} \qquad H\big|_{\{*\} \times I \cup \mathbb{D}^n \times \{1\}} \equiv x.$$

[34] The theorem is due to J.H.C. Whitehead, the inventor of CW-complexes, not George Whitehead, the author of [Whi78].

[35] Useful jargon: A homotopy $H : X \times I \to Y$ is a **homotopy rel** A (short for "relative to A") for some subset $A \subset X$ if the restricted maps $H_t|_A : A \to Y$ are the same for all $t \in I$.

Now foliate $\mathbb{D}^n \times I$ by a 1-parameter family of n-disks with identical boundary at $S^{n-1} \times \{0\}$, producing a homotopy from f to

$$g := H\big|_{S^{n-1}\times I \cup \mathbb{D}^n \times \{1\}},$$

which we can regard as a map $\mathbb{D}^n \to X$ since $S^{n-1} \times I \cup \mathbb{D}^n \times \{1\} \cong \mathbb{D}^n$. By construction, g has image in A, and each map in the new homotopy matches f at the boundary.
(Proof of the converse left as an exercise.)

- Compression lemma: Suppose (X, A) is a CW-pair, and (Y, B) is a pair of spaces such that $\pi_n(Y, B) = 0$ for every $n \geqslant 1$ and the map $\pi_0(B) \to \pi_0(Y)$ induced by the inclusion $B \hookrightarrow Y$ is surjective. Then any map of pairs $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $g : X \to Y$ with image contained in B.
Intuition: We would see this immediately (and with no assumptions about (X, A)) if we assumed that B is a deformation retract of Y, because we could then compose f with the deformation retraction. Requiring (X, A) to be a CW-complex instead of an arbitrary pair is the price we pay for weakening the assumption that there is a deformation retraction.
Proof: This follows a standard technique of constructing maps on a CW-complex X by starting with its 0-skeleton X^0 and then extending inductively to the other skeleta $X^0 \subset X^1 \subset X^2 \subset \ldots \subset X$. We claim there exists a sequence of maps $f =: f_{-1}, f_0, f_1, f_2, \ldots : X \to Y$ such that for every n,
  (1) $f_n(X^n) \subset B$;
  (2) f_n is homotopic rel $X^n \cup A$ to f_{n+1}.
The latter implies that there is a pointwise limit $g := \lim_{n\to\infty} g_n : X \to Y$ which matches f_n on the n-skeleton for each n, and we can construct a homotopy H from f to g rel A by piecing together the infinite sequence of homotopies $f_{-1} \rightsquigarrow f_0 \rightsquigarrow f_1 \rightsquigarrow \ldots$ such that $H(\cdot, 1 - 1/2^n) = f_{n-1}$ for each $n \geqslant 1$. (The resulting map is continuous because the topology of a CW-complex is defined the way that it is, i.e. the restriction to each finite-dimensional skeleton is continuous.) To prove the claim by induction, assuming f_{n-1} has already been constructed, first construct f_n and the homotopy $f_{n-1} \rightsquigarrow f_n$ on each n-cell $e^n \subset X\backslash A$ so that $f_n(e^n) \subset A$: this is possible for $n = 0$ because $\pi_0(A) \to \pi_0(X)$ is surjective, and for $n > 0$ it is possible due to Lemma 2 and the assumption $\pi_n(X, A) = 0$. Since $X^n \cup A \hookrightarrow X$ is a cofibration (Exercise 5.4), both f_n and the homotopy $f_{n-1} \rightsquigarrow f_n$ can then be extended to the rest of X.
- Proof of the cofibrant theorem ($\Rightarrow$): Assume $f : X \to Y$ is a weak homotopy equivalence. After replacing Y with the (unreduced) mapping cylinder $Z(f)$, which is (both strongly and weakly) homotopy equivalent to Y, and replacing $f : X \to Y$ with the inclusion $i_X : X \hookrightarrow Z(f)$, we can assume without loss of generality that f is the inclusion of a subspace $X \subset Y$. For any given CW-complex Q, applying the compression lemma to the map of pairs $(Q, \varnothing) \to (Y, X)$ then shows that the map $f_* : [Q, X] \to [Q, Y]$ is surjective. Injectivity amounts to the statement that if $\varphi, \psi : Q \to X$ are two maps that are homotopic as maps into Y, then they are also homotopic as maps into X, i.e. a homotopy $H : Q \times I \to Y$ can be modified without changing $H|_{Q\times\partial I}$ to produce a homotopy $Q \times I \to X$. Just apply the compression lemma to $H : (Q \times I, Q \times \partial I) \to (Y, X)$.
- Easy application of Whitehead's theorem: For any space X with the homotopy type of a CW-complex, X is contractible if and only if $\pi_n(X, x)$ vanishes for all $n \geqslant 0$ and $x \in X$.
Proof: Apply Whitehead's theorem to the unique map $X \to \{*\}$.
- Cautionary examples:

(1) Assuming X and Y are CW-complexes, just having isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for every $n \geqslant 0$ does not suffice to conclude that X and Y are homotopy equivalent. Counterexample: take $X := S^2 \times \mathbb{RP}^3$ and $Y := \mathbb{RP}^2 \times S^3$. Both are path-connected and have fundamental group $\mathbb{Z}_2$, and they also have the same universal cover $S^2 \times S^3$, implying via Exercise 7.5 that their higher homotopy groups are the same. But by a straightforward Topology 2 exercise, their homologies are different, so they are not homotopy equivalent. What we are missing in this example is an actual map $X \to Y$ to serve as the weak homotopy equivalence in Whitehead's theorem—the isomorphisms $\pi_n(X) \to \pi_n(Y)$ would need to come from such a map, otherwise the theorem does not apply. (In the application above for contractible spaces, we got lucky because the unique map $X \to \{*\}$ served as a weak homotopy equivalence, even though no such map was mentioned in the statement itself.)

(2) The **Warsaw circle** is a popular example of a non-contractible space X for which $\pi_n(X)$ vanishes for all $n \geqslant 0$; clearly, it must not have the homotopy type of a CW-complex. To construct it, start with the graph in $\mathbb{R}^2$ of the function $y = \sin(1/x)$ for $0 < x \leqslant 1$, the so-called ***topologist's sine curve***, and take its closure, which includes the compact interval $\{0\} \times [-1, 1]$. Now draw an embedded curve in $\mathbb{R}^2$, starting at the point on the graph with $x = 1$, then circling around to the other side of the y-axis without touching the closure of the graph, and rejoining it at the origin. No continuous path can move through the entirety of the portion of X that contains infinitely-many oscillations, and one can deduce from this that $\pi_n(X) = 0$ for all n. The fastest way I can think of to see that X is not contractible is through the observation that X is the intersection of a nested sequence $X_1 \supset X_2 \supset X_3 \supset \ldots \supset X$ of open neighorhoods that are all homotopy equivalent to S^1, and X can therefore be identified with the inverse limit of this sequence in Top. There are invariants in algebraic topology that can detect this fact, because they behave well with respect to inverse limits: singular cohomology is not one of them, but Čech cohomology is, so X cannot be contractible because $\check{H}^1(X;\mathbb{Z}) \cong \check{H}^1(S^1;\mathbb{Z}) \cong H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. A more elementary argument for the non-contractibility of X is outlined in Exercise 8.1 below.

Lecture 13 (6.06.2024): CW-approximation.

- Definition: Assume $n \geqslant 0$ is an integer, and all spaces and maps below are in the unpointed category.
  (1) A map $f : X \to Y$ is an n-**equivalence** if for every choice of base point $x \in X$, writing $y := f(x) \in Y$, the induced map $f_* : \pi_k(X, x) \to \pi_k(Y, y)$ is bijective for $0 \leqslant k < n$ and surjective for $k = n$. (In particular, f is a weak homotopy equivalence if and only if it is an n-equivalence for arbitrarily large n.)
  (2) A pair of spaces (X, A) is called n-**connected** if the inclusion map $A \hookrightarrow X$ is an n-equivalence.
  (3) A space X is called n-**connected** if $\pi_k(X) = 0$ for all $k = 0, \ldots, n$.[36]
- Remark 1: In the third definition, there is no need to mention base points, because "$\pi_0(X) = 0$" means that X is path-connected, and its homotopy groups are therefore independent of the base point up to isomorphism. Similarly, if X is path-connected, then there is no need to mention base points in defining what it means for $f : X \to Y$ to be an n-equivalence, as the surjection $\pi_0(X) \to \pi_0(Y)$ forces Y to be path-connected as well. Note

[36]As usual, for a group G, the statement "$G = 0$" should be understood as an abbreviation that means "G is a trivial group". If G is not a group but is a pointed set such as $\pi_0(X)$ or $\pi_1(X, A)$, then "$G = 0$" means that G has only one element.

however that f can be an n-equivalence without either of X or Y being path-connected, and (X, A) can be n-connected while X and A each have multiple path-components; in such cases, the detail that *all* choices of base points must be allowed is important (see the next remark).

- Remark 2: The long exact sequence of relative homotopy groups yields the following equivalent formulation of the second definition: *The pair (X, A) is n-connected if and only if for all choices of base point $x \in A$, $\pi_k(X, A, x) = 0$ for $k = 1, \dots, n$ and the map $\pi_0(A, x) \to \pi_0(X, x)$ induced by the inclusion is surjective.* There's a slight subtlety here due to the fact that the last three terms in the long exact sequence are not groups: for any fixed choice of base point $x \in A$, the vanishing of $\pi_1(X, A, x)$ implies that the preimage of the base point under the map $\pi_0(A, x) \to \pi_0(X, x)$ contains only the base point, but this does not imply in general that the map $\pi_0(A, x) \to \pi_0(X, x)$ is injective. (Exercise: Find a counterexample!) Inspecting the definitions of these sets, one finds however that $\pi_0(A, x) \to \pi_0(X, x)$ must indeed be injective if $\pi_1(X, A, x)$ vanishes for *all* choices of $x \in A$. There is of course nothing in the long exact sequence to guarantee the *surjectivity* of $\pi_0(A) \to \pi_0(X)$, so the latter needs to be mentioned explicitly whenever characterizing n-connected pairs in terms of the relative homotopy groups.
- Remark 3: X is n-connected if and only if the inclusion $\{x\} \hookrightarrow X$ of each point $x \in X$ is an n-equivalence. Or equivalently: X is n-connected if and only if it is path-connected and the inclusion $\{x\} \hookrightarrow X$ of *some* point $x \in X$ is an n-equivalence.
- Compression lemma (sharp version): Assume (X, A) is a CW-complex such that $\dim(X/A) \leqslant n$ for some integer $n \geqslant 0$, meaning that $X \backslash A$ contains no k-cells for $k > n$, and assume (Y, B) is an n-connected pair of spaces. Then every map of pairs $(X, A) \to (Y, B)$ is homotopic rel A to a map with image in B.
  Proof: The case $n = \infty$ was proved in the previous lecture, but the argument given there also proves this statement, one only needs to pay attention to where precisely the condition $\pi_k(Y, B) = 0$ is needed and for which values of k.
- Theorem 1: $\pi_k(S^n) = 0$ whenever $k < n$.
  Since $S^n \backslash \{*\} \cong \mathbb{R}^n$ is contractible, this follows easily from...
- Lemma: For $k < n$, every map $S^k \to S^n$ is homotopic (without loss of generality via a homotopy that is fixed on a neighborhood of some chosen base point) to one that is not surjective.
  Proof 1: By the simplicial approximation theorem (see e.g. [Wen23, Theorem 41.14]), $f : S^k \to S^n$ is homotopic to a simplicial map for suitable triangulations of S^k and S^n, and the image of this simplicial map is then contained in the k-skeleton of S^n, which cannot be everything if $n > k$.
  Proof 2: By standard perturbation results in differential topology (see [Hir94]), $f : S^k \to S^n$ is homotopic to a smooth map $g : S^k \to S^n$, and Sard's theorem then implies that almost every point $p \in S^n$ is a regular value of g, which means $g^{-1}(p) = \varnothing$ when $n > k$.
  In either version of the proof, it is an easy exercise to modify the argument so that the homotopy leaves f unchanged on some neighborhood of a chosen point.
- Theorem 2: If (X, A) is a CW-pair and $n \geqslant 0$ an integer such that all k-cells of X with $k \leqslant n$ are contained in A, then (X, A) is n-connected.
- Very useful corollary: For any CW-complex X and each $n \geqslant 0$, the inclusion of the $(n+1)$-skeleton into X induces an isomorphism $\pi_n(X^{n+1}) \cong \pi_n(X)$.
  Remark: This is a property that homology and cohomology also have, and it makes many computations more manageable. A word of caution, however: for a CW-complex X with finite dimension $\dim X \leqslant n$, it does *not* follow that $\pi_{n+1}(X) = 0$, quite unlike the situation

for homology and cohomology. (For instance, $\pi_3(S^2) \neq 0$, as shown in Exercise 7.8.) Computing higher homotopy groups would be much easier if this were true.

- Theorem 2 can be proved in three steps, with increasing generality:
  (1) Suppose $\ell > n$, $X^{\ell-1} \subset A$ and $X = A \cup e^\ell$ has only one cell outside of A, of dimension ℓ. For $k \leqslant n$, any map $f : (\mathbb{D}^k, S^{k-1}) \to (X, A)$ then descends to a map $S^k \cong \mathbb{D}^k/\partial\mathbb{D}^k \to X/A \cong S^\ell$, and since $\ell > k$, it follows that f is homotopic rel some neighborhood of $\partial\mathbb{D}^k$ to a map whose image misses at least one point $y \in e^\ell$. A deformation retraction of $\mathbb{D}^\ell\backslash\{y\}$ to $\partial\mathbb{D}^\ell$ then produces a homotopy of f rel S^{k-1} to a map with image contained in A, showing via Lemma 2 in the previous lecture that $\pi_k(X, A) = 0$. The surjectivity of $\pi_0(A) \to \pi_0(X)$ just means that every path-component of X contains points in A, and this is obvious.
  (2) Suppose X contains only finitely-many cells that are not in A. Then there is a finite sequence of cellular inclusions $A =: X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_N := X$ such that each X_{j+1} is obtained from X_j by attaching one cell of dimension greater than n. For any choice of base point, the induced maps $\pi_k(A) \to \pi_k(X)$ now factor into compositions of maps $\pi_k(X_j) \to \pi_k(X_{j+1})$ which are bijective for $k < n$ and surjective for $k = n$ due to Step 1.
  (3) In the general case, the surjectivity of $\pi_0(A) \to \pi_0(X)$ is still obvious because every path-component of X must contain a 0-cell, which is assumed to lie in A, thus it suffices to prove $\pi_k(X, A) = 0$ for all $k = 1, \ldots, n$ and all choices of base point; equivalently, one needs to prove that every map $f : (\mathbb{D}^k, S^{k-1}) \to (X, A)$ is homotopic rel S^{k-1} to a map into A. Let $X' \subset X$ denote the subcomplex consisting of all cells in A plus the (since $\mathbb{D}^k$ is compact) finitely many in $X\backslash A$ that intersect $f(\mathbb{D}^k)$. Step 2 now applies to (X', A) and proves $\pi_k(X', A) = 0$ for all choices of base point, which guarantees the existence of the desired homotopy.
- Cellular approximation theorem: For CW-pairs (X, A) and (Y, B), every continuous map of pairs $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $g : X \to Y$ such that for every $n \geqslant 0$, $g(X^n \cup A) \subset Y^n \cup B$.[37]
- Corollary 1: Every map of pairs between two CW-complexes is homotopic (as a map of pairs) to one that is cellular.
  Proof: Apply the theorem first to $f|_A : A \to B$, regarded as a map of pairs $(A, \varnothing) \to (B, \varnothing)$, thus making this restriction a cellular map. Since $A \hookrightarrow X$ is a cofibration, the resulting homotopy can be extended over X to define a homotopy from f to some map whose restriction to A is a cellular map $A \to B$. Applying the theorem again then modifies this outside of A to a cellular map $X \to Y$.
- Corollary 2: If two cellular maps $f : X \to Y$ are homotopic, then they are also *cellularly* homotopic, i.e. there exists a homotopy $H : X \times I \to Y$ from f to g that is also a cellular map with respect to the natural product cell decomposition of $X \times I$.
  Proof: Given any continuous homotopy $H : X \times I \to Y$ from f to g, apply the theorem to the map of pairs $H : (X \times I, X \times \partial I) \to (Y, Y)$, using the fact that for the obvious cell decomposition of I with a single 1-cell attached to two 0-cells at the end points, $X \times \partial I$ is a subcomplex of $X \times I$.
- Proof of the theorem: We construct a sequence of maps $f =: f_{-1}, f_0, f_1, f_2, \ldots : X \to Y$ such that for every n,
  (1) $f_n(e^n) \subset Y^n$ for every n-cell $e^n \subset X\backslash A$;
  (2) $f_n \underset{h}{\sim} f_{n+1}$ rel $A \cup X^n$.

[37] A slightly incorrect version of this statement appeared in this spot for a long time before it got corrected shortly after the end of the semester. Many thanks to the students who pointed out the error.

The condition $f_n(e^n) \subset Y^n$ can be achieved because by Theorem 2, $(X, A \cup X^n)$ is n-connected. As in our proof of the compression lemma in the previous lecture, there is then a pointwise limit $g := \lim_{n\to\infty} f_n$ that matches f_n on the n-skeleton for each n, and concatenating the infinite sequence of homotopies on successively smaller segments of the interval gives a homotopy from f to g rel A.

- Definition: A **CW-approximation** (X', φ) of a space X consists of a CW-complex X' and a weak homotopy equivalence $\varphi : X' \to X$.
- CW-approximation theorem:
  1. Every space X admits a CW-approximation (X', φ).
  2. If X is n-connected for some $n \geqslant 0$, then its CW-approximation X' can be chosen to have only one 0-cell and no k-cells for $k = 1, \dots, n$.
  3. Given spaces X, Y with CW-approximations (X', φ) and (Y', ψ) respectively, and a map $f : X \to Y$, there exists a unique homotopy class of maps $f' : X' \to Y'$ such that $f \circ \varphi \underset{h}{\sim} \psi \circ f'$.

     Corollary: The CW-approximation X' of any given space X is unique up to homotopy equivalence.
- Proof of (3): By the cofibrant theorem, the map $\psi_* : [X', Y'] \to [X', Y]$ is a bijection, and $[f'] := \psi_*^{-1}[f \circ \varphi]$ is thus uniquely determined.
- Setup for the proof of (1): The construction can be carried out separately for each path-component of X, so without loss of generality, assume X is path-connected, and fix an arbitrary choice of base point; path-connectedness will imply that nothing important depends on this choice. The space X' and map $\varphi : X' \to X$ are then constructed as the colimit of a sequence

$$\begin{tikzcd}
X_{(1)} \arrow[r, hook] \arrow[drrr, "\varphi^1"'] & X_{(2)} \arrow[r, hook] \arrow[drr, "\varphi^2"] & \dots \arrow[r, hook] & X_{(n)} \arrow[r, hook] \arrow[d, "\varphi^n"] & \dots \arrow[r, hook] & \operatorname{colim}_{n\to\infty} X_{(n)} =: X' \arrow[dll, "\varphi"] \\
& & & X & &
\end{tikzcd}$$

with the following propertes:
  1. Each $X_{(n)}$ is a CW-complex;
  2. Each $X_{(n+1)}$ is obtained from $X_{(n)}$ by attaching $(n+1)$-cells to the n-skeleton of $X_{(n)}$;
  3. The maps φ^n induce surjections $\varphi^n_* : \pi_k(X_{(n)}) \to \pi_k(X)$ for every k, which are also injective whenever $k < n$.

The first two conditions imply that the colimit X' can be realized as a CW-complex such that each $X_{(n)}$ is a subcomplex of X' containing its n-skeleton. It follows via Theorem 2 that the inclusion $X_{(n+1)} \hookrightarrow X'$ is an n-equivalence, so in light of the third condition, the diagram

$$\begin{tikzcd}
\pi_n(X_{(n+1)}) \arrow[rr, "\cong"] \arrow[dr, "\cong", "\varphi^{n+1}_*"'] & & \pi_n(X') \arrow[dl, "\varphi_*"] \\
& \pi_n(X) &
\end{tikzcd}$$

proves that φ is a weak homotopy equivalence.
- Initial step in the construction: For each $n \geqslant 1$, choose a set of pointed maps $\{f_\alpha : S^{k_\alpha} \to X\}_{\alpha \in J_n}$ with $k_\alpha := n$ such that the set of pointed homotopy classes $\{[f_\alpha] \in \pi_n(X) \mid \alpha \in J_n\}$ generates $\pi_n(X)$. Assuming the index sets J_m, J_n to be disjoint for $m \neq n$, write

$J := \bigcup_{n \in \mathbb{N}} J_n$ and define

$$X_{(1)} := \bigvee_{\alpha \in J} S^{k_\alpha} \xrightarrow{\varphi^1} X, \qquad \text{where} \qquad \varphi^1 := \bigvee_{\alpha \in J} f_\alpha.$$

We should understand $X_{(1)}$ as a CW-complex with the base point as its 0-skeleton, and for each $n \geqslant 1$ and $\alpha \in J_n$, an n-cell attached to the 0-skeleton. Since the classes $[f_\alpha]$ for all $\alpha \in J_n$ generate $\pi_n(X)$, the map $\varphi^1_* : \pi_n(X_{(1)}) \to \pi_n(X)$ is surjective for every n, and manifestly also bijective for $n = 0$.

- Inductive step:[38] Assume that $X_{(k)}$ and φ^k with the desired properties have already been constructed for all $k \leqslant n$. We now construct $X_{(n+1)}$ and φ^{n+1} by adding $(n+1)$-cells to $X_{(n)}$ and extending φ^n so as to destroy the kernel of the map to $\pi_n(X)$. This will leave the n-skeleton unchanged, so the inclusion $X_{(n)} \hookrightarrow X_{(n+1)}$ will induce an isomorphism $\pi_k(X_{(n)}) \cong \pi_k(X_{(n+1)})$ for each $k < n$, ensuring that the map to $\pi_k(X)$ remains an isomorphism. To define $X_{(n+1)}$, choose a collection of pointed maps $\{g_\alpha : S^n \to X_{(n)}\}_{\alpha \in K}$ such that the set $\{[g_\alpha] \in \pi_n(X_{(n)}) \mid \alpha \in K\}$ generates the kernel of $\varphi^n_* : \pi_n(X_{(n)}) \to \pi_n(X)$. By the cellular approximation theorem, we can assume after a homotopy of each g_α that it is a cellular map. (Note: We should use a cell decomposition of S^n with one 0-cell and one n-cell such that the 0-cell is its base point, which is therefore also a subcomplex; applying the cellular approximation theorem relative to that subcomplex then forces the resulting homotopy to be pointed.) Now define

$$X_{(n+1)} := X_{(n)} \cup \left(\bigcup_{\alpha \in K} e^{n+1}_\alpha \right),$$

meaning more precisely that for each $\alpha \in K$, we attach an $(n+1)$-cell e^{n+1}_α via the cellular attaching map $g_\alpha : S^n \to X^n_{(n)} \subset X_{(n)}$. This makes $X_{(n+1)}$ a CW-complex with $X_{(n)} \subset X_{(n+1)}$ as a subcomplex that contains its entire n-skeleton. Since each $\varphi^n_*[g_\alpha] \in \pi_n(X)$ vanishes by assumption, $\varphi^n : X_{(n)} \to X$ can now be extended over each of the new cells e^{n+1}_α by choosing an extension of $\varphi^n \circ g_\alpha : S^n \to X$ to $\mathbb{D}^{n+1}$, giving rise to a map $\varphi^{n+1} : X_{(n+1)} \to X$ that extends φ^n. It should be clear that $\varphi^{n+1}_* : \pi_k(X_{(n+1)}) \to \pi_k(X)$ is still surjective for every k, but the presence of the extra $(n+1)$-cells makes the maps $g_\alpha : S^n \to X_{(n)}$ nullhomotopic after composing them with the inclusion $X_{(n)} \hookrightarrow X_{(n+1)}$, and it follows that the kernel of $\varphi^n_* : \pi_n(X_{(n)}) \to \pi_n(X)$ is also annihilated by the map $\pi_n(X_{(n)}) \to \pi_n(X_{(n+1)})$. But the latter is also surjective since, by Theorem 2, $(X_{(n+1)}, X_{(n)})$ is n-connected, so this implies that $\varphi^{n+1} : \pi_n(X_{(n+1)}) \to \pi_n(X)$ is injective.
- Statement (2) in the theorem should now be obvious: if $\pi_k(X) = 0$ for $k = 1, \ldots, n$, then there is no need to include any cells of those dimensions in the construction described above. Note that if you combine this detail with Whitehead's theorem, you get some fairly non-obvious consequences by taking CW-approximations of spaces that are already CW-complexes, e.g. every n-connected CW-complex is homotopy equivalent to one whose n-skeleton is a single point.

[38] I am taking the liberty of writing up a slightly simpler version of this part of the proof than what I explained in the lecture. Instead of forcing pairs of maps $S^n \to X_{(n)}$ to become homotopic to each other by attaching reduced cylinders $Z_+ S^n$, this version focuses specifically on elements of the kernel of $\varphi^n_* : \pi_n(X_{(n)}) \to \pi_n(X)$, and adds $(n+1)$-cells in order to make individual maps $S^n \to X_{(n)}$ generating this kernel nullhomotopic. The argument in lecture was based mainly on [May99, §10.5], whereas this one is a hybrid of that with [Hat02, Prop. 4.13].

Suggested reading. Our treatment this week of the $\pi_1(X)$-action on $[Q,X]_+$ and the relationship between $[Q,X]_+$ and $[Q,X]_\circ$ was based in part on [DK01, §6.16], though the presentation by Davis and Kirk seems a bit more effortful because they avoid using knowledge of the transport functor. I do like the treatment of the cofibrant theorem and Whitehead's theorem that appears in [DK01, §7.9]; Davis and Kirk place this material right in the middle of their chapter on obstruction theory, but it does not actually require knowledge of the rest of that chapter. Up to some minor details that were altered in this writeup but not in the lecture itself, our treatment of cellular and CW-approximation followed [May99, §10.4 and §10.5], though most presentations of these topics seem to be quite similar, so you can read something nearly equivalent in e.g. [Hat02, §4.1]. One caveat about May's presentation: in place of the compression lemma, May prefers to use a more general technical result that he calls "HELP" (the Homotopy Extension and Lifting Property)—he surely has his reasons for this, but I personally prefer the compression lemma, and have not yet found understanding the HELP to be worth the effort that it requires.[39]

Exercises (for the Übung on 13.06.2024).

Exercise 8.1. Let $X \subset \mathbb{R}^2$ denote the Warsaw circle.

(a) Show that for any compact, path-connected and locally path-connected space Q, every map $f : Q \to X$ has its image contained in a subset of X homotopy equivalent to a compact interval; in particular, the image of f will not enter some region of X in which the oscillations become arbitrarily wild. (This proves that $\pi_n(X) = 0$ for all $n \geqslant 0$.)

(b) While part (a) shows that there are no interesting maps $S^n \to X$ for any $n \geqslant 0$, there is a fairly obvious map $f : X \to S^1$ that heuristically resembles a map of degree 1; more concretely, f sends the compact interval $\{0\} \times [-1,1] \subset X$ to a base point $* \in S^1$ and maps the rest of X bijectively to $S^1 \backslash \{*\}$. Show that f is not homotopic to a constant map, thus proving that X is not contractible.
Hint: If f were nullhomotopic, then it would admit a lift to the universal cover $\mathbb{R} \to S^1$. Note that you cannot conclude this from the usual lifting theorem in covering space theory, because X is not locally path-connected—however, you can still use the fact that the covering map $\mathbb{R} \to S^1$ is a fibration. Deduce a contradiction from the existence of this lift.

Remark: We will see later that there is a natural bijection between $[X,S^1]_\circ$ and the singular cohomology $H^1(X;\mathbb{Z})$ whenever X is a CW-complex. But the Warsaw circle clearly has $H^1(X;\mathbb{Z}) = 0$ (why?), thus demonstrating the failure of this correspondence in general for spaces that are not CW-complexes.

Exercise 8.2. We say that $p : E \to B$ has the homotopy lifting property (HLP) with respect to a pair of spaces (X, A) if the problem

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "\widetilde{H}_0 \cup \widetilde{h}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

is always solvable, i.e. given any homotopy $H : X \times I \to B$, along with a lift $\widetilde{H}_0 : X \to E$ of $H_0 = H(\cdot, 0)$ and a lifted homotopy $\widetilde{h} : A \times I \to E$ defined on the subset $A \subset X$, H admits a lift

[39] It is unclear to me whether this is really what May meant to say, but the exposition in [May99, §10.3] seems to suggest that one could deduce the basic result about inclusions of CW-subcomplexes being cofibrations (cf. Exercise 5.4(e)) as a special case of the HELP. That would be nice, but I don't think it's true; the reality is that the proof of the HELP (which May sketches in only two lines, using the words "induction over skeleta" to make it sound easy) inevitably requires choosing extensions of homotopies defined on subcomplexes, which is something of a headache if you haven't already proved beforehand that they are cofibrations.

$\widetilde{H} : X \times I \to E$ that matches $\widetilde{H}_0$ on $X \times \{0\}$ and $\widetilde{h}$ on $A \times I$. Show that the following conditions on $p : E \to B$ are all equivalent to the assumption that $p : E \to B$ is a Serre fibration:

(i) p has the HLP with respect to the disk $\mathbb{D}^n$ for every $n \geqslant 0$;
(ii) p has the HLP with respect to the one point space $\{*\}$ and the pair $(\mathbb{D}^n, S^{n-1})$ for every $n \geqslant 1$;
(iii) p has the HLP with respect to all CW-pairs (X, A).

Hint: You can get from (i) to (ii) via a clever choice of homeomorphism of pairs. Then use induction over the skeleta to get to (iii).

Exercise 8.3. The following application of Serre fibrations is borrowed from a famous paper of Gromov in symplectic geometry [Gro85]. All the spaces mentioned below have obvious topologies as subsets of finite-dimensional vector spaces, and with a little bit of differential geometry, one can show that they are all smooth submanifolds. It follows in particular that they are triangulable, and are thus CW-complexes; that is the only detail from the past two sentences that you'll actually need to know.

Fix a real vector space V of even dimension $2n$ for some $n \in \mathbb{N}$, and define the space of **complex structures** on V by

$$\mathcal{J}(V) := \left\{ J : V \to V \text{ linear} \;\middle|\; J^2 = -\mathbb{1} \right\}.$$

We wish to compare this with the space of **linear symplectic structures** on V, defined as

$$\Omega(V) := \left\{ \omega : V \oplus V \to \mathbb{R} \text{ bilinear} \;\middle|\; \omega(v,w) = -\omega(v,w) \text{ for all } v,w \text{ and } \omega(v,\cdot) \neq 0 \text{ for all } v \neq 0 \right\}.$$

We say that $J \in \mathcal{J}(V)$ is **tamed** by $\omega \in \Omega(V)$ if the condition

$$\omega(v, Jv) > 0 \text{ for all } v \neq 0$$

is satisfied. We say that J is **compatible** with ω if it is tame and additionally the positive bilinear form

$$\langle v, w \rangle := \omega(v, Jw)$$

is symmetric, thus defining a real inner product on V.

The following example shows that every $J \in \mathcal{J}(V)$ is compatible with some $\omega \in \Omega(V)$: regarding $\mathbb{C}^n$ as a real $2n$-dimensional vector space, multiplication by i defines a real-linear transformation that we can view as an element $i \in \mathcal{J}(\mathbb{C}^n)$, and with a bit of linear algebra, it is not hard to show that *all* complex structures on V are equivalent to this one via suitable choices of basis. One can then use the standard Hermitian inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$ to write down an example of a linear symplectic structure $\omega \in \Omega(\mathbb{C}^n)$ compatible with i, namely

$$\omega(v, w) := \operatorname{Re}\langle iv, w \rangle.$$

We will see in this exercise that up to homotopy, there is a natural homotopy equivalence $\mathcal{J}(V) \to \Omega(V)$ sending each J to an ω that tames it. This correspondence has important consequences linking symplectic and complex geometry: roughly speaking, it implies that all purely homotopy-theoretic questions about symplectic structures on smooth manifolds are equivalent to questions about almost complex structures.

In order to set up the correspondence, define the intermediate spaces

$$X^\tau(V) := \left\{ (\omega, J) \in \Omega(V) \times \mathcal{J}(V) \;\middle|\; J \text{ is tamed by } \omega \right\},$$
$$X(V) := \left\{ (\omega, J) \in \Omega(V) \times \mathcal{J}(V) \;\middle|\; J \text{ is compatible with } \omega \right\}.$$

(a) Convince yourself that the projection map $p_2 : X^\tau(V) \to \mathcal{J}(V) : (\omega, J) \mapsto J$ is a Serre fibration, and its fibers are contractible.
Hint: The contractibility of the fibers is the easy part—what kind of subset is the set of all $\omega \in \Omega(V)$ that tame a given $J \in \mathcal{J}(V)$? For proving that the projection is a Serre fibration,

first use Exercise 8.2 to make the condition you need to establish as simple as possible. It may then be helpful to observe that tameness is an open condition, and if you've seen partitions of unity before, you may also recognize that this is a good place to use one.

(b) Deduce that the map $p_2 : X^\tau(V) \to \mathcal{J}(V)$ is a homotopy equivalence.

Before we continue, you'll need to accept as a black box that the corresponding statements about the projection $p_2 : X(V) \to \mathcal{J}(V)$ are also true: it is a Serre fibration, and it has contractible fibers, implying that it is a homotopy equivalence. The contractibility of the fibers is quite easy to see; proving the homotopy lifting property takes a bit more work, since compatibility (unlike tameness) is not an open condition, but there are linear-algebraic tricks for dealing with this, the details of which would be too much of a digression. Those tricks also imply, in fact, that the other projection $p_1 : X(V) \to \Omega(V) : (\omega, J) \mapsto \omega$ is yet another Serre fibration with contractible fibers. Take these facts as given in the following.

(c) Prove that the inclusion $X^\tau(V) \hookrightarrow X(V)$ and the projection $p_1 : X^\tau(V) \to \Omega(V) : (\omega, J) \mapsto \omega$ are also homotopy equivalences.

(d) Deduce that for any given $\omega \in \Omega(V)$, the set of all $J \in \mathcal{J}(V)$ that are tamed by ω is contractible.

Exercise 8.4. Prove the following relative version of the CW-approximation theorem: for any pair of spaces (X, A), there exists a CW-pair (X', A') and a map of pairs $\varphi : (X', A') \to (X, A)$ such that both of the maps $X' \to X$ and $A' \to A$ defined by φ are weak homotopy equivalences. (What can you say in this case about the induced maps $\pi_k(X', A') \to \pi_k(X, A)$?)

9. Week 9

Lecture 14 (10.06.2024): Homotopy excision, part 1.

- Motivational question: What has stopped us so far from computing $\pi_n(S^n)$ by induction on n, the same way that one computes $H_n(S^n; \mathbb{Z})$? Most of the ingredients for such an argument are in place: we know the case $n = 1$, we have homotopy invariance, and we have long exact sequences of pairs in which $\pi_k(CX)$ will vanish for any cone CX. What's still missing is excision.
- The excision question: Given $B \subset \bar{B} \subset \mathring{A} \subset A \subset X$, when does the inclusion $(X\backslash B, A\backslash B) \hookrightarrow (X, A)$ induce an isomorphism on π_k?
  Equivalent formulation: Given subsets $A, B \subset X$ whose interiors cover X, and a base point in $A \cap B$, when is the map $\pi_k(A, A \cap B) \to \pi_k(X, B)$ induced by the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ an isomorphism?
- Definition: For $k \geqslant 2$, a **k-ad** of spaces $(X; A_1, \ldots, A_{k-1})$ consists of a space X and subsets $A_1, \ldots, A_{k-1} \subset X$. It is a **pointed k-ad** if it is equipped with a base point $* \in A_1 \cap \ldots \cap A_{k-1}$, thus making (X, A_j) a pointed pair for each $j = 1, \ldots, k-1$. A **map of k-ads** $f : (X; A_1, \ldots, A_{k-1}) \to (X'; A'_1, \ldots, A'_{k-1})$ is a map $f : X \to X'$ such that $f(A_j) \subset A'_j$ for each $j = 1, \ldots, k-1$. There are similarly obvious definitions for pointed maps of k-ads and homotopies of such maps, leading to categories of pointed or unpointed k-ads and associated homotopy categories. A k-ad for $k = 2$ is just a pair of spaces in the usual sense; for $k = 3$ and $k = 4$, they are called **triads** and **tetrads** respectively.[40]
- Homotopy excision theorem (Blakers-Massey): Suppose $(X; A, B)$ is a triad with $X = A \cup B$ and $C := A \cap B \neq \varnothing$, such that (A, C) is m-connected and (B, C) is n-connected for some

[40] In the lecture I defined the term "k-ad" to mean what I am actually calling a "$(k+1)$-ad" in this writeup. I'm pretty sure the terminology as written here is better, since it would seem silly for the terms "2-ad" and "triad" to be synonymous. In practice, we are only actually interested in pairs, triads and (occasionally) tetrads.

$m, n \geqslant 0$, and suppose also that either $m \geqslant 1$ or C is path-connected. Additionally, assume one of the following situations:
 (i) The interiors of A and B form an open covering of X.[41]
 (ii) X is a CW-complex with $A, B, C \subset X$ as subcomplexes.
Then for all choices of base point in C, the map $\pi_k(A, C) \to \pi_k(X, B)$ induced by the inclusion $(A, C) \hookrightarrow (X, B)$ is bijective for all $k = 1, \ldots, m + n - 1$ and surjective for $k = m + n$.

- Remarks: One could easily deduce case (ii) in the theorem from case (i), but I have chosen to state it separately because proving case (ii) directly is slightly easier, and anyway, the important applications we discuss will only require case (ii). In the lecture I neglected to mention the possibility of allowing (A, C) to be only 0-connected while C is path-connected; that is the assumption stated in [Hat02, Theorem 4.23], whereas [May99, §11.1] assumes (A, C) is at least 1-connected while allowing C to be disconnected. Note that while our assumptions technically allow $m = n = 0$, the statement is vacuous in that case due to the lack of definitions for relative π_0. The version stated in [May99, §11.1] also includes a conclusion about the induced maps $\pi_0(C) \to \pi_0(A)$ and $\pi_0(B) \to \pi_0(X)$, which I have not bothered to state here.
- Corollary (Freudenthal suspension theorem): For any well-pointed space X that is n-connected for some $n \geqslant 0$, the map
$$\pi_k(X) = [\Sigma^k S^0, X] \xrightarrow{\Sigma} [\Sigma^{k+1} S^0, \Sigma X] = \pi_{k+1}(\Sigma X)$$
defined via the reduced suspension functor $\Sigma := \Sigma_+$ is bijective for all $k = 0, \ldots, 2n + 2$ and surjective for $k = 2n + 1$.
- Proof of Freudenthal: Well-pointedness implies that up to homotopy equivalence, we can replace reduced suspensions and cones with their unreduced counterparts, which lend themselves better to excision arguments. We regard both the reduced and unreduced suspensions ΣX as quotients of $X \times I$ and decompose them as a union of two (reduced or unreduced) cones
$$\Sigma X = CX \cup C'X,$$
where CX is a quotient of $X \times [0, 1/2]$ and $C'X$ is a quotient of $X \times [1/2, 1]$, so $CX \cap C'X = X \times \{1/2\} = X$. The inclusions $X = X \times \{1/2\} \hookrightarrow CX \hookrightarrow \Sigma X$ and $X = X \times \{1/2\} \hookrightarrow C'X \hookrightarrow \Sigma X$ are automatically pointed maps in the reduced case, and in the unreduced case, we can place the base points on the cones and suspensions to make this so. Denote the summit of the unreduced bottom cone by $p \in C_\circ X \subset \Sigma_\circ X$. We then have a commuting diagram
$$\begin{tikzcd}
\pi_k(X) \arrow[rr, "\Sigma_+"] & & \pi_{k+1}(\Sigma_+ X) \arrow[d, "\cong"] \\
\pi_{k+1}(C'_+ X, X) \arrow[u, "\partial_*", "\cong"'] \arrow[rr] & & \pi_{k+1}(\Sigma_+ X, C_+ X) \\
\pi_{k+1}(C'_\circ X, X) \arrow[u, "q_*", "\cong"'] \arrow[r, "\cong"] & \pi_{k+1}(\Sigma_\circ X \backslash \{p\}, C_\circ X \backslash \{p\}) \arrow[r] & \pi_{k+1}(\Sigma_\circ X, C_\circ X) \arrow[u, "q_*", "\cong"']
\end{tikzcd},$$
in which all unlabeled arrows are induced by inclusions, and q denotes the quotient projections from unreduced to reduced cones/suspensions. The top vertical arrows are both bijections due to long exact sequences that contain homotopy groups of cones, while the

[41] Many authors call $(X; A, B)$ in this situation an **excisive triad**. I prefer to avoid this term for now, because when we talk about homology later on, I will want the word "excisive" to mean something a bit more general.

bottom vertical and bottom left horizontal arrows are bijections because they are induced by homotopy equivalences; in the case of the two maps labelled q_*, this follows from well-pointedness, and one can use the 5-lemma to deduce bijections of relative homotopy groups from bijections of absolute ones. It remains only to investigate the map on π_k induced by the inclusion

$$(\Sigma_\circ X \backslash \{p\}, C_\circ X \backslash \{p\}) \hookrightarrow (\Sigma_\circ X, C_\circ X).$$

Viewing $\Sigma_\circ X$ as the union of the interiors of $A := \Sigma_\circ X \backslash \{p\}$ and $B := C_\circ X$ with intersection $A \cap B = C_\circ X \backslash \{p\}$, the n-connectedness of X makes $(A, A \cap B) = (\Sigma_\circ X \backslash \{p\}, C_\circ X \backslash \{p\}) \underset{h.e.}{\simeq} (C'_\circ X, X)$ an $(n+1)$-connected pair since the inclusion $X \hookrightarrow C'_\circ X$ induces a surjection $\pi_m(X) \to \pi_m(C'_\circ X) = 0$ for all m and a bijection for $m \leqslant n$. Similarly, the inclusion

$$X \xrightarrow[h.e.]{\simeq} C_\circ X \backslash \{p\} \longrightarrow C_\circ X$$

induces surjections/bijections on π_m precisely when $C_\circ X \backslash \{p\} \hookrightarrow C_\circ X = B$ does, making the pair $(B, A \cap B)$ also $(n+1)$-connected. The stated result then follows from the excision theorem.

- Remark: There are easy counterexamples to the claim that $\pi_k(X) \cong \pi_{k+1}(\Sigma X)$ without some dimensional assumption relating k and the connectedness of X, and from this one deduces that the dimensional conditions in the excision theorem cannot be dropped, in contrast to the more powerful excision property satisfied by homology and cohomology. One such counterexample is

$$0 = \pi_2(\mathbb{R}) \cong \pi_2(S^1) \not\cong \pi_3(S^2),$$

where $0 = \pi_2(\mathbb{R}) \cong \pi_2(S^1)$ comes from viewing the contractible space $\mathbb{R}$ as a covering space of S^1 (see Exercise 7.5), and $\pi_3(S^2) \neq 0$ is deduced from the Hopf fibration (Exercise 7.8).
- Corollary: $\pi_n(S^n) \cong \mathbb{Z}$ for every $n \in \mathbb{N}$.
Proof: Since S^n is $(n-1)$-connected, Freudenthal implies that $\Sigma : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for every $n \geqslant 2$, and the problem is thus reduced to proving $\pi_2(S^2) \cong \mathbb{Z}$. For this, one can use the Hopf fibration $p : S^3 \to \mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\} \cong S^2$, whose fibers are homeomorphic to S^1 (see Exercise 7.8): we then have a long exact sequence

$$\ldots \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \xrightarrow{\partial_*} \pi_1(S^1) \longrightarrow \pi_1(S^3) \longrightarrow \ldots,$$

and since S^3 is 2-connected, $\partial_* : \pi_2(S^2) \to \pi_1(S^1) \cong \mathbb{Z}$ is therefore an isomorphism.
- Remark: Although Freudenthal does not imply it, one can deduce from the computation above that $\Sigma : \pi_1(S^1) \to \pi_2(S^2)$ is also an isomorphism. To see this, you should first recall what you've learned in previous courses about the integer-valued degree of maps $S^n \to S^n$ and convince yourself that for every $n \in \mathbb{N}$, the map

$$\pi_n(S^n) \to \mathbb{Z} : [f] \to \deg(f)$$

is a group homomorphism. Since Σ sends the identity map $S^1 \to S^1$ to the identity map $S^2 \to S^2$, the image under $\Sigma : \pi_1(S^1) \to \pi_2(S^2)$ of a generator of $\pi_1(S^1) \cong \mathbb{Z}$ is therefore a primitive element of $\pi_2(S^2) \cong \mathbb{Z}$, and thus a generator. One also deduces in this way that the map $[f] \to \deg(f)$ is in fact an isomorphism $\pi_n(S^n) \to \mathbb{Z}$ for every $n \in \mathbb{N}$.
- Lemma: Cases (i) and (ii) of the excision theorem both follow from the following variant of case (ii) with stronger hypotheses:
(ii)' X is a CW-complex with subcomplexes $C = A \cap B \subset A, B \subset X$ such that the m-skeleton of A and the n-skeleton of B are both contained in C.

Proof: The idea is to construct a CW-approximation of triads

$$(X'; A', B') \xrightarrow{\varphi} (X; A, B),$$

namely so that $(X'; A', B')$ is a triad of CW-complexes satisfying the strengthened hypothesis, and φ defines weak homotopy equivalences $X' \to X$, $A' \to A$, $B' \to B$ and $A' \cap B' =: C' \to C$. One constructs this by first constructing a CW-approximation C' of C, and then extending it to two CW-approximations of pairs $(A', C') \to (A, C)$ and $(B', C') \to (B, C)$ in the sense of Exercise 8.4. This is a straightforward extension of our proof of the CW-approximation theorem in Lecture 13, and the assumption that (A, C) is m-connected gives us the freedom to begin the induction with $(m+1)$-cells when expanding C' to A'; the n-connectedness of (B, C) has a similar implication for (B', C'). One then defines X' as the pushout $A' \cup_{C'} B'$, with $\varphi : X' \to X$ as the unique map extending the previously defined weak homotopy equivalences $A' \to A$ and $B' \to B$. What is not obvious from this construction is why $\varphi : X' \to X$ should be a weak homotopy equivalence, and in fact, this would not be true in general without the assumptions on $X = A \cup B$ stated in the excision theorem. Below we discuss cases (i) and (ii) separately.

- Why $\varphi : X' \to X$ is a weak homotopy equivalence in case (ii):
Since we are talking about CW-complexes, the word "weak" can be removed, as Whitehead implies that the maps $A' \to A$, $B' \to B$ and $C' \to C$ defined by φ are homotopy equivalences. Another helpful detail is that up to homotopy equivalence, the pushouts $X' = A' \cup_{C'} B'$ and $X = A \cup_C B$ can be replaced by homotopy pushouts, i.e. mapping cylinders, producing a diagram

$$\begin{tikzcd} Z(C' \hookrightarrow A', C' \hookrightarrow B') \arrow[r] \arrow[d] & Z(C \hookrightarrow A, C \hookrightarrow B) \arrow[d] \\ X' = A' \cup_{C'} B' \arrow[r, "\varphi"] & A \cup_C B = X \end{tikzcd}$$

in which the vertical arrows are quotient projections that collapse the cylinders. Equivalently, the map $Z(C \hookrightarrow A, C \hookrightarrow B) \to X$ is determined by a reinterpretation of the strictly commutative pushout diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \\ B \arrow[r, hook] & X \end{tikzcd}$$

as a diagram in hTop with the trivial homotopy between the two composed inclusions, and similarly for the map $Z(C' \hookrightarrow A', C' \hookrightarrow B') \to X'$. Maps from homotopy pushouts to pushouts defined in this way are not always homotopy equivalences, but these are, due to the fact that $C \hookrightarrow A$ and $C' \hookrightarrow A'$ are cofibrations (see Exercise 9.4). With this understood, we can interpret the map between the two mapping cylinders as induced by an isomorphism in the category of pushout diagrams discussed in Lecture 3, which implies that it is a homotopy equivalence.
- Why $\varphi : X' \to X$ is a weak homotopy equivalence in case (i):
After replacing the pushout $X' = A' \cup_{C'} B'$ with a homotopy pushout and citing Exercise 9.4, we are free to assume that X' is also covered by the interiors of A' and B'. We can then apply a general theorem that is stated and proved in [May99, §10.7], and I will attempt below to summarize a less technical variation on May's proof of that theorem. The main idea is one that you have seen before in the context of other excision theorems: subdivision.

- Weak homotopy theorem for excisive triads: Assume $f : (X'; A', B') \to (X; A, B)$ is a map of triads such that $X = \mathring{A} \cup \mathring{B}$, $X' = \mathring{A}' \cup \mathring{B}'$, and the restrictions of f to maps $A' \to A$, $B' \to B$ and $C' := A' \cap B' \to A \cap B =: C$ are all weak homotopy equivalences. Then $f : X' \to X$ is also a weak homotopy equivalence.
  Proof: We start with a standard mapping cylinder trick. . . the diagram
$$\begin{tikzcd} (X'; A', B') \arrow[rr, "f"] \arrow[dr, hook] & & (X; A, B) \\ & \Big(Z(f); Z\big(f^{-1}(A) \xrightarrow{f} A\big), Z\big(f^{-1}(B) \xrightarrow{f} B\big)\Big) \arrow[ur, "\simeq"', "h.e."] & \end{tikzcd}$$
  tells us that without loss of generality, we are free to assume f is an embedding. A quick remark about this: your first naive guess for the triad of mapping cylinders might have been
$$\Big(Z(f); Z\big(A' \xrightarrow{f} A\big), Z\big(B' \xrightarrow{f} B\big)\Big),$$
  but one cannot guarantee in general that the interiors of the two subsets in this triad cover $Z(f)$. Replacing A' and B' with the larger sets $f^{-1}(A)$ and $f^{-1}(B)$ fixes this problem, and does so without changing the homotopy type of the triad of mapping cylinders.
  With that out of the way, assume f is the inclusion of a subset $X' \subset X$ with $A' \subset A$ and $B' \subset B$. The goal is then to show that $\pi_k(X, X') = 0$ for every $k \in \mathbb{N}$ and every choice of base point in X', or equivalently, that every map
$$(I^k, \partial I^k) \xrightarrow{\varphi} (X, X')$$
  is homotopic rel ∂I^k to one whose image lies in X'. The assumption about open coverings makes possible the following: after subdividing I^k into sufficiently small subcubes and taking suitable unions of those cubes, we can equip the pair $(I^k, \partial I^k)$ with a cell decomposition in which it contains subcomplexes
$$K_C = K_A \cap K_B \subset K_A, K_B \subset K_A \cup K_B = I^k$$
  such that
$$\varphi(K_A) \subset A, \qquad \varphi(K_B) \subset B, \qquad \varphi(K_A \cap \partial I^k) \subset A', \quad \text{ and } \quad \varphi(K_B \cap \partial I^k) \subset B'.$$
  Having done this, we construct the desired homotopy of φ in three steps:
  (1) Homotop $\varphi|_{K_C}$ rel $K_C \cap \partial I^k$ to a map whose image lies in C'; this requires an induction over the skeleta using the compression lemma, which is applicable because $C' \hookrightarrow C$ is a weak homotopy equivalence. The homotopy can then be extended to the rest of I^k since $K_C \cup \partial I^k \hookrightarrow I^k$ is a cofibration.
  (2) Homotop $\varphi|_{K_A}$ rel $K_C \cup (K_A \cap \partial I^k)$ to a map whose image lies in A'; this follows by a similar argument as in step 1 and is possible because $A' \hookrightarrow A$ is a weak homotopy equivalence.
  (3) Homotop $\varphi|_{K_B}$ rel $K_C \cup (K_B \cap \partial I^k)$ to a map whose image lies in B'; likewise possible because $B' \hookrightarrow B$ is a weak homotopy equivalence.
- This lecture concluded with a definition of the homotopy groups of a triad, but I'm deferring that to the writeup of the next lecture, where they are actually used.

Lecture 15 (13.06.2024): Homotopy excision, part 2.

- Loose end from last time: We need to prove the homotopy excision theorem for case (ii)', meaning that $X = A \cup_C B$ is a CW-complex with subcomplexes A, B and $C = A \cap B$

such that A and B have only high-dimensional cells outside of C; specifically, $A^m \subset C$ and $B^n \subset C$.

- Definition: If (X, A) is a pointed pair of spaces, let $F(A \hookrightarrow X) \subset X^I$ denote the homotopy fiber of the inclusion, i.e. the space of paths in X that begin in A and end at the base point. We observe that any inclusion of pairs $(A, C) \hookrightarrow (X, B)$ gives rise to a natural inclusion of homotopy fibers $F(C \hookrightarrow A) \hookrightarrow F(B \hookrightarrow X)$. Recall that the relative homotopy groups were defined in Lecture 11 as
$$\pi_k(X, A) = \pi_{k-1}(F(A \hookrightarrow X))$$
for $k \geqslant 1$. For $k \geqslant 2$, we can now also define the homotopy groups of a pointed triad $(X; A, B)$ in any of the following equivalent ways:
$$\begin{aligned}\pi_k(X; A, B) &:= \pi_{k-2}\Big(F\big(F(C \hookrightarrow A) \hookrightarrow F(B \hookrightarrow X)\big)\Big) \\ &= \pi_{k-1}\big(F(B \hookrightarrow X), F(C \hookrightarrow A)\big) \\ &= \big[(I^k; I^{k-1}_A, I^{k-1}_B, \partial I^k \backslash (I^{k-1}_A \cup I^{k-1}_B)), (X; A, B, *)\big]_\circ ,\end{aligned}$$
where in the third line, we single out two specific $(k-1)$-dimensional faces of the boundary of the k-cube,
$$I^{k-1}_A := I^{k-2} \times \{0\} \times I, \qquad I^{k-1}_B := I^{k-1} \times \{0\}.$$
The description in the third line follows from that of the first line by identifying pointed maps $\Sigma^{n-2} S^0 = I^{k-2}/\partial I^{k-2} \to Y$ with maps of pairs $(I^{k-2}, \partial I^{k-2}) \to (Y, *)$ and interpreting paths in the path space X^I as maps $I^2 \to X$, thus giving
$$F\big(F(C \hookrightarrow A) \hookrightarrow F(B \hookrightarrow X)\big) = \big\{\text{maps } (I^2; \{0\} \times I, I \times \{0\}, \text{rest of } \partial I^2) \to (X; A, B, *)\big\}, \tag{9.1}$$
which is a pointed space with the constant map $I^2 \to * \in X$ as base point. There is an obvious group structure for $\pi_k(X; A, B)$ whenever $k \geqslant 3$, and it is abelian for $k \geqslant 4$, while $\pi_2(X; A, B)$ is only a pointed set.
- There is a more geometric characterization of $\pi_k(X; A, B)$ via maps defined on the pointed triad $(\mathbb{D}^k; \mathbb{D}^{k-1}_+, \mathbb{D}^{k-1}_-)$, where $S^{k-1} = \partial \mathbb{D}^k$ is decomposed as a union of two disks $\mathbb{D}^{k-1}_\pm$ that intersect at their common boundary $\partial \mathbb{D}^{k-1}_\pm = S^{k-2} \subset S^{k-1}$, which contains the base point. The basic observation is that if we quotient out the boundary region
$$K := \overline{\partial I^k \backslash (I^{k-1}_A \cup I^{k-1}_B)} \subset \partial I^k$$
on which the tetrad maps $I^k \to X$ described above are required to be constant, then one can choose a homeomorphism of I^k/K to $\mathbb{D}^k$ that identifies the resulting quotients of I^{k-1}_A and I^{k-1}_B with $\mathbb{D}^{k-1}_+$ and $\mathbb{D}^{k-1}_-$ respectively, giving
$$\pi_k(X; A, B) \cong [(\mathbb{D}^k; \mathbb{D}^{k-1}_+, \mathbb{D}^{k-1}_-), (X; A, B)]_+.$$
The group structure of $\pi_k(X; A, B)$ for $k \geqslant 3$ is less clear from this perspective, but it's good for pictures.
- For another useful characterization of $\pi_k(X; A, B)$, one can foliate the square I^2 by paths from $\{0\} \times I$ to $I \times \{0\}$, giving a new interpretation of the iterated homotopy fiber (9.1) as a space of paths in
$$P(A, B) := \big\{\gamma \in X^I \bigm| \gamma(0) \in A \text{ and } \gamma(0) \in B\big\} \subset X^I,$$
which contains a natural embedding of $C = A \cap B$ as the set of constant paths. Note that $P(A, B)$ has a natural description as the homotopy pullback of the inclusions $A \hookrightarrow X$ and

$B \hookrightarrow X$, appearing in the homotopy-commutative diagram

$$\begin{tikzcd} P(A,B) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"'] \arrow[dr, phantom, "\sim"] & A \arrow[d, hook] \\ B \arrow[r, hook] & X \end{tikzcd},$$

whereas $C = A \cap B$ can likewise be interpreted as the strict pullback of the same pair of inclusions. Elements of the space (9.1) are thus equivalent to paths in $P(A, B)$ that begin in $C \subset P(A, B)$ and end at the base point, thus

$$\pi_k(X; A, B) \cong \pi_{k-1}(P(A, B), C).$$

This is the characterization we will use in the proof of the excision theorem.

- The long exact sequence of relative homotopy groups of the pair $(F(B \hookrightarrow X), F(C \hookrightarrow A))$ can now be written as

$$\ldots \to \pi_3(X, B) \to \pi_3(X; A, B) \to \pi_2(A, C) \to \pi_2(X, B) \to \pi_2(X; A, B) \to \pi_1(A, C) \to \pi_1(X, B)$$

- Homotopy excision theorem (symmetric version): Assume the same hypotheses as in the homotopy excision theorem, except that arbitrary values $m, n \geqslant 0$ are allowed and there are no connectivity assumptions on C. Then for all choices of base point in $A \cap B$,

$$\pi_k(X; A, B) = 0 \qquad \text{for all } k = 2, \ldots, m + n.$$

- Remark: The additional hypothesis in the original excision theorem that either $m \geqslant 1$ or C is path-connected is needed in order to make the map $\pi_1(A, C) \to \pi_1(X, B)$ surjective, since the latter cannot be deduced from the long exact sequence above. Proving this surjectivity is a straightforward exercise that might remind you of the proofs of the simplest cases of the Seifert-van Kampen theorem (see Exercise 9.1). The rest of the theorem then follows via the long exact sequence from the symmetric version stated above.
- Setup for proving the symmetric excision theorem: Assuming $k \leqslant m + n$, we need to show that $\pi_{k-1}(P(A, B), C)$ vanishes for all choices of base point in C, or equivalently, that every map

$$f : (\mathbb{D}^{k-1}, S^{k-2}) \to (P(A, B), C)$$

is homotopic rel S^{k-2} to a map with image lying in C. By adjunction, such maps are equivalent to maps $\mathbb{D}^{k-1} \times I \to X$ that can only touch finitely many cells of X, so by induction, it suffices to consider the special case when A and B each contain only one cell in addition to C,

$$X := e^p \cup C \cup e^q,$$

where $A = C \cup e^p$, $B = C \cup e^q$, $p > m$ and $q > n$, hence $k + 1 < p + q$. Choosing points $x \in e^p$ and $y \in e^q$, there is now a deformation retraction implying that the inclusion

$$(X; A, B) \hookrightarrow (X; X \backslash \{y\}, X \backslash \{x\})$$

is a homotopy equivalence of triads, and we are therefore free to make the replacements

$$A := X \backslash \{y\}, \quad B := X \backslash \{x\}, \quad C := X \backslash \{x, y\}.$$

What comes next is the exceptionally clever part.

- Dimensional lemma (this is where the assumption $k + 1 < p + q$ is required): The map $f : \mathbb{D}^{k-1} \to P(A, B)$ is homotopic rel S^{k-2} to a map with the property that for each $z \in \mathbb{D}^{k-1}$, the path $f(z) \in P(A, B) \subset X^I$ does not hit both x and y.

- Proof of the theorem (assuming the dimensional lemma):
Since all of the paths $f(z) \in X^I$ miss at least one of x or y, we can choose a continuous function $u : \mathbb{D}^{k-1} \to I$ such that
$$u(z) = 0 \text{ if } f(z) \text{ hits } y, \qquad u(z) = 1 \text{ if } f(z) \text{ hits } x.$$
As a map $f : \mathbb{D}^{k-1} \to X^I$, f is clearly homotopic rel S^{k-2} to the map $f' : \mathbb{D}^{k-1} \to P(A, B)$ which sends each $z \in \mathbb{D}^{k-1}$ to the constant path living at the point $f(z)(u(z))$. We claim that this is in fact a homotopy of maps $\mathbb{D}^{k-1} \to P(A, B)$ and that f' takes values in C, meaning that none of the constant paths $f'(z)$ live at x or y. Indeed, one defines the homotopy $\{f_s : \mathbb{D}^{k-1} \to X^I\}_{s\in I}$ such that for each $s \in I$ and $z \in \mathbb{D}^{k-1}$, $f_s(z)$ is a reparametrization of $f(z)$ over subintervals that shrink to zero length as $s \to 1$. Now observe:
  - If $u(z) \notin \{0, 1\}$, then $f(z)$ misses both x and y, and therefore so does $f_s(z)$ for every $s \in I$.
  - If $u(z) = 0$, then $f(z)$ may hit y but it misses x, and $f'(z)$ lives at the point $f(z)(0) \in A = X\backslash\{y\}$. The fact that $f(z)$ misses x guarantees also $f_s(z)(1) \in B = X\backslash\{x\}$ for every s, and that $f'(z)$ is also contained in $C = X\backslash\{x, y\}$.
  - If $u(z) = 1$, one reaches a similar conclusion with the roles of x and y reversed.
- Proof of the dimensional lemma: All versions of this proof appeal at some stage to a general position argument, and there are various ways to carry out the details, but the quickest in my opinion is to make some use of smoothness. By adjunction, $f : \mathbb{D}^{k-1} \to P(A, B)$ is equivalent to a map of tetrads
$$(\mathbb{D}^{k-1} \times I; \mathbb{D}^{k-1} \times \{0\}, \mathbb{D}^{k-1} \times \{1\}, S^{k-2} \times I) \xrightarrow{\check{f}} (X; A, B, C),$$
and up to a C^0-small homotopy, we can assume via standard perturbation results in differential topology (see [Hir94]) that $\check{f}$ is a smooth on some neighborhood of the subsets $\check{f}^{-1}(x)$ and $\check{f}^{-1}(y)$, where it can be regarded as a map into the smooth manifold $e^p = \mathring{\mathbb{D}}^p$ or $e^q = \mathring{\mathbb{D}}^q$ respectively. The map
$$\mathbb{D}^{k-1} \times I^2 \xrightarrow{F} X \times X : (z, s, t) \mapsto \Big(\check{f}(z, s), \check{f}(z, t)\Big)$$
is then also smooth near $F^{-1}(x, y)$, and in light of the dimensional condition
$$\dim(\mathbb{D}^{k-1} \times I^2) = k + 1 < p + q = \dim(e^p \times e^q),$$
Sard's theorem implies that almost every point (x', y') near (x, y) is not in the image of F. Using isotopies of $\mathbb{R}^p$ and $\mathbb{R}^q$ near x and y respectively, we are free to assume after a further homotopy of f that (x, y) instead of (x', y') is a point that F misses, and f now has the desired property.

This completes the proof of the homotopy excision theorem. The application $\pi_n(S^n) \cong \mathbb{Z}$ was discussed in the previous lecture; the remainder of this lecture covers two more applications.

- Theorem 1: Given a set J and an integer $n \geqslant 2$, consider the pointed space
$$X := \bigvee_{\alpha\in J} S^n$$
and the inclusions $i^\alpha : S^n \hookrightarrow X$ associated to each $\alpha \in J$, which induce homomorphisms $i^\alpha_* : \pi_n(S^n) \to \pi_n(X)$ and therefore determine (via the universal property of the coproduct) a homomorphism
$$\Phi := \bigoplus_\alpha i^\alpha_* : \bigoplus_{\alpha\in J} \pi_n(S^n) \to \pi_n(X).$$

The map Φ is an isomorphism; in particular, $\pi_n(X)$ is thus isomorphic to the free abelian group generated by J.
Proof: Assume first that J is finite, and consider the product space $Y := \prod_{\alpha \in J} S^n$. Viewing S^n as a CW-complex with one 0-cell and one n-cell gives X a cell decomposition with one 0-cell and an n-cell associated to each $\alpha \in J$, and the product Y likewise inherits a product cell decomposition whose n-skeleton has a natural identification with the wedge sum X. Moreover, Y only has cells in dimensions that are divisible by n, so X is also its $(2n-1)$-skeleton, and it follows that (Y, X) is $(2n-1)$-connected. Since $n \geqslant 2$ and thus $2n - 1 \geqslant n + 1$, (Y, X) is also $(n+1)$-connected, so that the long exact sequence of the pair (Y, X) gives an isomorphism

$$0 = \pi_{n+1}(Y, X) \longrightarrow \pi_n(X) \xrightarrow{\cong} \pi_n(Y) \longrightarrow \pi_n(Y, X) = 0.$$

There is an easy general formula for homotopy groups of product spaces: the projections $p^\alpha : Y \to S^n$ for each $\alpha \in J$ induce homomorphisms $p^\alpha_* : \pi_n(Y) \to \pi_n(S^n)$ and thus (by the universal property of the product) a homomorphism

$$\Psi := \prod_\alpha p^\alpha_* : \pi_n(Y) \to \prod_{\alpha \in J} \pi_n(S^n),$$

which is easily shown to be an isomorphism in general. But since J is finite, the latter product is the same thing as the direct sum, and one can now check that Ψ and Φ are inverses.

The case where J is infinite follows easily from the finite case due to the fact that every map from $\mathbb{D}^k$ or S^k to X can intersect at most finitely many cells, which in this case means finitely many copies of S^n in the wedge sum, beyond the base point.

- Theorem 2: Assume $A \hookrightarrow X$ is a free cofibration, the pair (X, A) is m-connected and A is an n-connected space for some $m, n \geqslant 0$. Then for all choices of base points in A, the map

$$q_* : \pi_k(X, A) \to \pi_k(X/A)$$

induced by the quotient projection $q : (X, A) \to (X/A, A/A)$ is bijective for $k = 1, \ldots, m+n$ and surjective for $k = m + n + 1$.
Proof: Assuming $A \hookrightarrow X$ is a free cofibration gives us the freedom to use its unreduced homotopy cofiber $\operatorname{cone}(A \hookrightarrow X) = CA \cup_A X$ is a stand-in for the actual cofiber X/A, since the natural comparison map $\operatorname{cone}(A \hookrightarrow X) \to X/A$ is a homotopy equivalence (cf. Exercise 9.4). A more direct description of this homotopy equivalence is as the quotient projection

$$\operatorname{cone}(A \hookrightarrow X) = CA \cup_A X \xrightarrow{q} \operatorname{cone}(A \hookrightarrow X)/CA,$$

since the inclusion $X \hookrightarrow \operatorname{cone}(A \hookrightarrow X)$ descends to a natural homeomorphism of X/A to the quotient at the right. Consider the diagram

$$\begin{tikzcd}
\pi_k(X, A) \arrow[r] & \pi_k(\operatorname{cone}(A \hookrightarrow X), CA) \arrow[r, "q_*"] & \pi_k(\operatorname{cone}(A \hookrightarrow X)/CA) = \pi_k(X/A) \\
& \pi_k(\operatorname{cone}(A \hookrightarrow X)) \arrow[u, "\cong"] \arrow[ur, "q_*", "\cong"'] &
\end{tikzcd},$$

in which the unlabelled arrows are induced by inclusions, the up arrow is bijective due to a long exact sequence containing the homotopy groups of the contractible space CA, and the diagonal arrow labelled "q_*" is also bijective for the reasons stated above. The map $\pi_k(X, A) \to \pi_k(X/A)$ is consequently bijective or surjective if and only if the same holds for the map induced on π_k by the inclusion

$$(X, A) \hookrightarrow (\operatorname{cone}(A \hookrightarrow X), CA).$$

Viewing cone$(A \hookrightarrow X)$ as the union of CA with an open neighborhood of X, the excision theorem applies whenever $k \leqslant m + n + 1$ since (X, A) is m-connected and (CA, A) is $(n+1)$-connected.

Highlights from the Übung. It's worth mentioning two interesting results about Serre fibrations that were discussed in the Übung this week.

First, the long exact sequence of homotopy groups for a Serre fibration $p : E \to B$ with fiber inclusion $j : F = p^{-1}(*) \hookrightarrow E$ is a special case of an exact sequence of the form

$$\ldots \to [\Sigma^2 X, B] \xrightarrow{\partial_*} [\Sigma X, F] \xrightarrow{j_*} [\Sigma X, E] \xrightarrow{p_*} [\Sigma X, B] \xrightarrow{\partial_*} [X, F] \xrightarrow{j_*} [X, E] \xrightarrow{p_*} [X, B],$$

which can be defined for any CW-complex X, and makes sense in part because $\Sigma^n X$ is also a CW-complex for every $n \geqslant 0$. (This implies for instance via adjunction that $\Omega^n p : \Omega^n E \to \Omega^n B$ is also a Serre fibration for every $n \geqslant 0$.) The most non-obvious detail about this sequence is how to define the connecting maps ∂_*; our definition in the stricter setting of Hurewicz fibrations $p : E \to B$ relied on a homotopy class of maps $\delta : \Omega B \to F$ defined via the transport functor, but this only works if $p : E \to B$ has the homotopy lifting property with respect to ΩB, which is generally not a CW-complex. One can show however that for any Serre fibration, there is still a well-defined transport functor associated to homotopies $H : X \times I \to B$ whenever X is a CW-complex, and this is enough to give a definition of $\partial_* : [\Sigma X, B] \to [X, F]$ after interpreting pointed maps $\Sigma X \to B$ as pointed homotopies between the constant map $X \to B$ and itself; lifting such a homotopy gives a homotopy from the constant map $X \to E$ to a map that takes values in the fiber F, whose homotopy class is well defined because it comes from an application of the transport functor for homotopies of maps $X \to B$. Once the definition of ∂_* is in place, it is a straightforward exercise to verify the exactness of the sequence.

The second result to mention is the following, which is a nearly immediate consequence of the long exact sequence of a Serre fibration and Whitehead's theorem:

Theorem. *Suppose $f : X \to Y$ is a surjective Serre fibration: then f is a weak homotopy equivalence if and only if the fibers $f^{-1}(y)$ are weakly contractible for every $y \in Y$. In particular, if X, Y and the fibers $f^{-1}(y)$ are all homotopy equivalent to CW-complexes, we conclude that f is a homotopy equivalence if and only if every fiber $f^{-1}(y)$ is contractible.*

The surjectivity assumption is only meant to preclude silly examples in which the induced map $\pi_0(X) \to \pi_0(Y)$ fails to be surjective. This theorem is quite popular in several areas of differential geometry and topology, where almost all interesting spaces (even infinite-dimensional ones) have the homotopy type of CW-complexes, and we therefore obtain a practically checkable criterion for certain maps to be homotopy equivalences. It can be applied for instance in Exercise 8.3.

Suggested reading. The literature contains an abundance of unreadable proofs of the homotopy excision theorem, and what I eventually presented in lecture was something of a hybrid between [May99, §10.6 and Chapter 11] and [MV15, §4.4.1]. I made a conscious decision to use some black boxes from the smooth category in order to avoid getting bogged down with technical details in the proof of the dimensional lemma—in [Hat02, §4.2] and [tD08, §6.9], you can see some other ways of doing the same job without venturing into the smooth category, but the details are inevitably tedious. (To be fair, the proof of Sard's theorem is also somewhat tedious, though in principle it can be understood by any student with a solid first-year analysis background.)

Hatcher's treatment of the standard applications of excision is relatively readable, in spite of my minor complaint about failing to make clear why he can get away with using reduced and unreduced suspensions interchangeably. (Hint: It's because of cofibrations!)

I only recently discovered the book [MV15], which has some interesting things to say about the wider significance of the homotopy excision theorem from a more category-theoretic perspective.

For a book that tries to express everything in terms of square-shaped diagrams involving terms like "homotopy cocartesian," I find it surprisingly readable.

Exercises (for the Übung on 20.06.2024).

Exercise 9.1. Fill in the gaps between the proofs of the standard homotopy excision theorem and the symmetric version about the vanishing of the triad groups $\pi_k(X; A, B)$, i.e. show that under the given hypotheses, the map $\pi_1(A, C) \to \pi_1(X, B)$ is surjective for all choices of base points.

Exercise 9.2. Here's an exercise that we really should have done last week, but better late than never. Suppose $p : E \to B$ is a Serre fibration whose fiber inclusion $F \hookrightarrow E$ is pointed homotopic to a constant. Show that $\pi_1(F)$ is abelian and

$$\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$$

for every $n \geqslant 2$.
Hint: Turn the long exact sequence of homotopy groups into a split short exact sequence. You can construct a splitting out of a homotopy of $F \hookrightarrow E$ to the constant map.
Remark: One obtains some real-life applications of this formula from the quaternionic and octonionic versions of the Hopf fibration

$$S^3 \hookrightarrow S^7 \xrightarrow{p} S^4 \qquad \text{and} \qquad S^7 \hookrightarrow S^{15} \xrightarrow{p} S^8,$$

which are discussed e.g. in [Hat02, Examples 4.46 and 4.47]. *These imply that $\pi_7(S^4)$ and $\pi_{15}(S^8)$ each contain $\mathbb{Z}$ as direct summands.*

Exercise 9.3. The following example illustrates another contrast between homotopy groups and homology: the homotopy groups of a finite CW-complex need not be finitely generated. Let $n \geqslant 2$.

(a) Describe the universal cover of $S^1 \vee S^n$.
(b) Show that $\pi_n(S^1 \vee S^n)$ is a free abelian group with a countably infinite number of generators.
(c) Show however that $\pi_n(S^1 \vee S^n)$ contains an element whose orbit under the action of $\pi_1(S^1 \vee S^n)$ generates the whole group. In fancier language: $\pi_n(S^1 \vee S^n)$ is isomorphic to the free $\mathbb{Z}[\pi_1(S^1 \vee S^n)]$-module with one generator.

Exercise 9.4. A pushout square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "j_A"] \\ B \arrow[r, "j_B"] & X \end{tikzcd}$$

in Top or Top_* can always also be interpreted as a commuting diagram in hTop or hTop_* respectively, so that this data together with the trivial homotopy between $j_A \circ f$ and $j_B \circ g$ naturally determines a map $u : Z(f, g) \to X$ for which the diagram

$$\begin{tikzcd} A \arrow[d, hook, "i_A"'] \arrow[dr, "j_A"] & \\ Z(f,g) \arrow[r, "u"] & X \\ B \arrow[u, hook, "i_B"] \arrow[ur, "j_B"'] & \end{tikzcd}$$

commutes strictly. You should recall from Lecture 3: the map u is not uniquely determined by this diagram alone, because $Z(f, g)$ and the embeddings i_A and i_B do not actually define a pushout in hTop or hTop_*; nonetheless, the maps j_A, j_B and the choice of homotopy $j_A \circ f \rightsquigarrow j_B \circ g$ determine

u in a canonical way. In this particular setting, if we identify X with the space $A \cup_C B$ obtained by gluing A and B together along the maps f and g, then

$$Z(f, g) = A \cup_f (C \times I) \cup_g B \xrightarrow{u} A \cup_C B$$

becomes the obvious quotient projection that leaves A and B alone but flattens the cylinder $C \times I$ to C. This description strongly suggests that u should be a homotopy equivalence, though that isn't always true: you may have recognized this already, because if you take B to be a one point space, then $Z(f, g)$ becomes the mapping cone of $f : C \to A$, also known as its homotopy cofiber, while X becomes the actual cofiber $A/f(C)$, and u is now the natural comparison map from the homotopy cofiber to the cofiber of f, which we've seen is a homotopy equivalence if f is a cofibration, but not in general. This observation suggests a conjecture that turns out to be correct, and we used it in our proof of the homotopy excision theorem for CW-complexes: the natural map $Z(f, g) \to X$ from the homotopy pushout to the actual pushout is a homotopy equivalence whenever either f or g is a cofibration. Let's prove this.

(a) Back in Lecture 6, we proved a theorem stating that for any two fibrations $p : E \to B$ and $p' : E' \to B$ with the same base, if $f : E \to E'$ is both a homotopy equivalence and a map over B, then it is also a homotopy equivalence over B, with the important implication that it then also defines homotopy equivalences between the fibers of p and those of p'. The dual theorem is a fact that we have used several times behind the scenes, but never had time to discuss explicitly in lecture: Given a pair of cofibrations $j : A \hookrightarrow X$ and $j' : A \hookrightarrow X'$ with the same cobase, any map $f : X \to X'$ that is both a homotopy equivalence and a map *under* A is also a homotopy equivalence under A, and therefore descends to the cofibers as a homotopy equivalence $X/A \to X'/A$. Here, $f : X \to X'$ is called a "map under A" if $j' \circ f = j$, and the term "homotopy equivalence under A" implies that it has a homotopy inverse $g : X' \to X$ and homotopies of $g \circ f$ and $f \circ g$ to the identity through families of maps that all satisfy similar compatibility relations with respect to j and j'. The proof of the theorem is an application of the transport functor for cofibrations, completely analogous to the way that we proved the corresponding theorem about fibrations. If you have never yet thought through the reasons why it works, do so now.

(b) Consider a pair of pushout squares

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"] & A \arrow[d, "j_A"] \\ B \arrow[r, "j_B"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "f'"] \arrow[d, "g'"] & A' \arrow[d, "j'_A"] \\ B' \arrow[r, "j'_B"] & X' \end{tikzcd}$$

that have the same space C in the top left corner. In what sense does a pair of maps $\varphi : A \to A'$ and $\psi : B \to B'$ under C uniquely determine a map $X \to X'$?

(c) Notice that the double mapping cylinder $Z(f, g)$ can also be constructed by gluing the single mapping cylinder $Z(f) = A \cup_f (C \times I)$ to B along the map $C \times \{1\} = C \xrightarrow{g} B$, and thus also fits into an honest pushout square of the form

$$\begin{tikzcd} C \arrow[r, hook, "i_C"] \arrow[d, "g"] & Z(f) \arrow[d] \\ B \arrow[r, hook, "i_B"] & Z(f, g) \end{tikzcd} .$$

Use this observation together with the theorem described in part (a) to show that if $f : C \to A$ is a cofibration, then $u : Z(f, g) \to X$ is a homotopy equivalence.

Exercise 9.5. Recall from Exercise 6.6 that every pointed space X comes equipped with a canonical map $\Phi : X \to \Omega\Sigma X$. Show that the following statement is equivalent to the Freudenthal suspension theorem: For X well-pointed and n-connected, the map $\Phi : X \to \Omega\Sigma X$ is a $(2n+1)$-equivalence.

Exercise 9.6. Assume X is a CW-complex whose $(n-1)$-skeleton for some $n \geqslant 2$ is a single point, regarded in the following as a base point, and such that all $(n+1)$-cells $e_\beta^{n+1} \subset X^{n+1}$ have base-point preserving attaching maps $\varphi_\beta : S^n \to X^n$. Show that the map

$$\pi_n(X^n) \to \pi_n(X)$$

induced by the inclusion $X^n \hookrightarrow X$ is surjective, and its kernel is the subgroup generated by the homotopy classes of the attaching maps $[\varphi_\beta] \in \pi_n(X^n)$ for all $(n+1)$-cells e_β^{n+1}.
Hint: The group $\pi_{n+1}(X^{n+1}/X^n)$ is easy to compute. Can you replace $\pi_{n+1}(X^{n+1}, X^n)$ with $\pi_{n+1}(X^{n+1}/X^n)$ in the long exact sequence of (X^{n+1}, X^n)?

10. Week 10

Lecture 16 (17.06.2024): Hurewicz homomorphisms.

- In the following, $\mathsf{Top}^{\mathrm{rel}}$ denotes the category of pairs of spaces (X, A) with $A \subset X$, with maps of pairs as morphisms; taking homotopy classes of maps of pairs gives the associated homotopy category $\mathsf{hTop}^{\mathrm{rel}}$. Choosing a base point $* \in A$ and requiring maps of pairs (and homotopies thereof) to preserve the base point gives similar categories $\mathsf{Top}_*^{\mathrm{rel}}$ and $\mathsf{hTop}_*^{\mathrm{rel}}$. For convenience, we sometimes identify Top with the subcategory of $\mathsf{Top}^{\mathrm{rel}}$ consisting of pairs of the form $(X, \varnothing)$; similarly, Top_* embeds into $\mathsf{Top}_*^{\mathrm{rel}}$ as the pairs of the form $(X, \{*\})$.
- Axioms of a (**generalized**) **homology** or **cohomology theory**: A homology theory h_* is a collection of covariant functors $\{h_n : \mathsf{Top}^{\mathrm{rel}} \to \mathsf{Ab}\}_{n\in\mathbb{Z}}$ equipped with natural transformations $h_n(X, A) \stackrel{\partial_*}{\to} h_{n-1}(A)$ for each $n \in \mathbb{Z}$ and $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$ that satisfy the following set of axioms. (For a cohomology theory h^*, one instead has contravariant functors h^n and natural transformations $h^n(A) \stackrel{\partial^*}{\to} h^{n+1}(X, A)$ satisfying obvious analogues of the axioms—I will concentrate on the covariant case, and usually only mention the contravariant case when there is something non-obvious about it.)
  - (HTP) **homotopy**: The functors h_n descend to the homotopy category as functors $\mathsf{hTop}^{\mathrm{rel}} \to \mathsf{Ab}$.
  - (LES) **exactness**: For each $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$ and the natural inclusion maps $i : A \hookrightarrow X$ and $j : X = (X, \varnothing) \hookrightarrow (X, A)$, the sequence
  $$\ldots \longrightarrow h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X, A) \xrightarrow{\partial_*} h_{n-1}(A) \longrightarrow \ldots$$
  is exact.
  - (EXC) **excision**: If $X = A \cup B$ and there exists a continuous function $u : X \to I$ that equals 0 on $X \backslash B$ and 1 on $X \backslash A$, then the homomorphisms $h_n(A, A \cap B) \to h_n(X, B)$ induced by the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ are isomorphisms.[42]

[42] This formulation of the excision axiom is slightly nonstandard, as one would usually only require the weaker hypothesis that X is covered by the interiors of A and B. On the other hand, the stronger hypothesis is equivalent to the weaker one in every application of excision you're ever likely to think about, e.g. whenever X is metrizable, or in any other situation where Urysohn's lemma holds. It will be convenient to have the stronger hypothesis when we talk about homotopy-theoretic constructions of cohomology theories.

 – (ADD) **additivity**: For any collection of spaces $\{X_\alpha \in \mathsf{Top}\}_{\alpha \in J}$ and their natural inclusions $i^\alpha : X_\alpha \hookrightarrow \coprod_{\beta \in J} X_\beta$, the induced maps

$$\bigoplus_\alpha i^\alpha_* : \bigoplus_\alpha h_n(X_\alpha) \longrightarrow h_n\left(\coprod_\alpha X_\alpha\right)$$

 are isomorphisms. For a cohomology theory, one instead requires the induced maps

$$\prod_\alpha (i^\alpha)^* : h^n\left(\coprod_\alpha X_\alpha\right) \longrightarrow \prod_\alpha h^n(X_\alpha)$$

 to be isomorphisms.

The homology/cohomology groups $h_n(*)$ or $h^n(*)$ of a one-point space $*$ are called the **coefficient groups** of the theory.

- We call h_* or h^* an **ordinary** (or sometimes "classical") homology or cohomology theory if in addition to the axioms above, it satisfies
  – (DIM) **dimension**: $h_n(*) = 0$ (or $h^n(*) = 0$) for all $n \neq 0$.

  We will not normally assume the dimension axiom unless it is explicitly needed, though of course it is satisfied by the standard examples that you have seen before, namely singular homology and cohomology. An interesting example of a homology theory without the dimension axiom is bordism, whose coefficient groups $h_n(*)$ give the answers to nontrivial questions about the classification of closed n-manifolds up to the bordism equivalence relation, e.g. in oriented bordism theory, $h_4(*)$ is nontrivial due to the existence of closed oriented 4-manifolds that are not boundaries of any compact oriented 5-manifolds (one can use Poincaré duality and the Euler characteristic to show that $\mathbb{CP}^2$ is such a manifold).
- Some easy exercises:
  (1) For finite disjoint unions, one can derive (ADD) from (LES) and (EXC). This is why the additivity axiom did not appear in the foundational book by Eilenberg and Steenrod [ES52]. It was added later by Milnor [Mil62] in order to extend the computation of homology for CW-complexes from compact to infinite complexes.
  (2) Combining (ADD) with (LES) and the 5-lemma produces a variant of the additivity axiom for disjoint unions of pairs $\coprod_\alpha (X_\alpha, A_\alpha) := (\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha)$.
- The **reduced** groups associated to a homology or cohomology theory are defined via the unique map $\epsilon : X \to *$ to the one-point space as

$$\widetilde{h}_n(X) := \ker\left(h_n(X) \xrightarrow{\epsilon_*} h_n(*)\right) \subset h_n(X), \qquad \widetilde{h}^n(X) := \operatorname{coker}\left(h^n(*) \xrightarrow{\epsilon^*} h^n(X)\right) = h^n(X)/\operatorname{im}(\epsilon^*),$$

  and for pairs (X, A) with $A \neq \varnothing$, $\widetilde{h}_n(X, A) := h_n(X, A)$ or $\widetilde{h}^n(X, A) := h^n(X, A)$.
  Easy exercise (diagram chasing):
  (1) The homomorphisms induced by continuous maps $X \xrightarrow{f} Y$ restrict to maps $\widetilde{h}_n(X) \xrightarrow{f_*} \widetilde{h}_n(Y)$, or in the case of cohomology, descend to maps $\widetilde{h}^n(Y) \xrightarrow{f^*} \widetilde{h}^n(X)$. This makes the reduced theories into functors on $\mathsf{Top}^{\mathrm{rel}}$ (which automatically descend to functors on $\mathsf{hTop}^{\mathrm{rel}}$ and are therefore invariants of homotopy type).
  (2) The connecting homomorphisms $\partial_* : h_n(X, A) \to h_{n-1}(A)$ always have image in $\widetilde{h}_{n-1}(A)$, or in the case of cohomology, the maps $\partial^* : h^n(A) \to h^{n+1}(X, A)$ descend to well-defined maps $\widetilde{h}^n(A) \to h^{n+1}(X, A)$. It follows that the long exact sequences of pairs (X, A) also define exact sequences of the corresponding reduced groups.

- Any choice of map $i : * \hookrightarrow X$ defines a right-inverse of $\epsilon : X \to *$ and thus defines splittings of the short exact sequences
$$0 \to \widetilde{h}_n(X) \hookrightarrow h_n(X) \xrightarrow{\epsilon_*} h_n(*) \to 0, \qquad 0 \to h^n(*) \xrightarrow{\epsilon^*} h^n(X) \xrightarrow{\mathrm{pr}} \widetilde{h}^n(X) \to 0,$$
making each unreduced group (non-naturally) isomorphic to the direct sum of its reduced counterpart with the corresponding coefficient group. Note that since $h_n(*)$ can be nontrivial for any n, one does *not* generally conclude from this that the reduced and unreduced theories match outside of degree 0, unless the dimension axiom holds.
- Useful observation (equally valid for homology or cohomology): while $h_n(*)$ can in principle be nontrivial for any $n \in \mathbb{Z}$, $\widetilde{h}_n(*)$ vanishes by construction, and therefore so does $\widetilde{h}_n(X)$ whenever X is a contractible space. It follows via the reduced long exact sequence of the pair $(X, *)$ that the inclusion $(X, \varnothing) \hookrightarrow (X, *)$ induces a natural isomorphism
$$\widetilde{h}_n(X) \cong h_n(X, *) \qquad \text{for any pointed space } X.$$
- Exercise: For all spaces X and all generalized homology theories, there are natural isomorphisms $\widetilde{h}_n(X) \cong \widetilde{h}_{n+1}(\Sigma_\circ X)$.
Remark: In ordinary homology, one also has $H_n(X) \cong H_{n+1}(\Sigma_\circ X)$ for all $n \geqslant 1$, but that relies on the knowledge that $H_n(*) = 0$ for $n > 0$, so it is not always valid in generalized homology theories.
- Theorem: For any free cofibration $A \hookrightarrow X$, the quotient projection $(X, A) \to (X/A, A/A) = (X/A, *)$ induces natural isomorphisms[43]
$$h_n(X, A) \xrightarrow{\cong} h_n(X/A, *) \cong \widetilde{h}_n(X/A),$$
and similarly for cohomology theories.
Proof: We use the same setup as in Theorem 2 of the previous lecture, but with the added advantage that excision is valid without any dimensional conditions. The benefit of assuming $A \hookrightarrow X$ is a cofibration is that the projection of the unreduced mapping cone $\mathrm{cone}(A \hookrightarrow X) = CA \cup X$ to its quotient by $CA \subset \mathrm{cone}(A \hookrightarrow X)$ is a homotopy equivalence. The desired isomorphism then follows from the diagram
$$\begin{tikzcd} h_n(X, A) \arrow[r, "\cong"] & h_n\big(\mathrm{cone}(A \hookrightarrow X), CA\big) \arrow[r] & h_n\big(\mathrm{cone}(A \hookrightarrow X)/CA, *\big) \arrow[r, equal] & h_n(X/A, *) \\ & \widetilde{h}_n\big(\mathrm{cone}(A \hookrightarrow X)\big) \arrow[u, "\cong"] \arrow[r, "\cong"] & \widetilde{h}_n\big(\mathrm{cone}(A \hookrightarrow X)/CA\big) \arrow[u, "\cong"] \arrow[r, equal] & \widetilde{h}_n(X/A) \arrow[u, "\cong"] \end{tikzcd},$$
in which all maps are induced by the obvious inclusions or quotient projections, the up arrows are all isomorphisms due to long exact sequences, the leftmost arrow is an isomorphism by excision, and the map in the bottom row is an isomorphism by the homotopy axiom.
- Corollary (pointed additivity theorem): For any collection of well-pointed spaces $\{(X_\alpha, x_\alpha) \in \mathsf{Top}_*\}_{\alpha \in J}$ with inclusion maps $i^\alpha : X_\alpha \hookrightarrow \bigvee_{\beta \in J} X_\beta$, the induced homomorphisms
$$\bigoplus_\alpha i^\alpha_* : \bigoplus_\alpha \widetilde{h}_n(X_\alpha) \longrightarrow \widetilde{h}_n\left(\bigvee_\alpha X_\alpha\right)$$
are isomorphisms.
Proof: Well-pointedness implies that the inclusion $\coprod_\alpha \{x_\alpha\} \hookrightarrow \coprod_\alpha X_\alpha$ is a free cofibration,

[43] You may recognize this as a variant of a result about the homology of "good pairs" that is proved and used extensively in [Hat02]. Good pairs serve in elementary algebraic topology as an ad-hoc way of deriving some benefits from the theory of cofibrations without needing to delve into the homotopy-theoretic subtleties of that theory.

so combining the theorem with the relative version of the additivity axiom gives a sequence of natural isomorphisms

$$\bigoplus_\alpha \widetilde{h}_n(X_\alpha) \cong \bigoplus_\alpha h_n(X_\alpha, \{x_\alpha\}) \cong h_n\left(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}\right) \cong \widetilde{h}_n\left(\coprod_\alpha X_\alpha \Big/ \coprod_\alpha \{x_\alpha\}\right),$$

and the quotient at the right is precisely the wedge sum $\bigvee_\alpha X_\alpha$.

- Exercise (via the 5-lemma again): The pointed additivity theorem also generalizes to arbitrary wedge sums $\bigvee_\alpha (X_\alpha, A_\alpha) := (\bigvee_\alpha X_\alpha, \bigvee_\alpha A_\alpha)$ of **pointed pairs** $(X_\alpha, A_\alpha) \in \mathsf{Top}_*^{\mathrm{rel}}$, i.e. pairs with base points such that both of the inclusions $* \hookrightarrow A_\alpha$ and $* \hookrightarrow X_\alpha$ are free cofibrations.
- Setup for Hurewicz maps: In the following, assume $(Q, B) \in \mathsf{Top}_*^{\mathrm{rel}}$ is a well-pointed pair and is also a cogroup object in $\mathsf{hTop}_*^{\mathrm{rel}}$, so that for any other pointed pair $(X, A) \in \mathsf{Top}_*^{\mathrm{rel}}$, the set
$$[(Q, B), (X, A)]_+$$
of pointed homotopy classes of maps of pairs $(Q, B) \to (X, A)$ has a natural group structure.
- Example 1: For $n \geqslant 1$ and $A = \{*\} \subset X$, choosing
$$(Q, B) := (S^n, *) := \left(I^n/\partial I^n, \partial I^n/\partial I^n\right)$$
makes $[(Q, B), (X, A)]_+$ the group $\pi_n(X)$. The cogroup object structure of $(S^n, *)$ comes from the natural identification of $I^n/\partial I^n$ with the reduced suspension $\Sigma^n S^0$.
- Example 2: For $n \geqslant 2$, identify the cube I^{n-1} with $I^{n-1} \times \{0\}$ in the boundary of I^n and define
$$(Q, B) := (\mathbb{D}^n, S^{n-1}) := \left(I^n \Big/ \overline{\partial I^n \backslash I^{n-1}}, I^{n-1}/\partial I^{n-1}\right),$$
with the collapsed boundary of I^{n-1} understood as the base point $* \in S^{n-1} \subset \mathbb{D}^n$. As we've previously observed, this pair really is homeomorphic (though not in a canonical way[44]) to the standard presentation of $(\mathbb{D}^n, \partial\mathbb{D}^n)$, and the group $[(Q, B), (X, A)]_+$ is then $\pi_n(X, A)$. Our usual definition of the group structure on $\pi_n(X, A)$ does not refer to a cogroup object structure on $(\mathbb{D}^n, S^{n-1})$, but one can equivalently define this group structure in terms of a cogroup morphism
$$(\mathbb{D}^n, S^{n-1}) \xrightarrow{\mu} (\mathbb{D}^n \vee \mathbb{D}^n, S^{n-1} \vee S^{n-1}),$$
defined as a quotient projection after identifying the wedge sum with a quotient of $(\mathbb{D}^n, S^n)$ in which the subset $\{1/2\} \times I^{n-1} \subset I^n$ is also collapsed to a point.
- Theorem: For any homology theory h_* and any chosen class $c \in h_n(Q, B)$, the map
$$[(Q, B), (X, A)]_+ \xrightarrow{\Phi} h_n(X, A) : [f] \mapsto f_* c$$
is a group homomorphism.
Remark: This theorem does *not* say that $h_n(Q, B) \to h_n(X, A) : c \mapsto f_* c$ is a group homomorphism, which is obvious.
Proof: The product on $[(Q, B), (X, A)]_+$ is defined in terms of a coproduct morphism $\mu : (Q, B) \to (Q \vee Q, B \vee B)$ by
$$[f] * [g] = [(f \vee g) \circ \mu] \in [(Q, B), (X, A)]_+$$

[44]The lack of a *canonical* homeomorphism between S^n or $\mathbb{D}^n$ and the corresponding quotients of I^n means that we are abusing notation by writing these quotients in this way, but for present purposes, it is a very convenient abuse of notation.

for any two pointed maps of pairs $f, g : (Q, B) \to (X, A)$. One convenient feature of the category $\mathsf{Top}_*^{\mathrm{rel}}$ is that its terminal objects are also initial objects, namely the one-point pairs $(*, *)$, so the only possible choice for the identity morphism in the cogroup structure is $\epsilon : (Q, B) \to (*, *)$. The identity axiom for the cogroup structure then means that the diagram

$$\begin{tikzcd}
 & (Q \vee *, B \vee *) = (Q, B) \\
(Q, B) \arrow[ur, "\mathrm{Id}"] \arrow[r, "\mu"] \arrow[dr, "\mathrm{Id}"'] \arrow[r, phantom, shift left=3, "\sim"] \arrow[r, phantom, shift right=3, "\sim"] & (Q \vee Q, B \vee B) \arrow[u, "\mathrm{Id} \vee \epsilon =: p^1"'] \arrow[d, "\epsilon \vee \mathrm{Id} =: p^2"] \\
 & (* \vee Q, * \vee B) = (Q, B)
\end{tikzcd}$$

commutes up to pointed homotopy of maps of pairs. This implies the commutativity of the triangle at the left in the following diagram,

$$\begin{tikzcd}
h_n(Q, B) \arrow[r, "\mu_*"] \arrow[dr, "\Delta"'] & h_n(Q \vee Q, B \vee B) \arrow[r, "(f \vee g)_*"] \arrow[d, "\cong"', "(p^1_*, p^2_*)"] & h_n(X, A) \\
 & h_n(Q, B) \oplus h_n(Q, B) \arrow[ur, "f_* \oplus g_*"'] &
\end{tikzcd},$$

in which $\Delta(c) := (c, c)$ is the diagonal map, and the vertical map (p^1_*, p^2_*) is the inverse of the isomorphism $i^1_* \vee i^2_*$ that appears in the relative version of the pointed additivity theorem.[45] Feeding in $c \in h_n(Q, B)$ at the left and following the top row of the diagram produces $\Phi([f] * [g])$, while following instead the diagonal arrows from left to right produces $(f_* \oplus g_*)(c, c) = f_* c + g_* c = \Phi([f]) + \Phi([g])$.

- Now assume $h_* := H_*$ is an *ordinary* homology theory with coefficient group $H_0(*) = \mathbb{Z}$, e.g. it could be singular homology with integer coefficients, though for the main result stated below, it could also be a different theory. By one of the main theorems proved in Topology 2, this gives us the freedom to replace $H_*(X)$ for any CW-complex X with its *cellular* homology $H_*^{\mathrm{CW}}(X; \mathbb{Z})$ with integer coefficients. The quotients of cubes I^n that we decided to call S^n and $\mathbb{D}^n$ above have natural cell decompositions with a single n-cell, and the n-chains defined via that cell give us canonical generators[46]
$$[S^n] \in H_n^{\mathrm{CW}}(S^n, *; \mathbb{Z}) \cong \widetilde{H}_n^{\mathrm{CW}}(S^n) \cong \widetilde{H}_n(S^n) \cong H_n(S^n, *) \cong \mathbb{Z}$$
and
$$[\mathbb{D}^n] \in H_n^{\mathrm{CW}}(\mathbb{D}^n, S^{n-1}; \mathbb{Z}) \cong H_n(\mathbb{D}^n, S^{n-1}) \cong \mathbb{Z},$$
such that the connecting homomorphism in the long exact sequence of $(\mathbb{D}^n, S^{n-1})$ satisfies
$$\partial_* [\mathbb{D}^n] = [S^{n-1}].$$
These give rise to the absolute and relative **Hurewicz maps**
$$\pi_n(X) \xrightarrow{\Phi} H_n(X, *) \cong \widetilde{H}_n(X) : [f] \mapsto f_*[S^n],$$
$$\pi_n(X, A) \xrightarrow{\Phi} H_n(X, A) : [f] \mapsto f_*[\mathbb{D}^n].$$

[45] I have no idea whether this is useful, but the fact that (p^1_*, p^2_*) is an isomorphism requires only the finite version of the additivity axiom, which follows already from exactness and excision.

[46] I'm slightly concerned that I may not have paid enough attention to orientation conventions to guarantee that the relation $\partial_*[\mathbb{D}^n] = [S^{n-1}]$ holds without some annoying signs. If not, then one should make minor modifications to my definitions to make sure this relation holds.

The theorem above implies that these are group homomorphisms for $n \geqslant 1$ in the absolute case and $n \geqslant 2$ in the relative case, and it is easy to verify from the definitions that they satisfy a naturality condition with respect to pointed maps of pairs $(X, A) \to (Y, B)$. The formula $\partial_*[\mathbb{D}^n] = [S^{n-1}]$ implies moreover that they give rise to maps between the long exact sequences of homotopy and reduced homology groups of any pointed pair (X, A).

- Hurewicz theorem (absolute version): Suppose $n \geqslant 2$, H_* is an ordinary homology theory with coefficient group $\mathbb{Z}$, and X is an $(n-1)$-connected space with the homotopy type of a CW-complex. Then $\widetilde{H}_k(X) = 0$ for all $k < n$, and the Hurewicz map $\Phi : \pi_n(X) \to H_n(X)$ is an isomorphism.
- Remark: We did not assume in this statement that H_* must specifically be singular homology, but if one adds that assumption, then the theorem is also true for all $(n-1)$-connected spaces X, without needing to assume the homotopy type of a CW-complex. This is due to a special property that the singular homology and cohomology theories have: *weak* homotopy equivalences $X \to Y$ also induce isomorphisms $H_*(X) \xrightarrow{\cong} H_*(Y)$ and $H^*(Y) \xrightarrow{\cong} H^*(X)$, thus by CW-approximation, the theorem becomes valid for all spaces as soon as it is known for CW-complexes. Morally, the reason for the singular theory to be an invariant of *weak* homotopy type is that singular n-simplices $\sigma : \Delta^n \to X$ in any space X can be interpreted as maps of n-dimensional CW-complexes to X, and are therefore subject to the compression lemma. (For a complete proof based on this idea, see [Hat02, Prop. 4.21].) Invariance under weak homotopy equivalence does not follow from the Eilenberg-Steenrod axioms, and e.g. the example of the Warsaw circle (see Exercise 8.1) shows that it does not hold for Čech cohomology, which is nevertheless a completely valid ordinary cohomology theory.
- Remark: We excluded $n = 1$ in the statement of the Hurewicz theorem because the statement is not true in that case without modification, but you already know what is true instead, namely that for X path-connected, $\Phi : \pi_1(X) \to H_1(X)$ is surjective and descends to the abelianization of $\pi_1(X)$ as an isomorphism.
- Proof of the theorem: Using CW-approximation and Whitehead's theorem, we can replace X with a homotopy equivalent space that is a CW-complex consisting of a single 0-cell e^0, some n-cells e^n_α, some $(n+1)$-cells e^{n+1}_β with pointed attaching maps $\varphi_\beta : S^n \to X^n$, and arbitrary additional cells of dimension greater than $n+1$ whose presence does not affect the computation of either $\pi_n(X)$ or $\widetilde{H}_k(X)$ for $k \leqslant n$. Using cellular homology, the lack of cells of dimension $k = 1, \dots, n-1$ implies immediately that $\widetilde{H}_k(X) = 0$ for $k < n$. The n-skeleton X^n is now a wedge of n-spheres, and we used the homotopy excision theorem to compute π_n of such a space in Theorem 1 of the previous lecture: the answer was the free abelian group on the set of n-cells, which is simply the cellular n-chain group
$$\pi_n(X^n) \cong C_n^{\mathrm{CW}}(X; \mathbb{Z}),$$
with a precise isomorphism $\pi_n(X^n) \to C_n^{\mathrm{CW}}(X; \mathbb{Z})$ that sends the homotopy class of the characteristic map $(\mathbb{D}^n, S^{n-1}) \to (X^n, *)$ of each n-cell e^n_α to the corresponding generator of $C_n^{\mathrm{CW}}(X; \mathbb{Z})$. Note that since there are no $(n-1)$-cells, all chains in $C_n^{\mathrm{CW}}(X; \mathbb{Z})$ are cycles, and the resulting quotient projection $C_n^{\mathrm{CW}}(X; \mathbb{Z}) \to H_n^{\mathrm{CW}}(X; \mathbb{Z}) = \widetilde{H}_n^{\mathrm{CW}}(X; \mathbb{Z})$ fits together with the isomorphism $\pi_n(X^n) \to C_n^{\mathrm{CW}}(X; \mathbb{Z})$ described above into a commutative diagram
$$\begin{tikzcd} \pi_n(X^n) \arrow[r] \arrow[d, "\cong"] & \pi_n(X) \arrow[d, "\Phi"] \\ C_n^{\mathrm{CW}}(X; \mathbb{Z}) \arrow[r] & \widetilde{H}_n^{\mathrm{CW}}(X; \mathbb{Z}) \end{tikzcd},$$

where the map in the top row is induced by the inclusion $X^n \to X$ and Φ is the Hurewicz map (after identifying $\widetilde{H}_n^{\mathrm{CW}}(X;\mathbb{Z})$ with $\widetilde{H}_n(X)$). Since (X, X^n) is an n-connected pair, the map $\pi_n(X^n) \to \pi_n(X)$ in this diagram is surjective, and its kernel is described in Exercise 9.6: it is the subgroup generated by the attaching maps $[\varphi_\beta] \in \pi_n(X^n)$ of all the $(n+1)$-cells e_β^{n+1}. The isomorphism $\pi_n(X^n) \cong C_n^{\mathrm{CW}}(X;\mathbb{Z})$ identifies this subgroup with the image of the cellular boundary map $\partial^{\mathrm{CW}} : C_{n+1}^{\mathrm{CW}}(X;\mathbb{Z}) \to C_n^{\mathrm{CW}}(X;\mathbb{Z})$, which is the kernel of the quotient projection $C_n^{\mathrm{CW}}(X;\mathbb{Z}) \to \widetilde{H}_n^{\mathrm{CW}}(X;\mathbb{Z})$, thus implying that Φ is an isomorphism.

Lecture 17 (20.06.2024): From Hurewicz to Eilenberg-MacLane.

- Hurewicz theorem (relative version): Suppose $n \geqslant 2$, H_* is an ordinary homology theory with coefficient group $\mathbb{Z}$, $A \neq \varnothing$ is a simply connected space, and (X, A) is an $(n-1)$-connected pair of spaces with the homotopy type of a CW-pair. Then $H_k(X, A) = 0$ for all $k < n$, and the Hurewicz map $\Phi : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.
- Remark: As in the absolute version, CW-approximation allows us to lift the assumption about (X, A) being homotopy equivalent to a CW-pair if we use singular homology, which is (for slightly nontrivial reasons that are independent of the axioms) an invariant of weak homotopy type.
- Proof of relative Hurewicz: Assume without loss of generality that (X, A) is a CW-pair. Since (X, A) is $(n-1)$-connected and A is 1-connected, Theorem 2 from Lecture 15 implies that the quotient projection $(X, A) \to (X/A, *)$ is an $(n+1)$-equivalence, so in particular, the induced map $\pi_k(X, A) \to \pi_k(X/A)$ is an isomorphism for every $k \leqslant n$. The corresponding maps $H_k(X, A) \to \widetilde{H}_k(X/A)$ are likewise isomorphisms since $A \hookrightarrow X$ is a cofibration. Applying the absolute Hurewicz theorem to X/A, the result now follows from the naturality of the Hurewicz maps with respect to the quotient projection:

$$
\begin{tikzcd}
\pi_k(X, A) \arrow[r, "\cong"] \arrow[d] & \pi_k(X/A) \arrow[d, "\cong"] \\
H_k(X, A) \arrow[r, "\cong"] & \widetilde{H}_k(X/A)
\end{tikzcd}
\quad . \tag{10.1}
$$

- Remark: If A is path-connected but not simply connected, then the proof above fails because $(X, A) \to (X/A, *)$ is only an n-equivalence, so the map $\pi_n(X, A) \to \pi_n(X/A)$ is surjective but may not be injective. The diagram (10.1) then implies that the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is surjective and has the same kernel as $\pi_n(X, A) \to \pi_n(X/A)$. It is easy to see in fact that this kernel is sometimes nontrivial: in particular, it contains $[f] - [g]$ for any two maps of pairs $f, g : (\mathbb{D}^n, S^{n-1}) \to (X, A)$ that are *freely* but not necessarily pointed homotopic. By a mild extension of the discussion of free vs. pointed homotopy classes in Lecture 12, one can define a natural action of $\pi_1(A)$ on $\pi_n(X, A)$ that gives rise to a bijection
$$\pi_n(X, A)/\pi_1(A) \cong [(\mathbb{D}^n, S^{n-1}), (X, A)]_\circ.$$
It follows that the kernel of the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ contains the normal subgroup generated by all elements of the form $[f] - \gamma_\#[f]$ for $[f] \in \pi_n(X, A)$ and $[\gamma] \in \pi_1(A)$, and one can show (but we will not do so here) that the kernel is in fact precisely that subgroup. The Hurewicz map thus descends to an isomorphism
$$\pi_n'(X, A) \to H_n(X, A),$$

where $\pi'_n(X, A)$ denotes the quotient by the aforementioned subgroup: intuitively, this is the "largest" quotient of $\pi_n(X, A)$ in which freely homotopic maps of pairs are considered equivalent. We summarize:

- General Hurewicz theorem (see e.g. [Hat02, Theorem 4.37]): Suppose $n \geqslant 2$, H_* is an ordinary homology theory with coefficient group $\mathbb{Z}$, $A \neq \varnothing$ is a path-connected space, and (X, A) is an $(n-1)$-connected pair of spaces with the homotopy type of a CW-pair. Then $H_k(X, A) = 0$ for all $k < n$, and the Hurewicz map in degree n descends to an isomorphism $\Phi : \pi'_n(X, A) \to H_n(X, A)$.
- Remark: A minor modification to the statement above puts the absolute case $n = 1$ into this framework as well: the largest quotient $\pi'_1(X)$ of $\pi_1(X)$ in which freely homotopic loops are considered equivalent is the abelianization of $\pi_1(X)$.
- Homology Whitehead theorem: Assume X, Y are simply connected spaces that are homotopy equivalent to CW-complexes.[47] Then a map $f : X \to Y$ is a homotopy equivalence if and only if the induced maps $f_* : H_n(X) \to H_n(Y)$ are isomorphisms for all $n \geqslant 0$.
  Proof: By replacing Y with the mapping cylinder $Z(f)$, we can assume without loss of generality that $X \subset Y$ and f is the inclusion map, so f is a weak homotopy equivalence (and therefore also a homotopy equivalence by Whitehead's theorem) if and only if $\pi_n(Y, X)$ vanishes for every $n \geqslant 1$. Assuming $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for every n, the long exact sequence in homology gives $H_n(Y, X) = 0$ for every n, and (Y, X) is automatically 1-connected since X and Y are simply connected. We proceed by induction: if (Y, X) is $(n-1)$-connected for some $n \geqslant 2$, then Hurewicz implies $\pi_n(Y, X) \cong H_n(Y, X) = 0$, thus it is also n-connected.
- Remark: The theorem above admits various generalizations that relax the condition that X and Y are simply connected, though this condition must be handled with care, and the result is certainly false without some such assumption (cf. Exercise 10.2). One way to generalize is by assuming X and Y are **simple** spaces, which means that their fundamental groups are allowed to be nontrivial but they act trivially on the higher homotopy groups, the idea being that the quotient groups $\pi'_n(Y, X)$ appearing in the general Hurewicz theorem should then reduce to the usual $\pi_n(Y, X)$. The details of this generalization are not so straightforward, however, because the simplicity of X and Y for $X \subset Y$ does not immediately imply that $\pi_1(Y)$ also acts trivially on $\pi_n(Y, X)$. A proof of this version of the theorem is sketched e.g. in [Fra13, Prop. 6.6], but it uses a certain amount of technology that we haven't covered, such as Postnikov towers. In any case, the simply connected case is by far the most popular version of this theorem for applications, and if one really wants to remove assumptions on $\pi_1(X)$ and $\pi_1(Y)$, then the most natural language for it is homology with local coefficients.
- **Eilenberg-MacLane spaces**: For spaces X that satisfy $\pi_n(X) \neq 0$ for at most one value of $n \geqslant 0$, we will use the following notation:
  - Writing "$X = K(G, n)$" means X is a CW-complex (with a 0-cell as a base point) satisfying $\pi_n(X) \cong G$ and $\pi_k(X) = 0$ for all $k \neq n$.
  - Let us call X a **weak** $K(G, n)$ and write "$X = K^{\mathrm{w}}(G, n)$" if it satisfies the same condition on its homotopy groups as a $K(G, n)$, but without assuming that X is a CW-complex.

[47]Or if you want to use singular homology in particular, then you could instead just assume that X and Y are simply connected spaces that need not have anything to do with CW-complexes: the conclusion would then be that $f : X \to Y$ is a *weak* homotopy equivalence if the induced maps $H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$ are isomorphisms.

One finds minor differences in conventions in different books: not everyone requires a $K(G,n)$ to be a CW-complex, though most of the interesting applications add this assumption. The term "weak $K(G,n)$" is especially nonstandard (I made it up), but we'll find it useful. Note that $K(G,n)$ is not the name of a unique space, though we often abuse notation and pretend that it is. We will see in the next lecture that spaces with this name are at least unique up to homotopy equivalence.

- Theorem: Assume $n \geqslant 0$ is an integer and G is a pointed set which is also a group if $n \geqslant 1$ and abelian if $n \geqslant 2$. Then $K(G,n)$'s exist.
  Proof: For $n = 0$, just take $K(G,0)$ to be G with the discrete topology. For $n \geqslant 1$, G is assumed to be a group, and we start by choosing any generating subset $J \subset G$. The wedge sum $X^n := \bigvee_{\alpha \in J} S^n$ is then an n-dimensional CW-complex with $\pi_n(X^n)$ isomorphic to the free group on the set J if $n = 1$, or the free *abelian* group on J if $n \geqslant 2$. For $n = 1$, this follows from the Seifert-van Kampen theorem, and the $n \geqslant 2$ cases were the first application we carried out after proving the homotopy excision theorem. Since J generates G, the free (or free abelian) group on J comes with a surjective homomorphism $\rho : \pi_n(X^n) \to G$, and the next step is to attach enough $(n+1)$-cells to kill the kernel of that homomorphism, producing an $(n+1)$-dimensional CW-complex X^{n+1}, which is still $(n-1)$-connected and comes with a group isomorphism $\rho : \pi_n(X^{n+1}) \to G$. Next, attach enough $(n+2)$-cells to kill π_{n+1} completely; this does not change π_k for $k \leqslant n$, and produces an $(n+2)$-dimensional CW-complex that satisfies all of the previous conditions plus $\pi_{n+1}(X^{n+2}) = 0$. Now continue inductively by attaching higher-dimensional cells to kill all the groups π_k for $k > n$. (The result is typically an infinite-dimensional CW-complex if one does not get very lucky.)
- Example 1: S^1 is a $K(\mathbb{Z},1)$. (This is one of the lucky cases.)
- Example 2: It is instructive to think through the inductive procedure described above for constructing a $K(\mathbb{Z},2)$. Since $\mathbb{Z}$ requires only one generator, one can start with $X^2 := S^2$, having one 0-cell and one 2-cell. Now $\pi_2(S^2)$ is already the correct group, so there is no need to attach any 3-cells, but $\pi_3(S^2) \cong \mathbb{Z}$ is generated by the Hopf fibration $p : S^3 \to \mathbb{CP}^1 \cong S^2$, thus we can attach a single 4-cell along the map $p : S^3 \to \mathbb{CP}^1$ in order to kill it, producing $X^4 := S^2 \cup e^4$. You have probably seen an example before of a CW-complex with three cells having dimensions 0, 2 and 4: the most popular such example is $\mathbb{CP}^2$, and I will ask you to take it on faith for a moment that that is indeed what X^4 is. If you believe this, then the Hopf fibration $p : S^5 \to \mathbb{CP}^2$ with fibers S^1 shows that $\pi_4(\mathbb{CP}^2)$ is trivial, so we do not need any 5-cells, but the same fibration also proves $\pi_5(\mathbb{CP}^2) \cong \pi_5(S^5) \cong \mathbb{Z}$, so one must next attach a 6-cell to kill $\pi_5(\mathbb{CP}^2)$, producing a 6-dimensional CW-complex X^6 with one cell in each even dimension. Maybe you can now guess where this is leading...
- Theorem: $\mathbb{CP}^\infty = e^0 \cup e^2 \cup e^4 \cup \ldots = \operatorname{colim}_{n\to\infty} \mathbb{CP}^n$ is a $K(\mathbb{Z},2)$.
  Proof: Start with the corresponding infinite-dimensional colimit of spheres, i.e. the colimit of the sequence

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \ldots \hookrightarrow \operatorname{colim}_{n\to\infty} S^n =: S^\infty,$$

  where each S^n is the unit sphere in $\mathbb{R}^{n+1}$ and its inclusion into S^{n+1} is defined via the inclusion $\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \times \{0\} \hookrightarrow \mathbb{R}^{n+2}$. Each sphere thus appears inside the next one as an "equator", and it is easy to give every S^n a cell decomposition with two cells in each dimension so that these inclusions become cellular maps. We observe that since S^∞ has the same $(n+1)$-skeleton as S^{n+1} for any given $n \geqslant 0$, we have

$$\pi_n(S^\infty) \cong \pi_n(S^{n+1}) = 0,$$

i.e. S^∞ is weakly contractible, and by Whitehead's theorem therefore also contractible (though we will not need to know this below). Restricting attention to the odd dimensions, one also has Hopf fibrations $p : S^{2n+1} \to \mathbb{CP}^n$ which are compatible with a similar sequence of inclusions

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \ldots \hookrightarrow \operatorname{colim}_{n\to\infty} \mathbb{CP}^n =: \mathbb{CP}^\infty,$$

in which each $\mathbb{CP}^n$ is a quotient of $\mathbb{C}^{n+1}$ and the inclusion into $\mathbb{CP}^{n+1}$ is determined by $\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \{0\} \hookrightarrow \mathbb{C}^{n+2}$. Concretely, the sequence of Hopf fibrations for different values of n fit together into a diagram

$$\begin{tikzcd}
S^3 \arrow[r, hook] \arrow[d, "p"] & S^5 \arrow[r, hook] \arrow[d, "p"] & \ldots \arrow[r] & S^{2n+1} \arrow[r, hook] \arrow[d, "p"] & \ldots \\
\mathbb{CP}^1 \arrow[r, hook] & \mathbb{CP}^2 \arrow[r, hook] & \ldots \arrow[r] & \mathbb{CP}^n \arrow[r, hook] & \ldots
\end{tikzcd},$$

and by restriction to finite-dimensional subcomplexes, this uniquely determines a map

$$p : S^\infty \to \mathbb{CP}^\infty$$

whose fibers are also homeomorphic to S^1. The latter follows immediately from the fact that it is true in finite dimeneions, because every point in $\mathbb{CP}^\infty$ belongs to $\mathbb{CP}^n \subset \mathbb{CP}^\infty$ for some n sufficiently large, and the preimage of that subset under $p : S^\infty \to \mathbb{CP}^\infty$ is precisely $S^{2n+1} \subset S^\infty$. It is similarly easy to show that $p : S^\infty \to \mathbb{CP}^\infty$ is a Serre fibration (though I would not attempt to prove that it's a Hurewicz fibration, much less a fiber bundle!): to establish the homotopy lifting property with respect to disks $\mathbb{D}^k$, one can use the knowledge that every map $\mathbb{D}^k \times I \to \mathbb{CP}^\infty$ has image contained in a finite subcomplex, which in this case means $\mathbb{CP}^n$ for some $n < \infty$, and the condition then follows from the fact that $p : S^{2n+1} \to \mathbb{CP}^n$ is a fibration. The statement that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ now follows from the exact sequence

$$0 = \pi_n(S^\infty) \xrightarrow{p_*} \pi_n(\mathbb{CP}^\infty) \xrightarrow{\partial_*} \pi_{n-1}(S^1) \longrightarrow \pi_{n-1}(S^\infty) = 0.$$

- Theorem: For any CW-complex X, any integer $n \geqslant 0$ and any abelian group G, there exists a natural bijection

$$[X, K^{\mathrm{w}}(G, n)]_\circ \cong H^n_{\mathrm{CW}}(X; G)$$

identifying homotopy classes of maps from X to any weak $K(G, n)$ with cellular cohomology classes of degree n in X with coefficients in G.
Proof sketch: Let's abbreviate $Y := K^{\mathrm{w}}(G, n)$, and assume $n \geqslant 1$ (the case $n = 0$ is easy).
  (1) Every map $f : X \to Y$ is homotopic to a pointed map that is constant on X^{n-1}.
  Proof: Induction over the skeleta X^k for $k = 0, \ldots, n-1$, using the fact that $\pi_k(Y) = 0$.
  (2) Two maps $f, g : X \to Y$ are homotopic if and only if their restrictions fo X^n are homotopic.
  Proof: Inductively, assume $f_+ \underset{h}{\sim} f_-$ on X^k, for some $k \geqslant n$. Using the fact that $X^k \hookrightarrow X$ is a cofibration, we can then adjust f_- by a global homotopy so that $f_+ = f_-$ on X^k. Now for any $(k+1)$-cell $e^{k+1}_\alpha \subset X^{k+1}$ with characteristic map $\Phi_\alpha : (\mathbb{D}^{k+1}, S^k) \to (X^{k+1}, X^k)$, the maps $f_\pm \circ \Phi_\alpha$ match on S^k, so we can define a map $g : S^{k+1} \to Y$ by decomposing S^{k+1} into two disks $\mathbb{D}^{k+1}_+ \cup \mathbb{D}^{k+1}_-$ with common boundary $S^k = \mathbb{D}^{k+1}_+ \cap \mathbb{D}^{k+1}_- = \partial \mathbb{D}^{k+1}_\pm \subset S^{k+1}$ and setting

$$g := (f_+ \circ \Phi_\alpha) \cup (f_- \circ \Phi_\alpha) : \mathbb{D}^{k+1}_+ \cup_{S^k} \mathbb{D}^{k+1}_- = S^{k+1} \to Y.$$

Since $\pi_{k+1}(Y) = 0$, g can be extended to a map $\mathbb{D}^{k+2} \to Y$, and this extension gives rise to a homotopy $f_+ \circ \Phi_\alpha \rightsquigarrow f_- \circ \Phi_\alpha$ rel S^k. Doing this for all $(k+1)$-cells produces a homotopy from f_+ to f_- on X^{k+1}, and continuing inductively, one eventually constructs a global homotopy from f_+ to f_-.

(3) We associate an **obstruction cocycle**

$$\varphi_f \in C^n_{\mathrm{CW}}(X; G) = \mathrm{Hom}(C_n^{\mathrm{CW}}(X; \mathbb{Z}), G) = \{\text{functions } \{n\text{-cells of } X\} \to G\}$$

to each map $f : X \to Y$ such that $f|_{X^{n-1}} \equiv *$. For each n-cell $e^n_\alpha \in X$ with characteristic map $\Phi_\alpha : (\mathbb{D}^n, S^{n-1}) \to (X^n, X^{n-1})$, the map $f \circ \Phi_\alpha : \mathbb{D}^n \to Y$ sends S^{n-1} to the base point, so it represents a class in $\pi_n(Y) = G$, and we can set

$$\varphi_f(e^n_\alpha) := [f \circ \Phi_\alpha] \in \pi_n(Y) = G.$$

For simplicity, let's assume for the moment that $n \geqslant 2$, as we can then also characterize φ_f as follows. Since $f|_{X^{n-1}}$ is constant, it descends to a pointed map from the $(n-1)$-connected quotient space X/X^{n-1} to Y, and since X^n/X^{n-1} is a wedge of n-spheres containing one for each n-cell, we can identify $\pi_n(X^n/X^{n-1})$ with $C_n^{\mathrm{CW}}(X; \mathbb{Z})$ and define

$$C_n^{\mathrm{CW}}(X; \mathbb{Z}) \cong \pi_n(X^n/X^{n-1}) \xrightarrow{\varphi_f := f_*} \pi_n(Y) = G.$$

Observe that $f_* : \pi_n(X^n/X^{n-1}) \to \pi_n(Y)$ also factors through the map $\pi_n(X^n/X^{n-1}) \to \pi_n(X/X^{n-1})$ induced by inclusion, and we can now appeal to a fact that was also used in our proof of the Hurewicz theorem: the kernel of that map is precisely the image of the cellular boundary map $\partial^{\mathrm{CW}} : C^{\mathrm{CW}}_{n+1}(X; \mathbb{Z}) \to C_n^{\mathrm{CW}}(X; \mathbb{Z})$. Indeed, we have a diagram

$$\begin{tikzcd}
\pi_{n+1}(X^{n+1}/X^{n-1}, X^n/X^{n-1}) \arrow[r, "\partial_*"] \arrow[d, "\cong"] & \pi_n(X^n/X^{n-1}) \arrow[r] & \pi_n(X^{n+1}/X^{n-1}) \arrow[d, "\cong"] \\
\pi_{n+1}(X^{n+1}/X^n) & C_n^{\mathrm{CW}}(X; \mathbb{Z}) \arrow[u, "\cong"] \arrow[dr, "\varphi_f"] & \pi_n(X/X^{n-1}) \arrow[d, "f_*"] \\
C_{n+1}^{\mathrm{CW}}(X; \mathbb{Z}) \arrow[u, "\cong"] \arrow[ur, "\partial^{\mathrm{CW}}"] & & \pi_n(Y) = G
\end{tikzcd},$$

where the top row comes from the long exact sequence of a pair, and all unlabelled maps are induced by obvious inclusions or quotient projections. The vertical map in the upper right is bijective because $(X/X^{n-1}, X^{n+1}/X^{n-1})$ is an $(n+1)$-connected pair, while the one in the upper left is bijective due to Theorem 2 in Lecture 15 because the pair $(X^{n+1}/X^{n-1}, X^n/X^{n-1})$ is n-connected, X^n/X^{n-1} is $(n-1)$-connected, and $n + 1 < n + (n-1) + 1 = 2n$. The isomorphisms with cellular chain groups can be described explicitly in terms of the canonical isomorphisms $\pi_n(\mathbb{D}^n, S^{n-1}) = \pi_n(\mathbb{D}^{n+1}, S^n) = \mathbb{Z}$ and the maps $\Phi_\alpha : \mathbb{D}^n/S^{n-1} \to X^n/X^{n-1}$ and $\Phi_\beta : \mathbb{D}^{n+1}/S^n \to X^{n+1}/X^n$ induced by characteristic maps of n-cells e^n_α and $(n+1)$-cells e^{n+1}_β respectively: these give us

$$\bigoplus_\alpha (\Phi_\alpha)_* : \bigoplus_{e^n_\alpha \subset X^n} \pi_n(\mathbb{D}^n/S^{n-1}) = C_n^{\mathrm{CW}}(X; \mathbb{Z}) \xrightarrow{\cong} \pi_n(X^n/X^{n-1}),$$

$$\bigoplus_\beta (\Phi_\beta)_* : \bigoplus_{e^{n+1}_\beta \subset X^{n+1}} \pi_{n+1}(\mathbb{D}^{n+1}/S^n) = C_{n+1}^{\mathrm{CW}}(X; \mathbb{Z}) \xrightarrow{\cong} \pi_{n+1}(X^{n+1}/X^n).$$

The fact that the map $C^{\mathrm{CW}}_{n+1}(X; \mathbb{Z}) \to C_n^{\mathrm{CW}}(X; \mathbb{Z})$ in the diagram is ∂^{CW} can then be deduced by computing its matrix elements with respect to each choice of generators

e_β^{n+1} and e_α^n: composing the connecting morphism ∂_* with a characteristic map Φ_β produces a class represented by the corresponding attaching map $\varphi_\beta : S^n \to X^n$, whose coefficient for the generator e_α^n is then computed as the degree of its composition with the quotient projection $X^n \to X^n \backslash (X^n \backslash e_\alpha^n) = \mathbb{D}^n / S^{n-1}$, i.e. the incidence number of e_β^{n+1} and e_α^n. With this understood, the main message of the diagram is that

$$\varphi_f \circ \partial^{\mathrm{CW}} = 0,$$

or in other words, $\delta\varphi_f = 0 \in C^{n+1}_{\mathrm{CW}}(X; G)$, so φ_f is a cellular cocycle. Some modifications to this discussion are needed when $n = 1$: one issue is that the groups in the top row and $\pi_n(X/X^{n-1})$ can no longer be assumed abelian, but as luck would have it, the fact that $\pi_1(Y) = G$ is assumed abelian guarantees that $f_* : \pi_n(X/X^{n+1}) \to G$ descends to the abelianization, allowing us to replace all of the nonabelian groups with their abelianizations and still have a well-defined diagram. The other issue is that the quotient projection inducing the vertical map in the upper left corner is only a 2-equivalence when $n = 1$, so that the map induced on π_2 may not be bijective, but it is still surjective. This is enough to reach the same conclusion: $\varphi_f \circ \partial^{\mathrm{CW}}$ still vanishes, and φ_f is therefore a cocycle.

(4) Claim: Two maps $f, g : X \to Y$ that send X^{n-1} to the base point are (freely) homotopic if and only if the cocycles φ_f and φ_g represent the same class in $H^n_{\mathrm{CW}}(X; G)$.
Proof: If there is a homotopy $H : X \times I \to Y$ from f to g, then the same argument as in the first step allows us to assume without loss of generality that H is constant on $(X \times I)^{n-1} = X^{n-1} \times \partial I \cup X^{n-2} \times I$. It follows that for each $(n-1)$-cell $e_\beta^{n-1} \subset X$ with characteristic map $\Phi_\beta : (\mathbb{D}^{n-1}, S^{n-2}) \to (X^{n-1}, X^{n-2})$, the composition

$$(\mathbb{D}^{n-1} \times I, \partial(\mathbb{D}^{n-1} \times I)) \xrightarrow{\Phi_\beta \times \mathrm{Id}} (X^{n-1} \times I, X^{n-1} \times \partial I \cup X^{n-2} \times I) \xrightarrow{H} (Y, *)$$

represents a class

$$\psi_H(e_\beta^{n-1}) := [H \circ (\Phi_\beta \times \mathrm{Id})] \in \pi_n(Y) = G,$$

thus defining a cochain $\psi_H \in C^{n-1}_{\mathrm{CW}}(X; G)$. For each n-cell $e_\alpha^n \subset X$, the map $H \circ (\Phi_\alpha \times \mathrm{Id}) : \mathbb{D}^n \times I \to Y$ then implies a relation between elements of $\pi_n(Y) = G$ obtained by restricting this map to the various parts of $\partial(\mathbb{D}^n \times I)$, namely

$$\varphi_f(e_\alpha^n) - \varphi_g(e_\alpha^n) = \pm\psi_H(\partial e_\alpha^n),$$

thus $\varphi_f - \varphi_g = \pm\delta\psi_H$.
Conversely, for any given cochain $\psi \in C^{n-1}_{\mathrm{CW}}(X; G)$, one can define a homotopy $H : X^{n-1} \times I \to Y$ from f to g over the $(n-1)$-skeleton such that H is constant on $X^{n-2} \times I$ and $[H \circ (\Phi_\beta \times \mathrm{Id})] = \psi(e_\beta^{n-1})$ for each $e_\beta^{n-1} \subset X$; this is trivial since f and g are constant on the $(n-1)$-skeleton. The nontrivial part is to determine whether this homotopy can be extended to the n-skeleton of X: that is possible if and only if the resulting maps $\partial(\mathbb{D}^n \times I) \to Y$ associated to each $e_\alpha^n \subset X$ represent the trivial element of $\pi_n(Y) = G$, and this is equivalent to the condition $\varphi_f - \varphi_g = \pm\delta\psi$. Having extended the homotopy to the n-skeleton, it then extends globally due to step 2.

Suggested reading. Most of what we said about generalized homology theories this week follows easily from arguments that you probably already saw in Topology 2, but they are also covered e.g. in [tD08, Chapter 10], including the isomorphism $h_*(X, A) \cong \widetilde{h}_*(X/A)$ for cofibrations $A \hookrightarrow X$ (Proposition 10.4.5 in tom Dieck), which you probably hadn't seen before in precisely that form.

Most books prove the Hurewicz theorem specifically for singular homology, which I find slightly frustrating, because it doesn't strike me as the most natural context for it. From my perspective, Hurewicz is most naturally a theorem about the relationship between cellular homology and homotopy groups on CW-complexes—the oft quoted version for arbitrary spaces then follows by combining it with two other theorems that are of independent interest: CW approximation, and the theorem that singular homology is an invariant of weak homotopy type. (The latter is a theorem that we haven't had time for in this course, but e.g. the proof found in [Hat02, Prop. 4.21] is not difficult to follow, and a more formal treatment is given in [tD08, §9.5].) With that one caveat, the proof of Hurewicz in [Hat02, §4.2] is pretty close to what we did in lecture, and if you also want to understand the relative case when $\pi_1(A) \neq 0$, Hatcher will probably give you the shortest path from here to there. (Another proof of the general theorem can be found in [tD08, §20.1], and for $\pi_1(A) = 0$ it's also pretty similar to what we did, but tom Dieck's inductive argument for the case $\pi_1(A) \neq 0$ really *only* seems to make sense for singular homology.)

Our proof that the Hurewicz map is a homomorphism worked in a much more general context than can be found in most books, but the argument in the books is still essentially the same, e.g. [Hat02, Prop. 4.36] phrases it in terms of the explicitly-defined group structure of $\pi_n(X, A)$, but what Hatcher uses is really just the axiomatic properties of the underlying cogroup object structure of $(\mathbb{D}^n, S^{n-1})$ in $\mathsf{hTop}_*^{\mathrm{rel}}$. Replacing $(\mathbb{D}^n, S^{n-1})$ with an arbitrary well-pointed pair (Q, B) that is a cogroup object in $\mathsf{hTop}_*^{\mathrm{rel}}$ makes no substantive difference to the argument.

For proving the bijection $[X, K(G, n)] \cong H^n_{\mathrm{CW}}(X; G)$, several books (e.g. [Hat02, tD08]) take a less geometric and more axiomatic approach, which is different from what we've done so far, though it is on our agenda for next week.[48] A concise sketch of the more geometric approach appears in [May99, §18.5], and [DK01, §7.7] gives full details, though only after introducing some more general notions from obstruction theory that we have not yet discussed.

Exercises (for the Übung on 4.07.2024). Note that there will be no Übung on 27.06.2024, because it is a *dies academicus*. I may consider adding written solutions for some of these exercises if we end up with another backlog.

Exercise 10.1. Show that if X is a $K(G, 1)$ for some group G, then for each $n \geqslant 2$, π_n of its n-skeleton is a free abelian group.
Hint: Replace X with its universal cover, which also has a cell decomposition and has the same higher homotopy groups as X.

Exercise 10.2. A space is called **acyclic** if its reduced singular homology (with integer coefficients) vanishes. This is certainly true for contractible spaces, though acyclic spaces are not always contractible, e.g. [Hat02, Example 2.38] describes a compact 2-dimensional cell complex that has a nontrivial fundamental group but vanishing cellular homology. Assume in the following that X and Y are homotopy equivalent to CW-complexes.

(a) Show that if X is acyclic, then it is contractible if and only if it is simply connected.
(b) Show that the mapping cone of the unique map from X to a one-point space is contractible if and only if X is acyclic.
Remark: We showed in Exercise 2.3 that every homotopy equivalence has a contractible mapping cone. In light of the present exercise, the existence of acyclic but non-contractible spaces shows that the converse is false.
(c) Show that if X and Y are both simply connected, then every map $f : X \to Y$ with a contractible mapping cone is a homotopy equivalence.

[48] The following passage on p. 394 of [Hat02] boggles my mind: "It is possible to give a direct proof of the theorem, constructing maps and homotopies cell by cell. This provides geometric insight into why the result is true, but unfortunately the technical details of this proof are rather tedious. So we shall take a different approach..."

Exercise 10.3. Show that every closed and connected surface Σ with infinite fundamental group has a contractible universal cover, and is therefore a $K(G,1)$ for $G := \pi_1(\Sigma)$. Spaces X with $\pi_n(X) = 0$ for every $n > 1$ are also called **aspherical**.
Remark: If you remember the classification of surfaces, you may notice that this result applies to every closed and connected surface with only two exceptions, namely S^2 and $\mathbb{RP}^2$.

Exercise 10.4. For h_* a generalized homology theory with coefficient groups $G_n := h_n(*)$, compute $h_*(S^n)$.
Remark: As you might suspect, the computation of any generalized homology theory on a CW-complex X is determined by its coefficient groups G_n, as with ordinary homology, though the full story is more complicated due to the possibility of $h_k(S^n)$ and $h_k(\mathbb{D}^n, S^{n-1})$ being nontrivial in arbitrary degrees. In a nutshell, the collection of all the cellular homologies $H_*^{\mathrm{CW}}(X; G_n)$ for $n \in \mathbb{Z}$ determine $h_*(X)$ via the Atiyah-Hirzebruch spectral sequence.

Exercise 10.5. Assume (X, A) is a CW-pair, with A assumed path-connected but not simply connected, and assume also that all cells in $X \backslash A$ have dimension at least $n \geqslant 3$.[49] The following is one step in a proof of the general Hurewicz theorem, based on the same ideas as our proof in the absolute case. Show that $\pi_n(X^n \cup A, A)$ can be identified with the free abelian group on the set of n-cells in the universal cover of X (with its natural cell decomposition) whose projections are not in A. Can you also explicitly describe the action of $\pi_1(A)$ on $\pi_n(X^n \cup A, A)$ from this perspective?
Comment: Up to minor details, this is Lemma 4.38 in [Hat02], though I highly recommend trying to work it out yourself before reading what Hatcher says. The case $n = 2$ is more complicated, for reasons that you will notice when you try to answer the following quesiton: does the universal cover of A naturally live inside the universal cover of X?

Exercise 10.6. Show that for any CW-complex X of finite dimension $n \geqslant 1$, there is a natural bijection between the set of unpointed homotopy classes $[X, S^n]$ and $H^n(X; \mathbb{Z})$. Taking for granted that smooth manifolds are triangulable, deduce from this the **Hopf degree theorem**: For every closed, connected and oriented smooth n-manifold M, two maps $f, g : M \to S^n$ are homotopic if and only if $\deg(f) = \deg(g) \in \mathbb{Z}$.
Hint: S^n is not always a $K(\mathbb{Z}, n)$, but it has enough in common with a $K(\mathbb{Z}, n)$ for present purposes.
Remark: For a completely different proof of the Hopf degree theorem, working within the smooth category, see [Mil97].[50]

11. Week 11

Lecture 18 (24.06.2024): (Co-)homology on the pointed category.

- Clarification: When we say "$X = K(G, n)$" (or "$X = K^{\mathrm{w}}(G, n)$"), we mean that X is a pointed CW-complex (or in the weak case, just a pointed space) with $\pi_k(X) = 0$ for all $k \neq n$, and X is also equipped with the *additional data* of a choice of isomorphism $\pi_n(X) \xrightarrow{\cong} G$. This extra data was implicitly in the background of everything we said about $K^{\mathrm{w}}(G, n)$ in the previous lecture, e.g. the construction of cell complexes with the correct homotopy groups included a construction of such an isomorphism, and the correspondence between $[X, K^{\mathrm{w}}(G, n)]$ and $H^n_{\mathrm{CW}}(X; G)$ depends on having such an isomorphism at hand

[49] I have modified the statement of this exercise: The original version only assumed the weaker hypothesis that (X, A) is an $(n-1)$-connected CW-pair, but I realized after discussing this in the Übung that the statement may become false if $X \backslash A$ is allowed to contain $(n-1)$-cells. Up to homotopy equivalence, an $(n-1)$-connected CW-pair can of course be replaced by one that has the stronger property stated here, and that fact is useful in the proof of the general Hurewicz theorem.

[50] It is a law of nature that I will find an excuse to recommend that book at some point in every sufficiently advanced class that I teach...I suppose Functional Analysis may have been an exception.

so that the cellular cohomology of X with coefficients in $\pi_n(K^{\mathrm{w}}(G,n))$ can be identified with $H^n_{\mathrm{CW}}(X;G)$. A $K(G,n)$ with this extra structure is called a *polarized* $K(G,n)$ in [tD08]. I will not use this term, but we will always assume that the extra structure has been chosen, so that $\pi_n(K(G,n))$ and G may be treated as the same group whenever convenient.

- Recall: In the previous lecture, we constructed for every CW-complex X, every $n \geqslant 0$ and every abelian group G a bijective map[51]

$$[X, K^{\mathrm{w}}(G,n)] \xrightarrow{\cong} H^n_{\mathrm{CW}}(X;G) : [f] \mapsto [\varphi_f],$$

where the homotopy classes on the left hand side are unpointed, the representative $f : X \to K^{\mathrm{w}}(G,n)$ is chosen so that it maps the $(n-1)$-skeleton of X to the base point of $K^{\mathrm{w}}(G,n)$, and the obstruction cocycle $\varphi_f : C_n^{\mathrm{CW}}(X;\mathbb{Z}) \to G$ is defined on each n-cell $e^n_\alpha \subset X$ with attaching map $\Phi_\alpha : (\mathbb{D}^n, S^{n-1}) \to (X, X^{n-1})$ as

$$\varphi_f(e^n_\alpha) := [f \circ \Phi_\alpha] \in \pi_n(K^{\mathrm{w}}(G,n)) = G.$$

One can check from the definition that this bijection is natural with respect to X, so for any map $\psi : X \to Y$ between two CW-complexes—we can assume via cellular approximation that it is a cellular map without loss of generality and thus preserves $(n-1)$-skeleta—the bijections for X and Y fit into a diagram

$$\begin{tikzcd} {[Y, K^{\mathrm{w}}(G,n)]} \arrow[r, "\cong"] \arrow[d, "\psi^*"] & H^n_{\mathrm{CW}}(Y;G) \arrow[d, "\psi^*"] \\ {[X, K^{\mathrm{w}}(G,n)]} \arrow[r, "\cong"] & H^n_{\mathrm{CW}}(X;G) \end{tikzcd} .$$

In other words, if we ignore the fact that $H^n_{\mathrm{CW}}(X;G)$ is an abelian group and just regard it as a set, then our bijections define a natural isomorphism between the contravariant functors $\mathsf{CW} \to \mathsf{Set}$ defined by $[\cdot, K^{\mathrm{w}}(G,n)]$ and $H^n_{\mathrm{CW}}(\cdot;G)$, where $\mathsf{CW} \subset \mathsf{Top}$ denotes the subcategory whose objects are CW-complexes, with continuous maps as morphisms. Note that both functors also descend to functors $\mathsf{hCW} \to \mathsf{Set}$, where the homotopy category of CW-complexes hCW is defined as usual by replacing continuous maps with their homotopy classes.[52]

- Examples: The two explicit examples of Eilenberg-MacLane spaces we saw in the previous lecture now give natural bijections

$$H^1(X;\mathbb{Z}) \cong [X, S^1], \qquad H^2(X;\mathbb{Z}) \cong [X, \mathbb{CP}^\infty]$$

for every CW-complex X, and we observe that whenever X is compact, a map $X \to \mathbb{CP}^\infty$ is actually just a map $X \to \mathbb{CP}^n$ for some $n \gg 0$ sufficiently large. For any weak $K(G,n)$, we also have

$$[X, \Omega K^{\mathrm{w}}(G,n)] \cong H^{n-1}_{\mathrm{CW}}(X;G),$$

[51]It only just now occurred to me that we omitted the step of proving that the map $[X, K^{\mathrm{w}}(G,n)] \to H^n_{\mathrm{CW}}(X;G)$ is surjective, but this follows from similar arguments. In fact, given any cocycle $\varphi \in C^n_{\mathrm{CW}}(X;G)$, one can construct a map $f : X \to K^{\mathrm{w}}(G,n)$ with $f|_{X^{n-1}} \equiv *$ whose obstruction cocycle φ_f is φ. Proof: First extend the constant map $X^{n-1} \to K^{\mathrm{w}}(G,n)$ to each n-cell $e^n_\alpha \subset X$ so that $[f \circ \Phi_\alpha] = \varphi(e^n_\alpha) \in \pi_n(K^{\mathrm{w}}(G,n)) = G$. The condition $\delta\varphi(e^{n+1}_\beta) = \pm\varphi(\partial^{\mathrm{CW}} e^{n+1}_\beta) = 0$ for each $(n+1)$-cell $e^{n+1}_\beta \subset X$ then implies that f can also be extended to the $(n+1)$-skeleton, because for the attaching mapping $\varphi_\beta : S^n \to X^n$ of e^{n+1}_β, $f \circ \varphi_\beta : S^n \to K^{\mathrm{w}}(G,n)$ is nullhomotopic. Once this is done, f can be extended to all higher-dimensional skeleta because $\pi_k(K^{\mathrm{w}}(G,n)) = 0$ for $k > n$.

[52]A sensible alternative definition for the category CW would be to take *cellular* maps as morphisms, but the cellular approximation theorem makes this unnecessary: by our definition, $H^*_{\mathrm{CW}}(\cdot;G)$ is a well-defined functor on both CW and hCW because every morphism in CW is homotopic to a cellular map, and two cellular maps are homotopic if and only if they are cellularly homotopic.

because by adjunction,

$$\pi_k(\Omega K^{\mathrm{w}}(G,n)) = [\Sigma_+^k S^0, \Omega K^{\mathrm{w}}(G,n)]_+ = [\Sigma_+^{k+1} S^0, K^{\mathrm{w}}(G,n)]_+ = \pi_{k+1}(K^{\mathrm{w}}(G,n)),$$

proving that $\Omega K^{\mathrm{w}}(G,n)$ is always a weak $K(G,n-1)$. (The fact that $\Omega K(G,n)$ is not a CW-complex in any natural way was our main motivation to introduce the notion of *weak* $K(G,n)$'s.)

- Principle: The naturality of the bijection $[X, K^{\mathrm{w}}(G,n)] \xrightarrow{\cong} H^n_{\mathrm{CW}}(X;G)$ implies correspondences between structures defined on cellular cohomology and structures defined on the $K(G,n)$'s "up to homotopy". A few examples of this are illustrated below.
- Example 1: Since a $K(G,n)$ is in itself a CW-complex, we have

$$[K(G,n), K(G,n)] \cong H^n_{\mathrm{CW}}(K(G,n);G),$$

and if we assume the two spaces labelled $K(G,n)$ on the left hand side are the same one, then this determines a distinguished class

$$\iota_n \in H^n_{\mathrm{CW}}(K(G,n);G)$$

corresponding to the homotopy class of the identity map $K(G,n) \to K(G,n)$.
Exercise (using naturality): For any CW-complex X, the map

$$[X, K(G,n)] \to H^n_{\mathrm{CW}}(X;G) : [f] \mapsto f^*\iota_n$$

is precisely the natural bijection that we have already constructed; this is just another formula for it. One can therefore derive *all* cohomology classes of CW-complexes in a given degree n with given coefficients G from a single distinguished cohomology class of a $K(G,n)$!
(Note: One cannot similarly express the bijection $[X, K^{\mathrm{w}}(G,n)] \xrightarrow{\cong} H^n_{\mathrm{CW}}(X;G)$ via such a formula in general, because if $K^{\mathrm{w}}(G,n)$ is not a CW-complex, then we do not have a correspondence $[K^{\mathrm{w}}(G,n), K^{\mathrm{w}}(G,n)] \cong H^n(K^{\mathrm{w}}(G,n);G)$ and thus have no way to define a distinguished cohomology class ι_n in $K^{\mathrm{w}}(G,n)$.)
- Example 2: Suppose $\Phi : G \to H$ is a homomorphism of abelian groups. For any CW-complex X, this determines a natural chain map $\operatorname{Hom}(C_*^{\mathrm{CW}}(X;\mathbb{Z}),G) \xrightarrow{\Phi_*} \operatorname{Hom}(C_*^{\mathrm{CW}}(X;\mathbb{Z}),H)$ and therefore a natural transformation

$$H^n_{\mathrm{CW}}(\cdot;G) \xrightarrow{\Phi_*} H^n_{\mathrm{CW}}(\cdot;H),$$

where both sides are regarded as contravariant functors $\mathsf{hCW} \to \mathsf{Ab}$. Under the natural bijections, Φ therefore also defines a natural transformation

$$[\cdot, K^{\mathrm{w}}(G,n)] \xrightarrow{\Phi_*} [\cdot, K^{\mathrm{w}}(H,n)]$$

of contravariant functors $\mathsf{hCW} \to \mathsf{Set}$. Since any $K(G,n)$ is also an object of hCW, we then have in particular an induced map

$$[K(G,n), K(G,n)] \xrightarrow{\Phi_*} [K(G,n), K^{\mathrm{w}}(H,n)],$$

and assuming all the $K(G,n)$'s on both sides to be the same space, we obtain from this a distinguished (unpointed) homotopy class of maps $\varphi : K(G,n) \to K^{\mathrm{w}}(H,n)$, defined by

$$[\varphi] := \Phi_*[\mathrm{Id}] \in [K(G,n), K^{\mathrm{w}}(H,n)].$$

Using naturality, one then deduces:

- Theorem: Suppose Y is a $K(G,n)$ and Z is a weak $K(H,n)$ for two abelian groups G, H. Then every group homomorphism $\Phi : G \to H$ naturally determines a homotopy class of unpointed maps $\varphi : Y \to Z$ such that for all CW-complexes X, the diagram
$$\begin{tikzcd} {[X, K(G,n)]} \arrow[r, "\varphi_*"] \arrow[d, "\cong"] & {[X, K^{\mathrm{w}}(H,n)]} \arrow[d, "\cong"] \\ H^n_{\mathrm{CW}}(X; G) \arrow[r, "\Phi_*"] & H^n_{\mathrm{CW}}(X; H) \end{tikzcd}$$
commutes.
- Remark (pointed vs. unpointed): For $n \geqslant 1$, one could just as well regard $[\varphi] \in [K(G,n), K^{\mathrm{w}}(H,n)]$ in the theorem above as a pointed homotopy class, because the fact that $\pi_1(K^{\mathrm{w}}(H,n))$ is abelian (and in fact trivial if $n \geqslant 2$) implies via the theorem at the beginning of Lecture 12 that the natural map $[K(G,n), K^{\mathrm{w}}(H,n)]_+ \to [K(G,n), K^{\mathrm{w}}(H,n)]_\circ$ forgetting base points is a bijection. (Note that this depends on the fact that $K(G,n)$ is well-pointed, being a CW-complex with a 0-cell for a base point.) This is clear for $n \geqslant 2$ since $\pi_1(K^{\mathrm{w}}(H,n))$ is then trivial, while for $n = 1$, it follows from Exercise 11.1 below. For $n = 0$, one can carry out the same argument using pointed homotopy and reduced cohomology thanks to Exercise 11.2, giving rise to a pointed homotopy class of maps $K(G,0) \to K^{\mathrm{w}}(H,0)$; this just amounts to the statement that the unpointed map described above maps the base point of $K(G,0)$ to the same path-component as the base point of $K^{\mathrm{w}}(H,0)$, and is thus homotopic to a pointed map.
- Remark (induced maps on π_n): Exercise 11.3 below implies that for each $n \geqslant 1$, a map $\varphi : K(G,n) \to K^{\mathrm{w}}(H,n)$ is uniquely determined up to homotopy by the induced homomorphism $\varphi_* : G = \pi_n(K(G,n)) \to \pi_n(K^{\mathrm{w}}(H,n)) = H$. Exercise: For the map φ described above, the induced homomorphism on π_n is Φ. (One can also say this for $n = 0$, and the proof in that case is elementary.)
- Corollary 1: For every $n \geqslant 0$, one can define functors $\mathsf{Ab} \to \mathsf{hTop}$ and $\mathsf{Ab} \to \mathsf{hTop}_*$ that associate to each abelian group G a $K(G,n)$ and to each group homorphism $\Phi : G \to H$ the (unpointed or pointed) homotopy class of the map $\varphi : K(G,n) \to K(H,n)$ as explained in the theorem above and subsequent remarks.[53] These functors are not unique (since constructions of $K(G,n)$'s are not unique), but they are each canonical up to natural isomorphism.
- Corollary 2 (the case when Φ is an isomorphism): There is a natural pointed homotopy equivalence between any two spaces that are both $K(G,n)$'s for some $n \geqslant 0$ and abelian group G.
- Corollary 3 (using $\Phi = \mathrm{Id}$ and the fact that $\Omega K(G,n+1)$ is a weak $K(G,n)$): For any $K(G,n)$ and $K(G,n+1)$, there is a natural weak homotopy equivalence
$$K(G,n) \xrightarrow{\hat{e}} \Omega K(G,n+1)$$
defined up to pointed homotopy.
Remark: According to [Mil59], $\Omega K(G,n+1)$ is also homotopy equivalent to a CW-complex, so by Whitehead's theorem, $\hat{e}$ is actually a homotopy equivalence. This knowledge however is seldom actually useful.
- Trivial observation: Every abelian group G admits a unique group object structure (m, e, i) in the category Ab, and the group structure that it induces on G matches the native

[53] In the lecture I described this only as a functor to hTop, and replacing hTop with hTop_* makes no difference for $n \geqslant 1$ due to Exercises 11.1 and 11.2. For $n = 0$, one gets a slightly stronger and cleaner statement by working in the pointed category.

group structure of G. Indeed, a group object structure in Ab means that we have group homomorphisms

$$G \times G \xrightarrow{m} G, \qquad 0 \xrightarrow{e} G, \qquad G \xrightarrow{i} G,$$

where $0 \in \mathsf{Ab}$ denotes the trivial group, which is conveniently both an initial and a terminal object in Ab, so that there is only one possible choice of identity morphism $e : 0 \to G$. The identity axiom for a group object structure then means $m(g, 0) = m(0, g) = g$ for every $g \in G$, and since m is a homomorphism, it follows that

$$m(g, h) = m\big((g, 0) + (0, h)\big) = g + h$$

for every $g, h \in G$; the uniqueness of inverses now implies via the inverse axiom that $i : G \to G$ can only be the map $g \mapsto -g$. Note that it's important here to assume G is *abelian*, as the inverse map $i : G \to G$ would otherwise not be a morphism in the correct category.

- Another (almost) trivial observation: The functors from Ab to hTop or hTop_* in Corollary 1 preserve finite products, since $K(G, n) \times K(H, n)$ is clearly a $K(G \times H, n)$, and for the terminal object $0 \in \mathsf{Ab}$, one can take $K(0, n)$ to be $\{*\}$, which is a terminal object in both hTop and hTop_*. We can therefore feed the group object structure of G into such a functor and conclude:
- Corollary 4: For every $n \geqslant 0$ and abelian group G, every $K(G, n)$ has natural group object structures in hTop and hTop_* such that the bijection $[X, K(G, n)]_\circ \cong H^n_{\mathrm{CW}}(X; G)$ and its pointed variant $[X, K(G, n)]_+ \cong \widetilde{H}^n_{\mathrm{CW}}(X; G)$ outined in Exercise 11.2 become group isomorphisms.
Quick proof: One sees as follows that the group object structure of $K(G, n)$ determines the same group structure on $[X, K(G, n)]$ as $H^n_{\mathrm{CW}}(X; G)$. If we forget the group structure of $H^n_{\mathrm{CW}}(X; G)$ and just regard it as a set, then our natural bijection identifies the covariant functor $H^n_{\mathrm{CW}}(X; \cdot) : \mathsf{Ab} \to \mathsf{Set}$ with a composition $\mathsf{Ab} \to \mathsf{hTop} \to \mathsf{Set}$ of two covariant functors
$$G \xmapsto{K(\cdot, n)} K(G, n) \xmapsto{[X, \cdot]} [X, K(G, n)] \ ,$$
both of which preserve finite products, so feeding the group object structure of G into this composition produces the same result as feeding it into $H^n_{\mathrm{CW}}(X; \cdot) : \mathsf{Ab} \to \mathsf{Set}$. But $H^n_{\mathrm{CW}}(X; \cdot)$ is actually a functor $\mathsf{Ab} \to \mathsf{Ab}$, implying that this procedure makes $H^n_{\mathrm{CW}}(X; G)$ into a group object in Ab rather than just Set. According to the trivial observation above, there is only one such structure, and it reproduces the native group structure of $H^n_{\mathrm{CW}}(X; G)$.
- Remark: One can extend this line of thought quite a bit further and deduce e.g. that whenever G is a module over a commutative ring R, $K(G, n)$ naturally becomes an *R-module object* in hTop and hTop_* (figure out for yourself what the definition of that notion is). The existence of the cup product on cellular cohomology with coefficients in a ring R likewise corresponds to homotopy classes of maps $K(R, m) \wedge K(R, n) \to K(R, m + n)$ that can be used to give a purely homotopy-theoretic characterization of the cup product.
- Question motivated by the correspondence $[X, K(G, n)] \cong H^n(X; G)$:
What properties does a sequence $\{E_n\}$ of spaces need to have so that $h^n(X) := [X, E_n]$ defines a (generalized) cohomology theory? Note that if the E_n come with base points and we prefer to work with pointed homotopy, we can do this by replacing X with $X_+ = X \amalg \{*\}$, since $[X, E_n]_\circ \cong [X_+, E_n]_+$. One reason to prefer pointed spaces is that the bijection $[X, K(G, n)]_\circ \cong H^n_{\mathrm{CW}}(X; G)$ does not generalize in an obvious way to relative cohomology groups $H^n_{\mathrm{CW}}(X, A; G)$, and pointed spaces will turn out to provide an elegant remedy for

this. But the axioms we have for (co-)homology theories thus far do not pay any attention to base points, so we need a new set of axioms that does.

- Axioms of a (generalized) **pointed homology** or **cohomology** theory: A pointed homology theory $\widetilde{h}_*$ is a collection of covariant functors $\{\widetilde{h}_n : \mathsf{Top}_* \to \mathsf{Ab}\}_{n \in \mathbb{Z}}$ equipped with natural **suspension isomorphisms**

$$\widetilde{h}_n(X) \xrightarrow{\sigma_*} \widetilde{h}_{n+1}(\Sigma_+ X)$$

for each $n \in \mathbb{Z}$ and each well-pointed space $X \in \mathsf{Top}_*$, that satisfy the following set of axioms. (For cohomology, one speaks of contravariant functors $\widetilde{h}^n : \mathsf{Top}_* \to \mathsf{Ab}$, and I will again comment on this variation below only when there is something non-obvious to say.)

  - (HTP$_*$) **homotopy**: The functors $\widetilde{h}_n$ descend to the pointed homotopy category as functors $\mathsf{hTop}_* \to \mathsf{Ab}$.
  - (COF$_*$) **cofibration** (or **exactness**): For every pointed cofibration $A \hookrightarrow X$ of well-pointed spaces and every $n \in \mathbb{Z}$, the sequence

$$\widetilde{h}_n(A) \longrightarrow \widetilde{h}_n(X) \longrightarrow \widetilde{h}_n(X/A)$$

    induced by the inclusion $A \hookrightarrow X$ and quotient projection $X \to X/A$ is exact.
  - (ADD$_*$) **additivity**: For any collection of well-pointed spaces $\{X_\alpha \in \mathsf{Top}_*\}$ and their natural inclusions $i^\alpha : X_\alpha \hookrightarrow \bigvee_{\beta \in J} X_\beta$, the induced maps

$$\bigoplus_\alpha i^\alpha_* : \bigoplus_\alpha \widetilde{h}_n(X_\alpha) \longrightarrow \widetilde{h}_n\Big(\bigvee_\alpha X_\alpha\Big)$$

    are isomorphisms, or for a cohomology theory, the maps

$$\prod_\alpha (i^\alpha)^* : \widetilde{h}^n\Big(\bigvee_\alpha X_\alpha\Big) \longrightarrow \prod_\alpha \widetilde{h}^n(X_\alpha)$$

    are isomorphisms.
  - Remark 1: The notational resemblance to reduced (co-)homology is not accidental, and pointed theories are also sometimes called *reduced* theories, though the context is a bit different, as the reduced homologies in Lecture 16 were functors on $\mathsf{Top}^{\mathrm{rel}}$ rather than Top_*. A precise relationship between the two concepts denoted by $\widetilde{h}_*$ can be derived from the theorem below.
  - Remark 2: The definition above allows $\widetilde{h}_*$ to be defined for all pointed spaces, but only really assumes that it has good properties when the spaces are well-pointed. There are versions of this definition in which well-pointedness doesn't appear, but we will see below why it is important if one wants to have a nice relationship between pointed and unpointed theories. It is also often sensible to speak of (co-)homology theories defined on smaller subcategories of Top_* such as the category CW_* of pointed CW-complexes (which are all well-pointed), or to assume $\widetilde{h}_*$ is defined on all pointed spaces but only required to have good properties (notably the axioms (COF$_*$) and (ADD$_*$) and the existence of suspension isomorphisms) on CW-complexes.
  - Theorem (and of course there is also a variant for cohomology):
    (1) Any pointed homology theory $\widetilde{h}_*$ determines an unpointed homology theory by defining

$$h_n(X, A) := \widetilde{h}_n\Big(\mathrm{cone}_\circ(A \hookrightarrow X)\Big) = \widetilde{h}_n\Big(\mathrm{cone}_+(A_+ \hookrightarrow X_+)\Big)$$

for every $n \in \mathbb{Z}$ and $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$, with connecting homomorphisms given by

$$\begin{tikzcd} \widetilde{h}_n\big(\mathrm{cone}_+(A_+ \hookrightarrow X_+)\big) \arrow[r, "q_*"] \arrow[rr, bend left, "\partial_*"] & \widetilde{h}_n(\Sigma_+ A_+) \arrow[r, "\sigma_*^{-1}"] & \widetilde{h}_{n-1}(A_+) = \widetilde{h}_{n-1}\big(\mathrm{cone}_\circ(\varnothing \hookrightarrow A)\big), \end{tikzcd}$$

where $q : \mathrm{cone}_+(A_+ \hookrightarrow X_+) \to \mathrm{cone}_+(A_+ \hookrightarrow X_+)/X_+ = \Sigma_+ A_+$ is the quotient projection.

(2) Any unpointed homology theory h_* determines a pointed homology theory by defining

$$\widetilde{h}_n(X, *) := h_n(X, \{*\})$$

for every $n \in \mathbb{Z}$ and $(X, *) \in \mathsf{Top}_*$, with suspension isomorphisms defined for well-poined $(X, *) \in \mathsf{Top}_*$ in terms of the usual isomorphisms $\widetilde{h}_n(X) \cong \widetilde{h}_{n+1}(\Sigma_\circ X)$ for (unpointed) reduced homology, i.e.

$$\begin{tikzcd} h_n(X, \{*\}) \arrow[rrr, bend left, "\sigma_*"] & \widetilde{h}_n(X) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \widetilde{h}_{n+1}(\Sigma_+ X) \arrow[r, "\cong"] & h_{n+1}(\Sigma_+ X, \{*\}) \\ h_{n+1}(C_\circ X', X) \arrow[r, "\cong"] \arrow[ur, "\partial_*", "\cong"'] & h_{n+1}(\Sigma_\circ X, C_\circ X) & \widetilde{h}_{n+1}(\Sigma_\circ X) \arrow[l, "\cong"'] \arrow[u, "q_*", "\cong"'] & \end{tikzcd}$$

(Here, the connecting homomorphism ∂_* and maps induced by inclusions are isomorphisms due to the usual axioms of unpointed homology, and the quotient projection $q : \Sigma_\circ X \to \Sigma_+ X$ from unreduced to reduced suspension is an isomorphism so long as $(X, *)$ is well-pointed. For spaces that are not well pointed, this diagram still defines a map σ_*, but there is no guarantee that it is an isomorphism.)

- Remarks on the theorem:
  (i) We have seen the functor $\mathsf{Top} \to \mathsf{Top}_* : X \mapsto X_+ := X \amalg \{*\}$ before, and you will easily convince yourself that the unreduced mapping cone of $A \hookrightarrow X$ and the reduced mapping cone of $A_+ \hookrightarrow X_+$ are naturally pointed homeomorphic; the base point of $\mathrm{cone}_\circ(A \hookrightarrow X)$ is taken to be the summit of the cone since A and X are not equipped with base points of their own. You might be a bit surprised to see $\mathrm{cone}_\circ(\varnothing \hookrightarrow A)$ identified with A_+, but this becomes clear if you think of the mapping cone as a double mapping cylinder for maps $\varnothing \hookrightarrow A$ and $\varnothing \to *$.[54] The succinct expression for the absolute unpointed homology derived from a pointed homology theory thus becomes
  $$h_n(X) = \widetilde{h}_n(X_+).$$
  (ii) The pointed spaces $\mathrm{cone}_\circ(A \hookrightarrow X)$ and $\Sigma_+ A_+$ that appear in the definition of an unpointed homology theory derived from a pointed one are *always* well-pointed, for all $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$. This follows essentially from the fact that the inclusions of X, Y and $X \amalg Y$ into the unreduced double mapping cylinder of two maps $Z \to X$ and $Z \to Y$ is always a free cofibration. Analogously to the natural identification

[54] By the same token, the cone of the empty set is a one point space, not the empty set; in this way, $C_\circ : \mathsf{Top} \to \mathsf{Top}_*$ becomes a well-defined functor without needing to make an exception for the empty set. A similar but more jarring example is that the quotient $X/\varnothing$ is defined to be X_+, thus producing a well-defined functor $\mathsf{Top}^{\mathrm{rel}} \to \mathsf{Top}_* : (X, A) \mapsto X/A$. This makes sense if you think of X/A in general as the pushout of the maps $A \hookrightarrow X$ and $A \to *$, but we are still forced to accept a "quotient projection" $X \hookrightarrow X_+$ that is not surjective.

of $\mathrm{cone}_+(A_+ \hookrightarrow X_+)$ with $\mathrm{cone}_\circ(A \hookrightarrow X)$, the reduced suspension $\Sigma_+ A_+$ is closely related to the unreduced suspension of A, but is not exactly the same thing: it is the quotient of $\Sigma_\circ A$ in which the summits of both cones are identified, and therefore serve as a natural base point (which $\Sigma_\circ A$ on its own does not have).

(iii) Up to minor details (and natural isomorphisms), the correspondences $\widetilde{h}_* \mapsto h_*$ and $h_* \mapsto \widetilde{h}_*$ described in the theorem are inverse to each other. For instance, suppose $\widetilde{h}_*$ is given, and we define another homology theory for pointed spaces $(X, x) \in \mathsf{Top}_*$ by $\widehat{h}_n(X, x) := h_n(X, \{x\})$, where h_* has been defined from $\widetilde{h}_*$ via the stated prescription, thus

$$\widehat{h}_n(X, x) = \widetilde{h}_n\big(\mathrm{cone}_\circ(\{x\} \hookrightarrow X)\big) = \widetilde{h}_n\big(\mathrm{cone}_+(\{x, *\} \hookrightarrow X_+\big).$$

Inclusions of a single point $\{x\} \hookrightarrow X$ have the appealing property that their mapping cones are the same as their mapping cylinders, so there is an obvious homotopy equivalence $\mathrm{cone}_\circ(\{x\} \hookrightarrow X) = Z_\circ(\{x\} \hookrightarrow X) \to X$. Unfortunately, this is an *unpointed* homotopy equivalence (its obvious homotopy inverse $X \hookrightarrow \mathrm{cone}_\circ(\{x\} \hookrightarrow X)$ is not a pointed map), which is not quite good enough for the pointed homology theory $\widetilde{h}$. But we can get more mileage out of describing $\widehat{h}_n(X, x)$ in terms of the reduced cone: the latter is the homotopy pushout in the following diagram in hTop_*,

$$\begin{tikzcd}
\{x, *\} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\sim"] & X_+ \arrow[d] \\
* \arrow[r] & \mathrm{cone}_+(\{x, *\} \hookrightarrow X_+)
\end{tikzcd},$$

while the strict pushout in Top_* for the same two maps is a quotient of the coproduct $X_+ \vee \{*\} = X_+$ that identifies the two points $x, * \in X_+$, i.e. it is X, with $x \in X$ as the base point:

$$\begin{tikzcd}
\{x, *\} \arrow[r, hook] \arrow[d] & X_+ \arrow[d] \\
* \arrow[r] & (X, x)
\end{tikzcd}.$$

Viewing the latter as a diagram in hTop_* with the trivial homotopy then determines a pointed map $q : \mathrm{cone}_+(\{x, *\} \hookrightarrow X_+) \to X$, i.e. the quotient map that collapses the cylindrical part of the double mapping cylinder, and we know from Exercise 9.4 that q is a pointed homotopy equivalence if $\{x, *\} \hookrightarrow X_+$ is a pointed cofibration. The latter is true whenever $\{x\} \hookrightarrow X$ is a *free* cofibration, meaning (X, x) is well pointed, so if we are willing to restrict our attention to well-pointed spaces, then the pointed homotopy axiom now gives a natural isomorphism $\widetilde{h}_n(X, x) \cong \widehat{h}_n(X, x)$.

(iv) One can similarly (and more easily) see that going from an unpointed homology theory h_* to a pointed theory $\widetilde{h}_*$ and then back produces an unpointed theory

$$\check{h}_n(X, A) := \widetilde{h}_n(\mathrm{cone}(A \hookrightarrow X), *) := h_n(\mathrm{cone}(A \hookrightarrow X), \{*\})$$

that is naturally isomorphic to the original h_*. Indeed, the axioms of an unpointed homology theory make the following maps induced by inclusions into isomorphisms:

$$h_n(X, A) \xrightarrow{\cong} h_n(\mathrm{cone}(A \hookrightarrow X), CA) \xleftarrow{\cong} h_n(\mathrm{cone}(A \hookrightarrow X), *).$$

Here, the map on the left is a standard application of the homotopy and excision axioms, while the one on the right is an isomorphism due to the usual combination of homotopy invariance, exactness and the 5-lemma.

- One more remark: Defining $h_n(X, A)$ as the pointed homology of a homotopy cofiber $\operatorname{cone}(A \hookrightarrow X)$ is dual to the way we defined relative homotopy groups $\pi_n(X, A)$, namely as absolute homotopy groups of the homotopy fiber $F(A \hookrightarrow X)$. This is a hint that the role played by the Puppe fiber sequence in the background of the homotopy groups will be played in the homology/cohomology context by the Puppe *cofiber* sequence.
- Proof of the theorem: The pointed theory $\widetilde{h}_n(X, *) := h_n(X, \{*\})$ derived from an unpointed theory h_* is naturally isomorphic to the latter's *reduced* variant, and the axioms for $\widetilde{h}_*$ thus follow easily from the properties of reduced homology that we discussed in Lecture 16. The most interesting is perhaps the cofibration axiom, which follows directly from the exactness axiom of h_* due to the natural isomorphisms $\widetilde{h}_n(X/A) \cong h_n(X, A)$ that hold whenever $A \hookrightarrow X$ is a cofibration.

  If instead we start with a pointed theory $\widetilde{h}_*$ and then define h_n by the stated prescription, the homotopy axiom is easily verified because homotopies of maps of pairs $(X, A) \to (Y, B)$ induce pointed homotopies of maps $\operatorname{cone}(A \hookrightarrow X) \to \operatorname{cone}(B \hookrightarrow Y)$. The additivity axiom is, similarly, an almost immediate consequence of its pointed variant. The interesting part is proving excision and exactness. The next lemma explains why there was surprisingly no need to mention excision in the list of pointed axioms: defining $h_n(X, A)$ in terms of mapping cones makes excision an automatic consequence of the pointed homotopy axiom. We will then see that the exactness axiom follows easily from a combination of the cofibration and pointed homotopy axioms with the Puppe cofiber sequence.
- Lemma: Suppose $X = A \cup B$ and there exists a continuous function $u : X \to I$ that equals 0 on $X \backslash A$ and 1 on $X \backslash B$. Then the inclusion
$$\operatorname{cone}(A \cap B \hookrightarrow A) \hookrightarrow \operatorname{cone}(B \hookrightarrow X)$$
is a pointed homotopy equivalence.
Remark: This is the moment to recall that our statement of the excision axiom in Lecture 16 explicitly assumed the existence of such a function $u : X \to I$, which is a slightly stronger hypothesis than in standard versions of the axiom, though equivalent in all cases of practical interest. The inclusion of that hypothesis was specifically motivated by this lemma.
Proof of the lemma: Write points in $\operatorname{cone}(B \hookrightarrow X)$ as equivalence classes $[x, t]$ of tuples $(x, t) \in X \times I$, where we assume $x \in B$ for $t < 1$ and $[x, 0] = [y, 0]$ for all $x, y \in B$. The formula
$$g([x, t]) := [x, u(x)t]$$
then gives a well-defined pointed map $\operatorname{cone}(B \hookrightarrow X) \to \operatorname{cone}(A \cap B \hookrightarrow A)$ since it sends $[x, t]$ to the summit of the cone whenever $x \notin A$ and fixes $[x, 1]$ for every $x \in X \backslash B = A \backslash (A \cap B)$. Using a linear homotopy of $u : X \to I$ to the constant function with value 1, one similarly constructs families of pointed maps showing that g is a pointed homotopy inverse of the inclusion.
- Proof of exactness: For any given pair $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$, it suffices to prove that the top row of the following diagram is exact at each of its interior terms:

$$\begin{tikzcd}
h_n(A) \arrow[r] \arrow[d, equal] & h_n(X) \arrow[r] \arrow[d, equal] & h_n(X, A) \arrow[r, "\partial_*"] \arrow[d, equal] & h_{n-1}(A) \arrow[r] \arrow[d, equal] & h_{n-1}(X) \arrow[d, equal] \\
\widetilde{h}_n(A_+) \arrow[r] & \widetilde{h}_n(X_+) \arrow[r] & \widetilde{h}_n(\operatorname{cone}_+(A_+ \hookrightarrow X_+)) \arrow[dr, "q_*"'] & \widetilde{h}_{n-1}(A_+) \arrow[r] \arrow[d, "\sigma_*", "\cong"'] & \widetilde{h}_{n-1}(X_+) \arrow[d, "\sigma_*", "\cong"'] \\
 & & & \widetilde{h}_n(\Sigma_+ A_+) \arrow[r] & \widetilde{h}_n(\Sigma_+ X_+)
\end{tikzcd}$$

All pointed spaces appearing in this diagram are well pointed, all unlabelled maps are induced by obvious inclusions, and the diagram commutes due to the naturality of the suspension isomorphisms σ_* and the way that these isomorphisms determine the definition of ∂_*. The crucial observation is now that if we follow the bottom terms in the diagram from left to right but remove the functor $\widetilde{h}_n$, what we're looking at is the first five terms of the Puppe cofiber sequence for the pointed inclusion $A_+ \hookrightarrow X_+$:

$$A_+ \hookrightarrow X_+ \to \mathrm{cone}_+(A_+ \hookrightarrow X_+) \to \Sigma_+ A_+ \hookrightarrow \Sigma_+ X_+ \to \dots.$$

In particular, any consecutive three-term subsequence has the homotopy type of a pointed cofibration, so the exactness of the top row now follows directly from the pointed homotopy and cofibration axioms.

- Exercise: Work out the proof of the cohomological variant of the theorem, in particular the detail involving exactness of the unpointed theory. Since cohomology and homology are "dual" to each other, you may be intuitively expecting the Puppe fiber sequence to replace the cofiber sequence, but that intuition is wrong; the cofiber sequence is again what you need.

Lecture 19 (27.06.2024): Stable homotopy groups and spectra. Caveat: In this lecture, I am making less of an effort than usual to ensure that everything I'm saying is strictly correct and stated with all the necessary hypotheses (but not more than necessary). This is because different books often say slightly different things with subtly different assumptions (some of which they don't state explicitly), and the subtleties typically do not get explained in the books, so as a novice in homotopy theory, I have not yet had time to figure out what my official take on it is.

- Recall the Freudenthal suspension theorem: For $X \in \mathsf{Top}_*$ well-pointed and n-connected ($n \geqslant 0$), the map $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ defined via the reduced suspension functor Σ is bijective for all $k < 2n+1$, and surjective for $k = 2n+1$.
- Corollary 1: For any well-pointed space X and $k \geqslant 1$, $\Sigma^k X$ is $(k-1)$-connected. (Obvious for $k = 1$, then proof by induction on k using Freudenthal.)
- Corollary 2 (using Corollary 1): For any well-pointed space X and $n \geqslant 0$, the map

$$\pi_{n+k}(\Sigma^k X) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma^{k+1} X)$$

is an isomorphism whenever $n + k < 2(k-1) + 1 = 2k - 1$, i.e. for all $k > n + 1$.
In other words, the sequence

$$\pi_n(X) \xrightarrow{\Sigma} \pi_{n+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{n+2}(\Sigma^2 X) \longrightarrow \dots \longrightarrow \pi_{n+k}(\Sigma^k X) \longrightarrow \dots$$

eventually *stabilizes* if X is well pointed. The following definition treats this sequence as a direct system of groups and captures their isomorphism types for arbitrarily large k.
- Definition: For $n \in \mathbb{Z}$, the nth **stable homotopy group** of a pointed space X is

$$\pi_n^s(X) := \mathrm{colim}_{k \to \infty} \pi_{n+k}(\Sigma^k X).$$

Interpretation: Elements of the ordinary homotopy groups $\pi_n(X)$ are represented by pointed maps of spheres to X, and are considered equivalent if they are homotopic. The stable homotopy groups weaken this equivalence relation, and allow two non-homotopic maps of spheres to X to be called equivalent if they become homotopic after sufficiently many "stabilizations," meaning applications of the suspension functor.
- Remark: Since finitely-many terms in the direct system can be discarded without changing the direct limit, $\pi_n^s(X)$ is well defined for any $n \in \mathbb{Z}$, including $n < 0$. If X is well pointed, however, then the corollaries of Freudenthal above imply $\pi_n^s(X) = 0$ for $n < 0$, while for

$n \geqslant 0$, $\pi_n^s(X) \cong \pi_{n+k}(\Sigma^k X)$ for any k sufficiently large. Note also that since $\pi_k(X)$ is always abelian for $k \geqslant 2$, $\pi_n^s(X)$ is abelian for every $n \in \mathbb{Z}$.

- Functoriality: Each π_n^s is naturally a functor $\mathsf{Top}_* \to \mathsf{Ab}$, because pointed maps $f : X \to Y$ give rise to sequences of maps
$$\pi_{n+k}(\Sigma^k X) \xrightarrow{(\Sigma^k f)_*} \pi_{n+k}(\Sigma^k Y)$$
that determine a map $f_* : \pi_n^s(X) \to \pi_n^s(Y)$ between the corresponding colimits.
- Suspension isomorphisms: A natural homomorphism $\sigma_* : \pi_n^s(X) \to \pi_{n+1}^s(\Sigma X)$ is determined by the diagram
$$\begin{tikzcd} \ldots \arrow[r] & \pi_{n+k}(\Sigma^k X) \arrow[r, "\Sigma"] \arrow[d, "\Sigma"] & \pi_{n+k+1}(\Sigma^{k+1} X) \arrow[r] \arrow[d, "\Sigma"] & \ldots \arrow[r] & \pi_n^s(X) \arrow[d, "\sigma_*"] \\ \ldots \arrow[r] & \pi_{n+k+1}(\Sigma^{k+1} X) \arrow[r, "\Sigma"] & \pi_{n+k+2}(\Sigma^{k+2} X) \arrow[r] & \ldots \arrow[r] & \pi_{n+1}^s(\Sigma X) \end{tikzcd}.$$
But the direct systems on both rows are the same up to a shift, so their colimits are necessarily isomorphic, and σ_* is an explicit isomorphism; one can derive an inverse $\pi_{n+1}^s(\Sigma X) \to \pi_n^s(X)$ to σ_* by adding upward diagonal arrows representing identity maps to the diagram. The invertibility of σ_* thus follows from a purely formal argument that would work in any category, but one can also see it more explicitly from Freudenthal if X is well pointed, because then the vertical maps are all isomorphisms for sufficiently large k.
- Theorem: The stable homotopy groups $\pi_n^s : \mathsf{Top}_* \to \mathsf{Ab}$ satisfy the axioms of a pointed homology theory.
- Remark: The theorem seems to give hope that $\pi_n^s(X)$ could be computed for any CW-complex X, and in principle, such a computation will follow from the Atiyah-Hirzebruch spectral sequence if one can first compute all coefficient groups of the theory and then compute the cellular homology with respect to these coefficient groups. The unpointed homology theory determined by π_*^s according to the previous lecture has absolute homology groups $h_n(X) := \pi_n^s(X_+)$ for $X \in \mathsf{Top}$, so its coefficient group in degree $n \in \mathbb{Z}$ is
$$h_n(*) = \pi_n^s(S^0) = \operatorname{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k S^0) = \operatorname{colim}_{k\to\infty} \pi_{n+k}(S^k),$$
which is isomorphic to $\pi_{n+k}(S^k)$ for any k sufficiently large. We know that these groups are sometimes nontrivial for $n > 0$, thus h_* does not satisfy the dimension axiom, i.e. it is a *generalized* rather than an *ordinary* homology theory.
Bad news (or interesting news, depending on your point of view): The groups $\pi_n^s(S^0)$ are not known in general, and their computation is considered to be one of the great open problems of algebraic topology.
- Proof of the theorem: The axiom (HTP_*) is obvious from the definition. For the axiom (COF_*), we need to show that whenever $A \hookrightarrow X$ is a pointed cofibration of well-pointed spaces, the induced maps $\pi_n^s(A) \to \pi_n^s(X) \to \pi_n^s(X/A)$ form an exact sequence. This is a consequence of the long exact sequence of relative homotopy groups, because by Theorem 2 in Lecture 15, the vertical map (induced by a quotient projection) in the diagram
$$\begin{tikzcd} \pi_{n+k}(\Sigma^k A) \arrow[r] & \pi_{n+k}(\Sigma^k X) \arrow[r] \arrow[dr] & \pi_{n+k}(\Sigma^k X, \Sigma^k A) \arrow[d] \\ & & \pi_{n+k}(\Sigma^k X / \Sigma^k A) \cong \pi_{n+k}\big(\Sigma^k(X/A)\big) \end{tikzcd}$$
is an isomorphism whenever $n + k \leqslant 2k - 2$, meaning $k \geqslant n + 2$. The result then follows because exactness of sequences is preserved under direct limits. For the axiom (ADD_*),

one can start with the following lemma: If there is a collection $\{X_\alpha\}_{\alpha \in J}$ of well-pointed spaces such that the natural map

$$\bigoplus_\alpha \pi_n^S(X_\alpha) \longrightarrow \pi_n^S\Big(\bigvee_{\alpha \in J} X_\alpha\Big)$$

fails to be an isomorphism, then it also fails for some such collection in which J is finite. This is true for roughly the same reason why it sufficed to prove Theorem 1 in Lecture 15 for finite wedge sums: every compact subset of $\bigvee_\alpha X_\alpha$ intersects at most finitely many of the spaces X_α beyond their base points, and this applies in particular to the images of maps $S^{n+k} \to \bigvee_\alpha X_\alpha$ or homotopies between such maps. But recall that for finite collections, additivity in an unpointed homology theory follows already from the other axioms, namely (EXC) and (LES). Knowing this, one also deduces finite additivity in the pointed theory from the fact that it is satisfied in the unpointed theory. If this proof of additivity seems a bit cheap, I'll remind you that there is some serious machinery behind it: we're using the fact (proved in the previous lecture) that the unpointed homology theory determined by a pointed theory satisfies the exactness axiom, and the main result behind that was the Puppe cofiber sequence.[55]

- Definitions:
  (1) A **spectrum** E consists of a sequence of pointed spaces $\{E_n \in \mathsf{Top}_*\}$ that are defined for sufficiently large integers n and equipped with pointed maps $e : \Sigma E_n \to E_{n+1}$.
  (2) A spectrum E is an **Ω-spectrum** if the spaces E_n are defined for all $n \in \mathbb{Z}$ and are pointed CW-complexes, and the maps $\hat{e} : E_n \to \Omega E_{n+1}$ adjoint to $e : \Sigma E_n \to E_{n+1}$ are weak homotopy equivalences for all n.
- Remarks:
  (1) There is a major lack of consensus about the details of the two definitions given above. Many sources place stronger conditions on a spectrum and call the notions defined above *prespectra* to indicate the lack of stronger conditions. Some also would require all the spaces in a spectrum to be CW-complexes, which I have decided to require for Ω-spectra (because I can imagine needing the condition in that context) but not for spectra (because I don't see a need, even though the condition would hold for most of the examples we discuss). For Ω-spectra, some sources also drop the word "weak" in front of "homotopy equivalence"; actually, we could also do this without changing anything, since we are assuming the E_n are CW-complexes, and [Mil59] then tells us that the ΩE_{n+1} are also homotopy equivalent to CW-complexes. But that knowledge makes no practical difference in the main applications, so we're leaving in the word "weak".
  (2) While I do not see a need for our purposes to assume that the spaces E_n in a general spectrum are CW-complexes, I can vaguely imagine good reasons to assume that they are well pointed. I have not checked enough details to verify whether this condition would be truly useful or necessary, but for safety, you might want to assume that it holds, and it definitely does hold in all the examples that will be important for us.
  (3) It is conventional to assume that the structure maps $e : \Sigma E_n \to E_{n+1}$ in a spectrum are actual maps rather than just homotopy classes of maps, though for our purposes, having them defined only up to homotopy would be just as good. They can be

[55] In [Hat02, Prop. 4F.1], there is a different proof of finite additivity based on the product complex trick that we used in Theorem 1 of Lecture 15 to compute $\pi_n(S^n \vee \ldots \vee S^n)$. But either I'm missing something basic, or that trick is not actually applicable in the present situation and what it says there is complete nonsense. If you understand it, feel free to explain it to me.

defined canonically as maps in some of the examples below, while in others, only their homotopy classes are canonical, so that realizing them as maps requires some arbitrary choices.

(4) In the definition of spectra that I wrote down in lecture, E_n was required to be defined for all $n \in \mathbb{Z}$, but in fact this is not true in any of the examples below (they require $n \geqslant 0$), and it doesn't need to be. For the applications we want to discuss, it suffices if E_n is defined for every integer n greater than some threshold.

- Examples of spectra:
  - The **Eilenberg-MacLane spectrum** $K(G, \cdot)$ associated to an abelian group G consists of choices of a sequence of Eilenberg-MacLane spaces $\{K(G,n)\}_{n \geqslant 0}$ together with maps $e : \Sigma K(G,n) \to K(G,n+1)$ whose adjoints live in the canonical homotopy class of maps $\hat{e} : K(G,n) \to \Omega K(G,n+1)$ arising from the fact that $\Omega K(G,n+1)$ is a weak $K(G,n)$. These adjoints are weak homotopy equivalences, so if we extend the spectrum $K(G,\cdot)$ to $n < 0$ by formally defining $K(G,n) := \{*\}$ with the unique pointed maps $\Sigma\{*\} \to \{*\}$, $K(G,\cdot)$ becomes an Ω-spectrum. (Note that the based loop space of any discrete space is another one-point space, so the maps $K(G,n) \to \Omega K(G,n+1)$ are automatically also homotopy equivalences for $n < 0$.)
  - The **suspension spectrum** $\Sigma^\bullet X$ of any pointed space X can be defined as the sequence of spaces $\{\Sigma^n X\}_{n \geqslant 0}$ together with the identity maps $\Sigma\Sigma^n X \to \Sigma^{n+1} X$. An important special case is the **sphere spectrum** $\Sigma^\bullet S^0$, whose spaces are all spheres $\Sigma^n S^0 \cong S^n$. Suspension spectra are typically not Ω-spectra: for a given pointed CW-complex X, it is a straightforward exercise to show that $\Sigma^\bullet X$ is an Ω-spectrum if and only if the Freudenthal suspension theorem holds for X *without dimensional restrictions*, which is clearly false for the sphere spectrum since e.g. $\pi_2(S^1) \not\cong \pi_3(S^2)$. On the other hand, Freudenthal does imply (via Exercise 9.5) that for well-pointed X, the suspension spectrum $\Sigma^\bullet X$ is at least *asymptotically* an Ω-spectrum as $n \to \infty$, in the sense that $\hat{e} : \Sigma^n X \to \Omega\Sigma^{n+1} X$ is a $(2n-1)$-equivalence.
  - A special case of the following operation turns the sphere spectrum into an arbitrary suspension spectrum. The **smash product** of a spectrum $E = \{E_n\}$ with a pointed space X defines a new spectrum $E \wedge X$ with spaces $\{E_n \wedge X\}$ and maps

$$\begin{tikzcd} \Sigma(E_n \wedge X) \arrow[r] \arrow[rr, bend right, "e"'] & (\Sigma E_n) \wedge X \arrow[r, "e \wedge \mathrm{Id}"] & E_{n+1} \wedge X, \end{tikzcd}$$

where the first map in this composition is the canonical bijection $\Sigma(E_n \wedge X) = S^1 \wedge (E_n \wedge X) \to (S^1 \wedge E_n) \wedge X = (\Sigma E_n) \wedge X$, which is continuous because S^1 is friendly (cf. Exercise 6.1). By this definition, $\Sigma^\bullet S^0 \wedge X = \Sigma^\bullet X$.

- Proposition 1: There is a natural operation that converts any spectrum $E = \{E_n\}$ into an Ω-spectrum $E' = \{E'_n\}$.
Remark: I'm being intentionally vague about the precise meaning of the word "natural" in this statement. It has one, but defining it would require a proper definition of the category of spectra, which involves some subtle issues and we have no pressing need to get into them here.
Proof: I'll give a few details here that did not fit into the lecture. Let us pretend at first that we had not required the spaces in an Ω-spectrum to be CW-complexes, and just focus on constructing spaces E'_n with weak homotopy equivalences $E'_n \to \Omega E'_{n+1}$. The idea is to define E'_n for every $n \in \mathbb{Z}$ as the direct limit of the sequence

$$E_n \xrightarrow{\hat{e}} \Omega E_{n+1} \xrightarrow{\Omega\hat{e}} \Omega^2 E_{n+2} \xrightarrow{\Omega^2\hat{e}} \ldots \longrightarrow \operatorname{colim}_{k\to\infty} \Omega^k E_{n+k} =: E'_n,$$

with the understanding that if E_n is only defined for n larger than some threshold $n_0 \in \mathbb{Z}$, then we are free to begin the sequence with the term $\Omega^k E_{n+k}$ for any $k \geqslant n_0 - n$. A map $E'_n \to \Omega E'_{n+1}$ is then determined by the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^k E_{n+k} \arrow[r, "\Omega^k \hat{e}"] \arrow[d, "\Omega^k \hat{e}"] \arrow[rrr, bend left, "\psi_{k,n}"] & \Omega^{k+1} E_{n+k+1} \arrow[r] \arrow[d, "\Omega^{k+1} \hat{e}"] \arrow[rr, bend left, "\psi_{k+1,n}"] & \cdots \arrow[r] & E'_n \arrow[d, "\cong"] \\
\cdots \arrow[r] & \Omega(\Omega^k E_{(n+1)+k}) \arrow[r, "\Omega^{k+1} \hat{e}"] \arrow[drrr, bend right, "\Omega\psi_{k,n+1}"'] & \Omega(\Omega^{k+1} E_{(n+1)+(k+1)}) \arrow[r] \arrow[drr, "\Omega\psi_{k+1,n+1}"] & \cdots \arrow[r] & \operatorname{colim}_{k \to \infty} \Omega\big(\Omega^k E_{(n+1)+k}\big) \arrow[d] \\
& & & & \Omega E'_{n+1}
\end{tikzcd},$$

in which $\psi_{k,n} : \Omega^k E_{n+k} \to E'_n$ for each k, n denotes the natural map to the colimit, and the map $\operatorname{colim}_{k \to \infty} \Omega\big(\Omega^k E_{(n+1)+k}\big) \to \Omega E'_{n+1}$ comes from using the maps $\Omega\psi_{k,n+1}$ to view $\Omega E'_{n+1}$ as a target of the direct system in the second row. The first two rows are in fact the same direct system up to a shift, and therefore have isomorphic colimits, making the induced map $E'_n \to \operatorname{colim}_{k \to \infty} \Omega\big(\Omega^k E_{(n+1)+k}\big)$ a pointed homeomorphism; as in our construction of the suspension isomorphism for the stable homotopy groups, one could derive an inverse of this map by adding upward diagonal identity arrows to the diagram. It would be tempting to claim that the other vertical map $\operatorname{colim}_{k \to \infty} \Omega\big(\Omega^k E_{(n+1)+k}\big) \to \Omega E'_{n+1}$ is also a homeomorphism due to some general principle, e.g. that the functor Ω preserves colimits and $\Omega E'_{n+1}$ is therefore also a colimit of the second row—that sounds nice, but it is false, because right-adjoint functors such as Ω generally preserve limits, not colimits. The good news is that direct systems (over directed sets!) in the category Top_* have one or two extra properties that are useful for this situation, telling us that the passage to the colimit is sufficiently compatible with the functor Ω for our purposes: Exercise 11.4 implies that the map $\operatorname{colim}_{k \to \infty} \Omega\big(\Omega^k E_{(n+1)+k}\big) \to \Omega E'_{n+1}$ is at least a weak homotopy equivalence. We have therefore constructed weak homotopy equivalences $E'_n \to \Omega E'_{n+1}$, whose adjoint maps $\Sigma E'_n \to E'_{n+1}$ we take to be the structure maps of our new spectrum. There is clearly no reason to expect that the spaces E'_n constructed in this way are CW-complexes, but we can now get the desired Ω-spectrum by replacing them with CW-approximations (Exercise 11.4).

- Proposition 2: For any Ω-spectrum $E = \{E_n\}$, the spaces E_n admit unique group object structures in hTop_* such that for every CW-complex X, the composition

$$[X, E_n]_+ \xrightarrow{\Sigma} [\Sigma X, \Sigma E_n]_+ \xrightarrow{e_*} [\Sigma X, E_{n+1}]_+ \tag{11.1}$$

is a group isomorphism. Moreover, the resulting group objects in hTop_* are all abelian.
Remark: This proposition was my main motivation for insisting in the definition of Ω-spectra that the spaces E_n should be CW-complexes.
Proof: Since the E_n are CW-complexes, it will suffice to make them group objects in the subcategory $\mathsf{hCW}_* \subset \mathsf{hTop}_*$, which has the same morphisms as hTop_* but only CW-complexes as objects. For any CW-complex X and each $n \in \mathbb{Z}$, the composition in (11.1) is equivalent via adjunction to the map

$$[X, E_n]_+ \xrightarrow{\hat{e}_*} [X, \Omega E_{n+1}]_+,$$

which is a bijection due to the cofibrant theorem since $\widehat{e}$ is a weak homotopy equivalence.[56] If we use the natural group object structure of the loop space ΩE_{n+1} to make $[X, \Omega E_{n+1}]_+$ into a group, it follows that there is a unique group structure on $[X, E_n]$ making this bijection a group isomorphism, and the maps $[Y, E_n] \to [X, E_n]$ induced by pointed maps $X \to Y$ are then also group homomorphisms. Proposition 3 in Lecture 10 then produces on E_n a unique group object structure in hCW_* (and therefore also hTop_*) that reproduces this group structure on $[X, E_n]$ for every pointed CW-complex X. Defining this for every n, we then observe that the groups $[X, \Omega E_{n+1}] = [\Sigma X, E_{n+1}]$ are always abelian due to Proposition 4 in Lecture 11, because they inherit the same group structure from the cogroup object ΣX and the group object E_{n+1}. It follows that $[X, E_n]$ is also abelian for every pointed CW-complex X, and E_n is therefore an abelian group object.

- In the previous lecture, we equipped the Eilenberg-MacLane spaces $K(G, n)$ with the unique group object structures (in both hTop and hTop_*) for which the natural bijection $[X, K(G, n)]_\circ \to H^n_{\mathrm{CW}}(X; G)$ becomes a group isomorphism for every CW-complex X. Since $K(G, \cdot)$ is an Ω-spectrum, you should now be wondering whether that group object structure is the same as the one coming out of the construction above. This must be true if and only if it makes the bijective map $[X, K(G, n)]_+ \to [\Sigma X, K(G, n+1)]_+$ defined in (11.1) into a group isomorphism for every pointed CW-complex X. According to Exercise 11.2, that bijective map is equivalent to a map $\widetilde{H}^n_{\mathrm{CW}}(X) \to \widetilde{H}^{n+1}_{\mathrm{CW}}(\Sigma X)$, so there is an obvious candidate for what that isomorphism might be. I will leave it as an exercise to prove that that's what it is.
- Theorem: For any Ω-spectrum $E = \{E_n\}$,
$$\widetilde{h}^n(X) := [X, E_n]_+$$
with suspension maps σ_* defined by
$$\widetilde{h}^n(X) = [X, E_n]_+ \xrightarrow{\widehat{e}_*} [X, \Omega E_{n+1}]_+ = [\Sigma X, E_{n+1}]_+ = \widetilde{h}_{n+1}(\Sigma X)$$
defines a pointed cohomology theory on the subcategory $\mathsf{CW}_* \subset \mathsf{Top}_*$ of pointed CW-complexes.
Remark: The reason to restrict to CW-complexes in this statement is the suspension maps, because if we are only assuming that the maps $\widehat{e} : E_n \to \Omega E_{n+1}$ are weak homotopy equivalences, then the easy extension of the cofibrant theorem to pointed homotopy sets makes $\sigma_* : \widetilde{h}^n(X) \to \widetilde{h}^{n+1}(\Sigma X)$ as defined above a bijection whenever X is a CW-complex, but it would not tell us this for spaces that are not CW-complexes. It follows that the corresponding unpointed cohomology theory h_* will satisfy the unpointed axioms when restricted to CW-pairs, which is enough to do all the things that we usually do with axiomatic homology, e.g. computing it on all CW-complexes in terms of cellular cohomology and the coefficient groups $h^n(*)$. It is possible—but not obviously useful—that we could say more by using the knowledge from [Mil59] that ΩE_{n+1} is also homotopy equivalent to a CW-complex, but this would require some careful discussion of the distinction between pointed and unpointed homotopy equivalence, which I would rather not get into if I don't have a really good reason.
Proof of the theorem: Actually, it's obvious that the axioms of a pointed cohomology theory are satisfied. The most interesting one is (COF$_*$), which is now just a restatement

[56] Minor detail: The homotopy sets in the present context are pointed, while the cofibrant theorem as proved in Lecture 12 gave a bijection between *unpointed* homotopy sets. In fact, an easy modification of the same proof shows that if the weak homotopy equivalence is also a pointed map, then it also induces a bijection between pointed homotopy sets. This only requires applying the compression lemma in a slightly more careful way so that it produces pointed homotopies, which is possible because the base point of the CW-complex X is a subcomplex.

of the "main property" of cofibrations, i.e. that they contravariantly induce exact sequences of homotopy sets.

- Example: The Eilenberg-MacLane spectrum $K(G, \cdot)$ gives rise to a pointed cohomology theory $\widetilde{h}^n(X) = [X, K(G,n)]_+$ for CW-complexes X, and therefore also an unpointed theory given by $h^n(X) := \widetilde{h}^n(X_+) = [X_+, K(G,n)]_+ = [X, K(G,n)]_\circ$, so its coefficient groups are
$$h^n(*) = \widetilde{h}^n(S^0) = [S^0, K(G,n)]_+ = \pi_0(K(G,n)) \cong \begin{cases} G & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$
This computation shows that h^* satisfies the dimension axiom, and is therefore naturally isomorphic on CW-pairs (X, A) to the cellular cohomology $H^*_{\mathrm{CW}}(X, A; G)$. If we hadn't already given a direct proof that $[X, K(G,n)]_\circ$ and $H^n_{\mathrm{CW}}(X; G)$ are naturally isomorphic, we could have derived one in this way from the axiomatic framework: we would have instead had to give a more direct proof of the existence of natural weak homotopy equivalences $K(G,n) \to \Omega K(G, n+1)$, showing that $K(G, \cdot)$ is an Ω-spectrum. This is the approach taken in [Hat02, tD08].
- Remark: It is interesting to note that according to the *Brown representability theorem*, all pointed cohomology theories on CW-complexes arise from Ω-spectra as in the construction above. There is a proof of this theorem in [Hat02, §4.E].
- Definition: The **homotopy groups of a spectrum** $E = \{E_n\}$ are defined for each $n \in \mathbb{Z}$ by
$$\pi_n(E) := \operatorname{colim}_{k \to \infty} \pi_{n+k}(E_k),$$
i.e. the direct limit of the sequence of (mostly) abelian groups defined by
$$\begin{tikzcd} \dots \arrow[r] & \pi_{n+k}(E_k) \arrow[r] \arrow[dr, "\Sigma"] & \pi_{n+k+1}(E_{k+1}) \arrow[r] \arrow[dr, "\Sigma"] & \pi_{n+k+2}(E_{k+2}) \arrow[r] & \dots \\ & & \pi_{n+k+1}(\Sigma E_k) \arrow[u, "e_*"'] & \pi_{n+k+2}(\Sigma E_{k+1}) \arrow[u, "e_*"'] & \end{tikzcd}$$
For example, plugging in the suspension spectrum $\Sigma^\bullet X = \Sigma^\bullet S^0 \wedge X$ of a space X reproduces the stable homotopy groups of X. Since the stable homotopy groups define a homology theory, this hints at a more general way to construct homology theories out of spectra...
- Theorem: Any spectrum $E = \{E_n\}$ determines a pointed homology theory $\widetilde{h}_*$ defined by
$$\widetilde{h}_n(X) := \pi_n(E \wedge X),$$
with suspension isomorphisms $\sigma_* : \widetilde{h}_n(X) \to \widetilde{h}_{n+1}(\Sigma X)$ determined by the sequence of maps $\pi_{n+k}(E_k \wedge X) \to \pi_{n+k+1}(E_k \wedge \Sigma X)$ defined as the compositions
$$\pi_{n+k}(E_k \wedge X) \xrightarrow{\Sigma} \pi_{n+k+1}\big(\Sigma(E_k \wedge X)\big) \longrightarrow \pi_{n+k+1}(E_k \wedge \Sigma X),$$
in which the second homomorphism is induced by the canonical bijection $\Sigma(E_k \wedge X) = (E_k \wedge X) \wedge S^1 \to E_k \wedge (X \wedge S^1) = E_k \wedge (\Sigma X)$, which is continuous since S^1 is friendly.
- Corollary: Here is a somewhat exotic-looking formula for the singular homology of any CW-complex X with coefficients in an abelian group G:
$$H_n(X; G) \cong \pi_n(K(G, \cdot) \wedge X_+) = \operatorname{colim}_{k \to \infty} \pi_{n+k}\big(K(G,k) \wedge X_+\big).$$
- Brief remarks on the proof of the theorem: We proved the special case $E = \Sigma^\bullet S^0$ when we talked about the stable homotopy groups, and I would feel dishonest claiming that the general case is a completely straightforward extension of this, but indeed, most of the essential

ideas appeared there. I will admit that this is one point where I can clearly see the appeal of working in the compactly-generated category, though I would be surprised if it is truly necessary. For instance, in the proof that the suspension maps σ_* are isomorphisms, one needs to consider in particular whether the map $\pi_{n+k+1}\big(\Sigma(E_k \wedge X)\big) \to \pi_{n+k+1}(E_k \wedge \Sigma X)$ induced by the canonical bijection $(E_k \wedge X) \wedge S^1 \to E_k \wedge (X \wedge S^1)$ is an isomorphism. In the compactly-generated category this is obvious, because product topologies are adjusted in a way that makes the smash product associative without restrictions.[57]

- The point of expressing $H_n(X; G)$ as a homotopy group of a spectrum is not that this makes it easier to compute, but rather that the same perspective can lend us insight into the computation of *other* (generalized) homology theories. In particular, the coefficient groups of bordism theory are computed by expressing them as homotopy groups.

Suggested reading. Much of what we talked about this week is discussed in [tD08, §7.6 and §7.7], as well as [Hat02, §4.3 and §4.F]. I recommend reading Hatcher's take on homology theories derived from spectra (and the connection with stable homotopy groups in particular) before attempting to understand tom Dieck.

Exercises (also for the Übung on 4.07.2024).

Exercise 11.1. In this exercise, X is a CW-complex with a 0-cell as a base point, and G is a group, which need not generally be abelian.

(a) Suppose $f : X \to K^{\mathrm{w}}(G, 1)$ is a map that sends the 0-skeleton of X to the base point of $K^{\mathrm{w}}(G, 1)$, and $\gamma : (I, \partial I) \to (K^{\mathrm{w}}(G, 1), *)$ is a pointed loop whose homotopy class lies in the center of $G = \pi_1(K^{\mathrm{w}}(G, 1))$. Show that there exists a free homotopy $H : X \times I \to K^{\mathrm{w}}(G, 1)$ of f to itself such that for every 0-cell $x \in X$, $H(x, \cdot)$ is the path γ.

(b) Show that if G is abelian, then the action of $\pi_1(K^{\mathrm{w}}(G, 1)) = G$ on $[X, K^{\mathrm{w}}(G, 1)]_+$ is trivial; in particular, the natural map $[X, K^{\mathrm{w}}(G, 1)]_+ \to [X, K^{\mathrm{w}}(G, 1)]_\circ$ defined by forgetting base points is a bijection.
Hint: You can prove this by induction on the skeleta of X, but you might notice that the initial step in the induction is the tricky part. That's what part (a) is meant to help with.

Exercise 11.2. You may or may not have noticed, but the definition of the bijection $[X, K^{\mathrm{w}}(G, n)]_\circ \to H^n_{\mathrm{CW}}(X; G)$ given in lecture didn't strictly make sense in the case $n = 0$, since one cannot technically define an element of $\pi_0(K^{\mathrm{w}}(G, 0))$ as a homotopy class of maps $f \circ \Phi_\alpha : (\mathbb{D}^0, \partial\mathbb{D}^0) \to (K^{\mathrm{w}}(G, 0), *)$.

(a) Give a sensible definition of a natural bijection $[X, K^{\mathrm{w}}(G, 0)]_\circ \to H^0_{\mathrm{CW}}(X; G)$ for CW-complexes X and abelian groups G, and verify that it is indeed a bijection.

(b) Show that for *pointed* homotopy classes, there is similarly a natural bijection

$$[X, K^{\mathrm{w}}(G, n)]_+ \xrightarrow{\cong} H^n_{\mathrm{CW}}(X, \{*\}; G)$$

for all $n \geqslant 0$, abelian groups G and pointed CW-complexes X, thus identifying sets of pointed homotopy classes with *reduced* cellular cohomology.
Remark: The result of Exercise 11.1(b) is convenient to know for this, but it is also possible to solve this problem more directly and deduce Exercise 11.1(b) from it. Try to think about it from both perspectives.

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[57] My educated guess is that the map $(E_k \wedge X) \wedge S^1 \to E_k \wedge (X \wedge S^1)$ is at least a homotopy equivalence if X (and possibly also E_k?) is well pointed, and that would be good enough. Proofs of such things are unfortunately hard to find in the literature, but there is a hint in the comments at `https://math.stackexchange.com/questions/3353411/associativity-of-smash-product-up-to-homotopy-equivalence)`.

(c) Reprove the existence of the natural bijection $[X, K^{\mathrm{w}}(G,n)]_\circ \to H^n_{\mathrm{CW}}(X;G)$ as a corollary of the bijection in part (b).

Exercise 11.3. Assume $n \geqslant 1$ and G is an abelian group. Show that for any $(n-1)$-connected CW-complex X, the natural map

$$[X, K^{\mathrm{w}}(G,n)]_+ \to \mathrm{Hom}(\pi_n(X), G)$$

sending a pointed homotopy class $[f]$ to the induced homomorphism $f_* : \pi_n(X) \to \pi_n(K^{\mathrm{w}}(G,n))$ is a bijection.
Hint: Factor it through $H^n_{\mathrm{CW}}(X;G)$ and $\mathrm{Hom}(H_n^{\mathrm{CW}}(X;\mathbb{Z}), G)$, using the universal coefficient theorem and the Hurewicz theorem.

Exercise 11.4. The following clarifies two details in the prescription for constructing an Ω-spectrum $E' = \{E'_n\}$ out of an arbitrary spectrum $E = \{E_n\}$.

(a) Suppose $\{X_\alpha\}_{\alpha \in J}$ is a direct system in the category Top_* over a directed set $(J, \prec)$, with pointed maps $\varphi_{\beta\alpha} : X_\alpha \to X_\beta$ defined for every $\alpha \prec \beta$ in J, and let $\varphi_\alpha : X_\alpha \to X_\infty := \mathrm{colim}\{X_\alpha\}$ denote the natural maps to the direct limit. The maps $\Omega\varphi_{\beta\alpha} : \Omega X_\alpha \to \Omega X_\beta$ then similarly define a direct system $\{\Omega X_\alpha\}_{\alpha \in J}$, while the maps $\Omega\varphi_\alpha : \Omega X_\alpha \to \Omega X_\infty$ make ΩX_∞ a target of this system, so that by the universal property, there is a uniquely-determined pointed map

$$\mathrm{colim}\{\Omega X_\alpha\} \xrightarrow{u} \Omega X_\infty.$$

Show that u is a weak homotopy equivalence.
Hint: All colimits are coequalizers of coproducts, so X_∞ and $\mathrm{colim}\{\Omega X_\alpha\}$ are quotients of wedge sums, and any given map of a sphere (or homotopy of such maps) into such a space will then touch only finitely-many summands beyond the base point. Use adjunction to get rid of Ω, so that you only need to think about maps of spheres to X_α for some $\alpha \in J$.

(b) Show that for every spectrum $E = \{E_n\}$ with structure maps $e : \Sigma E_n \to E_{n+1}$, there exists a spectrum $E' = \{E'_n\}$ with structure maps $e' : \Sigma E'_n \to E'_{n+1}$ together with weak homotopy equivalences $\varphi_n : E'_n \to E_n$ such that the diagram

$$\begin{tikzcd} \Sigma E'_n \arrow[r, "\Sigma\varphi_n"] \arrow[d, "e'"] \arrow[dr, phantom, "\sim"] & \Sigma E_n \arrow[d, "e"] \\ E'_{n+1} \arrow[r, "\varphi_{n+1}"] & E_{n+1} \end{tikzcd}$$

commutes up to homotopy and the spaces E'_n are all pointed CW-complexes. Deduce that E' is then an Ω-spectrum if the adjoint maps $\hat{e} : E_n \to \Omega E_{n+1}$ are weak homotopy equivalences.

12. Week 12

Lecture 20 (1.07.2024): Fiber bundles.

- Definition: A map $p : E \to B$ is a **fiber bundle** (with **base** B, **total space** E and **fibers** $E_x := p^{-1}(x)$) if every point $x \in B$ has a neighborhood $\mathcal{U} \subset B$ on which there is a **local trivialization**, meaning a homeomorphism $\Phi : E|_{\mathcal{U}} := p^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ fitting into the diagram

$$\begin{tikzcd} E|_{\mathcal{U}} \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & \mathcal{U} \times F \arrow[dl, "\mathrm{pr}_1"] \\ & \mathcal{U} & \end{tikzcd},$$

where F can be any space and pr_1 denotes the projection map to the first factor. In particular, the local trivialization Φ maps the fiber E_y over each point $y \in \mathcal{U}$ homeomorphically to $\{y\} \times F$. The fiber bundle is said to be (globally) **trivial** if it admits such a trivialization with $\mathcal{U} := B$; similarly, the projection $B \times F \to B$ is often called the **trivial bundle** with fiber F.

- Remark: In the definition above, F does not absolutely need to be the same space for all local trivializations, though if B is at least connected, then it follows from the definition that all the fibers will at least be homeomorphic. This means that for fiber bundles over connected bases, there is no loss of generality if we always use a fixed space F as the so-called **standard fiber** for all local trivializations; we will typically do this.
- Definition: A **section** of a fiber bundle $p : E \to B$ is a map $s : B \to E$ such that $p \circ s = \mathrm{Id}_B$, i.e. the value of $s(x)$ at each point $x \in B$ lives in the fiber E_x. We denote the space of sections by

$$\Gamma(E) := \{\text{sections of } p : E \to B\},$$

and since $E|_{\mathcal{U}} \xrightarrow{p} \mathcal{U}$ is also naturally a fiber bundle for any subset $\mathcal{U} \subset B$, we can also speak of *sections of* E *over* $\mathcal{U}$ and write $\Gamma(E|_{\mathcal{U}})$ for the set of these. It is sometimes useful to put a topology on $\Gamma(E)$, and the appropriate choice of topology typically depends on the context; we will not need this, but we will want to talk about **homotopy classes of sections**, which are defined in an obvious way: $s_0, s_1 \in \Gamma(E)$ are homotopic if there is a homotopy $s : B \times I \to E$ from s_0 to s_1 such that $s_t : B \to E$ is a section for every $t \in I$. We will see in examples below that it is often a nontrivial question whether any globally-defined section exists, though they always exist locally: on any region $\mathcal{U} \subset B$ for which there exists a local trivialization $E|_{\mathcal{U}} \cong \mathcal{U} \times F$, one can write down a section just by choosing a map $\mathcal{U} \to F$, e.g. a constant map.
- Basic questions of *obstruction theory* for a fiber bundle $p : E \to B$:
  (1) Do there exist any (global) sections $s \in \Gamma(E)$?
  (2) How can we measure whether two sections $s_0, s_1 \in \Gamma(E)$ are homotopic?
- Here is a real-world example in which one would want to answer the questions above. Suppose M is a smooth manifold of even dimension $2n$. Its tangent bundle $TM = \bigcup_{x \in M} T_x M \to M$ is then an example of a fiber bundle, whose fibers are the tangent spaces (real $2n$-dimensional vector spaces), and various natural constructions that one can perform on vector spaces can also be used to create new fiber bundles out of TM. One such construction is to define

$$\mathcal{J}(TM) := \bigcup_{x \in M} \mathcal{J}(T_x M), \qquad \text{where } \mathcal{J}(T_x M) := \left\{ \text{linear maps } T_x M \xrightarrow{J} T_x M \;\middle|\; J^2 = -\mathbb{1} \right\},$$

a fiber bundle over M whose fiber $\mathcal{J}(T_x M)$ at each point $x \in M$ is the space of *complex structures* on the tangent space $T_x M$, i.e. choosing an element $J \in \mathcal{J}(T_x M)$ is equivalent to endowing the real $2n$-dimensional vector space $T_x M$ with a complex n-dimensional vector space structure in which J defines scalar multiplication by i. The sections of $\mathcal{J}(TM)$ are then known as **almost complex structures** on M. An almost complex structure exists naturally if M is a complex n-dimensional manifold, i.e. if it is covered by local $\mathbb{C}^n$-valued charts whose transition maps are holomorphic: the tangent spaces are then naturally complex vector spaces, so that multiplication by i defines a section $J \in \Gamma(\mathcal{J}(TM))$. One of the fundamental questions to ask in complex geometry is which real $2n$-dimensional smooth manifolds admit (not just *almost*) **complex structures**, meaning that they can arise as n-dimensional complex manifolds. It is a famously open question, for instance, whether S^6

admits a complex structure.[58] We see that the existence of a section $J \in \Gamma(\mathcal{J}(TM))$ is a clearly necessary condition for this. It is very far from being a sufficient condition: getting from an almost complex structure to a complex manifold structure is much harder, and is a fundamentally differential-geometric rather than topological problem, but the *almost* complex question must always be answered first, and it does have a well-understood answer using methods of algebraic topology. (Spoiler: S^6 has one, of course.)

One further remark on this: If two almost complex structures $J_0, J_1 \in \Gamma(\mathcal{J}(TM))$ are homotopic, then the homotopy theorem that we will discuss next time implies that the complex vector bundles (TM, J_0) and (TM, J_1) are isomorphic, which is a clearly necessary (but also not sufficient) condition for two complex manifolds (M, J_0) and (M, J_1) to be holomorphically diffeomorphic. The existence of a homotopy of sections from J_0 to J_1 is also something that can be clarified completely using algebro-topological methods.

- Preamble to a lengthy definition: The fiber bundles in the definition above do not have quite enough structure to be useful in most applications, but typical examples naturally have more structure, and the following discussion of G-bundles and structure groups provides a very general framework for encoding such structure. Several of the definitions could be simplified considerably if we only wanted to consider *effective* group actions, but that would exclude some important and interesting examples.
- Definitions: In the following, assume G is a topological group with identity element denoted by $e \in G$, and F is a topological space on which G acts continuously from the left. Some of the terminology used below is not completely standardized, but the definitions would get longer if I didn't use it.
  (1) A **bundle atlas** with **standard fiber** F on the fiber bundle $p : E \to B$ is a collection of local trivializations
$$\mathbf{\Phi} = \left\{ E_{\mathcal{U}_\alpha} \xrightarrow{\Phi_\alpha} \mathcal{U}_\alpha \times F \right\}_{\alpha \in J}$$
  such that the sets $\{\mathcal{U}_\alpha \subset B\}_{\alpha \in J}$ form an open covering of B.
  (2) A **system of G-valued transition functions** on B consists of an open covering $\{\mathcal{U}_\alpha \subset B\}_{\alpha \in J}$ and a collection of maps
$$\mathcal{T} = \left\{ \mathcal{U}_\alpha \cap \mathcal{U}_\beta \xrightarrow{g_{\beta\alpha}} G \right\}_{(\alpha,\beta) \in J \times J}$$
  that satisfy the following two conditions:
    (i) $g_{\alpha\alpha} = e$ on $\mathcal{U}_\alpha$ for each $\alpha \in J$;
    (ii) $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$ for each $\alpha, \beta, \gamma \in J$.
  Both relations together imply that the values of the functions $g_{\beta\alpha}$ and $g_{\alpha\beta}$ are inverse to each other at every point, and the second condition can thus be rewritten as
$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = e \qquad \text{on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma.$$
  In this form, it is popularly known as the **cocycle condition**, for reasons that might be apparent to you if you've seen Čech or sheaf cohomology before, and probably not if you haven't. Suffice it to say that in certain situations (which will not immediately concern us), this condition can be interpreted as saying that the collection of functions $g_{\beta\alpha}$ defines a 1-cocycle in some cochain complex.

  If a bundle atlas $\mathbf{\Phi}$ and system of transition functions $\mathcal{T}$ with the same underlying open cover $\{\mathcal{U}_\alpha \subset B\}_{\alpha \in J}$ are given, we will say that $\mathcal{T}$ defines **transition data** for $\mathbf{\Phi}$ if for every

[58] There are papers on the arXiv claiming both positive and negative answers to that question, some of them by quite famous people; see
https://mathoverflow.net/questions/1973/is-there-a-complex-structure-on-the-6-sphere.

$(\alpha, \beta) \in J \times J$, the so-called **transition map** $(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F$ defined via the diagram

$$\begin{tikzcd} & E|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[dl, "\Phi_\alpha"', "\cong"] \arrow[dr, "\Phi_\beta", "\cong"'] & \\ (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \arrow[rr] & & (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \end{tikzcd}$$

takes the form $(x, p) \mapsto (x, g_{\beta\alpha}(x)p)$. Notice that if the action $G \times F \to F$ is effective, then the local trivializations uniquely determine the functions $g_{\beta\alpha}$ via this relation; the existence of such functions $g_{\beta\alpha}$ then imposes a nontrivial condition on the local trivializations, since it will typically not be true that every map $(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times F$ arising from such a diagram can be expressed in terms of the G-action on F. But if that condition holds and the uniquely-determined functions $g_{\beta\alpha}$ exist, then the cocycle condition is automatically satisfied, and we now have a straightforward way to define what a *G-structure* on a fiber bundle $p : E \to B$ should be: it is a maximal bundle atlas with the property that the (uniquely-determined!) transition maps are all expressible as above in terms of the G-action on F. This definition does not require any explicit statement of the cocycle condition, and it makes sense in the effective case because the transition functions are uniquely determined: in particular, *any* bundle atlas with G-valued transition data then determines a unique one that is maximal. In the non-effective case, the transition data is no longer uniquely determined by the bundle atlas, and we therefore need the more cumbersome definitions below, whose purpose is mainly to clarify what it means for two bundle atlases together with choices of G-valued transition data on a fiber bundle to be equivalent. We will comment at the end of this lecture on what would be lost if we did not explicitly require the cocycle condition in this situation (cf. Theorem 1).

(3) A **morphism** $\mathcal{T}^1 \to \mathcal{T}^2$ between two systems of G-valued transition functions $\mathcal{T}^i = \left\{ g^{(i)}_{\beta\alpha} : \mathcal{U}^i_\alpha \times \mathcal{U}^i_\beta \to G \right\}_{(\alpha,\beta) \in J^i \times J^i}$ for $i = 1, 2$ consists of a collection of functions

$$\left\{ \mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta \xrightarrow{h_{\beta\alpha}} G \right\}_{(\alpha,\beta) \in J^1 \times J^2}$$

that satisfy the relations
  (iii) $h_{\alpha\beta} g^{(1)}_{\beta\gamma} = h_{\alpha\gamma}$ on $\mathcal{U}^2_\alpha \cap \mathcal{U}^1_\beta \cap \mathcal{U}^1_\gamma$ for each $\alpha \in J^2$ and $\beta, \gamma \in J^1$;
  (iv) $g^{(2)}_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma}$ on $\mathcal{U}^2_\alpha \cap \mathcal{U}^2_\beta \cap \mathcal{U}^1_\gamma$ for each $\alpha, \beta \in J^2$ and $\gamma \in J^1$.

Interpretation: Suppose G acts effectively on F, and $\mathcal{T}^1$ and $\mathcal{T}^2$ are the (uniquely-determined) transition data corresponding to two bundle atlases $\mathbf{\Phi}^1$ and $\mathbf{\Phi}^2$ respectively. The union of $\mathbf{\Phi}^1$ and $\mathbf{\Phi}^2$ then defines a new bundle atlas which may or may not admit G-valued transition data; if it does, then the resulting functions $h_{\beta\alpha}$ determined by transition maps $\Phi^2_\beta \circ (\Phi^1_\alpha)^{-1}$ for $\alpha \in J^1$ and $\beta \in J^2$ automatically satisfy the two relations above—they are a consequence of the cocycle condition for the enlarged system of transition functions—thus giving rise to a morphism $\mathcal{T}^1 \to \mathcal{T}^2$. In the non-effective case, the functions $h_{\beta\alpha}$ in this situation are not uniquely determined by the two bundle atlases, so it becomes necessary to impose conditions (iii) and (iv) explicitly.

(4) A **G-bundle atlas** for the fiber bundle $p : E \to B$ consists of the data $\mathcal{A} = (\mathbf{\Phi}, \mathcal{T}, \rho)$, where $\mathbf{\Phi}$ is a bundle atlas, $\mathcal{T}$ is a system of G-valued transition functions with the same underlying open cover of B, and ρ is a continuous left action of G on the standard fiber F of the bundle atlas, such that $\mathcal{T}$ and this action define transition data for $\mathbf{\Phi}$.

(5) Assume $p^i : E^i \to B$ for $i = 1, 2$ are fiber bundles with the same base B, and they have G-bundle atlases $\mathcal{A}^i = (\mathbf{\Phi}^i, \mathcal{T}^i, \rho)$ with the same standard fiber F and G-action ρ. A **G-bundle isomorphism** $(E^1, \mathcal{A}^1) \to (E^2, \mathcal{A}^2)$ then consists of a homeomorphism $\Psi : E^1 \to E^2$ and a morphism $\{h_{\beta\alpha}\} : \mathcal{T}^1 \to \mathcal{T}^2$ such that for every $\alpha \in J^1$ and $\beta \in J^2$, the map $(\mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta) \times F \to (\mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta) \times F$ determined by the diagram

$$\begin{tikzcd} E^1|_{\mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta} \arrow[r, "\Psi"] \arrow[d, "\Phi^1_\alpha", "\cong"'] & E^2|_{\mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta} \arrow[d, "\Phi^2_\beta", "\cong"'] \\ (\mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta) \times F \arrow[r] & (\mathcal{U}^1_\alpha \cap \mathcal{U}^2_\beta) \times F \end{tikzcd}$$

takes the form $(x, p) \mapsto h_{\beta\alpha}(x)p$.
Exercise: G-bundle isomorphisms deserve to be called "isomorphisms," i.e. they can be composed and inverted in a natural way.

(6) A **G-bundle** (or **fiber bundle with structure group** G) is a fiber bundle $p : E \to B$ equipped with an equivalence class of G-bundle atlases, where two G-bundle atlases $\mathcal{A}^1, \mathcal{A}^2$ are considered equivalent if there exists a G-bundle isomorphism $(E, \mathcal{A}^1) \to (E, \mathcal{A}^2)$ whose underlying homeomorphism $E \to E$ is the identity map.

Exercise: If the action $G \times F \to F$ is effective, then a G-bundle with standard fiber F has a maximal G-bundle atlas that contains all G-bundle atlases in the correct equivalence class.

- Examples: The general principle is that any structures on F that are preserved by the G-action get inherited by all the fibers of the bundle.
  (1) Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Taking standard fiber $F := \mathbb{K}^n$ and structure group $G := \mathrm{GL}(n, \mathbb{K})$ with its usual linear action on $\mathbb{K}^n$ makes all the fibers E_x of a G-bundle $p : E \to B$ into n-dimensional vector spaces over $\mathbb{K}$, and $p : E \to B$ is in this case called a (real or complex) **vector bundle** of **rank** n. One can specialize this example further:
    (a) Taking G to be one of the subgroups $\mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$ or $\mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C})$ means that the fibers E_x are equipped with inner products that vary continuously with x. Such a structure on a vector bundle is often called a **bundle metric**; in the case where E is the (real) tangent bundle of a smooth manifold M, it is a **Riemannian metric** on M.
    (b) In the case $\mathbb{K} = \mathbb{R}$, taking G to be the subgroup $\mathrm{GL}_+(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$ of matrices with positive determinant means that the fibers E_x are *oriented* real vector spaces, with orientations that vary continuously with x.
    (c) Here is the most popular non-effective example: Recall from Exercise 7.8 that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for all $n \geqslant 3$. The universal cover of $\mathrm{SO}(n)$ is a group called $\mathrm{Spin}(n)$ that comes with a degree 2 covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.[59] If $E \to B$ is a real vector bundle of rank n with an $\mathrm{SO}(n)$-structure, meaning it carries both a bundle metric and an orientation, then it may or may not also admit a **spin structure**, meaning a system of transition functions valued in $\mathrm{Spin}(n)$ that lift the given $\mathrm{SO}(n)$-valued transition functions while still satisfying the cocycle condition. Spin structures were originally motivated by quantum mechanics, but they now play an important role in differential topology due to invariants

[59]There is also a natural definition of $\mathrm{Spin}(2)$ that comes with a degree 2 covering map to $\mathrm{SO}(2)$; the difference is just that it is not the universal cover in that case, and in fact is isomorphic to $\mathrm{SO}(2) \cong S^1$. In spite of this isomorphism, it is not true that an $\mathrm{SO}(2)$-structure on a bundle can always be lifted to a $\mathrm{Spin}(2)$-structure, because not all maps to S^1 can be lifted to its double cover.

of smooth manifolds that are based on gauge-theoretic PDEs determined by a spin structure.

(2) Under suitable assumptions on the standard fiber F (e.g. compact and Hausdorff is good enough), *any* fiber bundle $p : E \to B$ with fibers homeomorphic to F can be regarded as a G-bundle with $G := \mathrm{Homeo}(F)$ acting on F in the obvious way. The reason this isn't true for arbitrary spaces F is that some conditions are required in order for $\mathrm{Homeo}(F)$ with the compact-open topology to be a topological group, and for its obvious action on F to be continuous. This is yet another problem that simply goes away if one decides to work in the compactly-generated category, accepting the subtle differences in standard definitions that come with that decision. But in practice, most interesting fiber bundles have strictly more structure than this, e.g. with structure groups that are finite-dimensional Lie groups, and are thus much nicer to work with than $\mathrm{Homeo}(F)$.

(3) There are interesting examples in which the standard fiber F can be identified with the group G, acting on itself by left multiplication. The extra structure this imparts upon the fibers E_x is then a right G-action that is free and transitive; it can be written in any choice of local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times G$ as the right action

$$(\mathcal{U}_\alpha \times G) \times G \to \mathcal{U}_\alpha \times G : (x, g)h := (x, gh)$$

defined on $\mathcal{U}_\alpha \times G$, and this formula determines a global action $E \times G \to E$ that is independent of choices of local trivializations because the action of G on itself by right multiplication commutes with its action by left multiplication. This type of G-bundle is called a **principal fiber bundle** (often abbreviated PFB).

(a) For any vector bundle $E \to B$ of rank n over $\mathbb{K}$, there is an associated principal $\mathrm{GL}(n, \mathbb{K})$-bundle $FE \to B$ called the **frame bundle** of E, whose fiber $F(E_x)$ over each point $x \in B$ is the set of bases of the vector space E_x with its obvious topology. Equivalently, the space $F(E_x)$ of **frames** for E_x can be identified with the space of linear isomorphisms $\phi : \mathbb{K}^n \to E_x$, which has an obvious right $\mathrm{GL}(n, \mathbb{K})$-action

$$\phi g := \phi \circ g$$

defined via composition of linear transformations.
Remark: This example shows clearly that global sections of a fiber bundle do not always exist. Taking E for instance to be the tangent bundle of S^2, a global section of the associated frame bundle $F(TS^2)$ would mean a pair of vector fields on S^2 that form a basis of the tangent space at every point. This would be a strong contradiction to the theorem that you cannot "comb the hair on a sphere".

(b) The Hopf fibrations $S^{2n+1} \to \mathbb{CP}^n$ are all principal S^1-bundles, where S^1 can be interpreted as a synonym for $\mathrm{U}(1)$ with its natural right action on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by scalar multiplication.

(c) The fibration $\mathrm{SO}(n) \to S^{n-1} : A \mapsto Ae_1$ in Exercise 7.8 is a principal $\mathrm{SO}(n-1)$-bundle if we identify $\mathrm{SO}(n-1)$ with the subgroup

$$\left\{ \begin{pmatrix} 1 & \\ & \mathbf{A} \end{pmatrix} \middle| \mathbf{A} \in \mathrm{SO}(n-1) \right\} \subset \mathrm{SO}(n),$$

acting on the total space $\mathrm{SO}(n)$ by right multiplication.

Remark: For many fibrations that are useful in computations, the easiest way to prove that they are fibrations is by proving that they are principal fiber bundles.

We'll discuss in the next lecture why that implies the homotopy lifting property. The proposition below reveals that proving something is a principal bundle is typically easier than writing down local trivializations: if one already sees a fiber-preserving right G-action that is free and transitive on every fiber, then one only still needs to check that *local sections* exist, which is often obvious.

(4) For any spaces B and F, one can define a **trivial G-bundle** $E := B \times F \xrightarrow{\mathrm{pr}_1} B$ by writing down the obvious bundle atlas with only one local (but actually global) trivialization, together with the obvious system of G-valued transition functions consisting only of the constant function $g_{\alpha\alpha}(x) := e \in G$ for all $x \in B$.

- Proposition: For any principle G-bundle $p : E \to B$ and open subset $\mathcal{U} \subset B$, there is a natural bijection between the set $\Gamma(E|_{\mathcal{U}})$ of sections of E over $\mathcal{U}$ and the set of G-bundle isomorphisms between $E|_{\mathcal{U}}$ and the trivial principal G-bundle $\mathcal{U} \times G \to \mathcal{U}$.
Quick proof: To turn a section $s : \mathcal{U} \to E$ into a trivialization $\Phi : E|_{\mathcal{U}} \to \mathcal{U} \times G$, one can exploit the free and transitive right G-action on the fibers and define
$$\Phi^{-1}(x, g) := s(x)g.$$
Remark: It follows that a principle bundle is trivial if and only if it admits a global section, so one should not expect principal bundles to admit global sections in typical cases. (By contrast, vector bundles always have a distinguished global section, the zero-section, but this is only because the fibers of a vector bundle are contractible spaces and thus topologically uninteresting. The existence of a global section tells us nothing about whether a vector bundle is trivial, and in fact, it is trivial if and only if its associated frame bundle—a principal bundle with the same structure group—is trivial.)
- Theorem 1: Given a continuous group action $G \times F \to F$ and a space B, there is a natural bijection between the following sets or equivalence classes:
  - G-bundles over B with standard fiber F, up to G-bundle isomorphism;
  - Systems $\mathcal{T}$ of G-valued transition functions on B, up to morphism $\mathcal{T}^1 \to \mathcal{T}^2$.

  I sometimes refer to elements in the second set as **abstract G-bundles**, as they contain much of the data that defines a fiber bundle, but without any well-defined total space or fibers. An abstract G-bundle thus encodes *many* possible fiber bundles that can have different fibers but the same structure group and transition functions.
Proof of the theorem (sketch): The details are straightforward once one has seen the recipe for constructing a G-bundle $E \to B$ with standard fiber F out of a given system $\{g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G\}_{(\alpha,\beta)\in J\times J}$ of G-valued transition functions. We define
$$E := \coprod_{\alpha\in J} (\mathcal{U}_\alpha \times F) \Big/ \sim,$$
with the equivalence relation defined by
$$\mathcal{U}_\alpha \times F \ni (x, p) \sim (x, g_{\beta\alpha}(x)p) \in \mathcal{U}_\beta \times F$$
for all $\alpha, \beta \in J$, $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $p \in F$. The fact that this really is an equivalence relation is due to the cocycle condition, and this is where we would lose something important if we had not explicitly assumed the cocycle condition in the setting of non-effective actions.
- Theorem 2: Every G-bundle $p : E \to B$ with standard fiber F can be recovered (up to G-bundle isomorphism) from a corresponding principal G-bundle $P \to B$ via the following prescription. We define $P \to B$ as the principal bundle constructed via Theorem 1 out of the right-action of G on itself and the same system of G-valued transition functions that E has. The original bundle E is then isomorphic to the so-called **associated bundle**

$P \times_G F \to B$ defined by

$$P \times_G F := (P \times F) \big/ G, \qquad \text{where} \qquad g(\phi, p) := (\phi g^{-1}, gp),$$

with the obvious projection $P \times_G F \to B$ determined by the principal bundle $P \to B$.
Proof sketch: Thanks to Theorem 1, one just needs to check that the associated bundle $P \times_G F$ can be presented as a G-bundle with standard fiber F and the same system of G-valued transition functions as the principal bundle P.

- Remark: The message of Theorem 2 is that a large number of the questions one would like to answer about general fiber bundles can be answered by focusing on the special case of principal bundles, which contain the fundamental genetic code of all interesting fiber bundles. Principal bundles give us the option of avoiding most discussions of local trivializations and transition functions, as these can be recovered fully from the data encoded in a free and transitive right G-action. For this reason, one finds many books that express definitions of fundamental notions such as *reduction of the structure group* and *spin structures* purely in terms of the global properties of principal bundles. For example, an alternative formulation of the definition I gave above for a G-bundle would first define a *principal* G-bundle to be a fiber bundle that is equipped with a fiber-preserving right G-action which is free and transitive on every fiber, and then say that an arbitrary fiber bundle $p : E \to B$ is a *G-bundle* if it is equipped with a fiber-preserving homeomorphism to the associated bundle $P \times_G F \to B$ for some principal G-bundle $P \to B$ and some left action of G on a space F. I tend to think that definitions in this form look rather mysterious and unrevealing, but they are also useful in practice.

Lecture 21 (4.07.2024): Obstruction theory. This lecture covers three topics that are somewhat separate, but the first two are prerequisites for the third. Topic 1 concerns pullbacks and the homotopy theorem for fiber bundles.

- Easy theorem: For any G-bundle $p : E \to B$ with transition functions $\{g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G\}$, the pullback $f^*E \to B'$ via a map $f : B' \to B$ is also naturally a G-bundle, with transition functions $\{g_{\beta\alpha} \circ f : f^{-1}(\mathcal{U}_\alpha) \cap f^{-1}(\mathcal{U}_\beta) \to G\}$.
Remark: One can also say this about general fiber bundles without mentioning the structure group, and the proof is easy because there is an obvious way to produce a local trivialization for f^*E out of any local trivialization of E. We sometimes call $f^*E \to B'$ the **induced fiber bundle** determined by $p : E \to B$ and the map $f : B' \to B$.
- Homotopy theorem for fiber bundles: For any G-bundle $p : E \to B$ whose base B is a paracompact space and any two homotopic maps $f, g : B' \to B$, there is a G-bundle isomorphism $f^*E \cong g^*E$.
- Remark: Paracompactness is a technical condition that arises whenever one needs to piece together local constructions into a global construction using a partition of unity. All metrizable spaces are paracompact, and I have never yet encountered a good reason to worry about spaces that aren't paracompact.
- Corollary 1: A fiber bundle over a paracompact base is a fibration.
Proof: The usual homotopy lifting property

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{H}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, "\widetilde{H}"] & B \end{tikzcd}$$

has the following interpretation: Every section $\widetilde{H}_0$ of the pullback bundle $H^*E \to X \times I$ restricted to $X \times \{0\}$ can be extended as a section to the rest of $X \times I$. This statement is obvious if H is a trivial homotopy of the form $H(x,t) = H_0(x)$, because the fibers $(H^*E)_{(x,t)} = E_{H(x,t)} = E_{H_0(x)}$ then depend on x but not t, so one can define $\widetilde{H}$ by $\widetilde{H}(x,t) := \widetilde{H}_0(x)$. But any map $H : X \times I \to B$ is homotopic rel $X \times \{0\}$ to such a trivial homotopy—just retract each of the paths $t \mapsto H(x,t)$ back to its starting point—so by the homotopy theorem, solving the problem in the case of a trivial homotopy also solves it in general.

- Corollary 2: Any G-bundle $E \to B$ over a contractible paracompact space B is isomorphic to a trivial G-bundle $B \times F \to B$.
- Proof of the homotopy theorem (sketch): I am summarizing an argument that is written up in full detail in [Cut21, Week 4: Fibrations III], with some additional background material in [Cut21, Week 4: Partitions of Unity]. It involves some subtleties that I will mostly suppress, but they are subtleties of *point-set* topology rather than algebraic topology. We will ignore the G-bundle structure and speak only of general fiber bundles, but it will be clear that if a G-bundle structure is present, then the construction respects it.

Note first that if $p : E \to B$ is a fiber bundle and $H : X \times I \to B$ is a homotopy, then the pullback bundles H_0^*E and H_1^*E are just the restrictions of the bundle $H^*E \to X \times I$ to the subsets $X \times \{0\}$ and $X \times \{1\}$, thus it will suffice to prove that two such restrictions of a bundle over a space of the form $X \times I$ are isomorphic. There is a slight subtlety here involving the hypotheses: the theorem assumes B is paracompact, but the domain $X \times I$ of an arbitrary homotopy $X \times I \to B$ need not be. One can deal with this using the notion of *numerably trivial* maps, which I'd rather not define here, but suffice it to say that every fiber bundle over a paracompact space is numerably trivial, every pullback of a numerably trivial fiber bundle is also numerably trivial, and this property will allow us to treat the pullback as if its base were paracompact. (The details behind this are nicely explained in [Cut21, Week 4: Fibrations III].) I'm therefore going to pretend in the following that there is never any need to worry about bases that are not paracompact.

With that understood, the result follows from a more technical statement: Given a fiber bundle $p : E \to B \times I$ with B paracompact, there exists an isomorphism of fiber bundles

$$E \cong \psi^* E \qquad \text{where} \qquad B \times I \xrightarrow{\psi} B \times I : (x,t) \mapsto (x,1).$$

More precisely, this means that there is a map $\Psi : E \to E$ sending each fiber $E_{(x,t)}$ homeomorphically to $(\psi^*E)_{(x,t)} = E_{\psi(x,t)} = E_{(x,1)}$, so it fits into the diagram

$$\begin{tikzcd} E \arrow[r, "\Psi"] \arrow[d, "p"] & E \arrow[d, "p"] \\ B \times I \arrow[r, "\psi"] & B \times I \end{tikzcd} .$$

One sometimes says in this situation that $\Psi : E \to E$ is a **bundle map** and that it **covers** the map $\psi : B \times I \to B \times I$. Restricting the bundle $p : E \to B \times I$ to $B \times \{0\}$, one obtains from Ψ an isomorphism of fiber bundles between the restrictions of E to $B \times \{0\}$ and $B \times \{1\}$, exactly what is needed for the homotopy theorem. The proof of the technical statement can be summarized as follows.

Step 1: Paracompactness implies that one can find a countable open covering $\{\mathcal{U}_n \subset B\}_{n \in \mathbb{N}}$ that is also **locally finite**, meaning that every point $x \in B$ has a neighborhood that intersects at most finitely-many of the sets $\mathcal{U}_n$, and moreover, it is possible to choose the sets $\mathcal{U}_n$ small enough so that E is trivializable over each of the sets $\mathcal{U}_n \times I$, giving local trivializations

$$E|_{\mathcal{U}_n \times I} \cong (\mathcal{U}_n \times I) \times F$$

for all $n \in \mathbb{N}$. That last detail is one of the subtle parts, because E of course admits local trivializations over all sufficiently small subsets of $B \times I$, but $\mathcal{U}_n \times I$ cannot be considered "small" so long as the entire interval I is involved. What one actually needs to do is choose neighborhoods of the form $\mathcal{U}_n \times (t_0 - \epsilon, t_0 + \epsilon) \subset B \times I$ on which E can be trivialized, exploit the compactness of I to establish a lower bound on the sizes $\epsilon > 0$ of subintervals for which this works, and then piece together finitely many trivializations on such subsets to form a trivialization on $\mathcal{U}_n \times I$. Let's assume this has been done. The other piece of auxiliary data that can now be chosen thanks to paracompactness is a collection of functions

$$\{\rho_n : B \to [0, \infty)\}_{n \in \mathbb{N}}$$

such that each ρ_n has its support contained in $\mathcal{U}_n$ and

$$\max_{n \in \mathbb{N}} \rho_n > 0.$$

Note that this maximum is well defined, and is in fact a continuous function on B, because by local finiteness, every point in B has a neighborhood on which only finitely-many of the functions ρ_n are nonzero. The sum $\sum_n \rho_n$ is a well-defined continuous function for the same reason, and if we were doing any of several things in differential geometry for which this kind of data is useful, we would now rescale the functions to assume $\sum_n \rho_n \equiv 1$ and call this collection a **partition of unity**. In the present context, we will do something slightly different but similar, and rescale them so that

$$\max_{n \in \mathbb{N}} \rho_n \equiv 1,$$

thus all of the ρ_n are now bounded above by 1, and at any given point, at least one of them equals 1.

Step 2: For each $n \in \mathbb{N}$, define $\psi_n : B \times I \to B \times I$ by

$$\psi_n(x, t) := (x, \max\{t, \rho_n(x)\}),$$

so ψ_n is the identity map outside of $\mathcal{U}_n \times I$. Then use the chosen local trivialization $E|_{\mathcal{U}_n \times I} \cong (\mathcal{U}_n \times I) \times F$ to define a bundle map $\Psi_n : E \to E$ covering ψ_n like so: we define it over $\mathcal{U}_n \times I$ as $\psi_n \times \mathrm{Id} : (\mathcal{U}_n \times I) \times F \to (\mathcal{U}_n \times I) \times F$ after identifying E on this region with the trivial bundle $(\mathcal{U}_n \times I) \times F$, and then extend it to the rest of $B \times I$ as the identity map.

Step 3: Thanks to local finiteness, the compositions

$$\psi_n \circ \psi_{n-1} \circ \ldots \circ \psi_1 : B \times I \to B \times I$$

have a well-defined and continuous limit as $n \to \infty$, since every point has a neighborhood on which at most finitely many of the maps appearing in any of these compositions are not the identity map. Moreover, since at least one of the functions ρ_n equals 1 at each point, the limit of these compositions is just $\psi(x, t) = (x, 1)$. We then obtain a bundle isomorphism Ψ covering ψ as a similar limit,

$$\Psi := \lim_{n \to \infty} (\Psi_n \circ \Psi_{n-1} \circ \ldots \circ \Psi_1).$$

Topic 2 concerns a special class of fiber bundles whose fibers are discrete abelian groups; we will need to use these below as "twisted coefficient groups" for a generalized version of cohomology.

- Definition: A **local system of abelian groups** $\{G_x\}_{x\in B}$ over a space B is a fiber bundle $p : \mathcal{G} \to B$ with fibers $p^{-1}(x) = G_x$ such that the standard fiber is a discrete abelian group on which the structure group of the bundle acts by group isomorphisms. Equivalently, a local system is a covering map $\mathcal{G} \to B$ whose fibers G_x are equipped with (abelian) group structures that vary continuously with x. (The precise meaning of the words "vary continuously with x" is best expressed in terms of local trivializations: every point $x \in B$ in the base of a local system $\mathcal{G} \to B$ has a neighborhood $\mathcal{U} \subset B$ on which $\mathcal{G}|_{\mathcal{U}}$ can be identified with $\mathcal{U} \times G$ for some fixed abelian group G carrying the discrete topology.)
- Example 1: For any topological n-manifold M, the orientation bundle $\{H_n(M, M\backslash\{x\}; \mathbb{Z})\}_{x\in M}$ is a local system over M whose fibers are the local homology groups, all isomorphic to $\mathbb{Z}$. This is why we called it a "bundle" in Topology 2 (cf. [Wen23, Lecture 52]).
- Example 2: When we discuss obstruction problems in the final part of this lecture, we need to consider local systems of the form $\{\pi_n(E_x)\}_{x\in B}$ whose fibers are homotopy groups of the fibers of a given fiber bundle $p : E \to B$. This requires an extra condition: we assume the fibers $E_x \cong F$ of the given bundle are **simple**, meaning that they are path-connected and have $\pi_1(F)$ acting *trivially* on $\pi_n(F)$ for all n. Recall that the natural action of $\pi_1(F)$ on itself is by conjugation, so simplicity implies among other things that $\pi_1(F)$ is abelian; more importantly, it implies that the natural map $\pi_n(F) \to [S^n, F]_\circ$ defined by forgetting base points is a bijection for all n. We will comment below on why, for most interesting applications of obstruction theory, it is not too restrictive to require that the fibers are simple.
- Proposition: Assume $p : E \to B$ is a fiber bundle with fibers $E_x \cong F$ that are simple, and also that the base B has the usual properties required in covering space theory.[60] Then the union of the groups $\{\pi_n(E_x)\}_{x\in B}$ can be given a natural topology that makes it a local system of abelian groups over B.
  Proof: For any point $x \in B$, choose a neighborhood $x \in \mathcal{U} \subset B$ such that there is a unique homotopy class of paths γ in $\mathcal{U}$ from x to any other point $y \in \mathcal{U}$. Such a path γ determines a homotopy $H : S^n \times I \to B$ of constant maps $H(x, t) := \gamma(t)$, and feeding this into the transport functor of $p : E \to B$ then gives a bijection
  $$\pi_n(E_x) \cong [S^n, E_x]_\circ \xrightarrow{H_\#} [S^n, E_y]_\circ \cong \pi_n(E_y),$$
  which depends only on $y \in \mathcal{U}$ and not the path γ. (One should check that the resulting bijection $\pi_n(E_x) \cong \pi_n(E_y)$ respects the group structure; I leave this as an exercise.) The resulting map
  $$\mathcal{U} \times \pi_n(E_x) \to \bigcup_{y\in\mathcal{U}} \pi_n(E_y)$$
  can be regarded as the inverse of a local trivialization.
- Example: A **trivial local system** over B is simply a product space $B \times G$, where G is an abelian group and carries the discrete topology. In this example, the fibers are the subsets $\{x\} \times G$, which are all copies of the same group, with the extra parameter $x \in B$ tacked on just for bookkeeping; to put it another way, a trivial local system is essentially the same thing as a perfectly ordinary abelian group. The next result comes as close as possible to identifying every local system over a sufficiently reasonable space with a trivial local system.

[60] I had to look them up: We want B to be path-connected and locally path-connected, and such that every point in B has a simply-connected neighborhood. Those are the conditions required in order to construct a universal cover of B.

- Proposition: Assume $\{G_x\}_{x\in B}$ is a local system of abelian groups over a path-connected space B with universal cover $f : \widetilde{B} \to B$. Then any choice of base points $* \in B$ and $* \in \widetilde{B}$ making f a pointed map determines an isomorphism of the local system $\{G_x\}$ with
$$(\widetilde{B} \times G)\Big/\pi_1(B),$$
where $G := G_*$ is the fiber of $\{G_x\}$ over the base point $* \in B$, and $\pi_1(B)$ acts on $\widetilde{B}$ in the usual way by deck transformations and on G by group isomorphisms. We regard $(\widetilde{B}\times G)/\pi_1(B)$ as a local system over B whose fiber at a point $x \in B$ is $(f^{-1}(x)\times G)/\pi_1(B)$, a discrete group that is isomorphic to G since $\pi_1(B)$ acts freely and transitively on the discrete set $f^{-1}(x)$.
Proof: This follows from standard covering space theory arguments. First define a map
$$\widetilde{B} \times G \xrightarrow{F} \bigcup_{x\in B} G_x$$
as follows: Given $(\widetilde{x}, g) \in \widetilde{B} \times G$, choose any path $\widetilde{\gamma}$ in $\widetilde{B}$ from $*$ to $\widetilde{x}$, let $\gamma := f \circ \widetilde{\gamma}$ denote the projected path in X from $*$ to $x := f(\widetilde{x})$, lift the latter to a path in the local system from $g \in G = G_*$ to some point $g' \in G_x$, and set $F(\widetilde{x}, g) := g'$. Under the usual identification of $\pi_1(B)$ with $f^{-1}(*)$, evaluating F at $f^{-1}(*) \times G$ defines an action of $\pi_1(B)$ on G by group isomorphisms, such that F descends to the quotient of $\widetilde{B} \times G$ by $\pi_1(B)$, and one then checks that the resulting map of the quotient to the given local system is a homeomorphism.
- Convenient fact: Every path-connected topological group G is simple.
Proof: We choose the base point of G to be the identity element $e \in G$. Given $[\gamma] \in \pi_1(G)$ and $[f] \in \pi_n(G)$, we can use the group structure to define a free homotopy from f to itself along γ by $H(x,t) := f(x)\gamma(t)$. If you recall how the action of π_1 on π_n is defined in general via the transport functor, the existence of this free homotopy implies $[\gamma] \cdot [f] = [f]$.
- Remark: I tried very hard to convince you in the previous lecture that many problems involving fiber bundles can be solved by considering only the special case of principal bundles. The fiber of a principal bundle is a topological group, and therefore always simple according to this result. It is not quite true that this is the *only* type of bundle one really has to worry about in obstruction-theoretic problems, but Exercise 12.2 below tells us that for a reasonable class of subgroups $H \subset G$, the quotient G/H will also be a simple space. This really does cover all examples that I can immediately think of good reasons to care about.

Topic 3: Obstruction theory for sections of fiber bundles.

- The general obstruction problem: Given a pair of spaces (X, A) and a fiber bundle $p : E \to X$, we consider the lifting/extension problem
$$\begin{tikzcd} A \arrow[r, "s_A"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \arrow[r, equal] \arrow[ur, dashed, "s"] & X \end{tikzcd}.$$
Interpretation: Given a section s_A of $E|_A$, can s_A be extended to a global section s of E? The questions of whether a global section exists and whether two given sections are homotopic can both be regarded as special cases of this problem.
- Standing assumptions: In the following, (X, A) is always a CW-pair, and the fibers $E_x \cong F$ of the bundle $p : E \to X$ are assumed to be simple. We can then try to solve the lifting/extension problem inductively over the skeleta of X, meaning that for each $n \geqslant 0$

in succession, we assume an extension of s_A to $A \cup X^{n-1}$ has already been found and try to extend it further to $A \cup X^n$:

$$\begin{tikzcd} A \cup X^{n-1} \arrow[r, "s_{n-1}"] \arrow[d, hook] & E \arrow[d, "p"] \\ A \cup X^n \arrow[r, hook] \arrow[ur, dashed, "s_n"] & X \end{tikzcd}$$

Notice: Extending a given section of $E|_A$ to $A \cup X^0$ is trivial, as the values of s_0 on 0-cells outside of A can be chosen arbitrarily. In the following we assume $n \geqslant 1$.

- Remark: The hypotheses in the setup can be relaxed a bit, e.g. it is possible to attack this problem when $p : E \to X$ is only a fibration and not a fiber bundle, but that would make some details trickier, and I am not aware of an interesting application that would require that level of generality.
- Idea: For each n-cell $e^n_\alpha \subset X \backslash A$ with characteristic map $\Phi_\alpha : (\mathbb{D}^n, S^{n-1}) \to (X, X^{n-1})$, the contractibility of $\mathbb{D}^n$ implies via the homotopy theorem that there exists a trivialization

$$\Phi_\alpha^* E \cong \mathbb{D}^n \times F.$$

The trivialization identifies sections over this region with maps to the fixed space F, so under that identification, exploiting the assumption that F is simple, we can define

$$\theta^n(e^n_\alpha) := [S^{n-1} \xrightarrow{\Phi_\alpha} X^{n-1} \xrightarrow{s_{n-1}} F] \in [S^{n-1}, F]_\circ = \pi_{n-1}(F).$$

We'd like to interpret θ^n defined in this way as a cochain in $C^n_{\mathrm{CW}}(X; \pi_{n-1}(F))$. Note that even in the case $n = 2$, $\pi_1(F)$ is a valid choice of coefficient group for cohomology, because simplicity implies that it is abelian.
- Problem: The homotopy class of $s_{n-1} \circ \Phi_\alpha|_{S^{n-1}}$ as a map $S^{n-1} \to F$ may depend on the choice of trivialization.
- Solution: The dependence on a choice can be eliminated by working in the universal cover $f : \widetilde{X} \to X$, with its induced cell decomposition, instead of directly in X. Set $\widetilde{A} := f^{-1}(A)$, choose base points $* \in X$ and $* \in f^{-1}(*) \subset \widetilde{X}$, let F denote the specific fiber E_* over the base point, and denote the pullback of E to the universal cover by $\widetilde{E} := f^* E \to \widetilde{X}$. For any $\widetilde{x} \in \widetilde{X}$, we can choose a path $\widetilde{\gamma} : I \to \widetilde{X}$ from $*$ to $\widetilde{x}$, let $\gamma := f \circ \widetilde{\gamma}$ denote the projected path in X from $*$ to $x := f(\widetilde{x})$, and trivialize E along γ, meaning we choose a trivialization of the pullback bundle $\gamma^* E \to I$, which is possible since I is contractible. This determines a homeomorphism $F = E_* \to E_x$, and since $\widetilde{\gamma}$ is unique up to homotopy, the properties of the transport functor imply that the homotopy class of the homeomorphism $F \to E_x$ depends only on the point $\widetilde{x} \in f^{-1}(x)$, not on any other choices. For every lift $\widetilde{e}^n_\alpha \subset \widetilde{X} \backslash \widetilde{A}$ of a given n-cell $e^n_\alpha \subset X \backslash A$, we can now use this procedure to fix a homeomorphism of F to the fiber over the center of $\widetilde{e}^n_\alpha$, and then move radially outward to define a trivialization of $\widetilde{E}$ over the rest of $\widetilde{e}^n_\alpha$, or more precisely, a trivialization of the pullback bundle $\widetilde{\Phi}^*_\alpha \widetilde{E} = \Phi^*_\alpha E \to \mathbb{D}^n$ induced by its characteristic map. Up to homotopy, the trivializations defined in this way are independent of choices, except that in general we may have different trivializations corresponding to different choices of lift to $\widetilde{X}$ for the same n-cell $e^n_\alpha \subset X \backslash A$. We thus have a well-defined cochain

$$\theta^n \in C^n_{\mathrm{CW}}(\widetilde{X}, \widetilde{A}; \pi_{n-1}(F)), \qquad \theta^n(\widetilde{e}^n_\alpha) := [s_{n-1} \circ \Phi_\alpha|_{S^{n-1}}] \in \pi_{n-1}(F),$$

with the understanding that the trivializations described above are used in order to view $s_{n-1} \circ \Phi_\alpha|_{S^{n-1}}$ as a map $S^{n-1} \to F$. Note that the same definition of θ^n also makes sense for n-cells in $\widetilde{A}$, but it vanishes automatically on these cells because an extension of

s_{n-1} over these cells exists. In general, $\theta^n(\tilde{e}^n_\alpha) = 0$ if and only if the section s_{n-1} can be extended over the particular n-cell e^n_α.

- Lemma: The cochain $\theta^n : C_n^{\mathrm{CW}}(\tilde{X}, \tilde{A}; \mathbb{Z}) \to \pi_{n-1}(F)$ is equivariant with respect to the natural action of $\pi_1(X)$ on both sides, where the action on $C_n^{\mathrm{CW}}(\tilde{X}, \tilde{A}; \mathbb{Z})$ is determined by the action of $\pi_1(X)$ on $\tilde{X}$ by deck transformations, and the action on $\pi_{n-1}(F)$ is the same one that appears in the isomorphism of local systems $\{\pi_{n-1}(E_x)\}_{x\in X} \cong (\tilde{X} \times \pi_{n-1}(F))/\pi_1(X)$.
- Definition: Given a CW-pair (X, A) with universal cover $f : \tilde{X} \to X$ and $\tilde{A} := f^{-1}(A)$, and a local system $\{G_x\} \cong (\tilde{X} \times G)/\pi_1(X)$ of abelian groups over X, the **cellular cohomology** $H^*_{\mathrm{CW}}(X, A; \{G_x\})$ of (X, A) with **local coefficients** in $\{G_x\}_{x\in X}$ can be defined as the homology of the subcomplex

$$\mathrm{Hom}_{\pi_1(X)}\left(C_*^{\mathrm{CW}}(\tilde{X}, \tilde{A}; \mathbb{Z}), G\right) \subset \mathrm{Hom}\left(C_*^{\mathrm{CW}}(\tilde{X}, \tilde{A}; \mathbb{Z}), G\right),$$

meaning we consider only cochains $\varphi : C_n^{\mathrm{CW}}(\tilde{X}, \tilde{A}; \mathbb{Z}) \to G$ that are $\pi_1(X)$-equivariant. (Exercise: The cellular boundary operator commutes with the $\pi_1(X)$-action, thus the group of equivariant cochains really is a subcomplex.)
- Fun fact: $H^*_{\mathrm{CW}}(X, A; \{G_x\})$ is naturally isomorphic to a corresponding version of singular cohomology with local coefficients, which can be defined in multiple equivalent ways. One version of the definition is phrased in terms of $\pi_1(X)$-equivariant singular cochains on the universal cover, closely analogous to the definition above. In another version, one avoids using the universal cover but instead generalizes the notion of a singular cochain on X as follows: instead of taking values in a fixed coefficient group G, an n-cochain φ associates to each singular simplex $\sigma : \Delta^n \to X$ a *lift* of σ that sends Δ^n to $\bigcup_{x\in X} G_x$, i.e. a section of the bundle of coefficient groups along σ. There is also an equivalent formulation of $H^*_{\mathrm{CW}}(X, A; \{G_x\})$ that avoids using the universal cover, but it looks somewhat less elegant: one has to fix a point x_α in each cell e^n_α and require the value $\varphi(e^n_\alpha)$ of a cochain φ to live in the fiber G_{x_α}. This makes the coboundary operator rather tricky to write down. In a very large number of important applications, however, one doesn't really need to know all this, because fortunate topological circumstances conspire to make sure that the local systems we need to worry about are trivial, which makes $H^*_{\mathrm{CW}}(X, A; \{G_x\})$ the same thing as the usual cellular cohomology with coefficients in a fixed abelian group.
- Main theorem:
  (1) The **obstruction cochain** $\theta^n \in C^n_{\mathrm{CW}}(X, A; \{\pi_{n-1}(E_x)\})$ defined via the prescription above is a cocycle, and thus represents a so-called **obstruction class**
  $$[\theta^n] \in H^n_{\mathrm{CW}}\big(X, A; \{\pi_{n-1}(E_x)\}\big),$$
  (2) The cocycle θ^n is determined by the homotopy class of the given section s_{n-1} of $E|_{A\cup X^{n-1}}$, which can be extended to $A \cup X^n$ if and only if $\theta^n = 0$.
  (3) There is a natural bijection between the set of all n-cocycles cohomologous to θ^n in the complex $C^*_{\mathrm{CW}}(X, A; \{\pi_{n-1}(E_x)\})$ and the set of all homotopy classes of sections of $E|_{A\cup X^{n-1}}$ that match s_{n-1} on $A \cup X^{n-2}$.

  In particular, it follows that $[\theta^n] = 0 \in H^n_{\mathrm{CW}}\big(X, A; \{\pi_{n-1}(E_x)\}\big)$ if and only if s_{n-1} can be modified on X^{n-1} (but without changing it on $A \cup X^{n-2}$) to produce a section that extends to $A \cup X^n$.
- Some brief comments on why the theorem is true:
  (1) $\theta^n(\partial e^{n+1}_\beta) = 0$ holds for every $(n+1)$-cell e^{n+1}_β due to roughly the same cancellation phenomenon that causes the cellular boundary map to satisfy $\partial^2 = 0$.

(2) This is clear from the definition of θ^n. Let me just emphasize that here we are talking about the vanishing of the *cochain* θ^n, rather than the cohomology class $[\theta^n]$ that it represents. The geometric meaning of the condition $[\theta^n] = 0$ is subtler than the meaning of $\theta^n = 0$, and is addressed by the third statement in the theorem.

(3) If s'_{n-1} is another section on $A \cup X^{n-1}$ that matches s_{n-1} on $A \cup X^{n-2}$, we can use it to define a cochain $\psi \in C^{n-1}_{\mathrm{CW}}(X, A; \{\pi_{n-1}(F)\})$ whose value on each lifted $(n-1)$-cell $\widetilde{e}^{n-1}_\gamma \subset \widetilde{X} \backslash \widetilde{A}$ is the homotopy class of a map $S^{n-1} \to F$ obtained by gluing together the disk maps $s_{n-1} \circ \Phi_\gamma : \mathbb{D}^{n-1} \to F$ and $s'_{n-1} \circ \Phi_\gamma : \mathbb{D}^{n-1} \to F$ along their matching boundaries, as usual with the understanding that a trivialization determined by a path from the base point to $\widetilde{e}^{n-1}_\gamma$ in $\widetilde{X}$ is used in order to regard s'_{n-1} and s_{n-1} in this region as F-valued maps. (Once again, the definition of $\psi(\widetilde{e}^{n-1}_\gamma)$ also makes sense for $\widetilde{e}^{n-1}_\gamma \subset \widetilde{A}$, but it vanishes automatically because s_{n-1} and s'_{n-1} are assumed to match on A.) For each lifted n-cell $\widetilde{e}^n_\alpha$, the difference between the values on $\widetilde{e}^n_\alpha$ of the two obstruction cocycles defined via s_{n-1} and s'_{n-1} is then $\pm\psi(\partial\widetilde{e}^n_\alpha)$. Conversely, given any $\psi \in C^{n-1}_{\mathrm{CW}}(X, A; \{\pi_{n-1}(F)\})$, s_{n-1} can be modified in the interior of each $(n-1)$-cell e^{n-1}_γ by performing a connected sum of each of the lifts $\widetilde{e}^{n-1}_\gamma$ with a sphere S^{n-1} and extending s_{n-1} over this sphere as a map $S^{n-1} \to F$ representing $\psi(\widetilde{e}^{n-1}_\gamma) \in \pi_{n-1}(F)$. Doing this produces a new section s'_{n-1} with a new obstruction cocycle that differs from θ^n by $\pm\delta\psi$.

- Theorem 2: If the fiber F is $(n-2)$-connected,[61] then every section s_A of $E|_A$ can be extended to $A \cup X^{n-1}$, and any two such extensions are homotopic rel A on $A \cup X^{n-2}$.
- Corollary: In the situation of Theorem 2, the obstruction class $[\theta^n] \in H^n_{\mathrm{CW}}\big(X, A; \{\pi_{n-1}(E_x)\}\big)$ in the main theorem can always be defined and is independent of the choice of extension of s_A to a section s_{n-1} on $A \cup X^{n-1}$, i.e. it depends only on the homotopy class of the original (unextended) section $s_A \in \Gamma(E|_A)$ and the topology of the bundle $E \to X$.
- Definition: The class $[\theta^n] \in H^n_{\mathrm{CW}}\big(X, A; \{\pi_{n-1}(E_x)\}\big)$ in the setting of Theorem 2 and its corollary is called the **primary obstruction class** for the problem of extending a given section s_A from A to X. The extension to $A \cup X^n$ is possible if and only if this cohomology class vanishes.
- Remark: If the primary obstruction class $[\theta^n]$ vanishes, then sections can be extended from A to $A \cup X^n$, but there may be *secondary* obstructions $[\theta^{n+1}] \in H^{n+1}_{\mathrm{CW}}\big(X, A; \{\pi_n(E_x)\}\big)$, $[\theta^{n+2}] \in H^{n+2}_{\mathrm{CW}}\big(X, A; \{\pi_{n+1}(E_x)\}\big)$ and so forth to finding further extensions over higher-dimensional skeleta, and the definitions of the secondary obstruction classes will typically depend on the choices of extensions to the previous skeleta. The good news is that there are many interesting applications in which these secondary obstructions vanish automatically, so that only the primary obstruction really needs to be understood.

Suggested reading. My favorite reference for the general theory of fiber bundles has traditionally been [Ste51], though looking at it again now, it does have a lot of antiquated notation that makes it an effort to read, and more seriously, it was evidently written before the importance of spin bundles was generally recognized, and thus does not adequately handle the case of non-effective group actions. (The treatment of this theory in [DK01, Chapter 4] acknowledges that non-effective actions are sometimes important, but still fails to discuss them adequately.) Most of the details that I had no time for in lecture are included in my lecture notes for Differential Geometry II from a few years ago; see [Wen21, Lectures 42 and 43].

[61] In the case $n = 1$, we understand this to be a vacuous condition, so the statement is always true.

The best exposition I have seen for the homotopy theorem on fiber bundles is in [Cut21, Week 4: Fibrations III], but proving this result with maximal generality requires a lot more effort than the special cases of it that are typically needed. This is a topic that can seem much easier if you are happy to work in the smooth category: for smooth fiber bundles, the homotopy theorem is a nearly obvious consequence of the existence of connections and parallel transport maps!

Davis and Kirk have an entire chapter [DK01, Chapter 5] on homology and cohomology with local coefficients, and you might also appreciate the slightly more concise treatment in [Hat02, §3.H]. In lecture we only scratched the surface of this topic, but fairly little of what needs to be said about it will seem truly surprising, and in practice, the need to actually compute cohomology with a nontrivial bundle of coefficient groups seems to arise quite rarely.

For obstruction theory, [DK01, Chapter 7] is a good read. Most of that chapter concentrates on less general obstruction problems involving maps and homotopies of maps between two spaces; one could express it in the language of fiber bundles, but the fiber bundle is assumed to be trivial, in which case there is no need for local systems of coefficients (at least if the target space is simple). After exploring the theory of primary and secondary obstruction classes in that less general setting, §7.10 adapts the discussion for sections of general fiber bundles, which actually requires almost no modifications to the theory beyond the use of local coefficients for cohomology.

Exercises (for the Übung on 11.07.2024).

Exercise 12.1. As mentioned in lecture, every fiber bundle with structure group G determines (up to isomorphism) a unique *principal* G-bundle that has the same system of G-valued transition functions. We discussed one concrete example of this correspondence: for a vector bundle $E \to B$ of rank n over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the corresponding principal $\mathrm{GL}(n, \mathbb{K})$-bundle is its frame bundle $FE \to B$, whose fibers are sets of bases for the fibers of E. See if you can give similar concrete descriptions of the principal bundles determined by the following:

(a) A real vector bundle of rank n equipped with a bundle metric, whose structure group is therefore $\mathrm{O}(n)$;
(b) A real vector bundle of rank n with structure group $\mathrm{SL}(n, \mathbb{R})$ acting linearly on the standard fiber.

Remark: Examples like these in which the fiber is a vector space and the group action is linear and effective can always be understood as geometrically meaningful subsets of the frame bundle $FE \to B$. A more abstract perspective is needed only when the action is not effective, e.g. for vector bundles equipped with a spin structure.

Exercise 12.2. Assume G is a topological group and $H \subset G$ a subgroup. We consider the set of left cosets $G/H = \{gH \mid g \in G\}$ as a topological space with the quotient topology. Notice that the right action $G \times H \to G$ of H on G by multiplication preserves the fibers of the quotient projection $p : G \to G/H$, and moreover, it acts freely and transitively on each fiber. It would be tempting to conclude immediately that $p : G \to G/H$ is a principal H-bundle, but part (a) below is a warning against being too hasty.

(a) Suppose G is the abelian Lie group $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, and $H \subset \mathbb{T}^2$ is the image of a homomorphism $\mathbb{R} \to \mathbb{T}^2$ of the form $t \mapsto [(pt, qt)]$ for some $p, q \in \mathbb{R}$. Show that if p/q is irrational, then the quotient map $p : G \to G/H$ is not a fiber bundle.
Hint: Fiber bundles always locally admit sections.
Remark: Here is an example of a topological subgroup that I would be even less enthusiastic to deal with in this context: $\mathrm{Diff}(M) \subset \mathrm{Homeo}(M)$ *for a compact smooth manifold M. Standard perturbation results imply that this subgroup is dense, which is very useful for some purposes, but not if you are hoping for a quotient space with nice properties.*

(b) Find an additional assumption about G and H that will allow you to conclude that the quotient map $p : G \to G/H$ is a principal H-bundle.
Remark: This is not exactly the answer I have in mind, but basic results in the theory of Lie groups imply that the necessary condition will hold whenever G is a Lie group and the subgroup $H \subset G$ is a topologically closed subset. The example in part (a) lacks the latter property.

(c) Under the assumption that $p : G \to G/H$ is a fiber bundle, and the additional assumption that the inclusion $H \hookrightarrow G$ induces a bijective map $\pi_0(H) \to \pi_0(G)$, show that the space G/H is simple.

(d) Just for fun, show that every path-connected space that is also a group object in hTop_* is also simple.
Comment: In fact, one doesn't need associativity or inverses for this, so it works for arbitrary H-spaces.

Exercise 12.3. I want to explain why we should care about obstruction problems in fiber bundles with fiber of the form G/H for a topological group G and subgroup $H \subset G$. The following definition gives a general framework for many situations in which one would like to endow a manifold with some extra geometric structure, but there might be topological reasons for that structure not to exist.

Recall that a G-bundle $p : E \to B$ comes with an equivalence class of G-bundle atlases $\mathcal{A} = (\boldsymbol{\Phi}, \mathcal{T}, \rho)$, which include a covering $\boldsymbol{\Phi}$ by local trivializations and a related system $\mathcal{T}$ of G-valued transition data. We say that the structure group G of this bundle can be **reduced** to H if the equivalence class contains a G-bundle atlas that is also an H-bundle atlas, meaning that its transition functions all take their values in the subgroup $H \subset G$. If this exists, then the resulting equivalence class of H-bundle atlases makes $p : E \to B$ into an H-bundle, and the equivalence class is called a **reduction** of the structure group to H. It is possible for a G-bundle to admit multiple distinct reductions, since the given equivalence class of G-bundle atlases may contain multiple H-bundle atlases that are not equivalent as H-bundle atlases, i.e. because the collection of G-valued functions $\{h_{\beta\alpha}\}$ needed for defining a morphism $\mathcal{T}^0 \to \mathcal{T}^1$ between two systems of H-valued transition functions cannot be made to take values in H. In case all this sounds too abstract, here are some concrete examples:

- Choosing an orientation of a real vector bundle $E \to B$ of rank n reduces its structure group from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{GL}_+(n, \mathbb{R})$. If B is connected and E is orientable, then there are exactly two inequivalent choices of such a reduction. Not all real vector bundles are orientable, so such reductions do not always exist.
- Choosing a bundle metric on a real vector bundle $E \to B$ of rank n reduces its structure group from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$. Adding an orientation then reduces the structure group further from $\mathrm{O}(n)$ to $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{GL}_+(n, \mathbb{R})$.
- Using the obvious identification $\mathbb{C}^n = \mathbb{R}^{2n}$, we can regard $\mathrm{GL}(n, \mathbb{C})$ as the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ consisting of real-linear maps that commute with multiplication by i (represented in this situation by a particular real $2n$-by-$2n$ matrix). For a real vector bundle $E \to B$ of rank $2n$, reducing its structure group from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{C})$ means choosing a complex structure, so that E becomes a complex vector bundle of rank n. Note that complex-linear maps $\mathbb{C}^n \to \mathbb{C}^n$ always correspond to real-linear maps $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that are orientation-preserving, thus $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}_+(2n, \mathbb{R})$, and every reduction to $\mathrm{GL}(n, \mathbb{C})$ therefore automatically also defines a reduction to $\mathrm{GL}_+(2n, \mathbb{R})$. This is just a fancy way of saying that every complex vector bundle is also a real vector bundle with a canonical orientation determined by its complex structure.

Assume in the following that G is an arbitrary topological group with subgroup $H \subset G$.

(a) Given a G-bundle $p : E \to B$ with standard fiber F and G-bundle atlas $\mathcal{A} = (\mathbf{\Phi}, \mathcal{T}, \rho)$, let $E^{G/H} \to B$ denote a G-bundle carrying the same G-valued transition data $\mathcal{T}$ but with the standard fiber F and G-action $\rho : G \times F \to F$ replaced by G/H and the obvious G-action $G \times G/H \to G/H$ defined by multiplication in G.[62] Show that if the structure group of $p : E \to B$ can be reduced to H, then $E^{G/H} \to B$ admits a global section.
Hint: There is some freedom in the choice of bundle atlas to use here. For the right choice, the statement becomes almost obvious.
Remark: Applying this to the tangent bundle $TM \to M$ of a smooth $2n$-manifold with structure group $G := \mathrm{GL}(2n, \mathbb{R})$ and subgroup $H := \mathrm{GL}(n, \mathbb{C})$, one can show that the associated bundle with fiber $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$ is naturally isomorphic to the bundle $\mathcal{J}(TM)$ of complex structures mentioned in Lecture 20.

(b) Show that if the conclusion of Exercise 12.2(b) holds for the subgroup $H \subset G$, then the converse of the statement in part (a) also holds: The structure group of $p : E \to B$ can be reduced to H whenever there exists a global section of the associated bundle $E^{G/H} \to B$.
Hint: Locally, sections of $E^{G/H}$ look like maps $\mathcal{U}_\alpha \to G/H$ defined on open subsets $\mathcal{U}_\alpha \subset B$, but Exercise 12.2(b) will allow you to represent these as maps $s_\alpha : \mathcal{U}_\alpha \to G$. How are different representatives s_α, s_β of this form related on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$?

(c) Prove that if the inclusion $H \hookrightarrow G$ is a homotopy equivalence and the conclusion of Exercise 12.2(b) holds, then every G-bundle over a CW-complex admits a reduction of its structure group to H.
Example: By polar decomposition, $\mathrm{GL}(n, \mathbb{R})$ admits a deformation retraction to $\mathrm{O}(n)$, and similarly, $\mathrm{GL}(n, \mathbb{C})$ admits a deformation retraction to $\mathrm{U}(n)$. This result thus implies the fact that every vector bundle over a CW-complex admits a bundle metric. (In differential geometry, one typically constructs bundle metrics more directly using partitions of unity.) Another example is important in symplectic geometry: One can show that the linear symplectic group $\mathrm{Sp}(2n)$ also contains $\mathrm{U}(n)$ as a homotopy equivalent closed subgroup, implying that symplectic vector bundles over CW-complexes can always be endowed with compatible complex structures (cf. Exercise 8.3).

(d) Show that every fiber bundle over a CW-complex with a weakly contractible structure group is trivial.

Exercise 12.4. Assume $p : E \to X$ is a fiber bundle and (X, A) is a pair of spaces. Given two sections $s_0, s_1 \in \Gamma(E)$ and a homotopy $\{h(\cdot, t) \in \Gamma(E|_A)\}_{t \in I}$ of sections over A from s_0 to s_1, the problem of extending h to a global homotopy $\{H(\cdot, t) \in \Gamma(E)\}_{t \in I}$ of sections from $H_0 := s_0$ to $H_1 := s_1$ is summarized by the diagram

$$\begin{tikzcd}
X \times \partial I \cup A \times I \arrow[r, "s_0 \cup s_1 \cup h"] \arrow[d, hook] & \mathrm{pr}_1^* E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r, equal] \arrow[ur, dashed, "H"] & X \times I \arrow[r, "\mathrm{pr}_1"] & X
\end{tikzcd},$$

where $\mathrm{pr}_1 : X \times I \to X$ denotes the obvious projection, the portion of the diagram on the right is a pullback square, and the portion on the left is a special case of the general obstruction problem we discussed in lecture, viewing homotopies of sections of E as sections of the pullback bundle

[62] Such a bundle can be constructed explicitly using Theorem 1 in Lecture 20. An alternative description of $E^{G/H}$ would be as the associated bundle $P \times_G (G/H)$ determined by the action $G \times G/H \to G/H$ and the (up to isomorphism) canonical principal G-bundle $P \to B$ such that $P \times_G F$ is isomorphic to E.

$\mathrm{pr}_1^* E \to X \times I$. If (X, A) is a CW-pair, then we can write the inductive version of this diagram as

$$\begin{tikzcd} X \times \partial I \cup \big((A \cup X^{n-1}) \times I\big) \arrow[r, "h_{n-1}"] \arrow[d, hook] & \mathrm{pr}_1^* E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times \partial I \cup ((A \cup X^n) \times I) \arrow[r, hook] \arrow[ur, dashed, "h_n"] & X \times I \arrow[r, "\mathrm{pr}_1"] & X \end{tikzcd} \tag{12.1}$$

for each $n \geqslant 0$, where by definition $X^{-1} := \varnothing$, and the problem is easily seen to be solvable for $n = 0$ if we assume the fibers $E_x \cong F$ are path-connected. Under the additional assumption that F is simple, the theorem from lecture characterizes the solvability of this problem for each $n \geqslant 1$ in terms of a relative cohomology class with local coefficients, but that characterization can be simplified somewhat in the situation at hand.

(a) Show that for each $n \geqslant 1$, there is a cohomology class

$$d^n(s_0, s_1; h) \in H^n_{\mathrm{CW}}(X, A; \{\pi_n(E_x)\})$$

that vanishes if and only if the problem (12.1) is solvable.

(b) Show that if the fiber F is $(n-1)$-connected, then the obstruction class $d^n(s_0, s_1; h)$ is determined by the homotopy classes of s_0 and s_1 and the given (unextended) homotopy h over A, and is independent of all other choices. It is in this case called the **primary obstruction** to the problem of extending the homotopy between s_0 and s_1 from A to X.

(c) Reinterpret the bijections $[X, K^{\mathrm{w}}(G, n)] \cong H^n_{\mathrm{CW}}(X; G)$ and $[X, S^n] \cong H^n_{\mathrm{CW}}(X; \mathbb{Z})$ (the latter for n-dimensional CW-complexes X according to Exercise 10.6) in the language of obstruction theory. What exactly does the cohomology class corresponding to a map $X \to K(G, n)$ or $X \to S^n$ obstruct?

13. Week 13

Lecture 22 (8.07.2024): The Euler class.

- Definition: Assume H^* is an ordinary cohomology theory and G is a topological group. A **characteristic class** for G-bundles assigns to every space X a function

$$\{G\text{-bundles over } X\} \Big/ \text{isomorphism} \xrightarrow{c} H^*(X)$$

that satisfies the following **naturality** property: for any map $f : X \to Y$ and G-bundle $E \to Y$,

$$c(f^* E) = f^* c(E) \in H^*(X).$$

- Remarks on the definition:
  (1) We did not specify in the definition what kinds of fibers the G-bundles can have, and that is because it doesn't matter. Theorems 1 and 2 in Lecture 20 imply that if there is a G-bundle isomorphism $E^1 \to E^2$ between two G-bundles with standard fiber F, then there is also a G-bundle isomorphism between any bundles obtained from E^1 and E^2 by keeping the same systems of G-valued transition data but replacing F with a different standard fiber F' that is acted upon by G. For example, if P^1 and P^2 are principal G-bundles (thus with standard fiber G), then a principal G-bundle isomorphism $\Psi : P^1 \to P^2$ is simply a fiber-preserving homeomorphism that is equivariant with respect to the right G-actions on P^1 and P^2, and for any other space F with a left G-action, Ψ determines a G-bundle isomorphism

$$P^1 \times_G F \to P^2 \times_G F : [\phi, p] \mapsto [\Phi(\phi), p]$$

between the corresponding associated G-bundles with standard fiber F. The standard characteristic classes (Chern, Stiefel-Whitney, Pontryagin and Euler) are all conventionally regarded as invariants of *vector* bundles with specific structure groups G, but it is sometimes more convenient to define them as invariants of *principal* G-bundles $P \to X$ and then let this determine the corresponding definition on all associated bundles via the rule $c(P \times_G F) := c(P)$. So for instance, if $G = \mathrm{GL}(n, \mathbb{R})$ and $E \to X$ is a real vector bundle, this amounts to defining $c(E) := c(FE)$ via the frame bundle $FE \to X$, which is a principal G-bundle. For the Euler class, we will give a more direct definition in terms of oriented real vector bundles, but if needed one could turn this into an equivalent definition of the Euler class for principal bundles with structure group $\mathrm{GL}_+(n, \mathbb{R})$.

(2) We will only attempt to define characteristic classes on bundles over CW-complexes, and will thus define them directly in cellular cohomology instead of working with a cohomology theory H^* defined on more general spaces. It should be mentioned that there also exist explicit (and quite elegant) constructions of certain characteristic classes in Čech cohomology, which make sense for bundles over arbitrary spaces, but naturality implies via the theorem below that they must match the classes that we construct in cellular cohomology. There are also characteristic classes that are defined in de Rham cohomology and thus make sense only in the smooth category—this subject is known as *Chern-Weil* theory, and defining characteristic classes in this way reveals a deep interplay between topology and curvature on smooth bundles with connections.

- Big Theorem: For every topological group G, there exists a CW-complex (the **classifying space** of G) and a principal G-bundle $EG \to BG$ (the **universal G-bundle**) such that for all CW-complexes X, the map

$$[X, BG] \to \{\text{principal } G\text{-bundles over } X\} \Big/ \text{isomorphism}$$

sending the homotopy class of a map $f : X \to BG$ to the isomorphism class of the pullback bundle $f^*EG \to X$ is a bijection.
- Remark: Although the Big Theorem is stated specifically for principal bundles, it follows immediately that all G-bundles with an arbitrary standard fiber F are pullbacks of the associated universal bundle $EG \times_G F$, and the homotopy set $[X, BG]$ thus similarly classifies all G-bundles with fiber F over X up to G-bundle isomorphism. We will make use of this in particular as a way to classify vector bundles.
- Remark: The existence of classifying spaces BG for arbitrary topological groups G is a result due to Milnor [Mil56], and we will not prove it in full generality, but will construct BG for a wide enough range of groups to understand the standard characteristic classes.
- Corollary: For G-bundles over CW-complexes, each characteristic class of G-bundles is determined by a single cohomology class $c(EG) \in H^*(BG)$.
Remark: With one caveat to be mentioned in the next remark, this is the way to prove that different versions of the same characteristic class constructed in different cohomology theories match on CW-complexes, and for this reason, quite a lot of attention in the theory of characteristic classes gets devoted to computing the cohomologies of the classifying spaces BG.
- Remark (just out of interest): As mentioned above, Chern-Weil theory produces smooth versions of the Chern, Euler and Pontryagin classes that are defined for smooth bundles over smooth manifolds and live in de Rham cohomology. One minor headache, however, is that one cannot directly use the Big Theorem above for proving that these classes match

their counterparts in cellular cohomology: the trouble is that while smooth manifolds are always CW-complexes, the relevant classifying spaces BG are *never* smooth manifolds—they are infinite-dimensional CW-complexes. In practice, one can get around this issue by using finite-dimensional *approximations* to BG, which are smooth manifolds: this is possible because classifying spaces can typically be described as colimits of sequences of smooth manifolds with increasingly large dimensions (we will see two examples at the end of this lecture). These manifolds "approximate" classifying spaces in the same sense that S^n approximates a $K(\mathbb{Z}, n)$ in Exercise 10.6, i.e. they do the same job if one restricts attention to CW-complexes up to a certain dimension.

- Question (motivation for the Euler class): Given a real vector bundle $E \to X$ of rank n, does E admit a section $s \in \Gamma(E)$ that is *nowhere zero*?
  Remark: There are many applications in which this is precisely the kind of question one wants to answer, i.e. whether there exist solutions to some equation of the form $s(x) = 0$. There are also nonlinear PDEs of geometric interest (especially in symplectic topology and gauge theory) that can be phrased as infinite-dimensional versions of this question, and the answer often depends on the underlying topology of the setting.
- Obstruction theory approach: The problem is to construct a global section $s \in \Gamma(\dot{E})$ of the fiber bundle
  $$\dot{E} := E \backslash X \to X,$$
  where we are identifying the base X with the image of the zero-section $X \to E$, thus the fibers of $\dot{E}$ are $\dot{E}_x = E_x \backslash \{0\}$. This is a fiber bundle with structure group $\mathrm{GL}(n, \mathbb{R})$ acting linearly on the standard fiber $\mathbb{R}^n \backslash \{0\}$, which has the homotopy type of S^{n-1}. Assume X is a CW-complex, and for simplicity also assume $n \geqslant 2$. The fiber $\mathbb{R}^n \backslash \{0\}$ is then $(n-2)$-connected and simple, so by the usual induction over skeleta, $\dot{E}$ admits a section s_{n-1} over X^{n-1}, and all such sections are homotopic over X^{n-2}. By the main theorem in the previous lecture, the extendability of s_{n-1} over the n-skeleton is determined by a primary obstruction class
  $$[\theta^n_E] \in H^n_{\mathrm{CW}}\big(X; \{\pi_{n-1}(E_x \backslash \{0\})\}_{x \in X}\big).$$
  A brief review of the definition: Choose a base point $* \in \widetilde{X} \xrightarrow{f} X$ in the universal cover of X and use a local trivialization to identify the fiber of $\dot{E}$ over that base point with $\mathbb{R}^n \backslash \{0\}$. Paths in $\widetilde{X}$ from the base point to lifts $\tilde{e}^n_\alpha \subset \widetilde{X}$ of the n-cells $e^n_\alpha \subset X$ then determine (up to homotopy) trivializations of $\dot{E}$ along the corresponding attaching map $\Phi_\alpha : (\mathbb{D}^n, S^{n-1}) \to (X, X^{n-1})$, and we use such trivializations to identify s_{n-1} over e^n_α with a function valued in $\mathbb{R}^n \backslash \{0\}$ in order to define the cellular n-cochain $\theta^n_E : C^{\mathrm{CW}}_n(\widetilde{X}; \mathbb{Z}) \to \pi_{n-1}(\mathbb{R}^n \backslash \{0\})$ by
  $$\theta^n_E(\tilde{e}^n_\alpha) := [s_{n-1} \circ \Phi_\alpha|_{S^{n-1}}] \in [S^{n-1}, \mathbb{R}^n \backslash \{0\}] = \pi_{n-1}(\mathbb{R}^n \backslash \{0\}).$$
  The way that the trivializations have been chosen makes this cochain equivariant with respect to the action of $\pi_1(X)$ via deck transformations on $\widetilde{X}$, so that θ^n_E actually belongs to the subcomplex defining the cohomology of X with coefficients in the local system $\{\pi_{n-1}(E_x \backslash \{0\})\}_{x \in X}$.
- New assumption: Suppose the vector bundle $E \to X$ is *oriented*, i.e. all its fibers are equipped with orientations, so the structure group has been reduced to $\mathrm{GL}_+(n, \mathbb{R})$. Choose the identification of the fiber of $\dot{E}$ over the base point with $\mathbb{R}^n \backslash \{0\}$ to be orientation preserving. This assumption removes local coefficients from the picture, because the action of $\pi_1(X)$ on $\pi_{n-1}(\mathbb{R}^n \backslash \{0\})$ is now determined by orientation-preserving linear transformations—these are all homotopic (as linear isomorphisms) to the identity, so the $\pi_1(X)$-action on

$\pi_{n-1}(\mathbb{R}^n \backslash \{0\})$ is trivial. This makes $\{\pi_{n-1}(E_x \backslash \{0\})\}_{x \in X}$ into a trivial local system, so the cohomology with coefficients in this local system is actually just cohomology with coefficients in the group $\pi_{n-1}(\mathbb{R}^n \backslash \{0\}) = \mathbb{Z}$.

- Definition: The **Euler class** of an oriented vector bundle $E \to X$ of rank n over a CW-complex is the primary obstruction class

$$e(E) := [\theta_E^n] \in H^n_{\mathrm{CW}}(X; \pi_{n-1}(\mathbb{R}^n \backslash \{0\})) = H^n_{\mathrm{CW}}(X; \mathbb{Z})$$

described above.

- Properties of the Euler class:
  (1) $e(E) = 0$ if and only if the bundle $E \to X$ admits a section that is nowhere zero on the n-skeleton X^n.
  (2) $e(\bar{E}) = -e(E)$, where $\bar{E}$ denotes the same vector bundle with reversed orientation.
  (3) (Naturality) For any oriented vector bundle $E \to Y$ and any map $f : X \to Y$ between CW-complexes, $e(f^*E) = f^*e(E)$.
  (4) (Whitney sum formula) For any two oriented vector bundles $E^1, E^2 \to X$,

$$e(E^1 \oplus E^2) = e(E^1) \cup e(E^2),$$

  where $E^1 \oplus E^2 \to X$ is the vector bundle whose fiber at each point $x \in X$ is the direct sum $E^1_x \oplus E^2_x$ with the orientation induced by the orientations of E^1_x and E^2_x.
  (5) For $X := M$ a closed oriented smooth manifold of dimension $n + k$ and $E \to M$ a smooth oriented vector bundle of rank n, the Poincaré dual of $e(E) \in H^n(M; \mathbb{Z})$ is the homology class represented by the smooth oriented submanifold $s^{-1}(0) \subset M$ for any section $s \in \Gamma(E)$ that is transverse to the zero-section:

$$\mathrm{PD}(e(E)) = [s^{-1}(0)] \in H_k(M; \mathbb{Z}).$$

  (Note that by the implicit function theorem, $s \in \Gamma(E)$ being transverse to the zero-section implies that $s^{-1}(0)$ is a submanifold, and it inherits a natural orientation from the orientations of M and E.) In the case $\dim M = n = \mathrm{rank}(E)$, this means

$$\langle e(E), [M] \rangle = \#s^{-1}(0),$$

  where the right hand side counts the isolated zeroes of $s \in \Gamma(E)$ with signs according to whether its linearization at these points preserves or reverses orientation. More generally, the latter formula holds for any section with isolated zeroes if (in the non-transverse case) each zero is assigned a suitable integer-valued weight, defined analogously to the *local degree* for maps between oriented manifolds of the same dimension (cf. [Wen23, Lecture 35]).
  (6) (Poincaré-Hopf theorem) For any closed oriented smooth manifold M,

$$\langle e(TM), [M] \rangle = \chi(M).$$

- Proofs of the properties:
  (1) Given the way that we defined $e(E)$ as an obstruction class, this is immediate from the main result of the previous lecture. Note: We are not claiming anything about the possibility of constructing nowhere-zero sections of E on $X \backslash X^n$. This is in general a nontrivial question, whose answer depends on secondary obstruction classes, though there are plenty of interesting situations (e.g. when $\dim X = n$) where those obstructions automatically vanish.
  (2) The sign arises from the fact that if we reverse the orientation of E, then we need to choose a different identification of the fiber of $\dot{E}$ over the base point with $\mathbb{R}^n \backslash \{0\}$.

(3) The map $f : X \to Y$ can be assumed cellular without loss of generality, and any section s_{n-1} of $E \to Y$ over Y^{n-1} then determines a section $s_{n-1} \circ f$ of $f^*E \to X$ over X^{n-1}. Using these specific sections in the definitions of the cocycles θ^n_E and $\theta^n_{f^*E}$ representing $e(E)$ and $e(f^*E)$ respectively then gives the relation

$$\theta^n_{f^*E}(e^n_\alpha) = \theta^n_E(f_* e^n_\alpha)$$

for all n-cells $e^n_\alpha \subset X$.

(4) Left as an exercise.

(5) We discuss the case $\dim M = n = \operatorname{rank}(E)$ and leave the rest as an exercise. Since M is a smooth manifold, it admits a triangulation, and given a section $s \in \Gamma(E)$ with isolated zeroes, we can choose that triangulation so that every n-simplex contains at most one zero of s, and only in its interior. Using the triangulation as a cell decomposition of M, we then have $\theta^n(e^n_\alpha) = 0$ for every n-simplex $e^n_\alpha \subset M$ on which s has no zero, and for n-simplices with a zero in the interior, $\theta^n(e^n_\alpha) \in \pi_{n-1}(\mathbb{R}^n \backslash \{0\})$ is represented by the restriction of s to the boundary of the n-simplex, expressed as a map $\partial\Delta^n \to \mathbb{R}^n\backslash\{0\}$ after choosing an oriented trivialization of E over the simplex. Under the canonical identification of $\pi_{n-1}(\mathbb{R}^n\backslash\{0\})$ with $\mathbb{Z}$, that is also precisely how the contribution of each individual zero to the signed and weighted count of zeroes $\#s^{-1}(0)$ is defined. The evaluation $\langle e(TM), [M]\rangle$ is the sum of all these contributions, since the fundamental class $[M]$ is represented by a sum of the n-simplices in the triangulation.

(6) In light of property (5), it suffices to construct a vector field $X \in \Gamma(TM)$ for which the signed count of zeroes of X is $\chi(M)$. We claim: For any given triangulation of M, regarded as a cell decomposition, there exists a vector field X that has exactly one zero in each k-cell for $k = 0, \dots, n$, and the sign it contributes to the signed count of zeroes is $(-1)^k$; consequently, the signed count of zeroes is $\chi(M)$. The construction of this vector field is best described with a picture that I cannot reproduce here: suffice it to say that it vanishes on every vertex of the triangulation, is tangent to every simplex, and flows inward toward a unique zero in the interior of each simplex. The sign $(-1)^k$ arises from the observation that at the unique zero in the interior of any given k-simplex, the linearization of the vector field has k negative eigenvalues with eigenvectors tangent to the k-cell, and $n-k$ positive eigenvalues with eigenvectors transverse to it; the sign of its determinant is therefore $(-1)^k$.

Further constructions of characteristic classes will be based on knowledge of classifying spaces, so the rest of this lecture begins this discussion.

- Theorem: Suppose $EG \to BG$ is a principal G-bundle whose total space EG is weakly contractible. Then for every CW-complex X and principal G-bundle $P \to X$, there is a unique homotopy class of maps $f : X \to BG$ such that the principal G-bundles P and f^*EG are isomorphic.
  Proof: Since G acts freely and transitively on the fibers of both P and EG, any G-equivariant map $P \to EG$ sends each fiber $P_x \cong G$ of P homeomorphically (and G-equivariantly) to some fiber $(EG)_{f(x)} \cong G$ of EG, thus defining a map $f : X \to BG$ along with a principal G-bundle isomorphism $P \to f^*EG$. The problem is therefore to show that there is a unique homotopy class of G-equivariant maps $P \to EG$. In fact, G-equivariant maps $\Psi : P \to EG$ are equivalent to global sections of the associated G-bundle

$$P \times_G EG := (P \times EG)\Big/ G \longrightarrow X,$$

where G acts on the product diagonally.[63] The fiber of $P \times_G EG$ at a point $x \in X$ is $(P_x \times EG)/G$, and we can associate to any equivariant map $\Psi : P \to EG$ a section $s \in \Gamma(P \times_G EG)$ defined at each point $x \in X$ by

$$s(x) := [\phi, \Psi(\phi)] \in (P_x \times EG)/G \qquad \text{for any} \qquad \phi \in P_x,$$

where the definition is independent of the choice of $\phi \in P_x$ since all possible choices are related by the G-action and Ψ is G-equivariant. Conversely, any section s determines an equivariant map Ψ in the same manner. Finally, we observe that the fibers of $P \times_G EG \to X$ are homeomorphic to EG, whose homotopy groups are all assumed to vanish, so the usual induction over the skeleta of X proves that this fiber bundle admits a global section, and moreover, all such sections are homotopic.

- Example 1: Let $G := \mathbb{Z}_2$. A principal $\mathbb{Z}_2$-bundle is just a covering map of degree 2. The infinite-dimensional CW-complex S^∞ is contractible (see Lecture 17), and is naturally a double cover of $\mathbb{RP}^\infty$, so we can set
$$E\mathbb{Z}_2 := S^\infty \to \mathbb{RP}^\infty =: B\mathbb{Z}_2.$$
- Example 2: Let $G := S^1 = \mathbb{R}/\mathbb{Z}$, or equivalently the unitary group $\mathrm{U}(1)$. The Hopf fibrations $p : S^{2n+1} \to \mathbb{CP}^n$ are all principal S^1-bundles, and taking the colimit as $n \to \infty$ (this was also discussed in Lecture 17) gives a Serre fibration
$$ES^1 = E\,\mathrm{U}(1) := S^\infty \xrightarrow{p} \mathbb{CP}^\infty =: B\,\mathrm{U}(1) = BS^1,$$
which is in fact also a principal S^1-bundle. To see this, we note that in light of the obvious fiber-preserving S^1-action that is free and transitive on the fibers, one only has to show that local sections of $p : S^\infty \to \mathbb{CP}^\infty$, as these can be combined with the S^1-action to define local trivializations (see the proposition in Lecture 20). Local sections of $p : S^\infty \to \mathbb{CP}^\infty$ can be constructed by taking colimits of local sections of the finite-dimensional Hopf bundles $S^{2n+1} \to \mathbb{CP}^n$ as $n \to \infty$.
- Remark: $\mathrm{U}(1)$ is the natural structure group for a complex vector bundle of rank 1 (also known as a **complex line bundle**) equipped with a Hermitian bundle metric, so $B\,\mathrm{U}(1) = \mathbb{CP}^\infty$ means that Hermitian line bundles over a CW-complex X are classified via the homotopy classes of maps $X \to \mathbb{CP}^\infty$, and every characteristic class for such bundles is a pullback of some class in $H^*(\mathbb{CP}^\infty)$. We will see next time that the bundle metric is superfluous in this discussion, and $\mathbb{CP}^\infty$ is also the classifying space of $\mathrm{GL}(1, \mathbb{C})$. As you may recall from Topology 2, the cohomology ring of $\mathbb{CP}^\infty$ with integer coefficients is easy to compute: it is generated by a single class in degree 2, and that class will give rise to the most popular characteristic class for complex vector bundles, the first Chern class.

Lecture 23 (10.07.2024): The Chern classes. All bases of bundles in this lecture are CW-complexes, and there is therefore no need to specify what cohomology theory H^* is being used, beyond its coefficient group.

Topic 1: We complete the construction of some useful classifying spaces that was begun in the previous lecture.

- Remark: The classifying space of a topological group G is not generally a unique space, but as long as it is a CW-complex, the theorem from last time about pullbacks of universal bundles determines its homotopy type (see Exercise 13.1). When we say that a particular

[63] In our usual construction of associated bundles $P \times_G F = (P \times F)/G$, G acts on P from the right and on F from the left, so that the equivalence relation on $P \times F$ can be expressed as $(\phi g, p) \sim (\phi, gp)$ for all $(\phi, p) \in P \times F$ and $g \in G$. In the present situation, EG comes with a right action instead of a left action, but we can choose to view it as a left action by defining $gp := pg^{-1}$ for $g \in G$ and $p \in EG$. The resulting equivalence relation on $P \times EG$ is thus $(\phi g, pg) = (\phi, p)$.

space "equals" BG, we simply mean that space is a CW-complex and is the base of a weakly contractible principal G-bundle, thus it is homotopy equivalent to any other space that we could conceivably choose to call BG.

- More constructions of classifying spaces: For $n \geqslant k \geqslant 0$ we define the **Grassmann manifold**
$$\mathrm{Gr}_k(\mathbb{R}^n) := \{k\text{-dimensional subspaces } V \subset \mathbb{R}^n\},$$
which has a **tautological vector bundle**
$$E^{k,n,\mathbb{R}} \to \mathrm{Gr}_k(\mathbb{R}^n), \qquad E_V^{k,n,\mathbb{R}} := V;$$
more formally, $E^{k,n,\mathbb{R}} := \left\{(V,v) \in \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \;\middle|\; v \in V\right\}$, which makes $E^{k,n,\mathbb{R}}$ a linear subbundle of the trivial vector bundle $\mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{Gr}_k(\mathbb{R}^n)$. Its frame bundle is a principal $\mathrm{GL}(k,\mathbb{R})$-bundle $\mathrm{St}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ called the **Stiefel manifold**, and can be described explicitly as
$$\mathrm{St}_k(\mathbb{R}^n) := \{\text{linearly-independent } k\text{-tuples of vectors in } \mathbb{R}^n\} \to \mathrm{Gr}_k(\mathbb{R}^n) : (v_1,\dots,v_k) \mapsto \mathrm{Span}\{v_1,\dots,v_k\}.$$
Example: $\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$, and $\mathrm{St}_1(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1}\backslash\{0\}$.
- Lemma: For any fixed $m, k \geqslant 0$, $\pi_m(\mathrm{St}_k(\mathbb{R}^n)) = 0$ for all n sufficiently large.
Proof: It is clearly true for $k = 1$ since $\mathrm{St}_1(\mathbb{R}^n)$ is homotopy equivalent to S^{n-1}. One can then prove the rest inductively because for each $k \geqslant 2$, there is a fiber bundle
$$\mathrm{St}_k(\mathbb{R}^n) \to \mathrm{St}_{k-1}(\mathbb{R}^n) : (v_1,\dots,v_k) \mapsto (v_1,\dots,v_{k-1})$$
with fibers homeomorphic to $\mathbb{R}^n\backslash\mathbb{R}^{k-1} \underset{h.e.}{\simeq} (\mathbb{R}^{k-1})^\perp\backslash\{0\} \cong \mathbb{R}^{n-k+1}\backslash\{0\} \underset{h.e.}{\simeq} S^{n-k}$. Whenever $n - k > m$, the long exact sequence of homotopy groups for this fibration gives an isomorphism
$$\pi_m(\mathrm{St}_k(\mathbb{R}^n)) \xrightarrow{\cong} \pi_m(\mathrm{St}_{k-1}(\mathbb{R}^n)).$$
- Example 3 (continuing the list of classifying spaces from last time): Taking the colimit of the bundles $\mathrm{St}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ as $n \to \infty$ gives a universal principal $\mathrm{GL}(k,\mathbb{R})$-bundle
$$E\,\mathrm{GL}(k,\mathbb{R}) := \mathrm{St}_k(\mathbb{R}^\infty) \to \mathrm{Gr}_k(\mathbb{R}^\infty) =: B\,\mathrm{GL}(k,\mathbb{R}).$$
The associated bundle with fiber $\mathbb{R}^k$ is then called the **universal k-plane bundle**
$$E^{k,\infty,\mathbb{R}} \to \mathrm{Gr}_k(\mathbb{R}^\infty),$$
and has the property that every real k-plane bundle (i.e. vector bundle of rank k) over a CW-complex X is isomorphic to $f^*E^{k,\infty,\mathbb{R}}$ for a unique homotopy class of maps $f : X \to \mathrm{Gr}_k(\mathbb{R}^\infty)$.
Example: Taking $k = 1$, we have
$$B\,\mathrm{GL}(1,\mathbb{R}) = \mathbb{RP}^\infty,$$
which matches $B\mathbb{Z}_2$, as we observed last time. This is not a coincidence (see Exercise 13.1(c)).
- Example 4: The discussion above also works if $\mathbb{R}$ is everywhere replaced by $\mathbb{C}$, thus giving a universal principal $\mathrm{GL}(k,\mathbb{C})$-bundle
$$E\,\mathrm{GL}(k,\mathbb{C}) := \mathrm{St}_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty) =: B\,\mathrm{GL}(k,\mathbb{C})$$
and a universal complex k-plane bundle
$$E^{k,\infty,\mathbb{C}} \to \mathrm{Gr}_k(\mathbb{C}^\infty),$$
of which all complex vector bundles of rank k over CW-complexes are pullbacks. Since $\mathrm{Gr}_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$, taking $k = 1$ gives
$$B\,\mathrm{GL}(1,\mathbb{C}) = BS^1 = \mathbb{CP}^\infty.$$

- Example 5: $\mathrm{Gr}_k(\mathbb{R}^n)$ has a double cover $\mathrm{Gr}_k^+(\mathbb{R}^n)$ whose elements are *oriented* k-dimensional subspaces, and since linearly-independent ordered k-tuples determine orientations, we have principal $\mathrm{GL}_+(k,\mathbb{R})$-bundles $\mathrm{St}_k(\mathbb{R}^n) \to \mathrm{Gr}_k^+(\mathbb{R}^n)$, which in the colimit as $n \to \infty$ give
$$E\,\mathrm{GL}_+(k,\mathbb{R}) := \mathrm{St}_k(\mathbb{R}^\infty) \to \mathrm{Gr}_k^+(\mathbb{R}^\infty) =: B\,\mathrm{GL}_+(k,\mathbb{R}).$$
- Example 6 (omitted from the actual lecture, but I'll add it here): Restricting to *orthogonal* k-tuples defines submanifolds of $\mathrm{St}_k(\mathbb{R}^n)$ and $\mathrm{St}_k(\mathbb{C}^n)$ that are naturally principal bundles over $\mathrm{Gr}_k(\mathbb{R}^n)$ or $\mathrm{Gr}_k(\mathbb{C}^n)$ with structure group $\mathrm{O}(k)$ or $\mathrm{U}(k)$ respectively, and the orthogonal version of $\mathrm{St}_k(\mathbb{R}^n)$ is also a principal $\mathrm{SO}(k)$-bundle over $\mathrm{Gr}_k^+(\mathbb{R}^n)$. The same arguments as above show that these Stiefel manifolds become weakly contractible in the colimit as $n \to \infty$, thus
$$B\,\mathrm{O}(k) = B\,\mathrm{GL}(k,\mathbb{R}) = \mathrm{Gr}_k(\mathbb{R}^\infty), \qquad B\,\mathrm{SO}(k) = B\,\mathrm{GL}_+(k;\mathbb{R}) = \mathrm{Gr}_k^+(\mathbb{R}^\infty), \qquad B\,\mathrm{U}(k) = B\,\mathrm{GL}(k,\mathbb{C}) = \mathrm{Gr}_k(\mathbb{C}^\infty).$$
For a general explanation of why certain classifying spaces of different but related groups turn out to be the same, see Exercise 13.1(c).

Topic 2: The first Chern class of a complex line bundle.

- Definition: On complex vector bundles of rank 1 (also known as complex **line bundles**) over CW-complexes X, the **first Chern class**
$$\{\text{complex line bundles over } X\} \big/ \text{isomorphism} \xrightarrow{c_1} H^2(X;\mathbb{Z})$$
is the unique characteristic class whose value $c_1(E^{1,\infty,\mathbb{C}}) \in H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ on the universal complex line bundle $E^{1,\infty,\mathbb{C}} \to \mathbb{CP}^\infty$ satisfies
$$\langle c_1(E^{1,\infty,\mathbb{C}}), [\mathbb{CP}^1]\rangle = -1,$$
where $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$ denotes the generator represented by the unique 2-cell in the usual cell decomposition of $\mathbb{CP}^\infty = e^0 \cup e^2 \cup e^4 \cup \ldots$.
Remark: The first property below gives some justification for the slightly counterintuitive appearance of a minus sign in this definition.
- Properties of c_1 on line bundles:
  (1) $c_1(L) = e(L)$, where the right hand side interprets the complex line bundle L as a real 2-plane bundle with the natural orientation determined by the complex structure.
  (2) $c_1(L \otimes L') = c_1(L) + c_1(L')$ for any two line bundles $L, L' \subset X$, where $L \otimes L' \to X$ is the line bundle whose fiber at each point $x \in X$ is the tensor product $L_x \otimes L'_x$.
  Cautionary remark: This formula for $c_1(L \otimes L')$ is easy to remember, but it is only valid when both bundles have rank 1. When we define c_1 for complex bundles of arbitrary rank, there will be a more complicated formula for $c_1(E \otimes F)$; see Exercise 13.5.

  Proofs:
  (1) Since all complex line bundles are pullbacks of $E^{1,\infty,\mathbb{C}} \to \mathbb{CP}^\infty$, it suffices to prove that $e(E^{1,\infty,\mathbb{C}}) \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ is also the generator that evaluates to -1 on $[\mathbb{CP}^1]$. In fact, the cohomology of $\mathbb{CP}^\infty$ in degree 2 is simply that of its 2-skeleton $\mathbb{CP}^1 \subset \mathbb{CP}^\infty$, thus it actually suffices to prove the same statement about the tautological line bundle $E^{1,2,\mathbb{C}} \to \mathbb{CP}^1$. This follows from the properties of the Euler class on smooth manifolds proved in the previous lecture, because $E^{1,2,\mathbb{C}} \to \mathbb{CP}^1$ admits a section for which the signed count of zeroes is -1; see Exercise 13.7.
  (2) Thanks to the first property, it suffices to prove $e(L \otimes L') = e(L) + e(L')$, where it should be stressed that the tensor product is complex, but the bundles on both sides are then interpreted as oriented real bundles of rank 2. Using our definition of the Euler class as a primary obstruction, we first choose nowhere-zero sections s, s' of L

and L' respectively over the 1-skeleton of X, and observe that the tensor product then inherits a nowhere-zero section

$$\sigma := s \otimes s' \in \Gamma\big((L \otimes L')|_{X^1}\big).$$

Along the boundary of any given 2-cell $e^2_\alpha \subset X$, any trivializations of L and L' determine a trivialization of $L \otimes L'$, so that s, s', σ all become identified with complex-valued functions having nonzero values related by $\sigma = ss'$. Plugging this information into the definition of the obstruction cochains for all three bundles, the relation

$$\theta^2_{L\otimes L'}(e^2_\alpha) = \theta^2_L(e^2_\alpha) + \theta^2_{L'}(e^2_\alpha) \in \pi_1(\mathbb{C}\backslash\{0\}) = \mathbb{Z}$$

follows because $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ is a topological group, and the definition of multiplication in its fundamental group $\pi_1(\mathbb{C}^*)$ can therefore be expressed in terms of the multiplication in $\mathbb{C}^*$ itself, namely as $[\gamma][\gamma'] := [\gamma\gamma']$ for two loops $\gamma, \gamma' : S^1 \to \mathbb{C}^*$, i.e. instead of concatenating them, we are multiplying their values pointwise.

- Theorem: Two complex line bundles L, L' over the same CW-complex X are isomorphic if and only if $c_1(L) = c_1(L')$.
  We give two proofs of this theorem: the second using a combination of big classification results we have in our toolbox, the first perhaps a bit more down-to-earth.
- Proof 1: A complex vector bundle isomorphism $\Phi : L \to L'$ between line bundles is the same thing as a nowhere-zero section of the complex line bundle $\mathrm{Hom}(L, L')$ whose fiber at each point $x \in X$ is the vector space of complex-linear maps $L_x \to L'_x$. Since c_1 of this bundle matches its Euler class, such a section will exist over the 2-skeleton of X if and only if $c_1(\mathrm{Hom}(L, L')) = 0$, but if it does, then it can also be extended inductively to a nowhere-zero section on all higher-dimensional skeleta as well, because the standard fiber of the bundle obtained by removing the zero section is $\mathbb{C}\backslash\{0\}$, and $\pi_n(\mathbb{C}\backslash\{0\}) = 0$ for all $n \geqslant 2$. To compute $c_1(\mathrm{Hom}(L, L'))$, we observe that there is a natural vector bundle isomorphism $\mathrm{Hom}(L, L') \cong L^* \otimes L$, where L^* denotes the dual bundle of L, whose fiber at each point $x \in X$ is the space of complex-linear maps $L_x \to \mathbb{C}$. Moreover, there is an obvious bundle isomorphism of $L^* \otimes L$ to the trivial complex line bundle, defined on each fiber as the map $L^*_x \otimes L_x \to \mathbb{C} : \lambda \otimes v \mapsto \lambda(v)$. The trivial bundle admits a nowhere-zero section and therefore has vanishing Euler class, hence also vanishing first Chern class, so we obtain the relation $c_1(L^*) + c_1(L) = 0$, and then conclude

  $$c_1\big(\mathrm{Hom}(L, L')\big) = c_1(L^* \otimes L') = c_1(L^*) + c_1(L') = -c_1(L) + c_1(L'),$$

  which vanishes if and only if $c_1(L) = c_1(L')$.
- Proof 2: Since $B\,\mathrm{GL}(1, \mathbb{C}) = \mathbb{CP}^\infty$, the two line bundles $L, L' \to X$ are both pullbacks of the tautological line bundle $E^{1,\infty,\mathbb{C}} \to \mathbb{CP}^\infty$ via maps $f, g : X \to \mathbb{CP}^\infty$, and they are isomorphic if and only if $f \underset{h}{\sim} g$. As it happens, $\mathbb{CP}^\infty$ is also a $K(\mathbb{Z}, 2)$, so we have a bijection $[X, \mathbb{CP}^\infty] \cong H^2(X; \mathbb{Z})$, which can be written explicitly as the map

  $$[X, \mathbb{CP}^\infty] \to H^2(X; \mathbb{Z}) : [h] \mapsto h^*\iota_2$$

  for a distinguished class $\iota_2 \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ that corresponds to the homotopy class of the identity map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty$. All other classes in $H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$ are necessarily pullbacks of ι_2 via maps $\mathbb{CP}^\infty \to \mathbb{CP}^\infty$, and it must therefore be a generator, implying $\iota_2 = \pm c_1(E^{1,\infty,\mathbb{C}})$, thus

  $$f \underset{h}{\sim} g \quad \Leftrightarrow \quad c_1(L) = f^*c_1(E^{1,\infty,\mathbb{C}}) = \pm f^*\iota_2 = \pm g^*\iota_2 = g^*c_1(E^{1,\infty,\mathbb{C}}) = c_1(L').$$

Topic 3: The Chern classes of complex vector bundles of arbitrary rank.

- Theorem/Definition: For all $k \in \mathbb{N}$, there exist unique characteristic classes $c_k(E) \in H^{2k}(X;\mathbb{Z})$ for complex vector bundles $E \to X$ of arbitrary rank over CW-complexes, such that the following conditions hold:
  (1) $c_1(E)$ matches the previous definition in the case $\operatorname{rank}(E) = 1$;
  (2) $c_k(E) = 0$ whenever $k > \operatorname{rank}(E)$;
  (3) The **total Chern class**
  $$c(E) := 1 + c_1(E) + c_2(E) + \ldots \in H^*(X;\mathbb{Z})$$
  satisfies the Whitney sum formula
  $$c(E \oplus F) = c(E) \cup c(F)$$
  for every pair of complex vector bundles E, F over the same base.
  (Note: The second condition makes the sum in the definition of $c(E)$ finite.)
- Corollary of the Whitney sum formula: The first Chern class is additive with respect to direct sums of vector bundles,
  $$c_1(E \oplus F) = c_1(E) + c_1(F).$$
- Proof sketch, part 1: The uniqueness of the Chern classes would be clear by an inductive argument if every vector bundle E of rank $n \geqslant 2$ contained a linear subbundle $F \subset E$ of strictly lower rank; one could then form a quotient vector bundle E/F with fibers E_x/F_x, and choose a bundle metric (which can always be done according to Exercise 12.3) in order to identify E/F with the orthogonal complement of F in E, giving rise to a splitting
  $$E \cong F \oplus (E/F).$$
  Unfortunately, not every vector bundle $E \to X$ can be split in this way, but the following shows that it always has useful pullbacks that can. We consider the **projectivization** of E, a fiber bundle
  $$\mathbb{P}(E) \xrightarrow{\pi} X$$
  whose fiber at each point $x \in X$ is the space $\mathbb{P}(E_x) \cong \mathbb{CP}^{n-1}$ of complex 1-dimensional subspaces of E_x. For each $x \in X$ and $\ell \in \mathbb{P}(E_x)$, the pullback vector bundle $\pi^*E \to \mathbb{P}(E)$ has fiber $(\pi^*E)_\ell = E_x$ containing $\ell \subset E_x$ as a distinguished 1-dimensional subspace, thus defining a complex line bundle $L \to \mathbb{P}(E)$ that is also a linear subbundle of π^*E and thus gives rise to a splitting
  $$\pi^*E \cong L \oplus (\pi^*E/L).$$
  Inductively, if the Chern classes have already been defined on bundles up to rank $n-1$, then this splitting determines $c(\pi^*E)$ uniquely, and we claim that in fact $c(E)$ is also uniquely determined by this. This requires a digression on the cohomology of fiber bundles.
- Digression (the Leray-Hirsch theorem): Suppose $\pi : E \to X$ is a fiber bundle over a CW-complex X, with a standard fiber F that is a *compact* CW-complex. Such bundles are assembled out of local pieces that look like product bundles $\mathbb{D}^n \times F \xrightarrow{\mathrm{pr}_1} \mathbb{D}^n$ for $n \geqslant 0$, so assume to start with that $E = \mathbb{D}^n \times F$, $X = \mathbb{D}^n$ and $\pi = \mathrm{pr}_1$. The cross product gives a homomorphism of $\mathbb{Z}$-graded abelian groups
  $$H^*(X;\mathbb{Z}) \otimes H^*(F;\mathbb{Z}) \xrightarrow{\times} H^*(X \times F;\mathbb{Z}),$$
  which can also be expressed in terms of the cup product as
  $$\varphi \otimes \psi \mapsto \varphi \times \psi = \pi^*\varphi \cup \mathrm{pr}_2^*\,\psi. \tag{13.1}$$
  Under some extra conditions, the Künneth formula will tells us that this is an isomorphism; the Künneth formula is usually stated for homology rather than cohomology because the

chain complexes need to be free, and singular cochain groups are *not* free due to the fact that duals of direct sums are generally direct products instead of direct sums. However, if we use cellular cohomology and assume both complexes are finitely generated, then the cochain groups are all finitely-generated free abelian groups, and the cellular cross product of cochains

$$C^*_{\mathrm{CW}}(X;\mathbb{Z}) \otimes C^*_{\mathrm{CW}}(F;\mathbb{Z}) \xrightarrow{\times} C^*_{\mathrm{CW}}(X \times F;\mathbb{Z})$$

becomes (obviously) an isomorphism. If we add the extra assumption that the cohomology $H^*(F;\mathbb{Z})$ is free, then the Tor terms vanish in the algebraic Künneth formula, implying that (13.1) is an isomorphism. The Leray-Hirsch theorem extends this conclusion to more general fiber bundles $\pi : E \to X$, but one needs a different way of writing (13.1) since there is generally no second projection map $\mathrm{pr}_2 : E \to F$. Observe however that in the setting of the product bundle, assuming X is path-connected, the image of $\mathrm{pr}_2^* : H^*(F;\mathbb{Z}) \to H^*(E;\mathbb{Z})$ is a subgroup of $H^*(E;\mathbb{Z})$ that pulls back isomorphically to $H^*(E_x;\mathbb{Z})$ under the inclusion $E_x \hookrightarrow E$ of each fiber. The right approach for the general case is to add the hypothesis that such a subgroup is given, and one obtains:
Leray-Hirsch theorem: Assume $\pi : E \to X$ is a fiber bundle over a CW-complex X, and $V \subset H^*(E;\mathbb{Z})$ is a finitely-generated free subgroup such that for every $x \in X$, the map

$$V \to H^*(E_x;\mathbb{Z})$$

induced by the inclusion $E_x \hookrightarrow E$ of the fiber is an isomorphism. Then the map

$$H^*(X;\mathbb{Z}) \otimes V \to H^*(E;\mathbb{Z}) : \varphi \otimes \psi \mapsto \pi^*\varphi \cup \psi$$

is also an isomorphism. In other words, any finite set of classes in $H^*(E;\mathbb{Z})$ that pull back to a basis of every fiber $H^*(E_x;\mathbb{Z})$ also form a basis of $H^*(E;\mathbb{Z})$ as a *module* over $H^*(X;\mathbb{Z})$.
Remark: The cohomological version of the Künneth formula described above is true (but harder to prove) under somewhat weaker assumptions that do not require F to be a compact cell complex, and for that reason, I stated the Leray-Hirsch theorem here without such an assumption, though that assumption would hold in any case for the application we need. Since singular cohomology is an invariant of weak homotopy type, one can also use CW-approximation to remove the assumption that X is a CW-complex. At this point in the course, some proofs of important theorems must be omitted due to lack of time, and this is one of them, but here is a very quick summary: the proof is by induction over the skeleta of X, using a relative version of the cohomological Künneth formula to extend the validity of the result from the $(n-1)$-skeleton to each n-cell.

- Proof sketch, part 2: We have a fiber bundle $\pi : \mathbb{P}(E) \to X$ whose fibers $\mathbb{P}(E_x) \cong \mathbb{CP}^{n-1}$ are compact cell complexes, and a complex line bundle $L \to \mathbb{P}(E)$ whose restriction to $\mathbb{P}(E_x)$ for each $x \in X$ is the tautological line bundle over $\mathbb{CP}^{n-1}$. Writing $i_x : \mathbb{P}(E_x) \hookrightarrow \mathbb{P}(E)$ for the inclusion, the definition and naturality of the first Chern class on line bundles now implies that $i_x^* c_1(L)$ generates $H^2(\mathbb{P}(E_x);\mathbb{Z}) \cong \mathbb{Z}$, and moreover, the classes

$$1, c_1(L), c_1(L)^2, \dots, c_1(L)^{n-1} \in H^*(\mathbb{P}(E);\mathbb{Z})$$

defined via the cup product pull back to each fiber $\mathbb{P}(E_x) \cong \mathbb{CP}^{n-1}$ as a basis for the free abelian group $H^*(\mathbb{P}(E_x);\mathbb{Z})$. The Leray-Hirsch theorem thus implies that

$$\underbrace{H^*(X;\mathbb{Z}) \oplus \ldots \oplus H^*(X;\mathbb{Z})}_{n} \to H^*(\mathbb{P}(E);\mathbb{Z}) : (\varphi_0, \dots, \varphi_{n-1}) \mapsto \sum_{k=0}^{n-1} \pi^*\varphi_k \cup c_1(L)^k$$

is an isomorphism of $\mathbb{Z}$-graded abelian groups. In particular, it follows that $\pi^* : H^*(X;\mathbb{Z}) \to H^*(\mathbb{P}(E);\mathbb{Z})$ is injective, so $c(E)$ is determined by $\pi^* c(E) = c(\pi^* E)$, which completes the proof of uniqueness for the Chern classes.

To show that classes $c_k(E)$ with the required properties actually exist independently of choices, one can apply the Leray-Hirsch theorem to the specific class $c_1(L)^n \in H^{2n}(\mathbb{P}(E);\mathbb{Z})$: the theorem makes this class uniquely representable as a sum for $k = 0,\dots,n-1$ of cup products of the classes $c_1(L)^k \in H^{2k}(\mathbb{P}(E);\mathbb{Z})$ with pullbacks of specific classes in $H^{2(n-k)}(X;\mathbb{Z})$. It turns out that we get the right result by defining $c_{n-k}(E)$ to be (up to a sign) the coefficients in that expression, i.e. we write $c_0(E) := 1 \in H^0(X;\mathbb{Z})$ and define $c_k(E) \in H^{2k}(X;\mathbb{Z})$ for $k = 1,\dots,n$ as the unique classes for which the relation

$$\sum_{k=0}^{n} (-1)^k \pi^* c_{n-k}(E) \cup c_1(L)^k = 0$$

holds. The rest of the proof is to verify that the formula $c(E \oplus F) = c(E) \cup c(F)$ now holds in general.

- Remark: If I had to name one specific reason why the theorem above is true, it would be the computation of $H^*(\mathbb{CP}^{n-1};\mathbb{Z})$: the Chern classes have the structure that they have mainly because that one particular cohomology ring has the structure that it has.

Suggested reading. For the Euler class, [DK01, §7.11] defines it in essentially the way that I did, as a primary obstruction class, though with fewer details. There are other ways to do it, the standard approach being to pull back the Thom class via the inclusion of the zero-section of $E \to X$ into its total space; this requires knowing more about the Thom class and the Thom isomorphism theorem, but the idea is summarized nicely in [Fre12, Lecture 8].

My presentation of classifying spaces was inspired in large part by [Fre12, Lecture 6], though Freed's treatment is a bit idiosyncratic, as he puts a lot of effort into constructing classifying spaces that are actually smooth (infinite-dimensional) manifolds instead of CW-complexes, e.g. instead of $\mathbb{CP}^\infty$, Freed's version of BS^1 is the projectivization of a complex infinite-dimensional separable Hilbert space—which is of course *homotopy equivalent* to $\mathbb{CP}^\infty$. It's an interesting read if you're curious about different ways of doing things, but in the end, most of the differences are essentially cosmetic. In [Ste51, Chapter 19], one also finds a relatively down-to-earth construction of classifying spaces for arbitrary compact Lie groups, and special cases of it are what we did in lecture.

My presentation of the Chern classes was also based largely on Freed's notes [Fre12, Lecture 7], which are apparently heavily influenced by [BT82], but I haven't read that portion of Bott and Tu. For a different approach that relies more heavily on computations of the cohomology of Grassmann manifolds, there is also [Hat, Chapter 3]. Complete proofs of the Leray-Hirsch theorem can be found in [Hat02, §4.D] or [tD08, §17.8].

Exercises (for the Übung on 18.07.2024).

Exercise 13.1. Let's talk about the functoriality of classifying spaces.

(a) Show that any two classifying spaces for the same topological group are canonically homotopy equivalent.
Hint: The universal bundle over one must be a pullback of the universal bundle over the other.

(b) Suppose $\Phi : H \to G$ is a homomorphism of topological groups which have universal bundles $EH \to BH$ and $EG \to BG$. Using Φ to define a left H-action on G, one obtains an associated H-bundle

$$EH \times_H G \to BH$$

with standard fiber G. Show that this fiber bundle also naturally has the structure of a principal G-bundle, and use this to define a canonical homotopy class of continuous maps $B\Phi : BH \to BG$ fitting into a diagram of the form

$$\begin{array}{ccc} EH & \longrightarrow & EG \\ \downarrow & & \downarrow \\ BH & \xrightarrow{B\Phi} & BG \end{array},$$

in which the top horizontal map is H-equivariant.

(c) Assume the homomorphism $\Phi : H \to G$ in part (b) is the inclusion of a subgroup $H \subset G$ such that the quotient projection $G \to G/H$ is a fiber bundle (cf. Exercise 12.2), and also that it is a homotopy equivalence. We saw in Exercise 12.3 that in this situation, every G-bundle admits a reduction of its structure group to H. Show that every principal G-bundle $P^G \to B$ must then contain an H-invariant $P^H \subset P^G$ that is a principal H-bundle over B, and that the inclusion

$$P^H \hookrightarrow P^G$$

is a weak homotopy equivalence. Deduce from this that any classifying space BG for G is also a classifying space for H.

Hint: The bundles can be trivialized along any map $(\mathbb{D}^n, S^{n-1}) \to (P^G, P^H)$.

Exercise 13.2. We have not yet gotten around to giving formal definitions of the direct sum and tensor product operations on vector bundles, so let's do that, and then prove the Whitney sum formula for the Euler class.

(a) Given vector bundles $E \to X$ and $F \to Y$ with structure groups G and H respectively, construct a vector bundle $E \boxplus F \to X \times Y$ whose fiber at each point $(x, y) \in X \times Y$ is the direct sum of vector spaces $E_x \oplus F_y$, and whose structure group is $G \times H$.

Remark: $E \boxplus F$ is sometimes called the **external direct sum** *(or external Whitney sum) of the bundles E and F. The ordinary Whitney sum of two bundles $E, F \to B$ over the same base can be obtained from the external sum as a pullback:*

$$E \oplus F = \Delta^*(E \boxplus F), \qquad \text{where} \qquad B \xrightarrow{\Delta} B \times B : x \mapsto (x, x),$$

thus its fibers are $(E \oplus F)_x = E_x \oplus F_x$, and $E \oplus F$ also inherits structure group $G \times H$ if E and F have structure groups G and H respectively.

(b) Given two vector bundles $E, F \to B$ over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, construct a vector bundle $E \otimes F \to B$ whose fiber at each point $x \in B$ is the tensor product of vector spaces $E_x \otimes F_x$. In the case where E and F both have rank 1, write down a formula for local transition functions of $E \otimes F$ determined by transition functions of E and F. (Such a formula was used implicitly in our proof in lecture that $c_1(L \otimes L') = c_1(L) + c_1(L')$ for complex line bundles.)

(c) The following is a preparatory lemma toward proving the Whitney sum formula: Suppose $f : (\mathbb{D}^m, S^{m-1}) \to (\mathbb{D}^m, S^{m-1})$ and $g : (\mathbb{D}^n, S^{n-1}) \to (\mathbb{D}^n, S^{n-1})$ are maps of pairs $(m, n \geqslant 1)$ whose restrictions to the boundary are maps of spheres with degrees $k, \ell \in \mathbb{Z}$ respectively. Regarding $\partial(\mathbb{D}^m \times \mathbb{D}^n)$ as a sphere of dimension $m + n - 1$, show that the product map

$$(\mathbb{D}^m \times \mathbb{D}^n, \partial(\mathbb{D}^m \times \mathbb{D}^n)) \xrightarrow{f \times g} (\mathbb{D}^m \times \mathbb{D}^n, \partial(\mathbb{D}^m \times \mathbb{D}^n))$$

restricts to the boundary as a map of degree $k\ell$.

Hint: I can think of a few possible approaches to this, but the simplest is perhaps to compute the degree of the induced map

$$S^{m+n} \cong (\mathbb{D}^m \times \mathbb{D}^n)/\partial(\mathbb{D}^m \times \mathbb{D}^n) \longrightarrow (\mathbb{D}^m \times \mathbb{D}^n)/\partial(\mathbb{D}^m \times \mathbb{D}^n) \cong S^{m+n}$$

as a sum of local degrees.

(d) For two oriented real vector bundles $E \to X$ and $F \to Y$ of ranks $m, n \geqslant 2$ respectively over CW-complexes, prove the formula
$$e(E \boxplus F) = e(E) \times e(F) \in H^{m+n}(X \times Y; \mathbb{Z}),$$
and deduce from this the Whitney sum formula
$$e(E \oplus F) = e(E) \cup e(F)$$
for two oriented bundles over the same base.
Hint: Our definition of the Euler class requires you to start by choosing nowhere-zero sections $s \in \Gamma(E|_{X^{m-1}})$ and $t \in \Gamma(F|_{Y^{n-1}})$. They can both be extended over the rest of X and Y respectively—you cannot assume that the extensions are nowhere zero, but you do obtain from this a nowhere-zero section $F(x, y) := (s(x), t(y))$ of $E \boxplus F$ over the $(m + n - 1)$-skeleton of $X \times Y$. Use the (easy) definition of the cross product on cellular cochains.

(e) In lecture, we did not give any definition for $e(E) \in H^1(X; \mathbb{Z})$ when $E \to X$ has rank 1, but this omission is forgivable: the structure group of an oriented real line bundle is the contractible group $\mathrm{GL}_+(1, \mathbb{R}) = (0, \infty)$, so by Exercise 12.3(d), such bundles are always trivial, and thus admit global sections. The only sensible definition of the Euler class in this case is therefore $e(E) := 0$. What content do the two formulas in part (d) now have if E or F is allowed to have rank 1? Are they still true?

Exercise 13.3. For an integer $k \geqslant 0$, let ϵ^k denote the (real or complex) trivial vector bundle of rank k over any given base. A vector bundle $E \to X$ is said to be **stably trivial** if $E \oplus \epsilon^k \cong \epsilon^n$ for some $k, n \geqslant 0$, and two vector bundles $E, F \to X$ (not necessarily of the same rank) are called **stably isomorphic** if $E \oplus \epsilon^k \cong F \oplus \epsilon^\ell$ for some $k, \ell \geqslant 0$. You will easily convince yourself that this defines an equivalence relation.

(a) Show that the Chern classes $c_k(E) \in H^{2k}(X; \mathbb{Z})$ depend only on the stable isomorphism class of E, and they vanish whenever E is stably trivial.
(b) Find some examples of real vector bundles (e.g. tangent bundles of familiar manifolds) that are nontrivial but stably trivial.
(c) Show that every (real or complex) vector bundle E over a compact CW-complex X has a **stable inverse**, meaning a vector bundle E^{-1} with the property that $E \oplus E^{-1}$ is stably trivial, and moreover, E^{-1} is unique up to stable isomorphism.
Hint: Since X is compact, a map from X to any classifying space BG takes values in a finite-dimensional subcomplex. Can you solve the problem for the universal vector bundle over that subcomplex?
(d) How are the Chern classes of a complex vector bundle $E \to X$ and its stable inverse E^{-1} related? Compute them for the stable inverse of the tautological line bundle $E^{1,n+1,\mathbb{C}} \to \mathbb{CP}^n$.

Exercise 13.4. You should probably not stress too much over the details of this, but I want to sketch one way of proving the formula
$$\mathrm{PD}(e(E)) = [s^{-1}(0)] \in H_{n-k}(M; \mathbb{Z})$$
for a smooth oriented vector bundle $E \to M$ of rank k over a smooth closed and oriented n-manifold, where $s \in \Gamma(E)$ is assumed to be any smooth section that is transverse to the zero-section. One thing you might vaguely recall in the historical background of Poincaré duality is the notion of a *dual cell decomposition*: any oriented triangulation of M determines a dual cell decomposition, which is not typically a triangulation, but has k-cells in bijective correspondence with the $(n - k)$-simplices

of the triangulation. More specifically, a k-cell in the dual decomposition can only intersect an m-simplex of the original triangulation if $k + m \geqslant n$, and it intersects a unique $(n-k)$-simplex of the triangulation at a single point lying in the interior of both (see e.g. [Wen23, Lecture 54], especially the picture on page 429). This correspondence between $(n-k)$-simplices and k-cells gives rise to an explicit isomorphism of chain complexes

$$C^*_{\mathrm{CW}}(M;\mathbb{Z}) \xrightarrow{\cong} C^{\Delta}_{n-*}(M;\mathbb{Z}), \tag{13.2}$$

in which the original triangulation is used on the right hand side and the dual cell decomposition on the left, and this isomorphism descends to cohomology/homology as the Poincaré duality map $H^*_{\mathrm{CW}}(M;\mathbb{Z}) \to H_{n-*}(M;\mathbb{Z})$.

In the situation at hand, transversality and the implicit function theorem make the zero-set $s^{-1}(0) \subset M$ a closed oriented submanifold of dimension $n-k$, and we can choose an oriented triangulation of M that contains in its $(n-k)$-skeleton an oriented triangulation of the submanifold $s^{-1}(0)$. Every point in $s^{-1}(0)$ thus belongs to a p-simplex of the triangulation for some $p \leqslant n-k$, and therefore also lies in a q-cell of the dual decomposition for some $q \geqslant n-p \geqslant k$, implying that s is nowhere zero on the $(k-1)$-skeleton of the dual decomposition. Moreover, the homology class $[s^{-1}(0)] \in H_{n-k}(M;\mathbb{Z})$ is represented in the oriented simplicial homology of the triangulation by a sum of the $(n-k)$-simplices that triangulate $s^{-1}(0)$, which corresponds under (13.2) to a sum of all the dual k-cells that intersect $s^{-1}(0)$. Think about this picture long enough to feel intuitively convinced that the preimage under (13.2) of this particular simplicial $(n-k)$-cycle is a cellular k-cocycle representing $e(E)$.

Remark: More modern (and less handwavy) proofs of the formula $\mathrm{PD}(e(E)) = [s^{-1}(0)]$ *typically express* $e(E)$ *in terms of the Thom class* $\tau \in H^k(E, E\backslash M;\mathbb{Z})$*, which gives an equivalent way of defining it—the Thom class also plays a key role in the intersection-theoretic interpretation of Poincaré duality.*

Exercise 13.5. A useful computational tool for Chern classes is known as the **splitting principle**, which says that a vector bundle $E \to X$ of arbitrary rank $n \geqslant 1$ can always be pulled back via some map $f : Y \to X$ so that its pullback splits into a direct sum of line bundles

$$f^*E \cong L_1 \oplus \ldots \oplus L_n$$

and the induced map $f^* : H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is injective. In the setting of a complex vector bundle, this has the consequence that the classes $c_1(L_1), \ldots, c_1(L_n) \in H^2(Y;\mathbb{Z})$ uniquely determine the total Chern class of E. We saw for instance in lecture that if one constructs the projectivization $\pi : \mathbb{P}(E) \to X$ of E, then the Leray-Hirsch theorem makes the map $\pi^* : H^*(X;\mathbb{Z}) \to H^*(\mathbb{P}(E);\mathbb{Z})$ injective, and the pullback bundle $\pi^*E \to \mathbb{P}(E)$ then has a splitting into the direct sum of a line bundle with another bundle of lower rank. Applying the same trick again $n-2$ more times gives a pullback that splits completely into line bundles. In practice, the splitting principle can be applied without knowing where such splittings actually come from—one only needs to know that they exist. The idea is to prove first that the desired statement holds for all bundles that can be split into line bundles, and then use the injectivity of f^* to deduce from this the general case.

(a) Show that the formula $c_1(L \otimes L') = c_1(L) + c_1(L')$ for two complex line bundles $L, L' \to X$ generalizes to bundles $E, F \to X$ of arbitrary ranks as

$$c_1(E \otimes F) = \mathrm{rank}(F)\, c_1(E) + \mathrm{rank}(E)\, c_1(F).$$

(b) Show that for the dual E^* of any complex vector bundle $E \to X$ and each $k \in \mathbb{N}$,

$$c_k(E^*) = (-1)^k c_k(E).$$

(c) Show that for a complex n-plane bundle E,
$$c_n(E) = e(E) \in H^{2n}(X;\mathbb{Z}),$$
where on the right hand side, E is regarded as an oriented real $2n$-plane bundle.

The following definition is needed for the rest of the exercise. The **determinant line bundle** $\det(E) \to X$ of a complex vector bundle $E \to X$ of any rank n can be defined by taking any atlas of local trivializations with $\mathrm{GL}(n,\mathbb{C})$-valued transition data for E, and keeping the same transition data but replacing the fiber $\mathbb{C}^n$ by $\mathbb{C}$, with $\mathrm{GL}(n,\mathbb{C})$ acting linearly on $\mathbb{C}$ via the group homomorphism
$$\det : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^* := \mathrm{GL}(1,\mathbb{C}).$$
Here are two other equivalent definitions of $\det(E) \to X$ that are more obviously independent of choices: for the first, use the frame bundle $FE \to X$ to identify E isomorphically with the associated bundle $FE \times_{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}^n$, then set $\det(E) = FE \times_{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}$ using the $\mathrm{GL}(n,\mathbb{C})$-action on $\mathbb{C}$ mentioned above. One gets an even more intrinsic definition by describing the fibers $\det(E_x) \cong \mathbb{C}$ of $\det(E) \to X$ as the top exterior powers $\Lambda^n(E_x)$ of the fibers of E. You can prove to yourself as a preliminary exercise that these three descriptions all define the same thing.

(d) Show that for any two complex vector bundles $E, F \to X$, $\det(E \oplus F) \cong \det(E) \otimes \det(F)$.

(e) Use the splitting principle to show that every complex vector bundle has the same first Chern class as its determinant line bundle.

Exercise 13.6. For two complex vector bundles $E, F \to X$ of the same rank $n \geqslant 1$ over a CW-complex, prove that the restrictions of E and F to the 2-skeleton are isomorphic if and only if $c_1(E) = c_1(F)$.

Remark: Recall that a compact connected surface Σ with $\partial\Sigma \neq \varnothing$ always admits a deformation retraction to its 1-skeleton, so it follows that all complex vector bundles over Σ are trivial, and so are complex vector bundles over S^1. We also conclude that for Σ a closed, connected and orientable surface, the isomorphism classes of complex vector bundles $E \to \Sigma$ are in bijective correspondence with the integers via $c_1(E) \in H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$.

Hint: Bundle isomorphisms $E \to F$ are the same thing as sections of a certain fiber bundle $\Theta \to X$ whose fiber at each point $x \in X$ is the space of invertible linear maps $E_x \to F_x$, thus its standard fiber can be taken to be $\mathrm{GL}(n,\mathbb{C})$, which admits a deformation retraction to $\mathrm{U}(n)$ by polar decomposition. You learned some things about $\pi_1(\mathrm{U}(n))$ in Exercise 7.8 that will be useful here. Another useful observation is that a section of $\mathrm{Hom}(E,F)$ induces a section of $\mathrm{Hom}(\det(E),\det(F))$ which is nonzero at precisely the points where the given linear map $E_x \to F_x$ is invertible. Use this to show that bundle isomorphisms $E \to F$ always exist over the 1-skeleton, and then to determine when such isomorphisms can be extended over 2-cells.

Exercise 13.7. Let $L := E^{1,2,\mathbb{C}} \to \mathbb{CP}^1$ denote the tautological line bundle, which is naturally a subbundle of the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}^2 \to \mathbb{C}^2$ since elements of $\mathbb{CP}^1$ are subspaces of $\mathbb{C}^2$; explicitly,
$$L = \left\{ (\ell, v) \in \mathbb{CP}^1 \times \mathbb{C}^2 \;\middle|\; v \in \ell \right\}.$$
As a complex 1-manifold, $\mathbb{CP}^1$ can be covered by coordinate charts that are related holomorphically: the standard construction denotes elements of $\mathbb{CP}^1$ by $[z_0 : z_1]$ (the equivalence class of $(z_0, z_1) \in \mathbb{C}^2\backslash\{0\}$ in its quotient by the action of $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$), and covers $\mathbb{CP}^1$ with two open subsets
$$\mathcal{U}_0 := \left\{ [z_0 : z_1] \in \mathbb{CP}^1 \;\middle|\; z_0 \neq 0 \right\}, \qquad \mathcal{U}_1 := \left\{ [z_0 : z_1] \in \mathbb{CP}^1 \;\middle|\; z_1 \neq 0 \right\}$$
on which the maps
$$\mathcal{U}_0 \xrightarrow{\varphi_0} \mathbb{C} : [z_0 : z_1] \mapsto z_1/z_0, \qquad \mathcal{U}_1 \xrightarrow{\varphi_1} \mathbb{C} : [z_0 : z_1] \mapsto z_0/z_1,$$

are homeomorhisms. On the same two open subsets, define local trivializations $L|_{\mathcal{U}_0} \to \mathcal{U}_0 \times \mathbb{C}$ and $L|_{\mathcal{U}_1} \to \mathcal{U}_1 \times \mathbb{C}$ that are related by a holomorphic transition function $g_{10} : \mathcal{U}_0 \cap \mathcal{U}_1 \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$. (This shows that $L \to \mathbb{CP}^1$ is not only a *complex* but also a *holomorphic* vector bundle, a notion that only makes sense when the base is a complex manifold.) Then write down an explicit section $s \in \Gamma(L)$ that equals zero at exactly one point, and show that this point contributes -1 to the signed count of zeroes that computes $\langle e(L), [\mathbb{CP}^1]\rangle$. The sign should be consistent with the convention that for complex-valued functions $f : \mathbb{C} \to \mathbb{C}$, regarded as sections of a trivial line bundle over $\mathbb{C}$, the zero at 0 of the section $f(z) = z^k$ for each $k \in \mathbb{N}$ contributes k, while $f(z) = \bar{z}^k$ contributes $-k$. (In general, the recipe is to examine the map $S^1 \to S^1$ obtained by normalizing the function along a small circle surrounding the isolated zero—one then measures which multiple of the canonical generator of $\pi_1(S^1)$ this map represents. For sections of an oriented real vector bundle over an oriented manifold, the contribution of each isolated zero is independent of choices of local chart and trivialization, so long as both respect the fixed orientations.)

Exercise 13.8. The **Stiefel-Whitney** classes $w_k(E) \in H^k(X; \mathbb{Z}_2)$ of a real vector bundle $E \to X$ over a CW-complex are defined analogously to the Chern classes, but with the universal complex line bundle $E^{1,\infty,\mathbb{C}} \to \mathbb{CP}^\infty$ replaced by its real counterpart $E^{1,\infty,\mathbb{R}} \to \mathbb{RP}^\infty$, which makes it natural to use $\mathbb{Z}_2$ coefficients since the cohomology ring $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2)$ has properties closely analogous to those of $H^*(\mathbb{CP}^\infty; \mathbb{Z})$. In particular, the first Stiefel-Whitney class $w_1(L) \in H^1(X; \mathbb{Z}_2)$ on real line bundles $L \to X$ is uniquely determined by naturality and the condition that $w_1(E^{1,\infty,\mathbb{R}}) \neq 0 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2$. Its definition on higher-rank real bundles, as well as the rest of the classes $w_k(E)$, can then be deduced via naturality from the Whitney sum formula

$$w(E \oplus F) = w(E) \cup w(F)$$

using the splitting principle (cf. Exercise 13.5), where we denote the **total Stiefel-Whitney class** $w(E) := 1 + w_1(E) + w_2(E) + \ldots \in H^*(X; \mathbb{Z}_2)$. As with the Chern classes, one requires $w_k(E) = 0$ for $k > \mathrm{rank}(E)$, so the sum in the total Stiefel-Whitney class is always finite.

(a) Show that two real line bundles $L, L' \to X$ over a CW-complex are isomorphic if and only if $w_1(L) = w_1(L')$. In particular, L is trivial if and only if $w_1(L) = 0$.
Hint: We sketched two proofs of the complex analogue of this statement in lecture, and both approaches are also options here. The easier approach is probably to make use of the double role that $\mathbb{RP}^\infty$ plays as $BGL(1, \mathbb{R})$ and $K(\mathbb{Z}_2, 1)$. But it is also possible to give a more direct definition of $w_1(L)$ as an obstruction to finding nowhere-zero sections of L over X^1.

(b) Show that for a real vector bundle $E \to X$ of arbitrary rank over a CW-complex, $w_1(E) = 0$ if and only if E is orientable.
Hint: There is also a real analogue of the determinant line bundle $\det(E) \to X$ introduced in Exercise 13.5.

(c) Show that two real vector bundles $E, F \to X$ of the same rank are isomorphic over the 1-skeleton if and only if $w_1(E) = w_1(F)$.

I mentioned that there are also constructions of certain characteristic classes in specific cohomology theories that do not require any cell decomposition. Here is a way to construct w_1 in Čech cohomology. For simplicity, assume the base X of our bundle $E \to X$ admits a so-called *good* open covering $\{\mathcal{U}_\alpha \subset X\}_{\alpha \in J}$, meaning that any nonempty finite intersection of sets in this covering is contractible.[64] For a given abelian group G, the Čech n-cochain group $\check{C}^n(X; G)$ with coefficients

[64]The existence of a good covering is a nontrivial condition, and it is also not strictly necessary, but it simplifies matters by avoiding the need for direct limits in our discussion of Čech cohomology. One can use convexity to show, for instance, that smooth manifolds always admit good coverings.

in G determined by this open covering consists of G-valued functions $\varphi \in \check{C}^n(X;G)$ defined on the set of $(n+1)$-tuples in J for which the corresponding sets in the open covering intersect nontrivially, that is,

$$\varphi(\alpha_0, \dots, \alpha_n) \in G \qquad \text{is defined whenever} \qquad \mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_n} \neq \varnothing.$$

A coboundary operator $\delta : \check{C}^n(X;G) \to \check{C}^{n+1}(X;G)$ on this complex is defined by

$$\delta\varphi(\alpha_0, \dots, \alpha_{n+1}) := (-1)^{n+1} \sum_{k=0}^{n+1} (-1)^k \varphi(\alpha_0, \dots, \widehat{\alpha}_k, \dots, \alpha_{n+1}),$$

where the hat on $\widehat{\alpha}_k$ means that that particular term is omitted. Algebraically, $\check{C}^*(X;G)$ is the (ordered) simplicial cochain complex of the so-called *nerve* (see [Wen23, Lecture 46]) of the open covering $\{\mathcal{U}_\alpha\}_{\alpha \in J}$, and under some conditions that we will not get into here, one can show that the resulting cohomology $\check{H}^*(X;G)$ is independent of the choice of open covering.

Now for a real n-plane bundle $E \to X$, suppose that our good covering $\{\mathcal{U}_\alpha\}_{\alpha \in J}$ comes with local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^n$ and a corresponding system of transition functions $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(n, \mathbb{R})$. We define from this data a Čech 1-cocycle $\varphi_E \in \check{C}^1(X;\mathbb{Z}_2)$ by

$$\varphi_E(\alpha, \beta) := \begin{cases} 0 & \text{if } \det g_{\alpha\beta} > 0, \\ 1 & \text{if } \det g_{\alpha\beta} < 0, \end{cases}$$

with the understanding that the signs of these determinants are constant on each $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ since using a good covering guarantees that this intersection is connected. The cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = e$ now lives up to its name: together with $g_{\alpha\alpha} = e$, it implies the relations

$$\varphi_E(\alpha, \beta) + \varphi_E(\beta, \alpha) = 0, \qquad \text{and thus} \qquad \delta\varphi_E(\alpha, \beta, \gamma) = \varphi_E(\beta, \gamma) - \varphi_E(\alpha, \gamma) + \varphi_E(\alpha, \beta) = 0.$$

(d) Show that changing the local trivializations $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{R}^n$ and consequently the transition functions $\{g_{\beta\alpha}\}$ alters the definition of the cocycle $\varphi_E \in \check{C}^1(X;\mathbb{Z}_2)$ by something that is a coboundary in the Čech cochain complex. It follows that its cohomology class

$$w_1(E) := [\varphi_E] \in \check{H}^1(X;\mathbb{Z}_2)$$

is independent of choices, and we take this to be the definition of the first Stiefel-Whitney class in Čech cohomology.

(e) Show that the structure group of the bundle $E \to X$ can be reduced to $\mathrm{GL}_+(n, \mathbb{R})$ if and only if the cocycle φ_E is a coboundary. This is the Čech version of the proof that $w_1(E)$ vanishes if and only if E is orientable.

(f) Show that under the natural isomorphism $\check{H}^*(X;\mathbb{Z}_2) \cong H^*_{\mathrm{CW}}(X;\mathbb{Z}_2)$ for any CW-complex X, the two definitions we have for $w_1(E)$ match.
Hint: You really just need to know three things, namely (1) naturality, (2) how to compute w_1 on a direct sum of line bundles, and (3) that the definitions match on the universal line bundle over $\mathbb{RP}^\infty$.

Remark: There is also a construction of w_2 in Čech cohomology, which gives a rather transparent proof that an oriented vector bundle E can be endowed with a spin structure if and only if $w_2(E) = 0$.

Exercise 13.9. The Leray-Hirsch theorem gives an isomorphism of $\mathbb{Z}$-graded abelian groups $H^*(E;\mathbb{Z}) \cong H^*(B;\mathbb{Z}) \otimes H^*(F;\mathbb{Z})$ for fiber bundles $E \to B$ with standard fiber F under certain conditions: in particular, the theorem requires the existence of a finitely-generated free subgroup of $H^*(E;\mathbb{Z})$ that pulls back isomorphically to every fiber. Find an example of a fiber bundle in which the latter condition does not hold, and $H^*(E;\mathbb{Z})$ is not isomorphic to $H^*(B;\mathbb{Z}) \otimes H^*(F;\mathbb{Z})$.

14. Week 14

Lecture 24 (15.07.2024): Bordism groups.

- Definition: The nth **unoriented bordism group** $\Omega_n^{\mathrm{O}}(X)$ of a space $X \in \mathsf{Top}$ for $n \geqslant 0$ consists of equivalence classes $[(M,f)]$ of **singular n-manifolds** (M,f) in X, meaning continuous maps $f : M \to X$ defined on arbitrary closed smooth n-manifolds M. The equivalence relation $(M,f) \sim (N,g)$ is called the **bordism** relation:[65] we say that (M,f) and (N,g) are **bordant** if there exists a continuous map $F : W \to X$ defined on a compact $(n+1)$-manifold W, along with a diffeomorphism $M \amalg N \cong \partial W$ that identifies $F|_{\partial W}$ with $f \amalg g$.
- Group structure of $\Omega_n^{\mathrm{O}}(X)$: It's important to be aware that in the definition of a singular n-manifold (M,f), we do not require M to be either connected or nonempty, and the empty set $\varnothing$ can be regarded as a smooth n-manifold for any $n \in \mathbb{Z}$. (Look at the definition of an n-manifold; you will see that this is true.[66]) We can thus make $\Omega_n^{\mathrm{O}}(X)$ into an abelian group by defining
$$[(M,f)] + [(N,g)] := [(M \amalg N, f \amalg g)], \qquad 0 := [(\varnothing, \cdot)],$$
where $\cdot$ denotes the unique map $\varnothing \to X$. Using $\varnothing$ as an identity element means that a singular n-manifold (M,f) represents the trivial bordism class if and only if M is diffeomorphic to the boundary of some compact $(n+1)$-manifold W such that f extends to a map $W \to X$. The next proposition shows that inverses always exist, thus $\Omega_n^{\mathrm{O}}(X)$ really is a group.
- Proposition: Every nontrivial element $[(M,f)] \in \Omega_n^{\mathrm{O}}(X)$ has order 2, i.e. it is its own inverse.
Proof: This is one of two conclusions that we can draw from the existence of the trivial homotopy $F : M \times I \to X : (x,t) \mapsto f(x)$, since $\partial(M \times I)$ is the disjoint union of $M \amalg M$ with $\varnothing$. (The slightly more obvious conclusion comes from writing $\partial(M \times I) = M \amalg M$ without mentioning the empty set: this is the reason why $(M,f) \sim (M,f)$, i.e. the bordism relation is reflexive.)
- Remark: I am leaving it as an exercise to verify that the bordism relation is also symmetric and transitive. The latter is the more interesting detail, as it requires gluing two bordisms together along diffeomorphic boundary components, which can always be done with the aid of smooth collar neighborhoods of the boundary.
- Remark: It is a legitimate question why we are requiring the domain of a singular n-manifold to be a *smooth* manifold, rather than just a topological manifold, since nothing in the discussion so far seems to depend at all on its smooth structure. The best answer I can give is that smooth manifolds have tangent bundles, which brings the theory of characteristic classes of vector bundles into the picture as a useful tool for computations. It is easy to define bordism groups with topological manifolds as domains, but harder to compute them, and understanding the smooth case is an essential prerequisite before the continuous case can be approached.[67]

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[65] In the older literature, it was more often (and sometimes still is) called the *cobordism* relation. At some point people seem to have decided that since bordism defines a homology theory and not a cohomology theory, it doesn't deserve a "co" after all.

[66] In fact, $\varnothing$ is even an n-manifold for $n < 0$, and there are many results I care deeply about whose proofs depend on the fact that it is the *only* one.

[67] For more on this, see `http://www.map.mpim-bonn.mpg.de/B-Bordism#Piecewise_linear_and_topological_bordism`, but I provide this link with the caveat that I personally understand a relatively small proportion of what it says.

- Definition: For $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$, the relative bordism group $\Omega_n^{\mathrm{O}}(X, A)$ modifies the definition of $\Omega_n^{\mathrm{O}}(X)$ so that representatives (M, f) of elements of $\Omega_n^{\mathrm{O}}(X, A)$ are maps of pairs $f : (M, \partial M) \to (X, A)$ defined on compact n-manifolds M that are allowed to have nonempty boundary. The bordism relation $(M, f) \sim (N, g)$ then becomes the existence of a map of pairs $F : (W, \partial_0 W) \to (X, A)$, whose domain is a smooth compact $(n + 1)$-manifold W with boundary
$$\partial W = \partial_+ W \cup \partial_0 W \cup \partial_- W$$
such that there is a diffeomorphism $M \amalg N \cong \partial_+ W \amalg \partial_- W$ identifying the restriction of F to this portion of ∂W with $f \amalg g$. Here, the subsets $\partial_+ W, \partial_- W, \partial_0 W \subset \partial W$ are assumed to be compact domains that intersect each other only along smooth closed $(n-1)$-dimensional submanifolds $N_\pm := \partial(\partial_\pm W)$,[68]
$$\partial_\pm W \cap \partial_0 W = N_\pm, \quad \text{thus} \quad \partial(\partial_0 W) = N_+ \amalg N_- \quad \text{and} \quad \partial_+ W \cap \partial_- W = \varnothing.$$
- There is an obvious way to define a boundary operator
$$\Omega_n^{\mathrm{O}}(X, A) \xrightarrow{\partial_*} \Omega_{n-1}^{\mathrm{O}}(A) : [(M, f)] \mapsto [(\partial M, f|_{\partial M})],$$
and a similarly obvious way to associate to any map of pairs $\varphi : (X, A) \to (Y, B)$ an induced homomorphism $\varphi_* : \Omega_n^{\mathrm{O}}(X, A) \to \Omega_n^{\mathrm{O}}(Y, B)$, making $\Omega_n^{\mathrm{O}} : \mathsf{Top}^{\mathrm{rel}} \to \mathsf{Ab}$ into a functor and ∂_* into a natural transformation.
- Easy theorem: The functors $\Omega_n^{\mathrm{O}} : \mathsf{Top}^{\mathrm{rel}} \to \mathsf{Ab}$ and boundary operators ∂_* define a (generalized) homology theory, and there is also a natural transformation
$$\Omega_n^{\mathrm{O}}(X, A) \to H_n(X, A; \mathbb{Z}_2) : [(M, f)] \mapsto f_*[M],$$
where $[M] \in H_n(M, \partial M; \mathbb{Z}_2)$ on the right hand side is the relative fundamental class of M.
- Remark: When I say "easy," I mean that it is substantially easier to prove that Ω_*^{O} satisfies the Eilenberg-Steenrod axioms (except for the dimension axiom) than to prove this about singular homology H_*, as most of the axioms are already almost obvious, and there is no need to bother with subdivision. Similarly, a *fundamental class* of a closed n-manifold M in the bordism group $\Omega_n^{\mathrm{O}}(M)$ can be defined with no effort at all: it is imply the bordism class of the identity map $M \to M$. Bordism theory only starts to become difficult when you want to compute it.
- Definition: The nth **oriented bordism group** $\Omega_n^{\mathrm{SO}}(X)$ is defined via the following modifications to the definition of $\Omega_n^{\mathrm{O}}(X)$. The manifolds M, N, W are all equipped with orientations, ∂W is equipped with the induced boundary orientation, and the diffeomorphism $M \amalg N \cong \partial W$ required for the bordism relation reverses the orientation of either M or N, e.g. we could write
$$M \amalg (-N) \cong \partial W$$
to indicate that the boundary orientation of ∂W matches the orientation of M but is opposite to the orientation of N. The group structure is defined in the same way as before,

[68]There is an equivalent formulation of this definition that I find more natural, but it takes a bit more effort to state because one must first define the notion of a smooth manifold with *boundary and corners*. Leaving that detail to your imagination, I would then describe W in the relative bordism relation as a manifold with boundary and corners whose boundary ∂W is the union of three "smooth faces" that intersect along two corners $N_+ = \partial_+ W \cap \partial_0 W$ and $N_- = \partial_- W \cap \partial_0 W$. This is in fact the picture you end up with if you try to imagine the relation between two homologous relative cycles in singular homology in terms of triangulated manifolds—corners arise quite naturally in that context, because the standard n-simplex has corners for any $n \geqslant 2$. A smooth manifold with boundary and corners is topologically still a manifold with boundary, though its smooth structure does detect the distinction between having corners or not. On the other hand, one can always "smooth the corners" to produce a smooth manifold with boundary (and no corners) that is canonical up to diffeomorphism, and this is why one obtains the same relative bordism relation regardless of whether the definition mentions corners or not.

but elements no longer necessarily have order 2; instead, inverses are obtained by reversing orientations,

$$-[(M,f)] = [(-M,f)],$$

which works because the boundary orientation of $\partial(M \times I)$ reverses the orientation of one of its components. The oriented bordism relation is then symmetric because the orientation of W can always be reversed, which reverses the orientation of its boundary. I leave it as an exercise to write down the definition of $\Omega_n^{\mathrm{SO}}(X,A)$ for a pair $(X,A) \in \mathsf{Top}^{\mathrm{rel}}$. The result is another generalized homology theory, with natural transformations

$$\Omega_n^{\mathrm{SO}}(X,A) \to H_n(X,A;\mathbb{Z}) : [(M,f)] \mapsto f_*[M]$$

defined via the integral fundamental class $[M] \in H_n(M,\partial M;\mathbb{Z})$.

- In keeping with the principle that a homology theory is essentially computable if and only if its coefficient groups are, we will focus mainly on the groups

$$\Omega_n^{\mathrm{O}} := \Omega_n^{\mathrm{O}}(*), \qquad \text{and} \qquad \Omega_n^{\mathrm{SO}} := \Omega_n^{\mathrm{SO}}(*).$$

  Elements $[M]$ of these groups are represented by smooth closed n-manifolds M, with $[M] = [N]$ if and only if the disjoint union of M and N is diffeomorphic to the boundary of a compact $(n+1)$-manifold W; in the case of Ω_n^{SO}, M, N, W are all required to be oriented and $M \amalg (-N) \cong \partial W$.
- Two easy observations:
  (1) If desired, it is possible to describe elements of Ω_n^{O} or Ω_n^{SO} using only manifolds that are *connected* and *nonempty*. This is because $S^n = \partial\mathbb{D}^{n+1}$, thus $[S^n] = 0$ in both Ω_n^{O} and Ω_n^{SO}, and moreover, the disjoint union $M \amalg N$ of two closed n-manifolds is bordant to the connected sum $M \# N$. The compact $(n+1)$-manifold W needed for this can be constructed by attaching an $(n+1)$-dimensional **1-handle** $\mathbb{D}^1 \times \mathbb{D}^n \cong I \times \mathbb{D}^n$ to a pair of disjoint n-disks in $(M \amalg N) \times \{1\} \subset \partial\left((M \amalg N) \times I\right)$.
  (2) $\Omega_*^{\mathrm{O}} := \bigoplus_{n=1}^{\infty} \Omega_n^{\mathrm{O}}$ and $\Omega_*^{\mathrm{SO}} := \bigoplus_{n=1}^{\infty} \Omega_n^{\mathrm{SO}}$ each have natural ring structures defined by

$$[M] \times [N] := [M \times N].$$

  The product is *graded* commutative since, for oriented manifolds, $N \times M$ carries the opposite orientation of $M \times N$ whenever both are odd-dimensional. In the unoriented theory this makes no difference, and "graded commutative" means the same thing as commutative since all nontrivial elements have order 2.
- Some low-hanging fruit:
  (0) $\Omega_0^{\mathrm{O}} \cong \mathbb{Z}_2$ and $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$, with a canonical generator furnished in both cases by a one point space. Compact oriented 0-manifolds are finite discrete sets with a sign ± 1 attached to each point; an explicit isomorphism $\Omega_0^{\mathrm{SO}} \to \mathbb{Z}$ is then obtained via a signed count of the points in such a set, which is well defined because the oriented boundary of I consiss of one positive point and one negative point.
  (1) $\Omega_1^{\mathrm{O}} = \Omega_1^{\mathrm{SO}} = 0$ because all closed 1-manifolds are finite disjoint unions of copies of S^1, and $S^1 = \partial\mathbb{D}^2$.
  (2) $\Omega_2^{\mathrm{SO}} = 0$ due to the classification of surfaces: Σ_g is the boundary of a compact region in $\mathbb{R}^3$ for each $g \geqslant 0$. In the unoriented case, we can immediately say that the Klein bottle $K^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2$ is nullbordant since every element in Ω_2^{O} is 2-torsion, hence $[K^2] = [\mathbb{RP}^2] + [\mathbb{RP}^2] = 0$, with an explicit filling W of $K^2 \cong \partial W$ obtained e.g. by attching a 1-handle $\mathbb{D}^1 \times \mathbb{D}^2$ to one side of $(\mathbb{RP}^2 \amalg \mathbb{RP}^2) \times I$ and attaching $\mathbb{RP}^2 \times I$ to the other side. Appealing again to the classification of surfaces, this implies that Ω_2^{O} cannot be larger than $\mathbb{Z}_2$. On the other hand, Theorem 1 below says that $[\mathbb{RP}^2] \neq 0 \in \Omega_2^{\mathrm{O}}$, hence $\Omega_2^{\mathrm{O}} \cong \mathbb{Z}_2$.

(3) The following result is substantially deeper than the others above: $\Omega_3^{\mathrm{SO}} = 0$ due to a theorem of Rokhlin showing that every oriented closed 3-manifold is the boundary of a compact oriented 4-manifold. The same is also true without orientations, thus $\Omega_3^{\mathrm{O}} = 0$.

- Theorem 1: $\mathbb{RP}^2$ is not the boundary of any compact 3-manifold.
Remarks: This implies the computation $\Omega_2^{\mathrm{O}} \cong \mathbb{Z}_2$ mentioned above. A proof using characteristic classes will be given below, but here is a quick sketch of a different proof: if W is a compact 3-manifold with $\partial W \cong \mathbb{RP}^2$, then gluing two copies of W together along their boundary produces a closed 3-manifold Y, whose Euler characteristic must be 0 as a consequence of Poincaré duality (with $\mathbb{Z}_2$ coefficients). But one can also compute $\chi(Y) = 2\chi(W) - \chi(\mathbb{RP}^2) = 2\chi(W) - 1$, thus $\chi(Y)$ cannot be even. (The same argument also works for $\mathbb{CP}^2$.)
- Theorem/Definition: There exists a unique sequence of characteristic classes

$$w_k(E) \in H^k(X; \mathbb{Z}_2), \qquad k \in \mathbb{N},$$

called the **Stiefel-Whitney classes** of real vector bundles $E \to X$ over CW-complexes, which have the following properties:

(1) For real line bundles $L \to X$, $w_1(L)$ is determined by naturality and the condition that w_1 is nonzero on the universal line bundle,

$$w_1(E^{1,\infty,\mathbb{R}}) \neq 0 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2;$$

(2) $w_k(E) = 0$ whenever $k > \mathrm{rank}_{\mathbb{R}}(E)$;

(3) The **total Stiefel-Whitney class** $w(E) := 1 + w_1(E) + w_2(E) + \ldots \in H^*(X; \mathbb{Z}_2)$ satisfies the Whitney sum formula

$$w(E \oplus F) = w(E) \cup w(F)$$

The proof of the existence and uniqueness of $w_k(E)$ for $\mathrm{rank}(E) = n > 1$ is closely analogous to the corresponding theorem about Chern classes: one can define a real version of the projectivized bundle $\pi : \mathbb{P}(E) \to X$, with fibers $\mathbb{P}(E_x) \cong \mathbb{RP}^{n-1}$, so that $\pi^*E \to \mathbb{P}(E)$ contains a tautological rank 1 subbundle $L \subset \pi^*E$, and the powers of its first Stiefel-Whitney class $w_1(L)^k \in H^k(\mathbb{P}(E); \mathbb{Z}_2)$ then determine the classes $w_k(E) \in H^k(X; \mathbb{Z}_2)$. Here one needs a version of the Leray-Hirsch theorem with $\mathbb{Z}_2$ coefficients, which works because the classes $w_1(L)^k$ form a finite basis for a free $\mathbb{Z}_2$-submodule of $H^*(\mathbb{P}(E); \mathbb{Z}_2)$ that pulls back isomorphically to the fibers $\mathbb{P}(E_x) \cong \mathbb{RP}^{n-1}$. The crucial detail is that $H^*(\mathbb{RP}^{n-1}; \mathbb{Z}_2)$ has nearly the same ring structure as $H^*(\mathbb{CP}^{n-1}; \mathbb{Z})$, only with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$.
- Some useful properties of w_1 for real vector bundles over CW-complexes (see Exercise 13.8):

(1) Two line bundles $L, L' \to X$ are isomorphic if and only if $w_1(L) = w_1(L')$.
Proof: $B\,\mathrm{GL}(1, \mathbb{R}) = B\mathbb{Z}_2 = \mathbb{RP}^\infty = K(\mathbb{Z}_2, 1)$, so we can write L and L' as pullbacks of the universal line bundle via maps $f, g : X \to \mathbb{RP}^\infty$, and they are isomorphic if and only if these maps are homotopic, which is true if and only if $f^*\iota_1 = g^*\iota_1 \in H^1(X; \mathbb{Z}_2)$ for the distinguished cohomology class $\iota_1 \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong [\mathbb{RP}^\infty, \mathbb{RP}^\infty]$ corresponding to the identity map $\mathbb{RP}^\infty \to \mathbb{RP}^\infty$. This class must be nontrivial in $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and thus matches the first Chern class of the universal line bundle, so the condition is equivalent to $c_1(L) = c_1(L')$.

(2) For two line bundles $L, L' \to X$, $w_1(L \otimes L') = w_1(L) + w_1(L')$.
(Take this on faith for the moment; a proof will appear below, though I neglected to mention it in the lecture itself.)

(3) A vector bundle E of arbitrary rank is orientable if and only if $w_1(E) = 0$.
Proof sketch: Define the determinant line bundle $\det(E) \to X$ of a real vector bundle

$E \to X$ analogously to the complex case (Exercise 13.5). One then has natural isomorphisms $\det(E \oplus F) \cong \det(E) \otimes \det(F)$, and in light of the formula $w_1(L \otimes L') = w_1(L) + w_1(L')$, the real analogue of the splitting principle using cohomology with $\mathbb{Z}_2$ coefficients can be used to prove just as in the complex case that

$$w_1(E) = w_1(\det(E)).$$

The bundle E is orientable if and only if $\det(E)$ is trivial, and since the latter is a line bundle, it is trivial if and only if $w_1(\det(E)) = 0$.

I will interrupt the lecture summary here to add some more details about the first Stiefel-Whitney class, as I realized while writing this up that one or two of its important properties do not follow so easily from what I said about it in lecture. In particular, the formula

$$w_1(L \otimes L') = w_1(L) + w_1(L')$$

seems obvious to me for various reasons, but I do not see an easy way to deduce it merely from naturality and the fact that $w_1(E^{1,\infty,\mathbb{R}}) \neq 0$. We proved the complex analogue of this formula in Lecture 23, using the identification of c_1 with the Euler class, which we had defined as an obstruction class. There is an analogous approach that works for the first Stiefel-Whitney class, based on an alternative definition of $w_1(L)$ that makes it the explicit answer to an obstruction-theoretic question.

The relevant question is: given a real line bundle $L \to X$, does L admit a section that is nowhere zero on the 1-skeleton of X? As in our definition of the Euler class, one can approach this question by removing the zero-section from L, thus defining a fiber bundle

$$\dot{L} \to X$$

with standard fiber $\mathbb{R}\backslash\{0\}$, whose sections are precisely the sections of L that never vanish. Notice: if we can define a section $s_1 \in \Gamma(\dot{L}|_{X^1})$ on the 1-skeleton, then in fact that section is automatically also extendable to all the higher-dimensional cells, because both path-components of the fiber $\mathbb{R}\backslash\{0\}$ are contractible, so that maps $S^{n-1} \to \mathbb{R}\backslash\{0\}$ can always be extended over $\mathbb{D}^n$ for $n \geqslant 2$. Thus the question we are actually asking is whether L admits a global nowhere-zero section, which on a line bundle is equivalent to asking whether the bundle is trivial. The only reason it might fail to be trivial is that sections of $\dot{L}$ defined on X^0 might fail to extend to X^1.

One slightly annoying detail: since the fibers $\mathbb{R}\backslash\{0\}$ of $\dot{L} \to X$ are not path-connected, this fiber bundle does not fit cleanly into the framework we developed for obstruction theory in Lecture 21. Nevertheless, it is not difficult to write down an obstruction cochain

$$\theta^1_L \in C^1_{\mathrm{CW}}(X; \mathbb{Z}_2)$$

that depends on a given section $s_0 \in \Gamma(\dot{L}|_{X^0})$ and vanishes if and only if s_0 can be extended to X^1. For each 1-cell $e^1_\alpha \subset X$ with characteristic map $\Phi_\alpha : (\mathbb{D}^1, S^0) \to (X, X^0)$, the bundle $\Phi^*_\alpha \dot{L}$ is trivializable and thus homeomorphic to $\mathbb{D}^1 \times (\mathbb{R}\backslash\{0\})$, so its total space has two path-components, and s_0 can be extended over e^1_α if and only if its values at the two boundary points $\partial\mathbb{D}^1 = S^0$ lie in the same component of $\mathbb{R}\backslash\{0\}$. The answer to this question is independent of the choice of trivialization, so we get a well-defined cochain $\theta^1_L : C_1^{\mathrm{CW}}(X; \mathbb{Z}) \to \mathbb{Z}_2$ by setting

$$\theta^1_L(e^1_\alpha) := \begin{cases} 0 & \text{if } s_0 \text{ extends over } e^1_\alpha, \\ 1 & \text{if not.} \end{cases}$$

One easily checks that $\theta^1_L(\partial e^2_\beta)$ for all 2-cells $e^2_\beta \subset X$, thus θ^1_L is a cocycle. It depends on the given section s_0 over the 0-skeleton, but if a different section s_0' is given, one can now define a 0-cochain

$\psi \in C^0_{\mathrm{CW}}(X;\mathbb{Z}_2)$ by

$$\psi(e^0_\alpha) := \begin{cases} 0 & \text{if } s_0 \text{ and } s_0' \text{ map } e^0_\alpha \text{ to the same component of the fiber,} \\ 1 & \text{if not.} \end{cases}$$

The difference between our cocycle θ^1_L and the new one defined using s_0' instead of s_0 is then $\pm\delta\psi$, thus the cohomology class

$$w_1(L) := [\theta^1_L] \in H^1(X;\mathbb{Z}_2)$$

is independent of the choice of section s_0 on X^0. It vanishes if and only if L admits a nowhere-zero section over the 1-skeleton, which is true if and only if L is a trivial bundle.

As with the Euler class, one can use cellular approximation to prove that w_1 under this definition satisfies naturality, so it is a well-defined characteristic class. I will take this opportunity to clarify a subtle detail that was glossed over in our definition of the Euler class: the obstruction-theoretic definition of the class $[\theta^1_L] \in H^1_{\mathrm{CW}}(X;\mathbb{Z}_2)$ leaves open the possibility that it might depend on the cell decomposition of X, and not just on the bundle $L \to X$. But naturality implies that it must indeed be independent: if X' denotes a second copy of X with a different cell decomposition, then for any cellular map $f : X \to X'$ homotopic to the identity, we have $f^*\theta^1_L = \theta^1_{f^*L}$, so that $f^* : H^1(X';\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2)$ maps $w_1(L)$ to $w_1(f^*L)$. Since $f^* = \mathrm{Id}$ and $f^*L \cong L$, this proves that the natural isomorphism between two versions of cellular cohomology defined with respect to two different cell decompositions identifies one version of $w_1(L)$ with the other. The same argument implies that our definition of the Euler class for an oriented vector bundle similarly does not depend on the cell decomposition.

So, why does our obstruction-theoretic definition of $w_1(L)$ match the definition we already had? By naturality, this must be true if and only if the new definition of w_1 matches the old one on the universal line bundle $E^{1,\infty,\mathbb{R}} \to \mathbb{RP}^\infty$. The latter is easy to check, because there are only two possible values that $w_1(E^{1,\infty,\mathbb{R}}) \in H^1(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2$ could take, and it will vanish if and only if $w_1(L)$ vanishes for all line bundles, which is clearly not true since there exist real line bundles that are not trivial. (Exercise: Describe a nontrivial real line bundle over S^1 whose total space is a Möbius band.)

With the new definition of $w_1(L)$ in place, a direct proof of the formula $w_1(L \otimes L') = w_1(L) + w_1(L')$ becomes possible: we use sections $s \in \Gamma(L)$ and $t \in \Gamma(L')$ that are nonvanishing on X^0 to produce a section $s \otimes t \in \Gamma(L \otimes L')$ that is likewise nonvanishing on X^0, and observe that $s \otimes t$ extends from X^0 to a given 1-cell e^1_α if and only if both s and t either do or do not extend. This produces the relation

$$\theta^1_{L\otimes L'}(e^1_\alpha) = \theta^1_L(e^1_\alpha) + \theta^1_{L'}(e^1_\alpha) \in \mathbb{Z}_2,$$

thus $w_1(L \otimes L') = w_1(L) + w_1(L')$.

Now that we have the tensor product formula and a direct obstruction-theoretic proof that L is trivial if and only if $w_1(L) = 0$, here is an easy second proof that two line bundles $L, L' \to X$ are isomorphic if and only if $w_1(L) = w_1(L')$. A bundle isomorphism $L \to L'$ is the same thing as a nowhere-zero section of the bundle $\mathrm{Hom}(L, L') \cong L^* \otimes L'$, and since we are talking about real bundles this time, $L^* \cong L$, with an explicit isomorphism provided by any choice of bundle metric.[69] Since all elements of $H^1(X;\mathbb{Z}_2)$ are 2-torsion, it follows that L and L' are isomorphic if and only if $0 = c_1(\mathrm{Hom}(L, L')) = c_1(L) + c_1(L') = -c_1(L) + c_1(L')$.

[69]Note that one cannot get away with this trick for complex vector bundles, and indeed, a complex vector bundle E is often not isomorphic to its dual bundle E^*, as evidenced by the fact that $c_1(E)$ is often not equal to $-c_1(E)$. The difference in the complex case is that Hermitian inner products are not complex-bilinear: they are instead complex-linear in one argument and antilinear in the other, so that the *real* vector bundle isomorphism $E \to E^*$ induced by any choice of complex bundle metric is fiberwise antilinear rather than linear, and is thus not an isomorphism of complex vector bundles.

Now back to the lecture summary.

- Definition: Let ϵ^k denote the trivial (real or complex) k-plane bundle over any given base. Two vector bundles $E, F \to X$, not necessarily of the same rank, are said to be **stably isomorphic** if there is a bundle isomorphism $E \oplus \epsilon^k \cong F \oplus \epsilon^\ell$ for some $k, \ell \geqslant 0$. Similarly, E is **stably trivial** if $E \oplus \epsilon^k$ is a trivial bundle for some $k \geqslant 0$. This definition makes sense for either real or complex bundles, and one often also considers versions with additional structure such as orientations, e.g. two oriented real bundles $E, F \to X$ are stably isomorphic (as oriented bundles) if there is an orientation-preserving bundle isomorphism $\epsilon^k \oplus E \cong \epsilon^\ell \oplus F$ for some $k, \ell \geqslant 0$.[70]
- Corollary of the formula $w(E \oplus F) = w(E) \cup w(F)$: Stably isomorphic real vector bundles have matching Stiefel-Whitney classes.
  Note: By Exercise 13.3, the same holds for stably isomorphic complex vector bundles and the Chern classes.
- Exercise: Show that the Euler class is *not* invariant under stable isomorphism of oriented vector bundles.
- Definition: The **Stiefel-Whitney numbers** of a closed smooth n-manifold M are numbers of the form
$$\langle w_{k_1}(M) \cup \ldots \cup w_{k_j}(M), [M] \rangle \in \mathbb{Z}_2$$
  for $k_1, \ldots, k_j \in \mathbb{N}$ with $k_1 + \ldots + k_j = n$, where $[M] \in H_n(M; \mathbb{Z}_2)$ is the unoriented fundamental class, and we abbreviate
$$w_k(M) := w_k(TM).$$
- Theorem 2: If M is the boundary of a compact $(n+1)$-manifold W, then all Stiefel-Whitney numbers of M vanish.
  Proof: Choose any vector field on W near $M = \partial W$ that points transversely outward at the boundary. This vector field spans a rank 1 subbundle of $TW|_M$ that is complimentary to TM, thus producing a bundle isomorphism
$$TW|_M \cong \epsilon^1 \oplus TM,$$
  so that $TW|_M$ and TM are stably isomorphic, and therefore have the same Stiefel-Whitney classes. Writing $i : M \hookrightarrow W$ for the inclusion, we now have $w_k(TM) = i^* w_k(TW)$ for each $k \in \mathbb{N}$, and thus
$$\begin{aligned}\langle w_{k_1}(TM) \cup \ldots \cup w_{k_j}(TM), [M] \rangle &= \langle i^* \left(w_{k_1}(TW) \cup \ldots \cup w_{k_j}(TW) \right), [M] \rangle \\ &= \langle w_{k_1}(TW) \cup \ldots \cup w_{k_j}(TW), i_*[M] \rangle = 0,\end{aligned}$$
  since $i_*[M] = 0 \in H_{n+1}(W; \mathbb{Z}_2)$.
- Proof of Theorem 1: Since $\mathbb{RP}^2$ is not orientable, $w_1(\mathbb{RP}^2) \neq 0 \in H^1(\mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and the ring structure of $H^*(\mathbb{RP}^2; \mathbb{Z}_2)$ then gives us a nonvanishing Stiefel-Whitney number:
$$\langle w_1(\mathbb{RP}^2) \cup w_1(\mathbb{RP}^2), [\mathbb{RP}^2] \rangle \neq 0.$$
- Some harder results about bordism groups:
  (4) $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$, with $[\mathbb{CP}^2]$ as a generator. It follows in particular that every closed oriented 4-manifold is either the boundary of a compact oriented 5-manifold or becomes such a boundary after taking its disjoint union (or connected sum) with finitely many copies of either $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$, the latter being the standard notation for $\mathbb{CP}^2$ with reversed orientation.

[70] Note that in talking about stably isomorphic oriented bundles, we need to be consistent about the order of the summands when writing $\epsilon^k \oplus E$ or $E \oplus \epsilon^k$, since the orientation of the direct sum sometimes depends on this choice.

(5) $\Omega_5^{\mathrm{SO}} \cong \mathbb{Z}_2$, with the nontrivial element represented by a particular fiber bundle over S^1 with fibers $\mathbb{CP}^2$, called the *Dold manifold*. This is the lowest dimension in which oriented bordism groups have torsion.
(6) Ω_6^{SO} and Ω_7^{SO} are both trivial, so closed oriented manifolds of dimensions 6 or 7 are always oriented boundaries. On the other hand, $\Omega_8^{\mathrm{SO}} \cong \mathbb{Z} \oplus \mathbb{Z}$, with a basis given by $[\mathbb{CP}^4]$ and $[\mathbb{CP}^2 \times \mathbb{CP}^2]$.
(7) Thom proved several important results in [Tho54], such as:
   (a) The converse of Theorem 2 is also true, and in fact: Two closed manifolds of the same dimension whose Stiefel-Whitney numbers all match are always bordant.
   (b) There is a graded ring isomorphism

$$\Omega_*^{\mathrm{O}} \cong \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, x_9, \ldots],$$

   where the generators x_k have degree k, and there is exactly one such generator for each $k \notin \{2^m - 1 \mid m \in \mathbb{N}\}$.
   (c) The natural transformation $\Omega_n^{\mathrm{O}}(X, A) \to H_n(X, A; \mathbb{Z}_2)$ is surjective for every $(X, A) \in \mathsf{Top}^{\mathrm{rel}}$ and $n \geqslant 0$. In other words, all singular homology classes with $\mathbb{Z}_2$ coefficients really can be represented via the fundamental classes of compact manifolds.
   (d) The natural transformation $\Omega_n^{\mathrm{SO}}(X, A) \to H_n(X, A; \mathbb{Z})$ is *not* generally surjective, but its rational version

$$\Omega_n^{\mathrm{SO}}(X, A) \otimes \mathbb{Q} \to H_n(X, A; \mathbb{Q})$$

   is. In other words, every singular homology class $c \in H_n(X, A; \mathbb{Z})$ has a positive multiple that can be represented by the fundamental class of a compact manifold.
   (e) There exists a graded ring isomorphism

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q} \to \mathbb{Q}[x_4, x_8, x_{12}, \ldots],$$

   where the generators x_{4n} can all be represented by complex projective spaces $\mathbb{CP}^{2n}$. Comment: The next lecture sketches a proof of this isomorphism and some of its applications.

- Stable classifying maps: Recall $B\,\mathrm{O}(n) = B\,\mathrm{GL}(n, \mathbb{R}) = \mathrm{Gr}_n(\mathbb{R}^\infty)$. The sequence of inclusions

$$\mathrm{O}(1) \overset{i}{\hookrightarrow} \mathrm{O}(2) \overset{i}{\hookrightarrow} \mathrm{O}(3) \hookrightarrow \ldots, \qquad i(\mathbf{A}) := \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{A} \end{pmatrix}$$

induces (according to Exercise 13.1) a sequence of maps between the corresponding classifying spaces, whose direct limit we will call $B\,\mathrm{O}$:

$$B\,\mathrm{O}(1) \xrightarrow{Bi} B\,\mathrm{O}(2) \xrightarrow{Bi} B\,\mathrm{O}(3) \longrightarrow \ldots \longrightarrow \mathrm{colim}_{n\to\infty} B\,\mathrm{O}(n) =: B\,\mathrm{O}.$$

Remark: We could equally well have started with a similar sequence of inclusions $\mathrm{GL}(n, \mathbb{R}) \hookrightarrow \mathrm{GL}(n+1, \mathbb{R})$ and obtained from it the same colimit of classifying spaces; the only real reason to call it $B\,\mathrm{O}$ instead of $B\,\mathrm{GL}(\cdot, \mathbb{R})$ is convention. On an intuitive level, calling the classifying spaces $B\,\mathrm{O}(n)$ rather than $B\,\mathrm{GL}(n, \mathbb{R})$ means that the vector bundles we are classifying carry bundle metrics as extra structure. We will not need to make any such choice of extra structure, though Exercise 12.3 guarantees that it is always possible to do so, and for certain purposes it is also convenient—but the fact that $B\,\mathrm{O}(n) = B\,\mathrm{GL}(n, \mathbb{R})$ gives a huge hint that important topological results that are proved using that extra structure will typically not depend on it.

- Exercise (based on Exercise 13.1): The pullback of the universal bundle $E^{n+1,\infty,\mathbb{R}}$ via the map $Bi : B\,\mathrm{O}(n) \to B\,\mathrm{O}(n+1)$ is naturally isomorphic to $\epsilon^1 \oplus E^{n,\infty,\mathbb{R}}$.
- Theorem 3: $B\,\mathrm{O}$ is a classifying space for real vector bundles over compact CW-complexes up to stable isomorphism. More precisely, recall that any real vector bundle $E \to X$ of rank n over a CW-complex is isomorphic to $\kappa_E^* E^{n,\infty,\mathbb{R}}$ for a unique homotopy class of maps $\kappa_E : X \to B\,\mathrm{O}(n)$, called the **classifying map** of E. Composing this map with the natural map of $B\,\mathrm{O}(n)$ to the colimit defines the so-called **stable classifying map**

$$\begin{tikzcd} X \arrow[r, "\kappa_E"'] \arrow[rr, bend left, "\kappa_E^s"] & B\,\mathrm{O}(n) \arrow[r] & B\,\mathrm{O}, \end{tikzcd}$$

  and the statement is that if X is compact, then two real vector bundles $E, F \to X$ of arbitrary ranks are stably isomorphic if and only if their stable classifying maps $\kappa_E^s, \kappa_F^s : X \to B\,\mathrm{O}$ are homotopic.
  Proof sketch: The classifying spaces $B\,\mathrm{O}(n)$ and their colimit $B\,\mathrm{O}$ are all CW-complexes, and the point of assuming X to be compact is that the image of any homotopy of stable classifying maps $X \times I \to B\,\mathrm{O}$ is then contained in a finite subcomplex, implying that it factors through a homotopy of maps $X \times I \to B\,\mathrm{O}(n)$ for sufficiently large n. With this understood, the result can be deduced from the previous exercise.
- Remark: Direct sums $E \oplus F$ define a natural commutative addition operation on the set $[X, B\,\mathrm{O}]$ of stable isomorphism classes of vector bundles over a compact CW-complex X, and Exercise 13.3 tells us that this even makes $[X, B\,\mathrm{O}]$ into a group, with the stable isomorphism class of a trivial bundle (corresponding to a constant classifying map $X \to B\,\mathrm{O}$) as identity element. One can deduce from this that $B\,\mathrm{O}$ is naturally a group object in hTop, and the resulting multiplication map $B\,\mathrm{O} \times B\,\mathrm{O} \to B\,\mathrm{O}$ up to homotopy can then be combined with the homological cross product $H_*(B\,\mathrm{O}; \mathbb{Z}_2) \otimes H_*(B\,\mathrm{O}; \mathbb{Z}_2) \xrightarrow{\times} H_*(B\,\mathrm{O} \times B\,\mathrm{O}; \mathbb{Z}_2)$ to make $H_*(B\,\mathrm{O}; \mathbb{Z}_2)$ into a ring.
- Yet another theorem due to Thom [Tho54]: There is a well-defined and *injective* graded ring homomorphism

$$\Omega_*^{\mathrm{O}} \xrightarrow{\Phi} H_*(B\,\mathrm{O}; \mathbb{Z}_2) : [M] \mapsto (\kappa_{TM}^s)_*[M]. \tag{14.1}$$

  Comment/apology: The symbol $[M]$ means two quite different things on the left and right sides of (14.1). On the left side, it is the unoriented bordism class $[M] \in \Omega_n^{\mathrm{O}}$ represented by a closed smooth n-manifold M, while on the right, it is the fundamental class $[M] \in H_n(M; \mathbb{Z}_2)$ of M in singular homology.
- Comments on the theorem: The deep part of the statement is that the homomorphism Φ is injective, but it is relatively easy to explain why the map is well defined and respects the ring structures. The latter is almost immediately clear, because the fundamental class of a disjoint union $M \amalg N$ can be understood as a sum of fundamental classes of its components, and the stable classifying map of $T(M \amalg N)$ is similarly a disjoint union of the stable classifying maps of TM and TN. The ring structure is also preserved, mainly because $T(M \times N)$ is naturally the direct sum of two subbundles TM and TN pulled back along the projections of $M \times N$ to M and N, so that the stable classifying maps are then related in a way that corresponds to the definition of the ring structure on $H_*(B\,\mathrm{O}; \mathbb{Z}_2)$. One then only has to show that $(\kappa_{TM}^s)_*[M] = 0 \in H_n(B\,\mathrm{O}; \mathbb{Z}_2)$ whenever M is diffeomorphic to the boundary of a compact $(n+1)$-manifold W. This is true for roughly the same reasons that the Stiefel-Whitney numbers of M vanish: the bundles TM and $TW|_M$ are stably isomorphic, and the relevant consequence of this in the present context is that the stable classifying maps κ_{TM}^s and $\kappa_{TW|_M}^s$ are homotopic. Without loss of generality, we

can then assume these two maps are equal, so that $\kappa^s_{TW} : W \to B\,\mathrm{O}$ is an extension of $\kappa^s_{TM} : M \to B\,\mathrm{O}$ to W. This implies that $[(M, \kappa^s_{TM})]$ is the trivial bordism class in $\Omega^{\mathrm{O}}_n(B\,\mathrm{O})$, and since $(\kappa^s_{TM})_*[M]$ is the image of this class under the natural transformation $\Omega^{\mathrm{O}}_n(B\,\mathrm{O}) \to H_n(B\,\mathrm{O}; \mathbb{Z}_2)$, it follows that $(\kappa^s_{TM})_*[M] = 0$.

The proof that Φ is also injective requires knowing more about the actual computation of Ω^{O}_*, and we'll discuss some aspects of the oriented analogue of this in the next lecture.

- Remark: We now have the following explanation for the converse of Theorem 2. One can use cell decompositions to compute $H^*(B\,\mathrm{O}(n); \mathbb{Z}_2)$ explicitly, and the result is quite simple: it is the polynomial ring $\mathbb{Z}_2[w_1, \ldots, w_n]$ generated by the Stiefel-Whitney classes $w_k(E^{n,\infty,\mathbb{R}}) \in H^k(B\,\mathrm{O}(n); \mathbb{Z}_2)$ of the universal n-plane bundle (see e.g. [Hat, Theorem 3.9]). This implies, on the one hand, that all possible characteristic classes of real vector bundles living in cohomology with $\mathbb{Z}_2$ coefficients are expressible in terms of the Stiefel-Whitney classes. It also gives us a criterion to judge when the homology class $(\kappa^s_{TM})_*[M] \in H_*(B\,\mathrm{O}; \mathbb{Z}_2)$ is trivial, namely by evaluating products of Stiefel-Whitney classes of universal bundles on it, which is equivalent to evaluating products of Stiefel-Whitney classes of TM on $[M]$. Since $\Phi : \Omega^{\mathrm{O}}_* \to H_*(B\,\mathrm{O}; \mathbb{Z}_2)$ is injective, it follows that any closed manifold whose Stiefel-Whitney numbers all vanish must represent the trivial element of Ω^{O}_*.

Lecture 25 (18.07.2024): Oriented bordism and signature. As sometimes happens with interesting topics at the end of the semester, what I managed to cover in this lecture was maybe about half of what I really wanted to. I've decided to handle the writeup like so: the lecture summary below is in the usual format, as a concise[71] summary of what was actually covered in the lecture. After the lecture summary, there will be a "computational appendix" to fill in some of the details that were skipped in the lecture.

Topic 1: The signature as a bordism invariant.

- Definition: The **signature** $\sigma(M) \in \mathbb{Z}$ of a closed oriented manifold M of dimension $4n$ is defined to be the signature of its **intersection form**
$$Q_M : H^{2n}(M; \mathbb{R}) \times H^{2n}(M; \mathbb{R}) \to \mathbb{R}, \qquad Q_M(\alpha, \beta) := \langle \alpha \cup \beta, [M] \rangle.$$
Recall that for a quadratic form Q on a real vector space V, the signature of Q is $m_+ - m_- \in \mathbb{Z}$, where $m_\pm$ are the dimensions of maximal subspaces of V on which Q is positive/negative-definite respectively. Equivalently, if one chooses an inner product on V in order to express Q as the symmetric bilinear map $Q(v, w) = \langle v, Aw \rangle$ defined via a symmetric linear transformation $A : V \to V$, then the signature is the number of positive eigenvalues of A minus the number of negative eigenvalues (each counted with multiplicity). In the examples below, we describe quadratic forms as symmetric matrices with respect to a choice of basis.
- Convention: For closed oriented manifolds M with dimension not divisible by 4, we define $\sigma(M) := 0$. This convention will be justified by the theorem below. Note that when $\dim M = 2k$ for k odd, the bilinear form Q_M on $H^k(M; \mathbb{R})$ can be defined as above, but it is antisymmetric rather than symmetric, making it a different kind of object algebraically.
- Example 1: $H_2(S^2 \times S^2)$ has two generators, represented by submanifolds $S^2 \times \{\mathrm{const}\}$ and $\{\mathrm{const}\} \times S^2$, which have a single positive transverse intersection with each other, but also have self-intersection 0 since e.g. there are two submanifolds of the form $S^2 \times \{\mathrm{const}\}$ that do not intersect each other. Identifying the intersection form via Poincaré duality with the homological intersection product (see e.g. [Wen23, Lecture 55]), $Q_{S^2 \times S^2}$ is thus

[71] My use of the word "concise" is consciously ironic. I am aware that my lecture summaries haven't actually been concise in quite some time.

represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has one positive and one negative eigenvalue, so $\sigma(S^2 \times S^2) = 0$.

- Example 2: For each $n \geqslant 1$, $H^{2n}(\mathbb{CP}^{2n}; \mathbb{R})$ is 1-dimensional with a positive-definite intersection form, so $\sigma(\mathbb{CP}^{2n}) = 1$.
- Example 3: Reversing the orientation of $\mathbb{CP}^{2n}$ does not change its cohomology ring but does change its fundamental class by a sign, thus $\sigma(\overline{\mathbb{CP}}^{2n}) = -1$.
- Theorem: The signature defines a ring homomorphism
$$\Omega_*^{\mathrm{SO}} \to \mathbb{Z} : [M] \mapsto \sigma(M).$$
Proof:
  - It respects the abelian group structure because the intersection form of a disjoint union $M \amalg N$ is the direct sum of the intersection forms of M and N, thus $\sigma(M \amalg N) = \sigma(M) + \sigma(N)$.
  - It respects the multiplicative structure mainly because of the Künneth formula, which tells us that for a closed oriented $4m$-manifold M and $4n$-manifold N, the map
$$\bigoplus_{k+\ell=2(m+n)} H^k(M; \mathbb{R}) \otimes H^\ell(N; \mathbb{R}) \to H^{2(m+n)}(M \times N; \mathbb{R}) : \alpha \otimes \beta \mapsto \pi_M^* \alpha \cup \pi_N^* \beta$$
is an isomorphism; here π_M, π_N are the obvious projections of $M \times N$ to M and N. Note that since M and N are both compact cell complexes and we are using field coefficients, this is one of the situations in which the Künneth formula works just as well for cohomology as homology. The consequence of this isomorphism for the intersection form is a direct sum decomposition
$$Q_{M \times N} \cong (Q_M \otimes Q_N) \oplus Q',$$
in which $(Q_M \otimes Q_N)$ is the quadratic form on $H^{2m}(M; \mathbb{R}) \otimes H^{2n}(N; \mathbb{R})$ defined by
$$(Q_M \otimes Q_N)(\alpha_M \otimes \alpha_N, \beta_M \otimes \beta_N) = Q_M(\alpha_M, \beta_M) Q_N(\alpha_N, \beta_N),$$
and Q' is the restriction of $Q_{M \times N}$ to the subspace made up of tensor products of cohomologies of degrees other than $2m$ on M and $2n$ on N. One can check that Q' is always representable as a block matrix with blocks of zeroes along the diagonal, and deduce from this that its signature is 0, while the signature of $Q_M \otimes Q_N$ is the product of the signatures of Q_M and Q_N, hence $\sigma(M \times N) = \sigma(M)\sigma(N)$.
  - Addendum on the multiplicative structure: If the dimension of $M \times N$ is divisible by 4 but the dimensions of M and N are not, then the summand Q' in the decomposition sketched above becomes the entirety of $Q_{M \times N}$, thus proving that $\sigma(M \times N) = 0$. This is why we can get away with the convention of defining $\sigma(M) = 0$ whenever $\dim M \notin 4\mathbb{Z}$.
  - To show that the map is well defined on bordism classes, it now suffices to show that $\sigma(\partial W) = 0$ for any compact oriented $(4n+1)$-manifold W. The key observation here is that the inclusion $i : \partial W \hookrightarrow W$ determines distinguished subspaces of $H^{2n}(\partial W; \mathbb{R})$ and $H_{2n}(\partial W; \mathbb{R})$ by
$$V := \ker \left(H_{2n}(\partial W; \mathbb{R}) \xrightarrow{i_*} H_{2n}(W; \mathbb{R}) \right) \subset H_{2n}(\partial W; \mathbb{R}),$$
$$V^\perp := \operatorname{im} \left(H^{2n}(W; \mathbb{R}) \xrightarrow{i^*} H^{2n}(\partial W; \mathbb{R}) \right) \subset H^{2n}(\partial W; \mathbb{R}),$$
where the notation for $V^\perp$ is motivated by its obvious identification with the annihilator of V under the natural isomorphism $H^{2n}(\partial W; \mathbb{R}) \cong \operatorname{Hom}_{\mathbb{R}}(H_{2n}(\partial W; \mathbb{R}), \mathbb{R})$. The

fact that $[\partial W]$ is in the kernel of $i_* : H_{4n}(\partial W;\mathbb{R}) \to H_{4n}(W;\mathbb{R})$ then implies that $Q_{\partial W}|_{V^\perp} = 0$, and for similar reasons, the homological intersection product of ∂W vanishes on V. One can deduce the latter either from formal considerations involving the way that Poincaré duality relates the long exact sequences in homology and cohomology of the pair $(W, \partial W)$, or alternatively, from the following more geometric argument. There is an intersection product that pairs relative classes $A \in H_{2n+1}(W, \partial W;\mathbb{R})$ and absolute classes $B \in H_{2n}(W;\mathbb{R})$, which corresponds under Poincaré duality to the relative cup product $H^{2n}(W;\mathbb{R}) \otimes H^{2n+1}(W, \partial W;\mathbb{R}) \to H^{4n+1}(W, \partial W;\mathbb{R})$. If B lies in the image of the map $i_* : H_{2n}(\partial W;\mathbb{R}) \to H_{2n}(W;\mathbb{R})$, then representing it with a closed $2n$-dimensional submanifold lying in ∂W reveals that the intersection product $A \bullet B$ matches the intersection product in ∂W of $\partial_* A \in H_{2n}(\partial W;\mathbb{R})$ and the class that B represents in $H_{2n}(\partial W;\mathbb{R})$. By the long exact sequence in homology, $\partial_* A$ can be any class in the kernel of the map $i_* : H_{2n}(\partial W;\mathbb{R}) \to H_{2n}(W;\mathbb{R})$, and if B is also assumed to lie in that kernel, then $A \bullet B$ obviously vanishes. One can deduce from all this that V is precisely the image of $V^\perp$ under the Poincaré duality isomorphism $H^{2n}(\partial W;\mathbb{R}) \to H_{2n}(\partial W;\mathbb{R})$, thus the dimensions of V and $V^\perp$ are both identical and complementary, implying $\dim V^\perp = \frac{1}{2}\dim H^{2n}(\partial W;\mathbb{R})$. Since the intersection form $Q_{\partial W}$ is nonsingular, this is in fact the largest possible dimension that a subspace on which $Q_{\partial W}$ vanishes could have. It implies that $Q_{\partial W}$ can be written in block matrix form as $\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$, and one can show by induction on the dimension of $H^{2n}(\partial W;\mathbb{R})$ that quadratic forms admitting such a representation always have signature 0.
- Corollary: All finite products of even-dimensional complex projective spaces represent primitive elements in Ω_*^{SO}.

Topic 2: The Pontryagin classes.

- Definition: The **complexification** of a real n-dimensional vector space V is the complex n-dimensional vector space $V^{\mathbb{C}}$ defined as the real $2n$-dimensional vector space $V^{\mathbb{C}} := V \oplus V$ endowed with the unique complex structure such that scalar multiplication by i satisfies
$$i(v,0) := (0,v), \quad i(0,v) := (-v,0) \qquad \text{for all} \qquad v \in V.$$
Equivalently, one can define $V^{\mathbb{C}}$ as the real tensor product $V \otimes \mathbb{C}$ (regarding $\mathbb{C}$ as a real 2-dimensional vector space), and define its complex structure by $i(v \otimes z) := v \otimes (iz)$. Complexification defines a covariant functor from $\mathbb{R}$-Vect to $\mathbb{C}$-Vect, and therefore also determines an operation turning any real vector bundle $E \to B$ of rank n into a complex vector bundle $E^{\mathbb{C}} \to B$ of rank n, whose fibers $E_x^{\mathbb{C}}$ are the complexifications of the fibers E_x for $x \in B$.
- Definition: For a real vector bundle $E \to X$ over a CW-complex and $k \in \mathbb{N}$, the kth **Pontryagin class** of E is
$$p_k(E) := (-1)^k c_{2k}(E^{\mathbb{C}}) \in H^{4k}(X;\mathbb{Z}).$$
Putting all Pontryagin classes together in a series produces the **total Pontryagin class** $p(E) := 1 + p_1(E) + p_2(E) + \ldots \in H^*(X;\mathbb{Z})$, which is always a finite sum since only finitely many of the Chern classes can be nonzero.
- Remark: When we defined Chern and Stiefel-Whitney classes, the procedure in both cases was to give a direct definition of the first class on line bundles, and then impose the Whitney sum formula as an axiom that determines the rest. Our definition of the Pontryagin classes does not follow this scheme, so one should not assume that they have completely analogous properties: we see below for instance that the Pontryagin classes of a line bundle are always trivial, and the Whitney sum formula is only conditionally satisfied. The main properties

of Pontryagin classes are deduced directly from the properties we have already established for the Chern classes, and this is also the way that many computations of $p_k(E)$ are carried out.

- Properties of Pontryagin classes:
  (0) The classes $c_{2k+1}(E^{\mathbb{C}}) \in H^*(X;\mathbb{Z})$ are 2-torsion for all real vector bundles E.
  Remark: This is not a property of Pontryagin classes so much as a justification of the choice not to include odd Chern classes in their definition, i.e. as 2-torsion elements, the odd Chern classes of a complexified bundle $E^{\mathbb{C}}$ carry less useful information about E than the even Chern classes. In particular, they disappear completely if one works in $H^*(X;\mathbb{Q})$ instead of $H^*(X;\mathbb{Z})$. (We will not need to use this, but it is possible in fact to express $c_{2k+1}(E^{\mathbb{C}})$ in terms of Stiefel-Whitney classes of E, thus these classes give nothing new—see e.g. [Hat, §3.2].)
  Proof: All real vector bundles E are isomorphic to their dual bundles E^*, because any choice of bundle metric gives rise to an explicit isomorphism $E \to E^* : v \mapsto \langle v, \cdot \rangle$. Complex vector bundles do not typically have this property, but since complexification is functorial, it follows that all complexifications of real vector bundles do, and Exercise 13.5(b) thus implies $c_{2k+1}(E^{\mathbb{C}}) = -c_{2k+1}(E^{\mathbb{C}})$.
  (1) $p_k(E) = 0$ whenever $2k > \mathrm{rank}(E)$.
  Proof: This follows immediately from the analogous property of Chern classes.
  (Remark: It follows that all Pontryagin classes of a real line bundle are trivial. This is sensible, since we already know that real line bundles are completely classified by the first Stiefel-Whitney class.)
  (2) The Whitney sum formula holds modulo 2-torsion elements:
  $$p(E \oplus F) = p(E) \cup p(F) \quad \text{mod 2-torsion in } H^*(\cdot;\mathbb{Z}),$$
  meaning more precisely that the formula $2p(E \oplus F) = 2p(E) \cup p(F)$ is always true. For the images of the Pontryagin classes under the natural transformation $H^*(\cdot;\mathbb{Z}) \to H^*(\cdot;\mathbb{Q})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, it follows that the usual version of the formula also holds,
  $$p(E \oplus F) = p(E) \cup p(F) \text{ in } H^*(\cdot;\mathbb{Q}),$$
  and for this reason, many of the important applications of Pontryagin classes work with rational instead of integer coefficients, though we will also see cases in which the sum formula holds over $\mathbb{Z}$ for other reasons.
  Proof: The formula follows from $c(E^{\mathbb{C}} \oplus F^{\mathbb{C}}) = c(E^{\mathbb{C}}) \cup c(F^{\mathbb{C}})$ if one pays careful attention to the signs and uses the fact that all terms containing odd Chern classes are 2-torsion.
  (3) The Pontryagin classes $p_k(E)$ depend only on the stable isomorphism class of the bundle E.
  Proof: This follows from the corresponding property of Chern classes since stably isomorphic real bundles have stably isomorphic complexifications. (Note that one cannot deduce it immediately from the Whitney sum formula for Pontryagin classes, since the formula does not hold unconditionally.)
- Remark: The sign in the definition $p_k(E) = (-1)^k c_{2k}(E^{\mathbb{C}})$ is a prevalent convention but not quite universal, e.g. the classic book [MS74] uses it, but [BT82] does not, though Bott and Tu are very good about including remarks to point out what differs as a result of not including the sign. Its main benefit is that it prevents some other annoying signs from appearing in certain computations mentioned below.

- Computing $p(E)$ for a complex vector bundle E:
A complex bundle $E \to X$ of rank n can be regarded as a real bundle of rank $2n$ by forgetting its complex structure, and this real bundle also has a complexification $E^{\mathbb{C}}$, which is a complex bundle of rank $2n$. It cannot be isomorphic to E since it has twice the rank, and it must also be isomorphic to its own dual, which need not be true for E, but the obvious guess is then that there should be a complex bundle isomorphism
$$E^{\mathbb{C}} \cong E \oplus E^*.$$
Exercise: This is true for any complex vector bundle E, and one can use a choice of Hermitian bundle metric to write down an explicit isomorphism. (For more on this, see the computational appendix.)
- Sample computation of Chern classes: Suppose $E \to X$ is a complex vector bundle of rank n that admits a splitting $E \cong \ell_1 \oplus \ldots \oplus \ell_n$ into a direct sum of line bundles, and abbreviate their first Chern classes by
$$x_j \in c_1(\ell_j) \in H^2(X;\mathbb{Z}).$$
The total Chern class of ℓ_j is then $1 + x_j \in H^*(X;\mathbb{Z})$, and the Whitney sum formula thus allows us to write the total Chern class of E as
$$c(E) = c(\ell_1) \cup \ldots \cup c(\ell_n) = \prod_{j=1}^{n}(1 + x_j) \in H^*(X;\mathbb{Z}).$$
Expanding this polynomial function of $x_1, \ldots, x_n$ gives formulas for each of the individual Chern classes of E: explicitly, $c_k(E)$ can then be written as the kth elementary symmetric polynomial in the variables $x_1, \ldots, x_n$, e.g.
$$c_2(E) = \sum_{i<j} x_i x_j := \sum_{i<j} c_1(\ell_i) \cup c_1(\ell_j) \in H^4(X;\mathbb{Z}).$$
Note that some of the terms one obtains when expanding the polynomial may turn out to be trivial when viewed as elements of $H^*(X;\mathbb{Z})$, e.g. the term $x_1 \ldots x_n \in H^{2n}(X;\mathbb{Z})$ will actually be 0 if $H^{2n}(X;\mathbb{Z}) = 0$.
- Sample computation of Pontryagin classes: For the same complex vector bundle E as in the previous sample computation, the isomorphism $E^{\mathbb{C}} \cong E \oplus E^*$ gives us a splitting of the complexification into a direct sum of $2n$ line bundles that come in pairs: $E^{\mathbb{C}} \cong (\ell_1 \oplus \ell_1^*) \oplus \ldots \oplus (\ell_n \oplus \ell_n^*)$. Writing $x_j = c_1(\ell_j)$ as before, Exercise 13.5(b) gives $c_1(\ell_j^*) = -x_j$, so the Whitney sum formula for Chern classes gives
$$c(E^{\mathbb{C}}) = \prod_{j=1}^{n}(1 + x_j)(1 - x_j) = \prod_{j=1}^{n}(1 - x_j^2) \in H^*(X;\mathbb{Z}).$$
Notice that this polynomial has no terms of odd degree, so in this situation, the odd Chern classes of $E^{\mathbb{C}}$ are not just 2-torsion but actually vanish—this is a consequence of the fact that E was not just a real vector bundle to start with, but a complex one. We can also now see the motivation for the sign $(-1)^k$ in our definition of $p_k(E)$: changing $-x_j^2$ to x_j^2 in the polynomial will change the sign of its terms of degree $2(2k+1)$ for all k while leaving terms of degree $4k$ unchanged, which is exactly what needs to happen in order to change $c(E^{\mathbb{C}})$ into $p(E)$, and we therefore have
$$p(E) = \prod_{j=1}^{n}(1 + x_j^2) \in H^*(X;\mathbb{Z}).$$

The class $p_k(E)$ can therefore be written as the kth elementary symmetric polynomial in the variables $x_1^2, \dots, x_n^2$.

- Chern classes of $T(\mathbb{CP}^n)$: Let $x \in H^2(\mathbb{CP}^n; \mathbb{Z})$ denote the generator with $\langle x, [\mathbb{CP}^1] \rangle = 1$, so that the tautological line bundle $\ell \to \mathbb{CP}^n$ has $c_1(\ell) = -x$. One can combine the first sample computation above with a well-chosen stable isomorphism of $T(\mathbb{CP}^n)$ to a sum of line bundles $(\ell^*)^{\oplus(n+1)}$, and since $c_1(\ell^*) = -c_1(\ell) = x$, this produces the explicit formula

$$\begin{aligned} c(T\mathbb{CP}^n) &= (1+x)^{n+1} \\ &= 1 + (n+1)x + \binom{n+1}{2} x^2 + \ldots + \binom{n+1}{2} x^{n-1} + (n+1)x^n \in H^*(\mathbb{CP}^n; \mathbb{Z}) \end{aligned}$$

for the total Chern class of $T\mathbb{CP}^n \to \mathbb{CP}^n$. It implies for instance that the first Chern class is characterized by $\langle c_1(\mathbb{CP}^n), [\mathbb{CP}^1] \rangle = n+1$, and if $n \geqslant 2$, the second Chern class is characterized by $\langle c_2(\mathbb{CP}^n), [\mathbb{CP}^2] \rangle = \binom{n+1}{2} = \frac{1}{2}n(n+1)$. The highest order term x^{n+1} in the expansion can be omitted because it lives in $H^{2n+2}(\mathbb{CP}^n; \mathbb{Z})$, which is trivial. See the computational appendix for further details on the stable isomorphism of $T(\mathbb{CP}^n)$ with $(\ell^*)^{\oplus(n+1)}$.

- Pontryagin classes of $T(\mathbb{CP}^n)$: Treating $T(\mathbb{CP}^n)$ as a real vector bundle but making use of its complex structure as in the second sample computation above, the aforementioned stable isomorphism gives the formula

$$p(T\mathbb{CP}^n) = (1+x^2)^{n+1} = 1 + (n+1)x^2 + \binom{n+1}{2} x^4 + \ldots,$$

thus characterizing $p_1(\mathbb{CP}^n) \in H^4(\mathbb{CP}^n; \mathbb{Z})$ by $\langle p_1(\mathbb{CP}^n), [\mathbb{CP}^2] \rangle = n+1$ if $n \geqslant 2$, and so forth. The main thing I want to point out about this formula for now is that all Pontryagin classes of $T\mathbb{CP}^n$ that could possibly be nontrivial (in light of its rank) actually are. Exactly which term in the expansion is the last to be written depends on whether n is even or odd, e.g. for the even-dimensional complex projective spaces, the nonvanishing term of highest degree is

$$p_n(\mathbb{CP}^{2n}) = \binom{2n+1}{n} x^{2n} \in H^{4n}(\mathbb{CP}^{2n}; \mathbb{Z}).$$

- Stable classifying maps for oriented vector bundles: Analogously to the unoriented case discussed at the end of the previous lecture, we have $B\,\mathrm{SO}(n) = B\,\mathrm{GL}_+(n, \mathbb{R}) = \mathrm{Gr}_n^+(\mathbb{R}^\infty)$, and every oriented n-plane bundle $E \to X$ over a CW-complex is isomorphic to the pullback via a **classifying map** $\kappa_E : X \to B\,\mathrm{SO}(n)$ of the **universal oriented n-plane bundle**

$$E^{n,+} \to B\,\mathrm{SO}(n).$$

The inclusions $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$ induce maps $B\,\mathrm{SO}(n) \to B\,\mathrm{SO}(n+1)$, and the **stable classifying map** κ_E^s of the oriented bundle $E \to X$ is then the composition of its classifying map κ_E with the natural map of $B\,\mathrm{SO}(n)$ to the colimit as $n \to \infty$:

$$\begin{tikzcd} X \arrow[r, "\kappa_E"'] \arrow[rr, bend left, "\kappa_E^s"] & B\,\mathrm{SO}(n) \arrow[r] & B\,\mathrm{SO} := \mathrm{colim}_{n\to\infty} B\,\mathrm{SO}(n). \end{tikzcd}$$

For X compact, we obtain a bijection between $[X, B\,\mathrm{SO}]$ and the set of stable isomorphism classes of oriented vector bundles over X. To clarify: two oriented bundles $E, F \to X$ are stably isomorphic (as oriented bundles) if for some $k, \ell \geqslant 0$, there is an orientation-preserving bundle isomorphism $\epsilon^k \oplus E \cong \epsilon^\ell \oplus F$, where trivial bundles ϵ^k are endowed with the canonical orientation determined by the standard Euclidean basis on each fiber. Here we need to be consistent about the order of the summands in expressions such as $\epsilon^k \oplus E$, since the induced orientation of the direct sum may depend on it.

- Proposition: Stable isomorphism classes of oriented tangent bundles determine a well-defined ring homomorphism
$$\Omega_*^{\mathrm{SO}} \xrightarrow{\Phi} H_*(B\,\mathrm{SO};\mathbb{Z}) : [M] \mapsto (\kappa^s_{TM})_*[M],$$
where once again the symbol $[M]$ means a bordism class on the left hand side and a fundamental class on the right hand side.
Comments on the proof: The definition of the natural ring structure on $H_*(B\,\mathrm{SO};\mathbb{Z})$ and the proof of the proposition both follow by essentially the same reasoning as in the unoriented case, discussed at the end of the previous lecture. Other than the fact that we now have a fundamental class with $\mathbb{Z}$ coefficients since M is oriented, the only truly new feature is that when $M = \partial W$ for a compact oriented $(n+1)$-manifold and M carries the *boundary orientation*, the stable isomorphism $TW|_M \cong \epsilon^1 \oplus TM$ therefore also respects orientations, which is why we can consider classifying maps valued in $B\,\mathrm{SO}$ instead of just $B\,\mathrm{O}$.
- Fact (see e.g. [Hat, Theorem 3.16(b)]): The ring $H^*(B\,\mathrm{SO};\mathbb{Q})$ is naturally isomorphic to the polynomial ring $\mathbb{Q}[p_1, p_2, p_3, \ldots]$ generated by the rational Pontryagin classes of universal oriented bundles of sufficiently large rank: $p_k := p_k(E^{n,+}) \in H^{4k}(B\,\mathrm{SO}(n);\mathbb{Q}) \cong H^{4k}(B\,\mathrm{SO};\mathbb{Q})$ for sufficiently large n. (This can be proved using explicit cell decompositions of Grassmann manifolds.)
Remark: One can deduce from homotopy exact sequences of fiber bundles that for any fixed k, the map $B\,\mathrm{SO}(n) \to B\,\mathrm{SO}(n+1)$ is k-connected for all n sufficiently large, and the Hurewicz theorem then gives isomorphisms $H_k(B\,\mathrm{SO}(n);\mathbb{Q}) \cong H_k(B\,\mathrm{SO}(n+1);\mathbb{Q}) \cong \ldots \cong H_k(B\,\mathrm{SO};\mathbb{Q})$, which dualize to isomorphisms on cohomology. All cohomology classes of $B\,\mathrm{SO}$ can therefore be described as cohomology class of $B\,\mathrm{SO}(n)$ for n large.
- Corollary (by pulling back universal bundles and evaluating cohomology on homology): For a closed oriented n-manifold M, the image of $\Phi([M]) \in H_n(B\,\mathrm{SO};\mathbb{Z})$ in $H_n(B\,\mathrm{SO};\mathbb{Q})$ vanishes if and only if all the Pontryagin numbers
$$\langle p_{k_1}(M) \cup \ldots \cup p_{k_j}(M), [M] \rangle \in \mathbb{Z}, \qquad 4(k_1 + \ldots + k_j) = n$$
of M vanish.
- Dual fact: $H_*(B\,\mathrm{SO};\mathbb{Q})$ has a basis consisting of all classes of the form $\Phi([M])$ (or rather their images in rational homology) for M ranging over the set of all finite products of even-dimensional complex projective spaces.
Comments on the proof: It should be clear that this set of homology classes has a natural bijective correspondence with the cohomology basis formed by all finite products of Pontryagin classes of oriented universal bundles. That it forms a basis of $H_*(B\,\mathrm{SO};\mathbb{Q})$ can thus be deduced from computations of Pontryagin classes of products of projective spaces.

Topic 3: The Pontryagin-Thom construction and computation of $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$.

- Fact (from differential topology): Given a smooth closed n-manifold M, for any $k > 0$ sufficiently large, there exists a unique smooth isotopy class of embeddings $M \hookrightarrow \mathbb{R}^{n+k}$.
Remark: This is sometimes called Whitney's embedding theorem, though it is the easier of two theorems that go by that name (the other one gives precise information about the possible value of k, which we do not need). The idea is simply that if one has enough dimensions to move around in, then by clever applications of Sard's theorem, any smooth map $M \to \mathbb{R}^{n+k}$ admits a small perturbation that is both injective and has injective derivatives

at all points (meaning it is also an immersion). Applying this to an arbitrary homotopy between two embeddings similarly turns the homotopy into a family of embeddings.

- Consequence: The **normal bundle** of $M \hookrightarrow \mathbb{R}^{k+n}$ is defined to be the quotient vector bundle
$$\nu^k M := T\mathbb{R}^{n+k}|_M \Big/ TM = \epsilon^{n+k}/TM \to M.$$
Isotopies of embeddings give rise to isomorphic normal bundles, thus the so-called **stable normal bundle** of M, meaning the stable isomorphism class of $\nu^k M$, depends only on the manifold M and not on its embedding into $\mathbb{R}^{n+k}$, nor even on the number k. If M is oriented, then $\nu^k M$ inherits an orientation, and the stable normal bundle is a stable isomorphism class of oriented vector bundles.
- Pontryagin-Thom construction: According to the tubular neighborhood theorem, a neighborhood of M in $\mathbb{R}^{n+k}$ can be identified diffeomorphically with the total space of the normal bundle $\nu^k M$. Let $S^{n+k} := \mathbb{R}^{n+k} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{R}^{n+k}$, and define the **Thom space** of the universal oriented k-plane bundle likewise as its one-point compactification
$$M\,\mathrm{SO}(k) := E^{k,+} \cup \{\infty\}.$$
As an oriented vector bundle, $\nu^k M$ has a classifying map $\kappa_{\nu^k M} : M \to B\,\mathrm{SO}(k)$, and the isomorphism $\nu^k M \cong \kappa^*_{\nu^k M} E^{k,+}$ thus gives rise to a pullback square
$$\begin{tikzcd} \nu^k M \arrow[r] \arrow[d] & E^{k,+} \arrow[d] \\ M \arrow[r, "\kappa_{\nu^k M}"] & B\,\mathrm{SO}(k) \end{tikzcd}.$$
Identifying $\nu^k M$ with a neighborhood of M in S^{n+k}, let
$$S^{n+k} \xrightarrow{f^k_M} M\,\mathrm{SO}(k)$$
denote the continuous extension of the map $\nu^k M \to E^{k,+}$ in the pullback square that sends all of $S^{n+k} \backslash \nu^k M$ to the point at infinity in the Thom space.
- Observe: The total space of the stabilized universal bundle $\epsilon^1 \oplus E^{k,+}$ is topologically $\mathbb{R} \times E^{k,+}$, and adding a point at infinity to it is equivalent to taking the reduced suspension of $M\,\mathrm{SO}(k)$. The top horizontal map in the pullback square
$$\begin{tikzcd} \epsilon^1 \oplus E^{k,+} \arrow[r] \arrow[d] & E^{k+1,+} \arrow[d] \\ B\,\mathrm{SO}(k) \arrow[r] & B\,\mathrm{SO}(k+1) \end{tikzcd}$$
thus extends to a natural map
$$\Sigma\big(M\,\mathrm{SO}(k)\big) \xrightarrow{e} M\,\mathrm{SO}(k+1),$$
and the relation between the maps $f^k_M : S^{n+k} \to M\,\mathrm{SO}(k)$ for different values of k is then $f^{k+1}_M = e \circ \Sigma f^k_M$. The collection of spaces $M\,\mathrm{SO} := \{M\,\mathrm{SO}(k)\}_{k \geqslant 0}$ together with the maps e described above is called a **Thom spectrum** (cf. Lecture 19).
- Pontryagin-Thom theorem: There is a natural isomorphism
$$\Omega^{\mathrm{SO}}_n \to \pi_n(M\,\mathrm{SO}) := \operatorname{colim}_{k\to\infty} \pi_{n+k}(M\,\mathrm{SO}(k))$$
sending each bordism class $[M] \in \Omega^{\mathrm{SO}}_n$ of closed oriented n-manifolds M to the stable homotopy class of the map $f^k_M : S^{n+k} \to M\,\mathrm{SO}(k)$ described above.

- Remark: A similar theorem applies to the unoriented bordism groups, with the oriented Thom spectrum $M\,\mathrm{SO}$ replaced by a similar construction formed out of Thom spaces of unoriented universal bundles. One can also define other bordism theories in which the smooth manifold M is endowed with different kinds of "stable tangential structure," of which orientations are the simplest example. For all such theories, there is a version of the Pontryagin-Thom theorem identifying it with the generalized homology theory of a Thom spectrum corresponding to the appropriate sequence of classifying spaces.
- About the proof of Pontryagin-Thom: Here I shall repeat the advice I often give, that you should spend your weekend reading [Mil97], which gives a beautiful explanation of one of the simpler variants of the Pontryagin-Thom construction, computing homotopy sets $[S^{n+k}, S^k]$ in terms of *framed cobordism* classes of smooth n-dimensional submanifolds $M \subset S^{n+k}$. Once you've understood that special case, it is not hard to guess how the more general construction works. The inverse of the map $[M] \mapsto [f^k_M]$ can be described as follows: Given a map $f : S^{n+k} \to M\,\mathrm{SO}(k)$ for large k, one can assume after a homotopy of f that it satisfies a transversality condition making the preimage $f^{-1}(B\,\mathrm{SO}(k))$ of the zero-section $B\,\mathrm{SO}(k) \subset E^{k,+} \subset M\,\mathrm{SO}(k)$ into a smooth oriented closed submanifold $M \subset S^{n+k}$ of codimension k. Moreover, for a generic homotopy $H : S^{n+k} \times I \to M\,\mathrm{SO}(k)$ between two such maps, $H^{-1}(B\,\mathrm{SO}(k))$ then gives an oriented bordism between the corresponding closed submanifolds.
- Application: Taking the tensor product with $\mathbb{Q}$ in order to kill torsion, one obtains a chain of isomorphisms

$$\begin{aligned}\Omega^{\mathrm{SO}}_n \otimes \mathbb{Q} &\cong \pi_{n+k}(M\,\mathrm{SO}(k)) \otimes \mathbb{Q} \qquad (\text{for } k \gg 0)\\ &\cong H_{n+k}(M\,\mathrm{SO}(k);\mathbb{Q}) \cong H_n(B\,\mathrm{SO}(k);\mathbb{Q}) \cong H_n(B\,\mathrm{SO};\mathbb{Q}),\end{aligned}$$

  which (up to minor details such as replacing TM with $\nu^k M$) turns out to be the same thing as the rational version of the map $\Omega^{\mathrm{SO}}_n \to H_n(B\,\mathrm{SO};\mathbb{Z}) : [M] \mapsto (\kappa^s_{TM})_*[M]$. Here are some very brief explanations for the isomorphisms:
  - $\Omega^{\mathrm{SO}}_n \otimes \mathbb{Q} \cong \pi_{n+k}(M\,\mathrm{SO}(k)) \otimes \mathbb{Q}$ follows from the Pontryagin-Thom theorem, after showing that $\pi_{n+k}(M\,\mathrm{SO}(k))$ for $k \gg 0$ is isomorphic to the colimit as $k \to \infty$.
  - The passage from π_{n+k} to H_{n+k} comes from a rational version of the Hurewicz theorem, which holds for a certain range of dimensions due to computations showing that homotopy groups of spheres in certain dimensions are always finite (see e.g. [tD08, §20.8]).
  - $H_{n+k}(M\,\mathrm{SO}(k);\mathbb{Q}) \cong H_n(B\,\mathrm{SO}(k);\mathbb{Q})$ is a version of the Thom isomorphism theorem, which follows relatively easily from a cell decomposition of $M\,\mathrm{SO}(k)$ in which every n-cell of $B\,\mathrm{SO}(k)$ must be multiplied by a k-cell (due to the fibers of the k-plane bundle $E^{k,+} \to B\,\mathrm{SO}(k)$) to produce an $(n+k)$-cell of $M\,\mathrm{SO}(k)$.
- Corollaries:
  (1) A closed oriented manifold M satisfies $k[M] = 0 \in \Omega^{\mathrm{SO}}_*$ for some $k \neq 0 \in \mathbb{Z}$ if and only if all of its Pontryagin numbers vanish.
  (2) $\Omega^{\mathrm{SO}}_* \otimes \mathbb{Q}$ is the polynomial ring generated by the oriented bordism classes of all even-dimensional complex projective spaces.
- Another corollary: If $\Psi : \Omega^{\mathrm{SO}}_* \to \mathbb{Q}$ is a ring homomorphism satisfying $\Psi([\mathbb{CP}^{2n}]) = 1$ for every $n \in \mathbb{N}$, then it also satisfies $\Psi([M]) = \sigma(M)$ for every closed oriented smooth manifold M.

Topic 4: The Hirzebruch signature formula. (Just a very rough sketch—see the computational appendix for more details.)

- Idea of genera: There is an algebraic recipe to produce from any formal power series of the form $f(z) = 1 + a_1 z + a_2 z^2 + \dots$ with rational coefficients $a_j \in \mathbb{Q}$ a ring homomorphism $\Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ that takes the form
$$[M] \mapsto \langle F_n(p_1(M), \dots, p_n(M)), [M] \rangle$$
for manifolds M of dimension $4n$, where F_n is a polynomial in n variables with rational coefficients.
- The L-genus: Hirzebruch found a specific power series $f(z) := \frac{\sqrt{z}}{\tanh \sqrt{z}} = 1 + \frac{1}{3}z - \frac{1}{45}z^2 + \dots$ for which the value taken by the resulting ring homomorphism $\Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ on every even-dimensional complex projective space is 1. The result is a sequence of polynomials $L_n \in \mathbb{Q}[p_1, \dots, p_n]$ such that the formula
$$\langle L_n(p_1(M), \dots, p_n(M)), [M] \rangle = \sigma(M)$$
holds for all closed oriented $4n$-manifolds M, where the left hand side is an invariant known as the **L-genus** of M. This is the *Hirzebruch signature theorem.* The polynomial L_n can be computed from the power series; explicit formulas in the cases $n = 1, 2$ are
$$L_1(p_1) = \frac{1}{3}p_1, \qquad L_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2).$$
- Remark: One powerful aspect of the signature theorem is that the definition of the L-genus provides no hint that its values should be integers, but they must be, since the signature is. In Lecture 1, we sketched an argument that uses this fact to prove that certain smooth manifolds homeomorphic to S^7 cannot also be diffeomorphic to S^7, because if they were, then there would exist a closed smooth 8-manifold M with $\sigma(M) = 8$ and $p_1(M) = 0$, implying
$$\langle L_2(p_1(M), p_2(M)), [M] \rangle = \frac{7}{45} \langle p_2(M), [M] \rangle = 8,$$
which is numerically impossible.

Epilogue to the lecture summary: I was asked in the first lecture whether there is any intuitive way to understand why the fact that $45 \cdot 8$ isn't divisible by 7 should be relevant to the existence of exotic spheres. I would say that the particular numbers arising here are happy accidents, but one can point to a couple of big ideas that make it seem nearly *inevitable* for such an argument to work. The first is the Pontryagin-Thom construction, which identifies the seemingly exotic bordism ring $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ with something much more familiar, the cohomology of a single CW-complex $B\,\mathrm{SO}$, thus providing a manageable list of generators for $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ that are spaces we understand very well. The second big idea is the algebraic recipe for producing ring homomorphisms $\Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ out of polynomial funtions of characteristic classes; this was only vaguely mentioned in the lecture, but we'll say more about it in the computational appendix below. Once the generators of $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ are understood, it is not difficult to show that there is a unique power series $f \in \mathbb{Q}[[z]]$ generating a sequence of polynomials that produce exactly the result one wants, namely a ring homomorphism $\Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ that equals 1 on $[\mathbb{CP}^{2n}]$ for every n, and therefore must equal the signature in general. In theory, it could have turned out that these polynomials all have integer coefficients, which would have been very unlucky—the signature formula would then still have nontrivial content, but it would not provide such an obvious criterion to rule out certain combinations of classes in $H^{4k}(M; \mathbb{Z})$ arising as the Pontryagin classes of a smooth manifold M. On the other hand, the algebraic construction naturally gives rational coefficients for the polynomials—there is no evident reason why they should turn out to be integers, and in the absence of such a reason, it seems nearly impossible for the signature formula not to give us something nontrivial and useful, independently of the precise numbers that turn out to deliver the final step in the proof.

Computational appendix. I want to go into some more detail now on a few things that had to be glossed over in the final lecture, though some of them were discussed a bit in the final Übung. It probably would have required at least one extra lecture to do all of this properly.

A caveat about the splitting principle. Let me first point out a mental trap that is a bit too easy for beginners learning about characteristic classes to fall into.

We discussed the splitting principle for Chern classes in Exercise 13.5, and as was also mentioned in Exercise 13.8, it works equally well for Stiefel-Whitney classes. What about Pontryagin?

There do exist versions of the splitting principle for Pontryagin classes, e.g. [LM89, Proposition 11.2] states a result that applies specifically to *oriented* vector bundles of even rank. Such results are typically more complicated and less obviously useful than the naive statement one might expect, namely that everything important about Pontryagin classes can be deduced from bundles that split into direct sums of real line bundles. The latter is simply not true: you should immediately feel suspicious when you notice that for any real vector bundle $E \to X$ that splits into line bundles $\ell_1 \oplus \ldots \oplus \ell_n$, the Pontryagin classes of the line bundles are all trivial, so applying the Whitney sum formula with rational coefficients, one deduces that the Pontryagin classes of E in $H^*(X;\mathbb{Q})$ must also all vanish. They do not have to vanish in $H^*(X;\mathbb{Z})$, because the Whitney sum formula does not hold for Pontryagin classes with integer coefficients, but it still holds up to 2-torsion, leading to the conclusion that $p_k(E) \in H^{4k}(X;\mathbb{Z})$ is 2-torsion for all k. This is not implausible—and in fact it is true—because split bundles $E = \ell_1 \oplus \ldots \oplus \ell_n$ are somewhat special. But after Exercise 13.5, you may have developed the intuition that whatever holds for split bundles holds for all of them, due to the splitting principle; in the world of Chern classes, this is largely true, and very useful. If it were true for Pontryagin classes, then we would have just proved that they are always 2-torsion and always vanish over rational coefficients, which would be completely absurd—among other things, the Hirzebruch signature formula then could not be true unless every closed oriented manifold had signature 0.

So, why doesn't the naive version of the splitting principle work for Pontryagin classes? Given a vector bundle $E \to X$, the splitting principle provides a map $f : Y \to X$ with two essential properties: first, that the pullback f^*E splits into a sum of line bundles, and second, that $f^* : H^*(X) \to H^*(Y)$ is injective on cohomology with the appropriate choice of coefficients. I mention the coefficients because that is the detail that makes this fail for the Pontryagin classes. To see why, let's clarify why it does *not* fail for Stiefel-Whitney classes: arguing as in the complex case, one can produce from any real vector bundle $E \to X$ of rank n a real projectivization $\pi : \mathbb{P}(E) \to X$, which is a fiber bundle with fibers $\mathbb{RP}^{n-1}$. Pulling back E along π produces a bundle with a distinguished rank 1 subbundle $\ell \subset \pi^*E$, so that π^*E is the direct sum of ℓ with another subbundle of strictly smaller rank, and repeating the same trick then produces further pullbacks that split into more summands, until eventually all the summands are line bundles. This procedure thus gives what we need if and only if the map π^* on cohomology is injective, and for the purposes of Stiefel-Whitney classes, that specifically means

$$H^*(X;\mathbb{Z}_2) \xrightarrow{\pi^*} H^*(\mathbb{P}(E);\mathbb{Z}_2)$$

is injective. This is true due to the Leray-Hirsch theorem, which is applicable because $H^*(\mathbb{RP}^{n-1};\mathbb{Z}_2)$ has the same simple polynomial ring structure as $H^*(\mathbb{CP}^{n-1};\mathbb{Z})$, with powers of Stiefel-Whitney classes of the tautological line bundle serving as a preferred basis: as a consequence, the classes $w_1(\ell)^k \in H^k(\mathbb{P}(E);\mathbb{Z}_2)$ for $k = 0,\ldots,n-1$ span a finitely-generated free $\mathbb{Z}_2$-submodule of $H^*(\mathbb{P}(E);\mathbb{Z}_2)$ that pulls back isomorphically to the cohomology of every fiber.

In order for the same trick to produce a splitting principle for Pontryagin classes, we would need the map

$$H^*(X;\mathbb{Z}) \xrightarrow{\pi^*} H^*(\mathbb{P}(E);\mathbb{Z})$$

to be injective, or at the very least, we would need the same map on cohomology with rational coefficients to be injective, in which case the splitting principle would still tells us something about $p_k(E) \in H^{4k}(X;\mathbb{Q})$. But neither is injective in general: in the first place, there is no hope of applying the Leray-Hirsch theorem with $\mathbb{Z}$ coefficients when $n \geqslant 3$, because $H^*(\mathbb{RP}^{n-1};\mathbb{Z})$ then has torsion in degree 2 and is thus not a *free* $\mathbb{Z}$-module. Even if we work with $\mathbb{Q}$ coefficients, we would also need to find a subspace of $H^*(\mathbb{P}(E);\mathbb{Q})$ that pulls back isomorphically to $H^*(\mathbb{RP}^{n-1};\mathbb{Q})$, and there is no obvious reason for such a subspace to exist. It does exist for trivial reasons when n is odd, because $\mathbb{RP}^{n-1}$ is then non-orientable and $H^*(\mathbb{RP}^{n-1};\mathbb{Q})$ is thus nontrivial only in degree 0. This implies a very limited version of the splitting principle: for any real vector bundle $E \to X$ of odd rank n, there exists a map $f : Y \to X$ such that $f^* : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is injective and f^*E splits into the sum of a line bundle with another bundle of even rank $n-1$. This splitting does not tell much about the Pontryagin classes that we didn't already know: it tells us that $p_k(E) = 0$ for $2k > n-1$, but since n is odd, that condition also means $2k > n$, so we already knew $p_k(E) = 0$ before the splitting. To get a full splitting into line bundles, one would need the projectivization trick to work also when n is even, but $H^*(\mathbb{RP}^{n-1};\mathbb{Q})$ is then nontrivial in degree $n-1$, so this requires finding a class in $H^{n-1}(\mathbb{P}(E);\mathbb{Q})$ that evaluates nontrivially on the fibers.

Here is an explicit example with $n = 2$ in which no such class exists, and the naive splitting principle clearly fails. Consider the tautological complex line bundle

$$\ell \to \mathbb{CP}^k, \qquad k \in \mathbb{N},$$

which we can also regard as an oriented real 2-plane bundle. Choosing a Hermitian bundle metric, we can also assume that it comes with a system of U(1)-valued transition functions $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{U}(1)$, or equivalently, SO(2)-valued transition functions, when viewing ℓ as a real bundle. One obtains the real projectivization $\mathbb{P}(\ell) \to \mathbb{CP}^k$ by keeping the same SO(2)-valued transition functions and replacing the standard fiber $\mathbb{R}^2$ with $\mathbb{RP}^1$, which SO(2) acts on linearly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [t_0 : t_1] = [at_0 + bt_1 : ct_0 + dt_1].$$

If we identify $\mathbb{RP}^1$ with $S^1 \cong \mathrm{U}(1)$ via the homeomorphism

$$\begin{aligned} \mathbb{RP}^1 &\xrightarrow{\cong} \mathrm{U}(1), \\ [\cos(\theta) : \sin(\theta)] &\mapsto e^{2i\theta}, \end{aligned}$$

the SO(2)-action on $\mathbb{RP}^1$ now becomes the action of $\mathrm{U}(1) \cong \mathrm{SO}(2)$ on U(1) defined by

$$e^{i\alpha} \cdot e^{2i\theta} := e^{2i(\theta+\alpha)} = (e^{i\alpha})^2 e^{2i\theta},$$

so another way to construct $\mathbb{P}(\ell)$ is therefore as the fiber bundle with U(1)-valued transition functions $g_{\beta\alpha}^2 : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{U}(1)$ and standard fiber U(1), with the structure group U(1) acting on the standard fiber by left multiplication. This makes it a principal U(1)-bundle: specifically, it is the principal U(1)-bundle $P \to \mathbb{CP}^k$ whose associated complex line bundle $P \times_{\mathrm{U}(1)} \mathbb{C}$ is $\ell \otimes \ell$, as the latter has transition functions obtained as the squares of the transition functions of ℓ. This bundle is closely related to the Hopf fibration

$$S^{2k+1} \xrightarrow{p} \mathbb{CP}^k,$$

which can viewed as the unit circle bundle consisting of all unit vectors in fibers of ℓ, and is thus the orthogonal frame bundle of ℓ; equivalently, it is the principal U(1)-bundle over $\mathbb{CP}^k$ whose associated complex line bundle is ℓ. One then obtains P from the Hopf fibration as a quotient, in which all circle fibers are quotiented by the antipodal map: globally, this means quotienting the total space S^{2k+1} itself by the antipodal map, hence

$$\mathbb{P}(\ell) \cong S^{2k+1}/\mathbb{Z}_2 = \mathbb{RP}^{2k+1}.$$

Now the problem is clear: $H^1(\mathbb{RP}^{2k+1};\mathbb{Q}) = 0$ for every $k \in \mathbb{N}$, so there is no class in $H^1(\mathbb{P}(\ell);\mathbb{Q})$ that evaluates nontrivially on the fibers $\mathbb{RP}^1 \cong \mathbb{P}(\ell_x) \subset \mathbb{P}(\ell)$, and the Leray-Hirsch theorem therefore cannot be applied. You can convince yourself via the following exercise that this is not just a failure of one particular proof:

Exercise. Show that for every $k \geqslant 2$, $p_1(\ell) \in H^4(\mathbb{CP}^k;\mathbb{Z})$ of the tautological complex line bundle $\ell \to \mathbb{CP}^k$, viewed as a real 2-plane bundle, is not torsion.

Of course, splitting higher-rank bundles into line bundles is still a useful trick and does frequently get applied for proving things about Pontryagin classes: one only needs to apply it to the Chern classes of the complexification.

The complexification of a complexification. In order to understand why $E^{\mathbb{C}} \cong E \oplus E^*$ for every complex vector bundle E, it is useful to define the **conjugate** $\bar{E}$ of a complex vector bundle. Backing up a bit, we can associate to every complex vector space V another complex vector space $\bar{V}$, which is defined as the same set and the same *real* vector space, but with an extra sign inserted in front of the definition of scalar multiplication by i. It is convenient to use the following notation for this: we let

$$V \to \bar{V} : v \mapsto \bar{v}$$

denote the identity map, and adopt the convention that an element of $\bar{V}$ should always be written as $\bar{v}$ to distinguish it from the corresponding element $v \in V$. The definition of complex scalar multiplication on $\bar{V}$ is then determined by scalar multiplication on V via the formula

$$\lambda\bar{v} := \overline{\bar{\lambda}v}, \qquad \lambda \in \mathbb{C},\ v \in V,$$

where for $\lambda := a + ib$ with $a, b \in \mathbb{R}$, $\bar{\lambda} := a - ib$ is the usual notion of complex conjugation. This notation has the virtue that many of the formulas one writes with it look automatically true, though one must often keep in mind (as in the above definition) that the bar can have slightly different literal meanings when placed over two symbols that appear right next to each other.

Any complex vector bundle $E \to X$ now has a conjugate complex vector bundle $\bar{E} \to X$ whose fibers $\bar{E}_x$ are the conjugates of the fibers E_x. There is an obvious bijection

$$E \to \bar{E} : v \mapsto \bar{v},$$

and it is an isomorphism of *real* vector bundles, but not a *complex* bundle isomorphism because it is complex *antilinear* on each fiber, not complex linear. Any complex vector bundle atlas for E gives rise to a bundle atlas for $\bar{E}$ such that the two systems of transition functions are related to each other by complex conjugation. It will sometimes happen that E and $\bar{E}$ are isomorphic as complex vector bundles, but not via the tautological map $v \mapsto \bar{v}$. The real benefit of the conjugate bundle is that we can now express the natural complex analogue of the statement that any bundle metric on a *real* vector bundle determines an isomorphism to its dual bundle:

Proposition 14.1. *Any Hermitian bundle metric $\langle\ ,\ \rangle$ on a complex vector bundle E determines a complex vector bundle isomorphism*[72]

$$\bar{E} \xrightarrow{\cong} E^* : \bar{v} \mapsto \langle \bar{v}, \cdot \rangle.$$

[72] I am using the convention that Hermitian inner products are complex antilinear in the first argument and complex linear in the second.

□

With that understood, the real reason behind the isomorphism $E^{\mathbb{C}} \cong E \oplus E^*$ is that for any complex vector space V, the formula

$$V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} V \oplus \bar{V} : v \otimes z \mapsto (zv, z\bar{v})$$

defines an isomorphism of complex vector spaces, where as usual, we forget the complex structure of V on the left hand side and regard it as a *real* vector space in order to define its complexification $V^{\mathbb{C}}$. Indeed, this map is clearly complex linear, and if we write $V^{\mathbb{C}}$ as $V \oplus V$ instead of $V \otimes_{\mathbb{R}} \mathbb{C}$, it takes the form

$$V \oplus V \to V \oplus \bar{V} : (v, w) \mapsto (v + iw, \overline{v - iw}),$$

which is clearly invertible. For any complex vector bundle E, one can use this formula on all fibers at once to define an explicit complex bundle isomorphism $E^{\mathbb{C}} \cong E \oplus \bar{E}$, and choosing a Hermitian bundle metric then identifies the latter with $E \oplus E^*$.

The stable isomorphism class of $T(\mathbb{CP}^n)$. In the last lecture, we mentioned that there is a stable isomorphism of complex vector bundles

$$T(\mathbb{CP}^n) \cong \underbrace{\ell^* \oplus \ldots \oplus \ell^*}_{n+1} \qquad \text{(stably)},$$

which—in light of the stable isomorphism invariance of the Chern and Pontryagin classes—easily gives rise to explicit formulas for the total Chern and Pontryagin classes of $T(\mathbb{CP}^n)$, namely

$$c(T\mathbb{CP}^n) = (1 + x)^{n+1}, \qquad p(T\mathbb{CP}^n) = (1 + x^2)^{n+1},$$

with $x \in H^2(\mathbb{CP}^n; \mathbb{Z})$ as usual denoting the generator with $\langle x, [\mathbb{CP}^1] \rangle = 1$. One way to see the stable isomorphism is as follows.

The manifold $\mathbb{CP}^n$ is defined as the quotient of the manifold $\mathbb{C}^{n+1} \backslash \{0\}$ by the smooth (and free and proper) action of the Lie group $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ by scalar multiplication. In differential geometry, there is a useful general theorem about quotients of manifolds M by free and proper smooth actions of a Lie group G: it tells us firstly that at each point $x \in M$, the orbit $G \cdot x \subset M$ is a smooth submanifold, and secondly, that the quotient M/G has a natural smooth manifold structure for which the tangent space $T_{[x]}(M/G)$ is naturally isomorphic to the obvious quotient of tangent spaces

$$T_{[x]}(M/G) \cong T_x M \Big/ T_x(G \cdot x).$$

Applying this to $\mathbb{CP}^n$, we observe that for any point $z \in \mathbb{C}^{n+1} \backslash \{0\}$ representing a point $[z] \in \mathbb{CP}^n$, the tangent space $T_z(\mathbb{C}^{n+1} \backslash \{0\})$ is canonically isomorphic to $\mathbb{C}^{n+1}$, while the tangent space at z to its $\mathbb{C}^*$-orbit is precisely the 1-dimensional subspace $[z] \in \mathbb{CP}^n$, which we can equivalently choose to call the fiber $\ell_{[z]} \subset \mathbb{C}^{n+1}$ at $[z] \in \mathbb{CP}^n$ of the tautological line bundle $\ell \to \mathbb{CP}^n$. We therefore have a natural isomorphism

$$T_{[z]}\mathbb{CP}^n \cong T_z(\mathbb{C}^{n+1} \backslash \{0\}) \Big/ \ell_{[z]} = \mathbb{C}^{n+1} / \ell_{[z]},$$

in which the right hand side can be identified with the fiber at $[z] \in \mathbb{CP}^n$ of the quotient vector bundle ϵ^{n+1}/ℓ. The only catch is that this identification of $T_{[z]}\mathbb{CP}^n$ with $\mathbb{C}^{n+1}/\ell_{[z]}$ depends on the choice of point $z \in \ell_{[z]} \backslash \{0\}$ representing the equivalence class $[z]$. Indeed, any other representative of the same equivalence class can be written as $\lambda z \in \mathbb{C}^{n+1} \backslash \{0\}$ for some $\lambda \in \mathbb{C}^*$, and for any tangent vector $X \in T_z(\mathbb{C}^{n+1} \backslash \{0\})$ representing an equivalence class $[X] \in T_z(\mathbb{C}^{n+1} \backslash \{0\}) / \ell_{[z]} = T_{[z]}\mathbb{CP}^n$, multiplying $X \in \mathbb{C}^{n+1}$ by λ gives an equivalence class of tangent vectors at λz,

$$[\lambda X] \in T_{\lambda z}(\mathbb{C}^{n+1} \backslash \{0\}) \Big/ \ell_{[z]} = T_{[z]}\mathbb{CP}^n,$$

which necessarily corresponds to the same tangent vector in $T_{[z]}\mathbb{CP}^n$ as $[X]$. What this really means is that there is a natural complex vector bundle isomorphism

$$\mathrm{Hom}(\ell, \epsilon^{n+1}/\ell) \xrightarrow{\cong} T(\mathbb{CP}^n),$$

defined at each point $[z] \in \mathbb{CP}^n$ by associating to each complex-linear map $\Phi : \ell_{[z]} \to \mathbb{C}^{n+1}/\ell_{[z]}$ the tangent vector in $T_{[z]}\mathbb{CP}^n$ corresponding to the equivalence class

$$[\Phi(z)] \in T_z(\mathbb{C}^{n+1}\backslash\{0\})\Big/\ell_{[z]}$$

for any chosen representative $z \neq 0 \in \ell_{[z]}$ of $[z]$. The result is independent of the choice of representative, because all other choices are of the form λz for $\lambda \in \mathbb{C}^*$, and Φ is complex linear.

That was the tricky part: the rest only requires minimal cleverness. We recall that since ℓ is a line bundle, its tensor product with its dual ℓ^* is trivial, and the direct sum of ϵ^{n+1}/ℓ with ℓ is likewise a trivial bundle ϵ^{n+1}. Thanks to the isomorphism $T(\mathbb{CP}^n) \cong \mathrm{Hom}(\ell, \epsilon^{n+1}/\ell) \cong \ell^* \otimes (\epsilon^{n+1}/\ell)$, we can then write

$$\epsilon^1 \oplus T(\mathbb{CP}^n) \cong (\ell^* \otimes \ell) \oplus \big(\ell^* \otimes (\epsilon^{n+1}/\ell)\big) \cong \ell^* \otimes \big(\ell \oplus (\epsilon^{n+1}/\ell)\big) \cong \ell^* \otimes \epsilon^{n+1} \cong (\ell^*)^{\oplus(n+1)}.$$

That is the stable isomorphism that was claimed.

Multiplicative sequences and genera. Here are some more details about the "algebraic recipe" that leads to the L-genus appearing in the Hirzebruch signature formula. The idea in the background can be expressed in considerably more generality: suppose in particular that we have a sequence of characteristic classes

$$q_k \in H^{mk}(X; R), \qquad k \in \mathbb{N}$$

defined for a given class of vector bundles $E \to X$ that is closed under the direct sum operation, where $m \geqslant 1$ is a fixed integer and R is a commutative ring with unit. We will assume that the "total q-class"

$$q(E) := 1 + q_1(E) + q_2(E) + q_3(E) + \dots$$

satisfies the Whitney sum formula

$$q(E \oplus F) = q(E) \cup q(F)$$

for all pairs of bundles on which these classes are defined. The Stiefel-Whitney, Chern and Pontryagin classes are all examples of this setup if one chooses the correct class of vector bundles and the correct coefficients, e.g. the Chern classes require the bundles to be complex, and the Pontryagin classes require coefficients in $R := \mathbb{Q}$ if we want the Whitney sum formula to hold without restriction. We do *not* need to assume that only finitely-many of the classes $q_k(E)$ are nonzero for any given bundle E, so the total q-class may be a formal series rather than a finite sum, but since only finitely-many of its terms have degrees satisfying any given upper bound, the Whitney sum formula still gives well-defined relations between the q-classes of E, F and $E \oplus F$, which can be written in the form

$$q_n(E \oplus F) = \sum_{k+\ell=n} q_k(E) \cup q_\ell(F)$$

if we allow the summation to range over integers $k, \ell \geqslant 0$ and adopt the convention

$$q_0(E) := 1 \in H^0(X; R) \qquad \text{for all } E \to X.$$

The idea now is to produce new characteristic classes as the homogeneous terms in formal sums of the form

$$K(E) = 1 + K_1(q_1(E)) + K_2(q_1(E), q_2(E)) + K_3(q_1(E), q_2(E), q_3(E)) + \dots,$$

where for each $n \in \mathbb{N}$, $K_n \in R[x_1, \dots, x_n]$ is a polynomial in n variables with coefficients in R, having the correct form so that plugging in the cohomology classes $q_1(E), \dots, q_n(E)$ and interpreting products as cup products produces a cohomology class of degree mn, the same degree as $q_n(E)$. This is a purely algebraic condition on the polynomials K_n, e.g. it means that K_1, K_2, K_3 must have the form

$$K_1(x_1) = ax_1, \qquad K_2(x_1, x_2) = bx_1^2 + cx_2, \qquad K_3(x_1, x_2, x_3) = dx_1^3 + ex_1x_2 + fx_3$$

for some coefficients $a, b, c, d, e, f \in R$. Once again, the formal sum $K(E)$ is not required to be finite, but for each individual $n \geqslant 1$, one obtains in this way a characteristic class $K_n(E) := K_n(q_1(E), \dots, q_n(E))$ living in $H^{mn}(X; R)$, defined on the same class of vector bundles $E \to X$ for which the classes q_n are defined. Here is the important part: we would also like to choose the polynomials K_n such that the formula

$$K(E \oplus F) = K(E) \cup K(F) \tag{14.2}$$

is satisfied for all pairs of bundles $E, F \to X$ on which the q-classes are defined. This too can be interpreted as a purely algebraic condition: given that the total q-class satisfies the Whitney sum formula, (14.2) will be satisfied whenever the polynomials $K_n \in R[x_1, \dots, x_n]$ satisfy a particular sequence of algebraic relations, the first two of which are

$$K_1(x_1 + y_1) = K_1(x_1) + K_1(y_1),$$
$$K_2(x_1 + y_1, x_2 + x_1y_1 + y_2) = K_2(x_1, x_2) + K_1(x_1)K_1(y_1) + K_2(y_1, y_2),$$

and for arbitrary $n \in \mathbb{N}$, the nth relation in the sequence can be written as

$$K_n\left(x_1 + y_1, \dots, \sum_{k+\ell=n} x_k y_\ell\right) = \sum_{k+\ell=n} K_k(x_1, \dots, x_k) K_\ell(y_1, \dots, y_\ell), \tag{14.3}$$

where the summations range over integers $k, \ell \geqslant 0$ and we write $x_0 = y_0 = K_0 := 1$. Any sequence of polynomials $K_n \in R[x_1, \dots, x_n]$ defined for all $n \in \mathbb{N}$ and satisfying these relations is called a **multiplicative sequence**, and it gives rise to a sequence of characteristic classes satisfying (14.2).

We can already see that there exists at least one multiplicative sequence: taking $K_n(x_1, \dots, x_n) := x_n$ for each $n \in \mathbb{N}$ makes $K(E)$ just another way of writing the total q-class, which satisfies (14.2) by assumption, and the relations (14.3) are satisfied trivially. We will see below that there are many more multiplicative sequences beyond this one.

Since $K(E)$ cannot contain any information that isn't already in the q-classes, one may naturally wonder what is to be gained from rewriting them in this way. The answer is that under some reasonable assumptions, every multiplicative sequence defines a ring homomorphism on a corresponding bordism theory.

To see how this works, let's assume for concreteness that the q-classes are the Pontryagin classes $p_k \in H^{4k}(X; \mathbb{Q})$ with rational coefficients, so every multiplicative sequence $K_n \in \mathbb{Q}[x_1, \dots, x_n]$ for $n \in \mathbb{N}$ then gives rise to a sequence of characteristic classes $K_n(E) \in H^{4n}(X; \mathbb{Q})$ defined for all real vector bundles $E \to X$ over CW-complexes, and expressed as polynomial functions of the Pontryagin classes of E. Note that unlike the Pontryagin classes themselves, it is possible (depending on the choice of polynomials) for infinitely-many of the classes $K_n(E)$ to be nonzero on any given bundle, if its base is infinite-dimensional. However, we will specifically consider tangent bundles of closed oriented manifolds M, because these have fundamental classes $[M] \in H_*(M; \mathbb{Q})$ on which $K(TM)$ can be evaluated to produce a numerical invariant, called the K**-genus** of M:

$$K(M) := \langle K(TM), [M] \rangle \in \mathbb{Q}.$$

The convention in this definition is that for a formal series $x := x_0 + x_1 + x_2 + \dots$ of cohomology classes $x_k \in H^k(X)$, evaluation $\langle x, c \rangle$ on a homogeneous homology class $c \in H_k(X)$ means $\langle x_k, c \rangle$. Concretely, the K-genus of a manifold M of dimension $4n$ is thus given by

$$K(M) = \langle K_n(p_1(M), \dots, p_n(M)), [M] \rangle,$$

and $K(M)$ is defined to vanish whenever $\dim M \notin 4\mathbb{Z}$, since $K(TM)$ then has no terms of degree equal to $\dim M$.

Theorem 14.2. *For any multiplicative sequence $\{K_n\}$ with rational coefficients, the K-genus defines a ring homomorphism*

$$\Omega_*^{\mathrm{SO}} \to \mathbb{Q} : [M] \mapsto K(M).$$

Proof. The K-genus for manifolds M of any given dimension is a specific Pontryagin number of M, and thus vanishes whenever M is the boundary of a compact oriented manifold. The relation $K(M \amalg N) = K(M) + K(N)$ follows from naturality, since the pullback of $T(M \amalg N)$ via the inclusion of each of its components is the tangent bundle of that component. From these two facts, we see that $K(M)$ depends in general only on the bordism class $[M] \in \Omega_*^{\mathrm{SO}}$, and defines a group homomorphism. The interesting part is the ring structure: given two closed oriented manifolds M and N, the fundamental class $[M \times N]$ is the homological cross product $[M] \times [N]$, while the tangent bundle $T(M \times N)$ has fibers $T_{(x,y)}(M \times N) = T_x M \oplus T_y N$ and is thus the direct sum

$$T(M \times N) = \pi_M^* TM \oplus \pi_N^* TN,$$

with π_M, π_N denoting the projections of $M \times N$ to its factors. Using the multiplicative relation and naturality, we therefore have

$$\begin{aligned}
K(M \times N) &= \langle K(T(M \times N)), [M \times N] \rangle = \langle K(\pi_M^* TM \oplus \pi_N^* TN), [M] \times [N] \rangle \\
&= \langle \pi_M^* K(TM) \cup \pi_N^* K(TN), [M] \times [N] \rangle \\
&= \langle K(TM) \times K(TN), [M] \times [N] \rangle \\
&= \langle K(TM), [M] \rangle \cdot \langle K(TN), [N] \rangle = K(M)K(N),
\end{aligned}$$

where it should be noted that no annoying signs appear in the last step because the K-classes have nontrivial terms only in even degrees. □

If the goal is to find a ring homomorphism $\Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ that matches the signature, then the to-do list should now be clear: we need to find a multiplicative sequence $\{K_n\}$ with rational coefficients for which the K-genus of $\mathbb{CP}^{2n}$ is 1 for every $n \in \mathbb{N}$. You will notice that the one concrete example of a multiplicative sequence that we've already seen does *not* do the trick: taking $K_n(x_1, \dots, x_n) = x_n$ for every n, the K-genus of $\mathbb{CP}^{2n}$ becomes its top-dimensional Pontryagin number

$$\langle p_n(\mathbb{CP}^{2n}), [\mathbb{CP}^{2n}] \rangle = \binom{2n+1}{n},$$

which is interesting and nontrivial, but it is not 1. This motivates the search for other multiplicative sequences, and the following algebraic result provides them.

Theorem 14.3. *For any commutative ring R and any formal power series $f \in R[[z]]$ of the form $f(z) = 1 + a_1 z + a_2 z^2 + \dots$ with coefficients $a_j \in R$, there exists a unique multiplicative sequence $\{K_n \in R[x_1, \dots, x_n]\}_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, the coefficient of x_1^n in K_n is a_n.*

Another useful way to characterize the relationship in Theorem 14.3 between the power series $f(z) = 1 + a_1 z + a_2 z^2 + \dots$ and the polynomials K_n is as follows: the characteristic classes $K_n(p_1(E), \dots, p_n(E)) \in H^{4n}(X; \mathbb{Q})$ determined by f have the property that for any bundle $E \to X$ whose rational Pontryagin classes $p_k(E) \in H^{4k}(X; \mathbb{Q})$ vanish for all $k \geqslant 2$, $K(E)$ is given by evaluating the power series on $p_1(E)$:

$$K(E) = f(p_1(E)) = 1 + a_1 p_1(E) + a_2 p_1(E)^2 + a_3 p_1(E)^3 + \dots \tag{14.4}$$

From this perspective, it should not be surprising that the polynomials K_n are all uniquely determined by f, and e.g. whenever E is a complex bundle (regarded as a real bundle for the purposes of Pontryagin classes), one can use the splitting principle for complex bundles to derive from this a more general formula expressing $K(E)$ purely in terms of f. As should be clear from the statement, however, Theorem 14.3 is a purely algebraic result that requires no knowledge of characteristic classes, neither in the statement nor in the proof.

I will say only a few things about the proof: first, if you try deriving formulas for the first few elements in a multiplicative sequence K_n with prescribed initial terms $K_n(x_1, \dots, x_n) = a_n x_1^n + \dots$, you will already develop some intuition that the theorem must be true. The full proof is a bit tedious, but not fundamentally difficult: the main thing it relies on is a standard result about the elementary symmetric polynomials, namely that every symmetric polynomial in n variables can be expressed uniquely as a polynomial function of elementary symmetric polynomials. A readable complete proof may be found in [MS74, Lemma 19.1].

The L-genus. So how does one produce a multiplicative sequence of polynomials $L_n \in \mathbb{Q}[x_1, \dots, x_n]$ such that feeding them with Pontryagin classes produces a genus L with $L(\mathbb{CP}^{2n}) = 1$ for all n?

According to Theorem 14.3, every multiplicative sequence can be encoded by a power series $f(z) = 1 + a_1 z + a_2 z^2 + \dots$, in this case with rational coefficients since we are working with Pontryagin classes, so we can try to compute $L(\mathbb{CP}^{2n})$ from this power series. In fact, this is not difficult: since the Pontryagin classes are invariant under stable isomorphism, the same will be true of the L-classes, and we are therefore free to replace $T\mathbb{CP}^{2n}$ with $(\ell^*)^{\oplus(2n+1)}$, where ℓ^* is the dual of the tautological complex line bundle $\ell \to \mathbb{CP}^{2n}$. Viewed as a real bundle of rank 2, ℓ^* has a complexification isomorphic to $\ell \oplus \ell^*$ and thus has total Pontryagin class

$$p(\ell^*) = 1 + x^2,$$

with $x \in H^2(\mathbb{CP}^{2n}; \mathbb{Z})$ as usual denoting the generator with $\langle x, [\mathbb{CP}^1] \rangle = 1$. In particular, $p_1(\ell^*) = x^2 \in H^4(\mathbb{CP}^{2n}; \mathbb{Z})$ and $p_k(\ell^*) = 0$ for all $k \geqslant 2$, which puts us in the situation of (14.4), and we can therefore write

$$L(\ell^*) = f(p_1(\ell^*)) = f(x^2) = 1 + a_1 x^2 + a_2 x^4 + \dots .$$

Using the multiplicative property, the result for $T(\mathbb{CP}^{2n})$ is

$$L(T\mathbb{CP}^{2n}) = L((\ell^*)^{\oplus(2n+1)}) = [f(x^2)]^{2n+1},$$

which is another power series in x^2 with rational coefficients. Evaluating it on $[\mathbb{CP}^{2n}] \in H_{4n}(\mathbb{CP}^{2n}; \mathbb{Q})$ picks out the x^{2n} term in this series, and since $\langle x^{2n}, [\mathbb{CP}^{2n}] \rangle = 1$, the L-genus of $\mathbb{CP}^{2n}$ will then be characterized by

$$L(\mathbb{CP}^{2n}) = b_n,$$

where

$$[f(z^2)]^{2n+1} = (1 + a_1 z^2 + a_2 z^4 + \dots)^{2n+1} = 1 + b_1 z^2 + b_2 z^4 + \dots + b_n z^{2n} + \dots .$$

Proposition 14.4. *There is a unique power series $f \in \mathbb{Q}[[z]]$ for which the corresponding genus $L(\mathbb{CP}^{2n}) = b_n$ computed above is 1 for every $n \in \mathbb{N}$.*

It is not difficult to prove this proposition by an elementary recursive algorithm, and anyone with enough perserverance (or a computer) can calculate the coefficients in the series as far as their patience will allow. I personally have just enough patience to tell you that it begins with

$$f(z) = 1 + \frac{1}{3} z - \frac{1}{45} z^2 + \dots, \tag{14.5}$$

which is something that Wikipedia also could have told you. This is already enough information to compute what is truly important for our purposes: the polynomials $L_1(x_1)$ and $L_2(x_1, x_2)$, the second of which plays a starring role in the proof that exotic 7-spheres exist. Knowing the coefficient of the linear term in f, Theorem 14.3 implies that L_1 can only be

$$L_1(x_1) = \frac{1}{3}x_1.$$

The second polynomial must then be of the form $L_2(x_1, x_2) = -\frac{1}{45}x_1^2 + bx_2$ for some $b \in \mathbb{Q}$, and one deduces from the $n = 2$ case of the multiplicative relation (14.3) that $b = \frac{7}{45}$, giving

$$L_2(x_1, x_2) = \frac{1}{45}(7x_2 - x_1^2).$$

These two polynomials give the formulas we've seen for the L-genera of closed oriented manifolds of dimensions 4 and 8 respectively, namely

$$L(M^4) = \frac{1}{3}\langle p_1(M), [M]\rangle, \qquad L(M^8) = \frac{1}{45}\langle 7p_2(M) - p_1(M)^2, [M]\rangle.$$

The Hirzebruch signature theorem for dimensions 4 and 8 thus tells us that these rational numbers must always actually be integers, because they are equal to $\sigma(M)$.

I should point out: the L-classes $L_n(p_1(E), \ldots, p_n(E)) \in H^{4n}(X; \mathbb{Q})$ are well-defined characteristic classes for every real vector bundle $E \to X$ over a CW-complex, and they live generally in $H^*(X; \mathbb{Q})$, not in $H^*(X; \mathbb{Z})$. It is a special feature of tangent bundles $E = TM$ of closed oriented $4n$-manifolds that the L-class in degree $4n$ can be lifted to $H^{4n}(M; \mathbb{Z})$, despite having rational coefficients in its definition.

One last thing: for the topological applications we have discussed, we had no need to worry about whether the formal power series $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ converges, but for the particular power series giving rise to the L polynomials, it turns out that it does converge on a neighborhood of $0 \in \mathbb{C}$ to a holomorphic function that can be written in closed form, and that function is

$$f(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}}.$$

(Yes, that's a holomorphic function near $0 \in \mathbb{C}$: the presence of two square roots instead of just one makes it single-valued, and the apparent singularity at the origin is removable. Some routine computations with familiar power series will convince you that its Taylor series looks like (14.5).) The standard trick to see that this is the right function is based on Cauchy integration in the complex plane. Recall that for any meromorphic function on a punctured neighborhood of $0 \in \mathbb{C}$, the coefficient of $1/z$ can be extracted via the integral

$$\frac{1}{2\pi i}\oint\left(\frac{b_n}{z^n} + \frac{b_{n-1}}{z^{n-1}} + \ldots + \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + \ldots\right) dz = b_1,$$

where the symbol $\oint$ means integration along a small counterclockwise circle in $\mathbb{C}$ surrounding the origin. This follows from Cauchy's theorem, because all terms in the series other than b_1/z are derivatives of holomorphic functions on the punctured plane, but the antiderivative of $1/z$ is multivalued, producing $\oint \frac{dz}{z} = 2\pi i$. For the holomorphic function $f(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}}$ and each $n \in \mathbb{N}$, we are looking for the coefficient of z^{2n} in the function

$$[f(z^2)]^{2n+1} = \left(\frac{z}{\tanh z}\right)^{2n+1},$$

or to put it a bit more simply, the coefficient of z^n in $[f(z^2)]^{n+1}$ for each $n \geqslant 2$ even. This coefficient is the integral

$$\frac{1}{2\pi i}\oint \frac{1}{z^{n+1}}\left(\frac{z}{\tanh z}\right)^{n+1} dz = \frac{1}{2\pi i}\oint \frac{dz}{(\tanh z)^{n+1}} = \frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta^{n+1}(1-\zeta^2)}$$
$$= \frac{1}{2\pi i}\oint \frac{1}{\zeta^{n+1}}(1+\zeta^2+\zeta^4+\ldots)\, d\zeta,$$

where we've used the substitution $\zeta := \tanh z$, which is a diffeomorphism on a neighborhood of $0 \in \mathbb{C}$ and thus does not meaningfully change the path of integration. This last integral extracts the coefficient of $\frac{1}{\zeta}$ in the meromorphic function $\frac{1}{\zeta^{n+1}}(1+\zeta^2+\zeta^4+\ldots)$, which is 1 if n is even, and 0 otherwise.

Suggested reading. The relatively concise treatment of bordism theory in [tD08, Chapter 20] is good for getting yourself oriented in the subject, though of course, following all the details would require several results (e.g. the rational Hurewicz theorem and the cohomology of Hopf algebras) from earlier chapters that you probably haven't read. Davis and Kirk [DK01, Chapter 8] may be more helpful for understanding the Pontryagin-Thom construction, and they also talk about more general "flavors" of bordism theory besides Ω_*^{O} and Ω_*^{SO}, including manifolds with "stable complex structures" and such things.

For the basics on Stiefel-Whitney and Pontryagin classes, I will go ahead and recommend the old standard [MS74], which is now available on the internet in a collaboratively produced fully typeset TeX version, in case you are allergic to math books written with typewriters. (I have however noticed one or two typos in the TeX version that do not appear in the original.) I especially recommend Milnor and Stasheff for their treatment of multiplicative sequences (Chapter 19), without which I would still be intensely confused about the L-genus. The last few chapters of Bott and Tu [BT82] are also quite helpful, with the caveat that they use a different sign convention for Pontryagin classes, and it is not always easy to tell whether the cohomology theory they are using at any given moment is singular or de Rham.

Exercises. Are you kidding?

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