HOLOMORPHIC CURVES IN SYMPLECTIC AND CONTACT GEOMETRY
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Instructor

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Description

Pseudoholomorphic curves, also called *J–holomorphic* or simply *holomorphic curves* for short, are solutions to a first order elliptic PDE that generalizes the Cauchy-Riemann equation for Riemann surfaces mapping into "almost" complex manifolds. Gromov showed in his famous 1985 paper that a natural setting in which to study these objects is furnished by *symplectic manifolds*: these are even-dimensional manifolds that look locally like the Hamiltonian "phase space" of classical mechanics. Since symplectic manifolds are all locally the same (quite unlike the situation in Riemannian geometry), most interesting symplectic questions are of a more *global* nature, and on this subject very little was known until Gromov's work revealed that the compactness properties of holomorphic curves encode global symplectic invariants, implying for instance the celebrated "non-squeezing" theorem. Some years later, holomorphic curves also found application in *contact geometry*, the odd-dimensional analogue. Contact manifolds arise naturally as regular level surfaces of Hamiltonians in symplectic manifolds, or as boundary components of *symplectic cobordisms*: the latter furnish a natural setting in which to study *punctured* holomorphic curves with cylindrical ends, producing a more TQFT-style picture that yields invariants in contact topology.

In *dimension four*, the theory of holomorphic curves has a distinctive flavor due to intersection theory: the algebraic intersections of holomorphic curves are always positive, and are thus easy to control. The resulting interaction between topology and analysis produces much stronger compactness results, with beautiful applications, e.g. in some situations, the existence of a single holomorphic curve implies a geometric decomposition of its ambient manifold, essentially determining that manifold up to symplectomorphism.

The goal of this course is to develop enough of the basic theory of holomorphic curves to understand some of its classic applications, with emphasis on the 4–dimensional case. The first two thirds of the course will focus on closed holomorphic curves in closed symplectic manifolds, culminating in the proof of Gromov's non-squeezing theorem and McDuff's classification of rational and ruled symplectic 4–manifolds using embedded holomorphic spheres. The last third will then extend these ideas to the more general context of punctured holomorphic curves in symplectic cobordisms, with applications to 3–dimensional contact topology, including obstructions to symplectic filling and the Weinstein conjecture. Along the way, we'll discuss a number of important technical details, including the implicit function theorem in Banach manifolds, Fredholm theory and transversality, "bubbling off" analysis, positivity of intersections and the adjunction formula. Fundamental concepts from symplectic and contact geometry (e.g. Hamiltonian vector fields, Darboux's theorem and Moser deformation arguments, Maslov index) will also be introduced as needed.

Literature

The main text for the course will be an extensive set of lecture notes by the instructor. Aside from these, the book of McDuff and Salamon [MS04] is essential for anyone serious about the subject; most of the topics in the first 2/3 of the course are treated there, though not always in exactly the same way that we'll treat them in the lecture. Another book containing much of the same material is [94]. A surprisingly large proportion of it all originates in Gromov's paper [Gro85], which is difficult for beginners but worth rereading once every few years. Additional references for various specific topics will be recommended as needed; several are cited in the syllabus below.

Prerequisites

- Differential Geometry (manifolds, vector fields, vector bundles, differential forms and Stokes theorem)
- Functional Analysis (linear operators on Banach spaces, spectral theory, familiarity with Sobolev spaces)
- Algebraic Topology (fundamental group, homology and cohomology of manifolds, Poincaré duality, first Chern class)

Syllabus

The following is meant as a week-by-week breakdown of the topics to be covered. It is only tentative.

1. Introduction, some basics of symplectic and contact topology [MS98]
2. Nonlinear Cauchy-Riemann operators and linearizations, linear elliptic regularity, unique continuation and intersections of J–holomorphic curves
3. Banach manifolds and bundles [Lan93, Lan99], local existence and regularity of J–holomorphic curves
4. Fredholm theory, Riemann-Roch, Teichmüller spaces
5. Sard-Smale theorem, transversality for somewhere injective J–holomorphic curves
6. Sketch of Deligne-Mumford compactification [SS92], bubbling off analysis
7. Gromov compactness, non-squeezing theorem
8. Special properties in dimension 4: automatic transversality, positivity of intersections, adjunction formula
9. Exceptional spheres and blowups, classifying rational and ruled symplectic 4–manifolds [McD90]
10. Contact manifolds, symplectizations, symplectic cobordisms, stable Hamiltonian structures, punctured holomorphic curves [EGH00]
11. Fredholm and compactness theory in symplectic cobordisms [Wena, BEH+03]
12. Automatic transversality and intersection theory of punctured holomorphic curves [Wena, Sie08, Sie]
13. Applications to symplectic filling of contact 3–manifolds, the Weinstein conjecture [Wenb]
14. Additional topics as time permits

References

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[Sie08] R. Siefring, *Relative asymptotic behavior of pseudoholomorphic half-cylinders*, Comm. Pure Appl. Math. **61** (2008), no. 12, 1631–1684.

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[Wena] C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*. To appear in Comment. Math. Helv., Preprint arXiv:0802.3842.

[Wenb] ———, *Strongly fillable contact manifolds and J–holomorphic foliations*. To appear in Duke Math. J., Preprint arXiv:0806.3193.