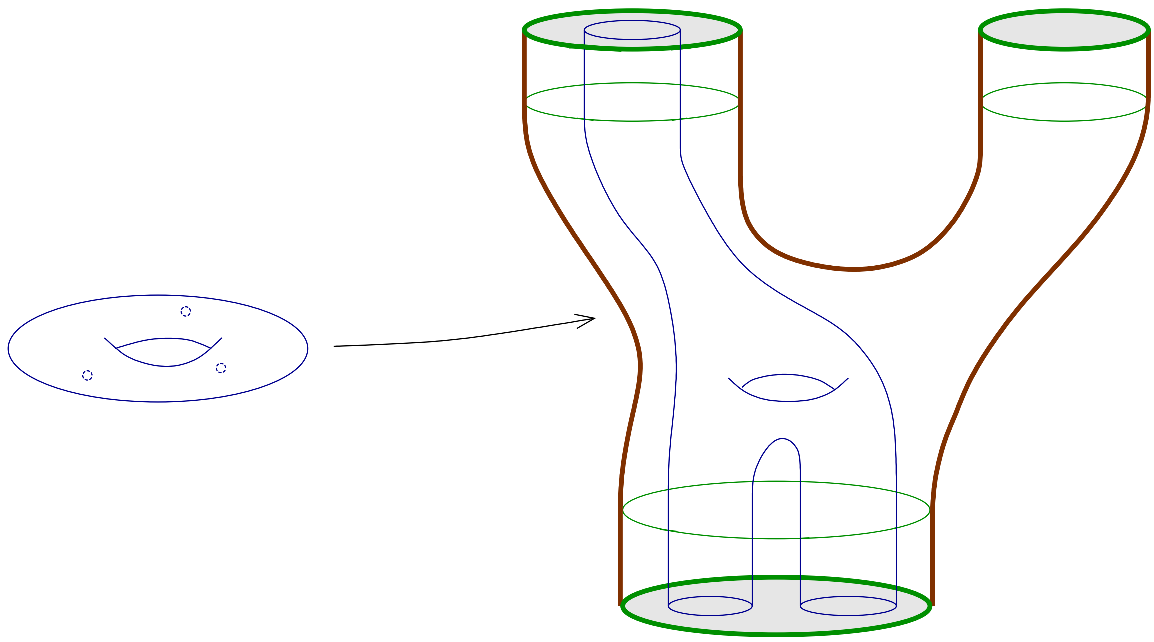
On Contact Topology, Symplectic Field Theory and the PDE That Unites Them



Chris Wendl

University College London

Slides available at:

http://www.homepages.ucl.ac.uk/~ucahcwe/publications.html#talks

How are the following related?

Problem 1 (dynamics):
If $H(q_1, p_1, \ldots, q_n, p_n)$ is a time-independent Hamiltonian and $H^{-1}(c)$ is convex, does

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

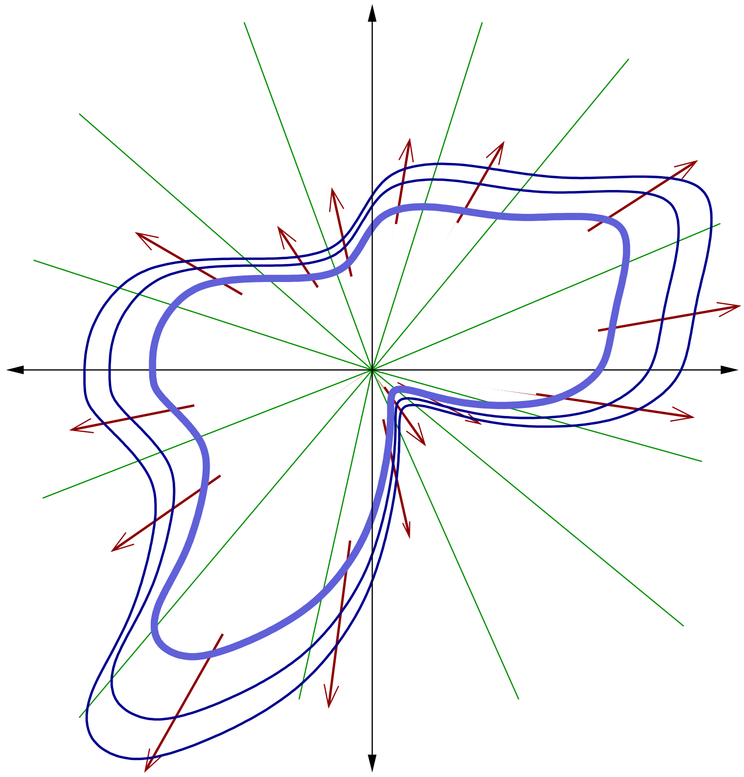
have a periodic orbit in $H^{-1}(c)$?

Problem 2 (topology):
Is a given closed manifold M the boundary of any compact manifold W?
How unique is W?

Problem 3 (complex geometry / PDE):
Given a Riemann surface Σ and complex manifold W, what is the space of holomorphic maps $\Sigma \to W$?
(Finite dimensional? Smooth? Compact?)

Problem 4 (mathematical physics):
How trivial is my TQFT?

Theorem (Rabinowitz-Weinstein '78).
Every **star-shaped** *hypersurface in* $\mathbb{R}^{2n}$ *admits a periodic orbit.*



Definition. A **symplectic structure** on a $2n$-dimensional manifold W is a system of local coordinate systems $(q_1, p_1, \ldots, q_n, p_n)$ in which Hamilton's equations are invariant.
It carries a natural volume form:

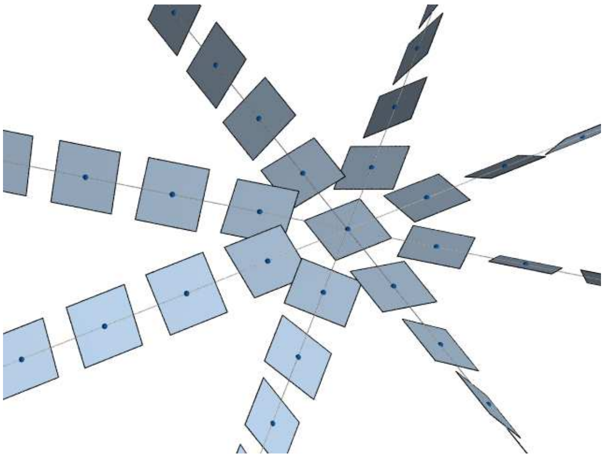
$$dp_1\, dq_1 \ldots dp_n\, dq_n.$$

∂W is **convex** if it is transverse to a vector field Y that *dilates* the symplectic structure.

$M := \partial W$ convex $\rightsquigarrow$ **contact structure**

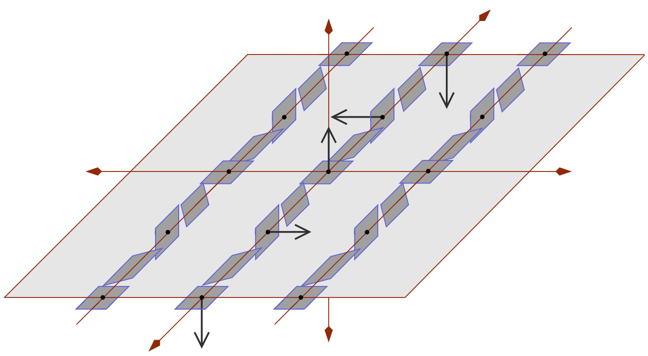
$$\xi \subset TM,$$

a field of tangent hyperplanes that are "locally twisted" (*maximally nonintegrable*),



and transverse to the **Reeb** (i.e. Hamiltonian) vector field.

Example: $T^3 := S^1 \times S^1 \times S^1$



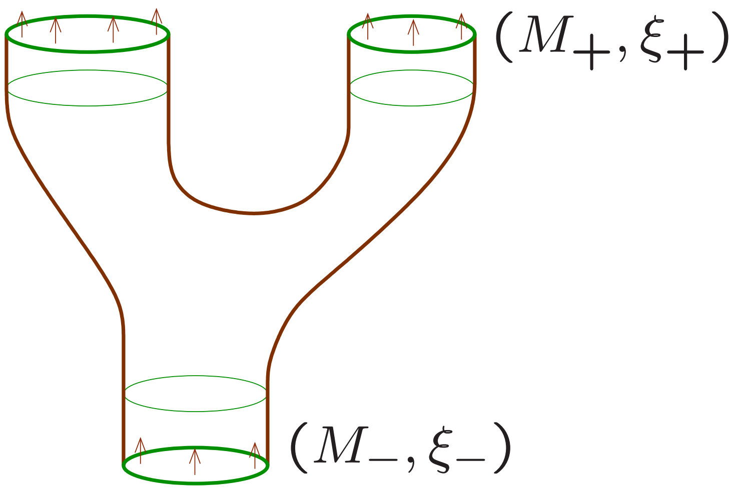
$=$ boundary of $T^2 \times \mathbb{D} = D^*T^2 \subset T^*T^2$.

Some hard problems in contact topology

1. Classification of contact structures: given ξ_1, ξ_2 on M, is there a diffeomorphism $\varphi : M \to M$ mapping ξ_1 to ξ_2?

2. Weinstein conjecture: Every Reeb vector field on every closed contact manifold has a periodic orbit?

3. Partial orders: say $(M_-, \xi_-) \prec (M_+, \xi_+)$ if there is a (symplectic, exact or Stein) cobordism between them.

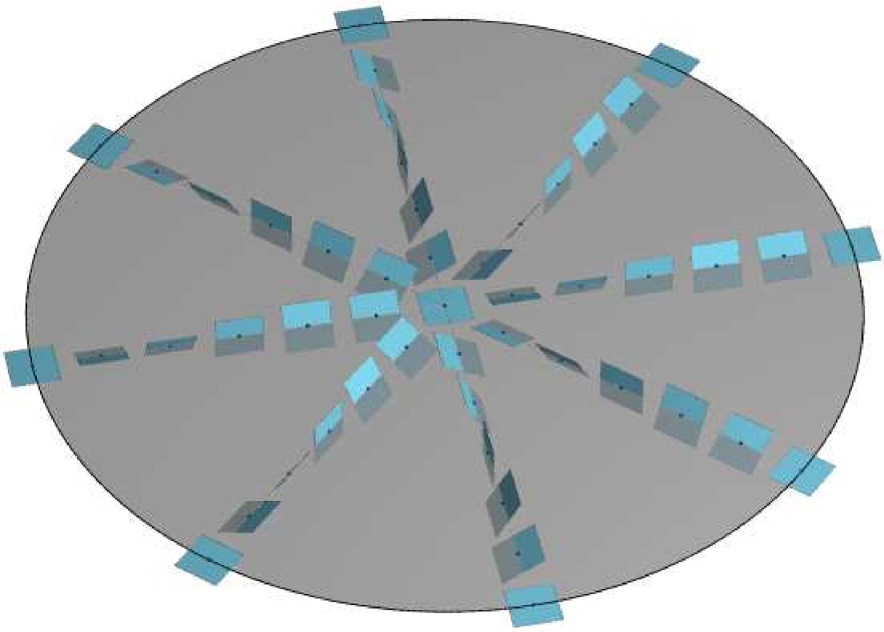


When is $(M_-, \xi_-) \prec (M_+, \xi_+)$?
When is $\emptyset \prec (M, \xi)$? (Is it *fillable*?)

Overtwisted vs. tight

Theorem (Eliashberg '89). *If ξ_1 and ξ_2 are both overtwisted, then $(M, \xi_1) \cong (M, \xi_2) \Leftrightarrow \xi_1$ and ξ_2 are homotopic.*

"Overtwisted contact structures are flexible."



Theorem (Gromov '85 and Eliashberg '89). ξ *overtwisted* $\Rightarrow (M, \xi)$ *not fillable.*

Non-overtwisted contact structures are called **"tight"** .

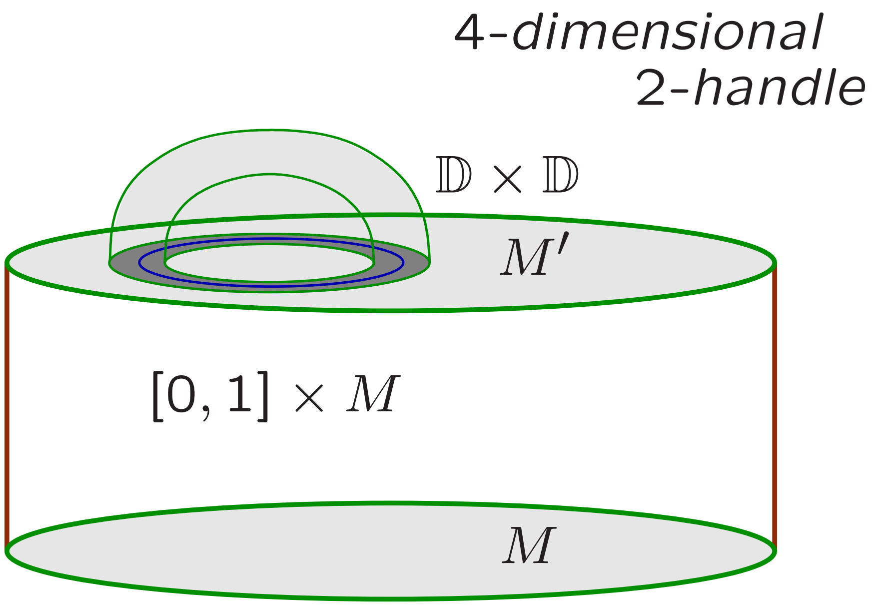
They are not fully understood.

Conjecture.

Suppose $(M,\xi) \xrightarrow{\text{contact surgery}} (M',\xi')$.

Then (M,ξ) *tight* $\Rightarrow$ (M',ξ') *tight*.

Surgery $\rightsquigarrow$ handle attaching cobordism:



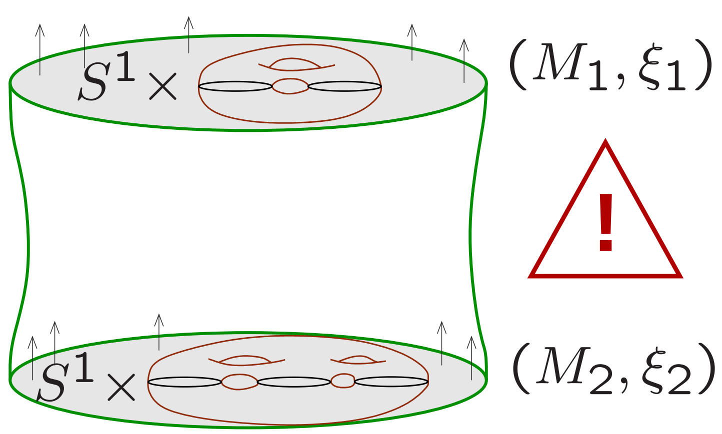
$$\partial(([0,1] \times M) \cup (\mathbb{D} \times \mathbb{D})) = -M \sqcup M'$$

Recent results: $\exists$ "degrees of tightness".

Theorem (Latschev-W. 2010). *There exists a numerical contact invariant* $\mathrm{AT}(M, \xi) \in \mathbb{N} \cup \{0, \infty\}$ *such that:*

- $(M_-, \xi_-) \prec (M_+, \xi_+) \Rightarrow$
  $\mathrm{AT}(M_-, \xi_-) \leq \mathrm{AT}(M_+, \xi_+)$
- $\mathrm{AT}(M, \xi) = 0 \Leftrightarrow$
  (M, ξ) *is algebraically overtwisted*
- (M, ξ) *fillable* $\Rightarrow \mathrm{AT}(M, \xi) = \infty$
- $\forall k$, $\exists (M_k, \xi_k)$ *with* $\mathrm{AT}(M_k, \xi_k) = k$.

Corollary:
$(M_k, \xi_k) \xrightarrow{\text{contact surgery}} (M_\ell, \xi_\ell) \Rightarrow \ell \geq k$.



Symplectic Field Theory

(Eliashberg-Givental-Hofer '00 + Cieliebak-Latschev '09)

(M, ξ) with Reeb vector field $\rightsquigarrow$
$\mathcal{P} :=$ {periodic Reeb orbits on M}.

$\mathcal{A} :=$ graded commutative algebra with unit and generators $\{q_\gamma\}_{\gamma \in \mathcal{P}}$.

$\mathcal{W} :=$ {formal power series $F(q_\gamma, p_\gamma, \hbar)$} with,

$$[p_\gamma, q_{\gamma'}] = \delta_{\gamma,\gamma'}\hbar.$$

$F \in \mathcal{W}$, substitute $p_\gamma := \hbar \frac{\partial}{\partial q_\gamma} \rightsquigarrow$ operator

$$D_F : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]]$$

"Theorem": There exists $\mathcal{H} \in \mathcal{W}$ with $\mathcal{H}^2 = 0$ such that $D_{\mathcal{H}}(1) = 0$ and

$$H_*^{\text{SFT}}(M, \xi) := H_*(\mathcal{A}[[\hbar]], D_{\mathcal{H}}) := \frac{\ker D_{\mathcal{H}}}{\operatorname{im} D_{\mathcal{H}}}$$

is a **contact invariant**.

Symplectic cobordism $(M_-, \xi_-) \prec (M_+, \xi_+)$
$\rightsquigarrow$ **natural map**

$$H_*^{\text{SFT}}(M_+, \xi_+) \to H_*^{\text{SFT}}(M_-, \xi_-)$$

preserving elements of $\mathbb{R}[[\hbar]]$.

Example

If no periodic orbits, then $H_*^{\text{SFT}}(M,\xi) = \mathbb{R}[[\hbar]]$.

Definition (Latschev-W.).

We say (M,ξ) has **algebraic k-torsion** if $[\hbar^k] = 0 \in H_*^{\text{SFT}}(M,\xi)$.

$$\boxed{\text{AT}(M,\xi) := \sup\left\{k \;\middle|\; [\hbar^{k-1}] \neq 0 \in H_*^{\text{SFT}}(M,\xi)\right\}}$$

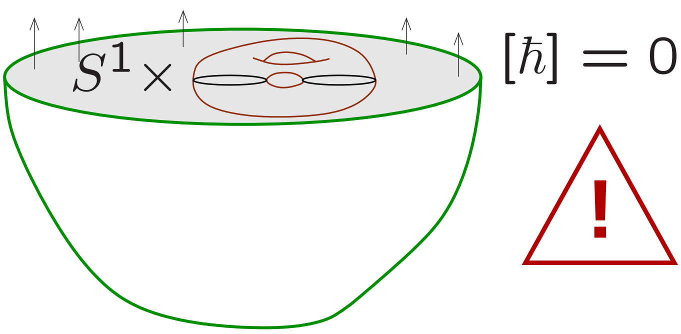
Example

Overtwisted $\Rightarrow$

all "interesting" contact invariants vanish:

$$H_*^{\text{SFT}}(M,\xi) = \{0\} \Rightarrow [1] = 0 \Rightarrow \text{AT}(M,\xi) = 0.$$

Theorem. *Algebraic k-torsion* $\Rightarrow$ *not fillable.*



$[\hbar] = 0 \in H_*^{\text{SFT}}(M,\xi)$

$\downarrow$

$[\hbar] \neq 0 \in H_*^{\text{SFT}}(\emptyset)$

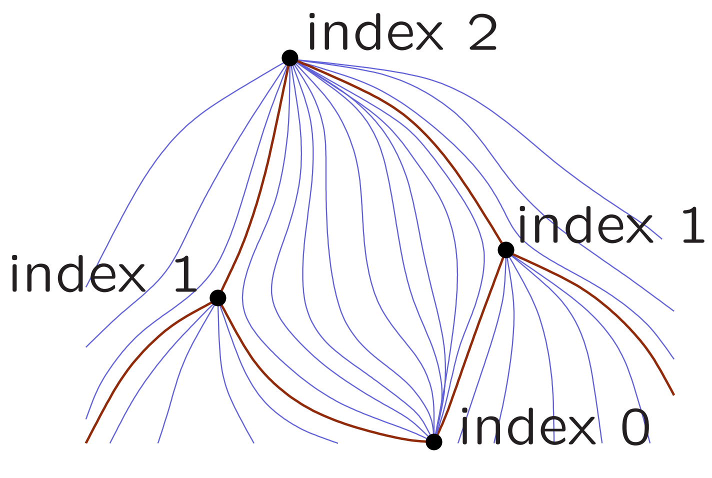
A beautiful idea (Witten '82 + Floer '88):

(X, g) Riemannian manifold, $f : X \to \mathbb{R}$ generic Morse function. Then singular homology

$$H_*(X; \mathbb{Z}) \cong H_*\left(\mathbb{Z}^{\#\,\mathrm{Crit}(f)}, d_f\right),$$

where d_f counts rigid gradient flow lines,

$$\dot{x}(t) + \nabla f(x(t)) = 0.$$



SFT of $(M, \xi = \ker \alpha)$:
"∞-dimensional Morse theory" for the *contact action functional*

$$\Phi : C^\infty(S^1, M) \to R : x \mapsto \int_{S^1} x^* \alpha,$$

with $\mathrm{Crit}(\Phi) = \{$periodic Reeb orbits$\}$.

Gradient flow:

Consider 1-parameter families of loops $\{u_s \in C^\infty(S^1, M)\}_{s\in\mathbb{R}}$ with

$$\partial_s u_s + \nabla\Phi(u_s) = 0.$$

$\rightsquigarrow$ cylinders $u : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ satisfying the *nonlinear Cauchy-Riemann equation*

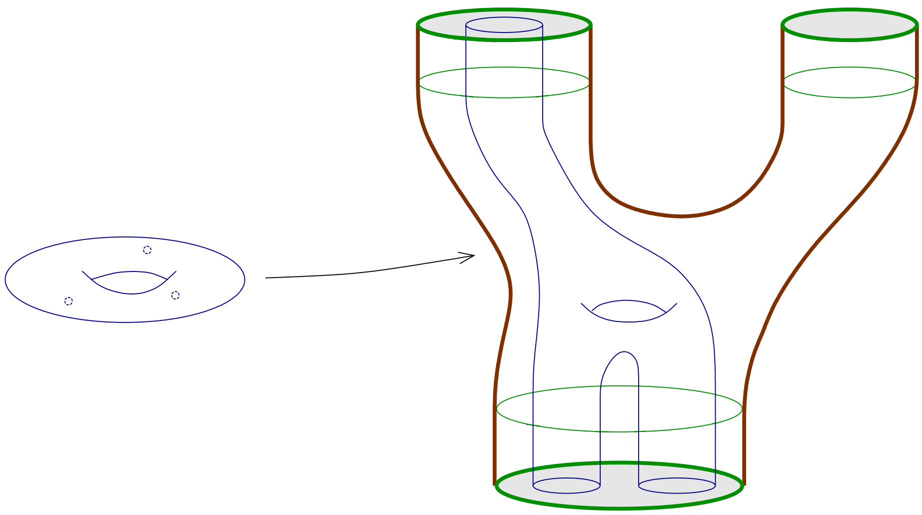
$$\partial_s u + J(u)\,\partial_t u = 0$$

for an almost complex structure J on $\mathbb{R} \times M$.

For a symplectic cobordism W and Riemann surface Σ, consider J-holomorphic curves

$$u : \Sigma \setminus \{z_1, \dots, z_n\} \to W$$

approaching Reeb orbits at the punctures.

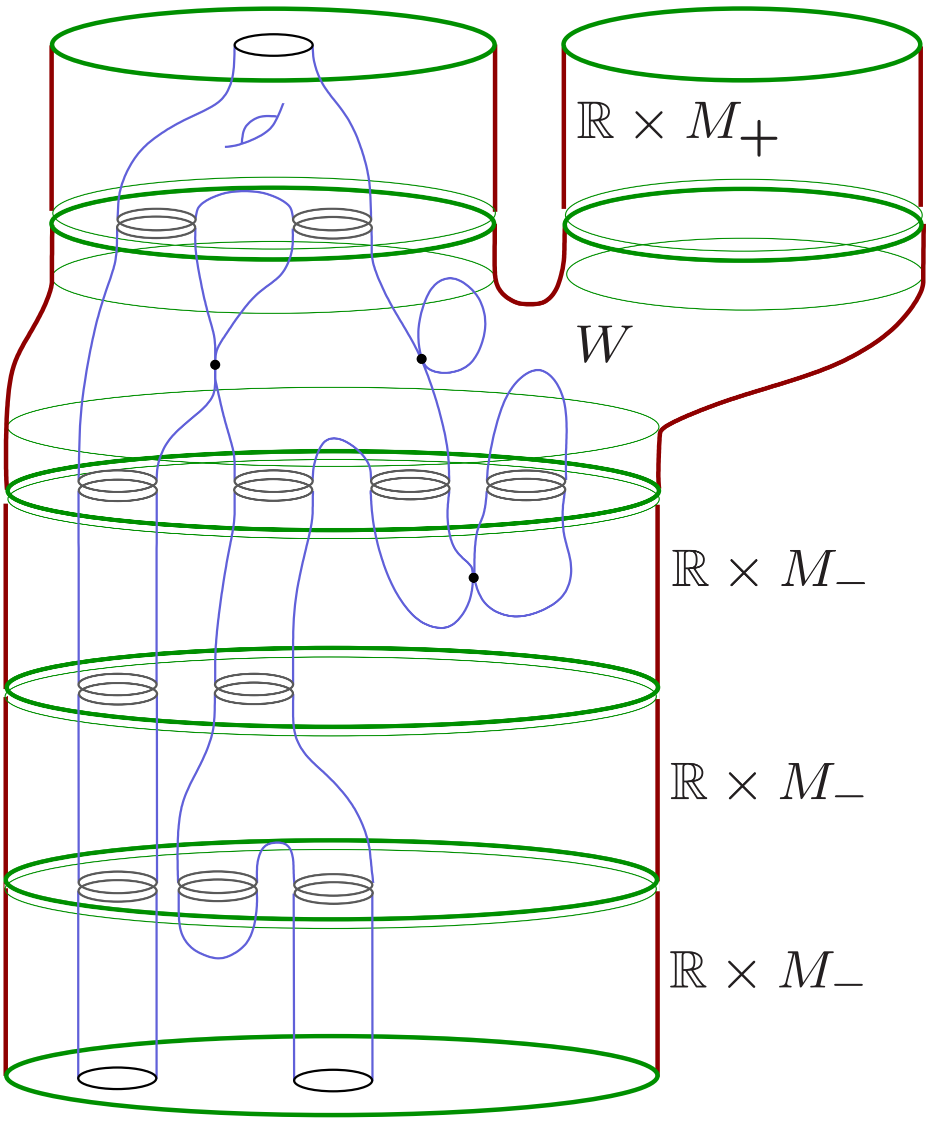


The Cauchy-Riemann equation is **elliptic**:

$$\|u\|_{W^{1,p}} \leq \|u\|_{L^p} + \|\partial_s u + i\,\partial_t u\|_{L^p}$$

$\Rightarrow$ Spaces of holomorphic curves are (often)

- smooth finite-dimensional manifolds,
- compact up to *bubbling* / *breaking*.

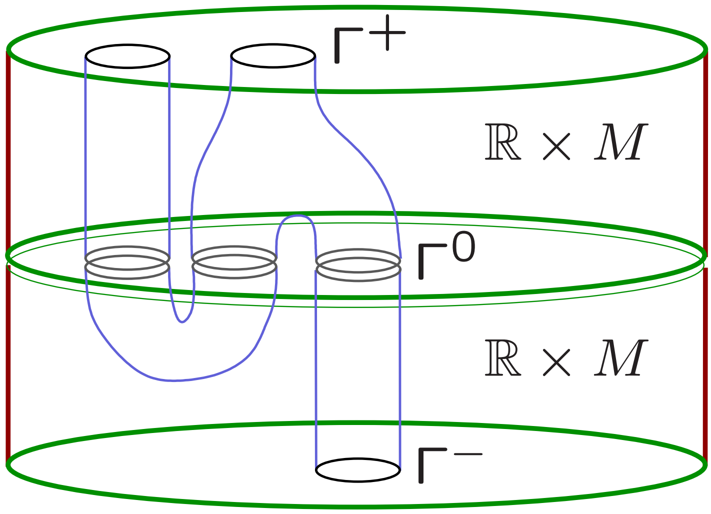


Definition of $\mathcal{H}$

$\Gamma^\pm := (\gamma_1^\pm, \ldots, \gamma_{k_\pm}^\pm)$ lists of Reeb orbits

$\mathcal{M}_g(\Gamma^+, \Gamma^-) := \{$ rigid J-holomorphic curves in $\mathbb{R} \times M$ with genus g, ends at $\Gamma^\pm\} \big/$ parametrization

$$\mathcal{H} := \sum_{g, \Gamma^+, \Gamma^-} \# \left(\mathcal{M}_g(\Gamma^+, \Gamma^-)/\mathbb{R} \right) \hbar^{g-1} q^{\Gamma^-} p^{\Gamma^+}$$

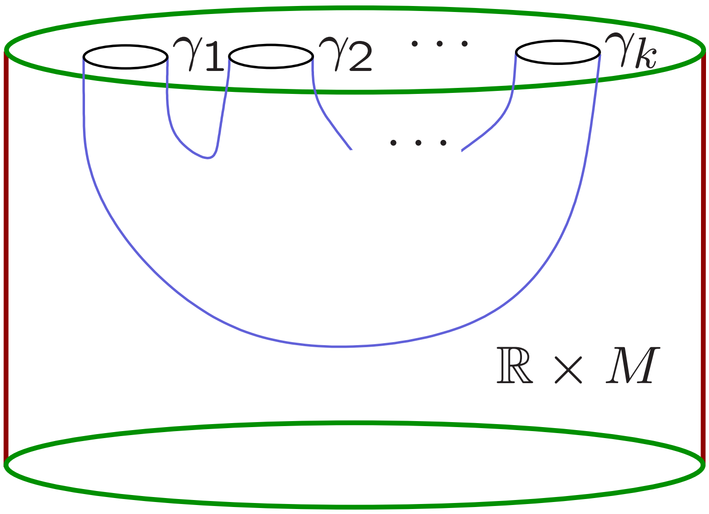


SFT compactness theorem:
$\overline{\mathcal{M}}_g(\Gamma^+, \Gamma^-) = \{J$-holomorphic buildings$\}$

$\mathcal{H}^2$ counts the boundary of a 1-dimensional space $\Rightarrow \mathcal{H}^2 = 0$.

Example

Suppose $\mathbb{R} \times M$ has exactly one rigid J-holomorphic curve, with genus 0, no negative ends, and positive ends at orbits $\gamma_1, \ldots, \gamma_k$.



Then

$$\mathcal{H} = \hbar^{-1} p_{\gamma_1} \ldots p_{\gamma_k}.$$

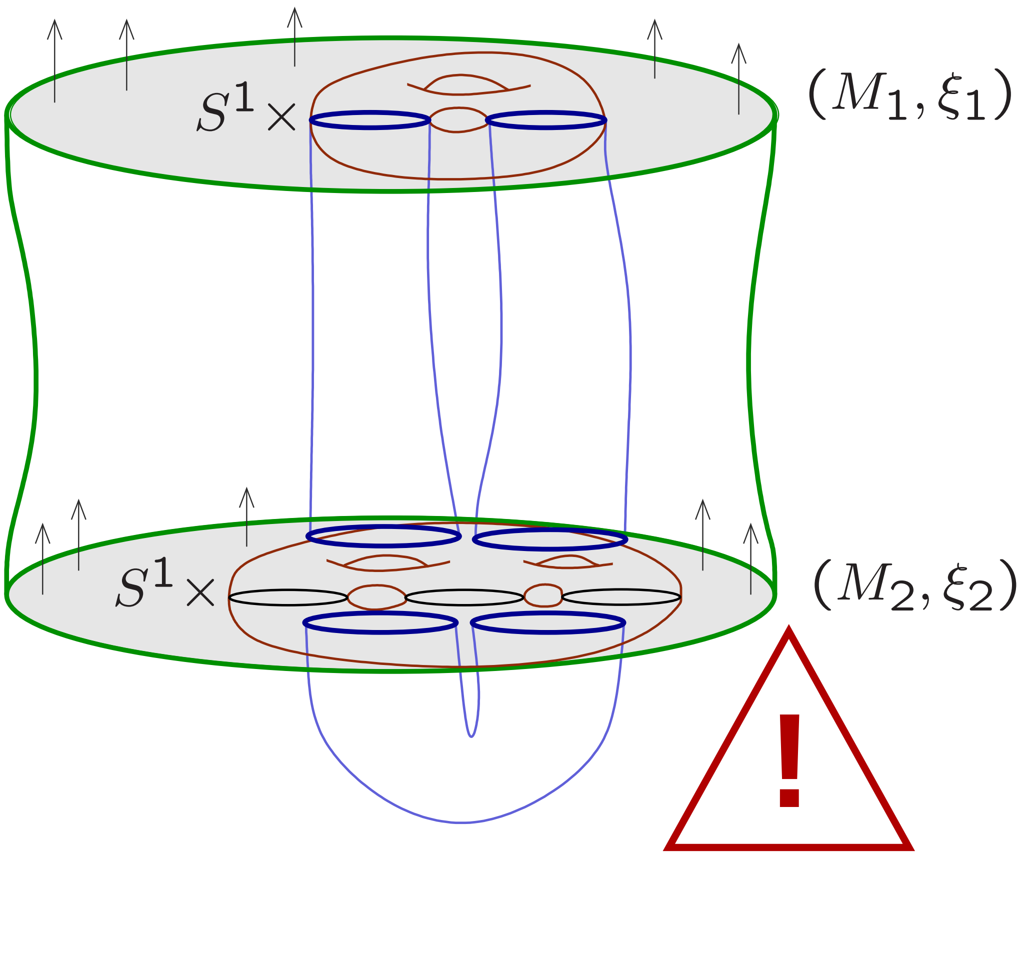
Substituting $p_{\gamma_i} = \hbar \dfrac{\partial}{\partial q_{\gamma_i}}$ gives

$$D_{\mathcal{H}}\left(q_{\gamma_1} \ldots q_{\gamma_k}\right) = \hbar^{k-1}$$

$\Rightarrow [\hbar^{k-1}] = 0 \in H_*^{\mathsf{SFT}}(M, \xi)$

$\Rightarrow \mathsf{AT}(M, \xi) \leq k - 1.$

Why $(M_2, \xi_2) \prec (M_1, \xi_1)$ is not true:



Some open questions and partial answers

1. What geometric conditions correspond to $\text{AT}(M, \xi) = k$?

   - Overtwistedness, Giroux torsion, planar torsion:
     C. Wendl, *A hierarchy of local filling obstructions for contact 3-manifolds*, Preprint 2010, arXiv:1009.2746.

2. Interesting examples beyond dimension 3?

   - Higher-dimensional overtwisted disks:
     K. Niederkrüger, *The plastikstufe—a generalization of the overtwisted disk to higher dimensions*, Algebr. Geom. Topol. **6** (2006), 2473-2508.
     F. Bourgeois, K. Niederkrüger, *PS-overtwisted contact manifolds are algebraically overtwisted*, in preparation.
   - Higher-dimensional Giroux torsion:
     P. Massot, K. Niederkrüger and C. Wendl, *Weak and strong fillability of higher dimensional contact manifolds*, to appear in Invent. Math., Preprint 2011, arXiv:1111.6008.

3. Can contact structures with $\text{AT}(M, \xi) \geq k$ be classified???

   - Overtwisted contact structures are flexible:
     Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623-637.
   - Coarse classification—finitely many have $\text{AT}(M, \xi) \geq 2$:
     V. Colin, E. Giroux and K. Honda, *Finitude homotopique et isotopique des structures de contact tendues*, Publ. Math. Inst. Hautes Études Sci. **109** (2009), no. 1, 245-293.

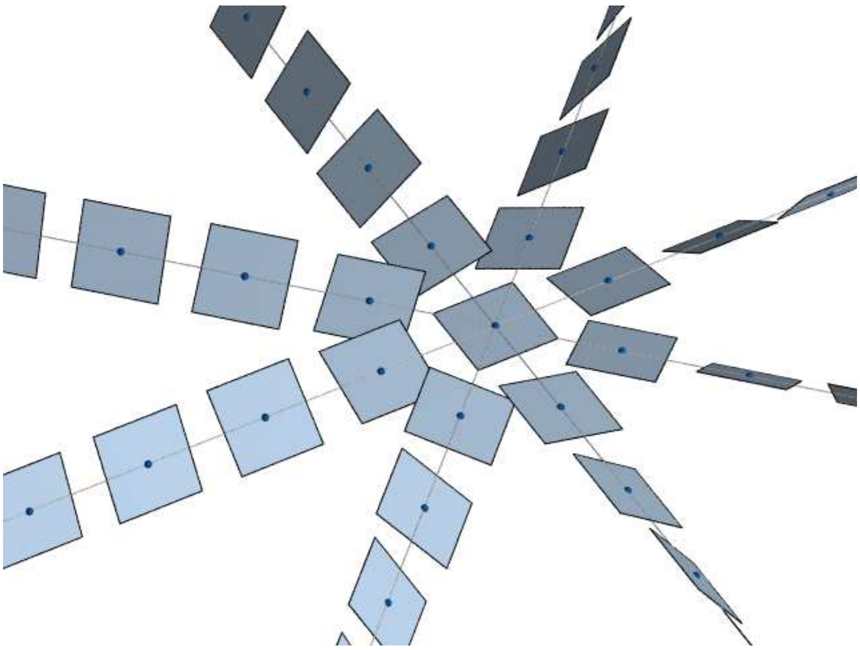
Main reference

- Janko Latschev and Chris Wendl, *Algebraic torsion in contact manifolds*, Geom. Funct. Anal. **21** (2011), no. 5, 1144-1195, with an appendix by Michael Hutchings.

Acknowledgment

Contact structure illustrations by Patrick Massot:

`http://www.math.u-psud.fr/~pmassot/`



These slides are available at:

`http://www.homepages.ucl.ac.uk/~ucahcwe/publications.html#talks`