ASYMPTOTIC MEAN-SQUARE STABILITY OF TWO-STEP METHODS FOR STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS

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Abstract.

We deal with linear multi-step methods for SDEs and study when the numerical approximation shares asymptotic properties in the mean-square sense of the exact solution. As in deterministic numerical analysis we use a linear time-invariant test equation and perform a linear stability analysis. Standard approaches used either to analyse deterministic multi-step methods or stochastic one-step methods do not carry over to stochastic multi-step schemes. In order to obtain sufficient conditions for asymptotic mean-square stability of stochastic linear two-step-Maruyama methods we construct and apply Lyapunov-type functionals. In particular we study the asymptotic mean-square stability of stochastic counterparts of two-step Adams-Bashforth- and Adams-Moulton-methods, the Milne-Simpson method and the BDF method.

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1 Introduction

Our aim is to study when a numerical approximation generated by a stochastic linear two step scheme shares asymptotic properties of the exact solution of an SDE of the form

$$\mathrm{d}X(t) = f(t, X(t))\,\mathrm{d}t + G(t, X(t))\,\mathrm{d}W(t)\,,\ t \in \mathcal{J}\,,\ X(t_0) = X_0\,, \tag{1.1}$$

where $\mathcal{J} = [t_0, \infty)$, $f : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$. Later we consider also complex-valued functions f, G, X. The driving process W is an m-dimensional Wiener process on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\in\mathcal{J}}$. The initial value X_0 is a given $\mathcal{F}_{t_0}$-measurable random variable (it can be deterministic data, of course), independent of the Wiener process and with finite second moments. We assume that there exists a path-wise unique strong solution $X = \{X(t),\ t \in \mathcal{J}\}$ of Equation (1.1) and we indicate the

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dependence of this solution upon the initial data by writing $X(t) \equiv X(t; t_0, X_0)$. The numerical methods to be considered are generally drift-implicit linear two-step Maruyama methods with constant step-size h which for (1.1) take the form

$$
\begin{aligned}
(1.2) \quad & \alpha_2 X_{i+1} + \alpha_1 X_i + \alpha_0 X_{i-1} \\
&\qquad = h[\beta_2\, f(t_{i+1}, X_{i+1}) + \beta_1\, f(t_i, X_i) + \beta_0\, f(t_{i-1}, X_{i-1})] \\
&\qquad\quad + \gamma_1\, G(t_i, X_i)\, \Delta W_i + \gamma_0\, G(t_{i-1}, X_{i-1})\, \Delta W_{i-1},
\end{aligned}
$$

for $i = 1, 2, \ldots$, where $t_i = i \cdot h,\ i = 0, 1, \ldots$, and $\Delta W_i = W(t_{i+1}) - W(t_i)$. For normalization we set $\alpha_2 = 1$. We require given initial values $X_0, X_1 \in L_2(\Omega, \mathbb{R}^n)$ that are $\mathcal{F}_{t_0}$- and $\mathcal{F}_{t_1}$-measurable, respectively. We emphasize that an explicit discretization is used for the diffusion term. For $\beta_2 = 0$, the stochastic multi-step scheme (1.2) is explicit, otherwise it is drift-implicit. See also [3, 4, 7, 8, 9, 13, 14, 17, 18].

Given a reference solution $X(t; t_0, X_0)$ of (1.1), the concept of *asymptotic mean-square stability in the sense of Lyapunov* concerns the question whether or not solutions $X^{D_0}(t) = X(t; t_0, X_0 + D_0)$ of (1.1) exist and approach the reference solution when t tends to infinity. The distance between $X(t)$ and $X^{D_0}(t)$ is measured in the mean-square sense, i.e. in $L_2(\Omega)$, and the terms $D_0 \in L_2(\Omega)$ are small perturbations of the initial value X_0 .
Already in the deterministic case it is a difficult problem to answer the question when numerical approximations share asymptotic stability properties of the exact solution *in general*. Including stochastic components into the problem does not simplify the analysis. In deterministic numerical analysis, the first step in this direction is a *linear stability analysis*, where one applies the method of interest to a linear test equation and discusses the asymptotic behaviour of the resulting recurrence equation (see e.g. [10]). Well-known notions like A-stability of numerical methods refer to the stability analysis of linear test equations. In this article we would like to contribute to the linear stability analysis of stochastic numerical methods and thus we choose as a test equation the linear scalar complex stochastic differential equation

$$
(1.3) \quad \mathrm{d}X(t) = \lambda X(t)\,\mathrm{d}t + \mu X(t)\,\mathrm{d}W(t), \quad t \geq 0, \quad X(0) = X_0, \quad \lambda, \mu, X_0 \in \mathbb{C},
$$

with the complex geometric Brownian motion as exact solution. In complex arithmetic we denote by $\bar{\eta}$ the complex conjugate of a complex scalar $\eta \in \mathbb{C}$. The method (1.2) applied to (1.3) takes the following form:

$$
\begin{aligned}
(1.4) \quad X_{i+1} + \alpha_1 X_i + \alpha_0 X_{i-1} &= h[\beta_2\, \lambda\, X_{i+1} + \beta_1\, \lambda\, X_i + \beta_0\, \lambda\, X_{i-1}] \\
&\quad + \gamma_1\, \mu\, X_i\, \Delta W_i + \gamma_0\, \mu\, X_{i-1}\, \Delta W_{i-1}, \qquad i \geq 1,
\end{aligned}
$$

where $\alpha_2 = 1$. Subsequently we will assume that the parameters in (1.4) are chosen such that the resulting scheme is mean-square convergent (see [8]). Then the coefficients of the stochastic terms have to satisfy $\gamma_1 = \alpha_2 = 1$ and $\gamma_0 = \alpha_1 + \alpha_2$. We will in particular discuss stochastic versions of the explicit and implicit

Adams methods, i.e., the Adams-Bashforth and the Adams-Moulton method, respectively, the Milne-Simpson method and the BDF method.

Section 2 contains definitions of the notions of stability discussed in this article. In Section 3 we will discuss the difficulties that arise when applying standard approaches for performing a linear stability analysis either for stochastic one-step methods or for deterministic multi-step schemes to the case of stochastic multi-step methods. In Section 4 we give a Lyapunov-type theorem ensuring asymptotic mean-square stability of the zero solution of a class of stochastic difference equations under sufficient conditions on the parameters. The method of construction of Lyapunov functionals [15] is briefly sketched. Then we construct appropriate Lyapunov functionals in four different ways and thus obtain four sets of sufficient conditions on the parameters. In Section 5 results for stochastic linear two-step methods, in particular the explicit and implicit Adams methods, the Milne-Simpson method and the BDF method are presented. Section 6 summarizes our findings and points out open problems.

2 Basic notions

We will be concerned with *mean-square stability* of the solution of Equation (1.1), with respect to perturbations D_0 in the initial data X_0. We here recall various definitions, which are based on those given in [11].

Definition 2.1. *The reference solution X of the SDE (1.1) is termed*

1. mean-square stable*, if for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that the solution $X^{D_0}(t)$ exists for all $t \geq t_0$ and*

$$\mathbb{E}|X^{D_0}(t) - X(t)|^2 < \epsilon$$

whenever $t \geq t_0$ and $\mathbb{E}|D_0|^2 < \delta$;

2. asymptotically mean-square stable*, if it is mean-square stable and if there exists a $\delta \geq 0$ such that whenever $\mathbb{E}|D_0|^2 < \delta$*

$$\mathbb{E}|X^{D_0}(t) - X(t)|^2 \to 0 \quad \textit{for} \quad t \to \infty.$$

It is well known for which parameters $\lambda, \mu \in \mathbb{C}$ the solutions

$$X(t) = X_0 e^{(\lambda - \frac{1}{2}|\mu|^2)t + \mu W(t)} \tag{2.1}$$

of the linear test equation (1.3) approach zero in the mean-square sense. The following result can be found e.g. in [1, pp. 189–190], [11, 12, 20, 21]. Its proof uses the fact, that $\mathbb{E}e^{\mu W(t) - \frac{1}{2}|\mu|^2 t} = 1$.

Theorem 2.1. *The zero solution of (1.3) is asymptotically mean-square stable if*

$$Re(\lambda) < -\frac{1}{2}\,|\mu|^2\,. \tag{2.2}$$

Now we formulate analogous definitions for the discrete recurrence equation (1.2) with solution $\{X_i\}_{i=0}^{\infty}$. We denote by $\{X_i\}_{i=0}^{\infty} = \{X_i(X_0, X_1)\}_{i=0}^{\infty}$ the reference solution and by $\{X_i^{D_0,D_1}\}_{i=0}^{\infty} = \{X_i(X_0 + D_0,\, X_1 + D_1)_{i=0}^{\infty}\}$ a solution of (1.2) where the initial values have been perturbed.

Definition 2.2. *The reference solution* $\{X_i\}_{i=0}^{\infty}$ *of (1.2) is said to be*

1. mean-square stable *if, for each* $\varepsilon > 0$*, there exists a value* $\delta > 0$ *such that, whenever* $\mathbb{E}(|D_0|^2 + |D_1|^2) < \delta$*,*

$$\mathbb{E}|X_i^{D_0,D_1} - X_i|^2 < \varepsilon\,, \quad i = 1, 2, \ldots\,;$$

2. asymptotically mean-square stable *if it is mean-square stable and if there exists a value* $\delta > 0$ *such that, whenever* $\mathbb{E}(|D_0|^2 + |D_1|^2) < \delta$*,*

$$\mathbb{E}|X_i^{D_0,D_1} - X_i|^2 \to 0 \quad as \quad i \to \infty\,.$$

Recall that applying a convergent stochastic two-step method (1.2) to our test equation (1.3) results in the stochastic *difference* equation (1.4) with $\gamma_1 = 1$, $\gamma_0 = 1 + \alpha_1$. For simplification in our subsequent analysis we rewrite (1.4) as

$$X_{i+1} \;=\; a\, X_i \;+\; c\, X_{i-1} \;+\; b\, X_i\, \xi_i \;+\; d\, X_{i-1}\, \xi_{i-1}\,, \tag{2.3}$$

$$a \;=\; \frac{-\alpha_1 + \lambda h\ \beta_1}{1 - \lambda h\ \beta_2}, \qquad c \;=\; \frac{-\alpha_0 + \lambda h\ \beta_0}{1 - \lambda h\ \beta_2}, \tag{2.4}$$

$$b \;=\; \frac{\mu h^{\frac{1}{2}}}{1 - \lambda h\ \beta_2}, \qquad d \;=\; \frac{\mu h^{\frac{1}{2}}\ (1{+}\alpha_1)}{1 - \lambda h\ \beta_2}, \tag{2.5}$$

where $\xi_{i-1} = h^{-\frac{1}{2}} \Delta W_{i-1}$, and $\xi_i = h^{-\frac{1}{2}} \Delta W_i$ are $\mathcal{N}(0,1)$ Gaussian random variables, independent of each other. Obviously this recurrence equation admits the zero solution $\{X_i\}_{i=0}^{\infty} = \{0\}_{i=0}^{\infty}$, which will be the reference solution in the subsequent stability analysis.

We would like to add a remark concerning the choice of the linear test equation (1.3). The scalar linear test equation (1.3) is less significant for SDEs than the corresponding scalar linear test equation is for ODEs. Generally, it is not possible to decouple systems of linear equations of the form

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + \sum_{r=1}^{m} B_r X(t)\,\mathrm{d}Wr(t),$$

where X is a vector-valued stochastic process, to scalar equations, since the eigenspaces of the matrices $A, B_1, \ldots, B_m$ may differ. The results for the scalar test equation are thus only significant for linear systems where the matrices $A, B_1, \ldots, B_m$ are simultaneously diagonalizable with constant transformations. We refer to [19] for a linear stability analysis of one-step methods applied to systems of linear equations. As a first step in the area of linear stability analysis for stochastic multi-step schemes we restrict our attention to scalar linear test equations (see Section 6).

3 Review of standard approaches for a linear stability analysis

Several approaches for an investigation in linear stability analysis exist, either for stochastic one-step methods or for stochastic multi-step methods. However, it turns out that standard methods can not easily be extended to the case of stochastic multi-step methods. We here describe the difficulties arising with standard approaches.

3.1 The approach for stochastic linear one-step schemes

In [12] the author considers the stochastic θ-method and investigates its asymptotic mean-square stability properties. The method, applied to the test equation (1.3) has the form

$$X_{i+1} = X_i + \theta\, h\lambda\, X_{i+1} \; + (1-\theta)\; h\lambda\; X_i \; + \; \sqrt{h}\, \mu\, X_i\; \xi_i,$$

where $\theta \in [0,1]$ is a fixed parameter. Rewritten as a one-step recurrence it reads

$$X_{i+1} = (\widetilde{a} + \widetilde{b}\,\xi_i)\; X_i, \tag{3.1}$$

$$\text{where} \qquad \widetilde{a} = \frac{1+(1-\theta)\lambda h}{1-\theta\lambda h}, \qquad \widetilde{b} = \frac{\mu h^{\frac{1}{2}}}{1-\theta\lambda h}.$$

Squaring both sides of Equation (3.1) and taking the expectation, one obtains an *exact* one-step recurrence for $\mathbb{E}|X_i|^2$, i.e.,

$$\mathbb{E}|X_{i+1}|^2 = (|\widetilde{a}|^2 + |\widetilde{b}|^2)\; \mathbb{E}|X_i|^2, \tag{3.2}$$

which allows a direct derivation of conditions for asymptotic mean-square stability of the zero solution. For comparison we now apply this approach to the stochastic multi-step method. Squaring both sides of (2.3) yields

$$|X_{i+1}|^2 = |a+b\xi_i|^2\; |X_i|^2 + |c+d\xi_{i-1}|^2\; |X_{i-1}|^2 + 2\Re\{(a+b\xi_i)X_i\; \overline{(c+d\;\xi_{i-1})X_{i-1}}\},$$

and the last term is not so easily resolved. Either one resorts to inequalities, such as $2AB \le A^2 + B^2$, or one follows the recurrence further down. The latter approach provides for $\lambda, \mu \in \mathbb{R}$ an exact recurrence of the form

$$\begin{aligned}\mathbb{E}|X_{i+1}|^2 \;&=\; (|a|^2 + |b|^2)\; \mathbb{E}|X_i|^2 + (|c|^2 + |d|^2)\mathbb{E}|X_{i-1}|^2 \\ &\quad +2a(ac+bd)\sum_{j=0}^{i-2} c^j \mathbb{E}|X_{i-j}|^2 + 2a\mathbb{E}(c+d\xi_0)X_0X_1.\end{aligned}$$

In any case one does not immediately obtain conditions for asymptotic mean-square stability of the zero solution as in the case of the one-step recurrence (3.2).

3.2 The approach for deterministic two-step schemes

When the schemes (1.2) are applied to deterministic ordinary differential equations they are reduced to well-known deterministic linear two-step schemes. With $\mu=0$ one recovers the deterministic linear test equation $x'(t)=\lambda x(t),\ t>0$, and obtains the recurrence (2.3) with $b=d=0$, i.e.

$$X_{i+1} = a\ X_i \ +\ c\ X_{i-1}, \quad i=1,2,\ldots, \tag{3.3}$$

with coefficients a, c from (2.4). The recurrence (3.3) may be considered as a scalar, linear, homogeneous difference equation with constant coefficients. Its (complex) solutions form a two-dimensional linear space spanned by

$$X_i = c_1\zeta_1^i + c_2\zeta_2^i \qquad \text{or} \qquad X_i = c_1\zeta_3^i + c_2 i\zeta_3^i, \qquad i=0,1,\ldots,$$

if either ζ_1, ζ_2 are the two distinct roots or if ζ_3 is the double root of the characteristic polynomial

$$\psi(\zeta) = \zeta^2 - a\zeta - c \tag{3.4}$$

of the difference equation. Therefore, all solutions of the difference equation (3.3) approach zero for $i\to\infty$ if and only if the roots of the characteristic polynomial (3.4) lie inside the unit circle of the complex plane. Equivalently, this result is obtained by reformulating the two-step recurrence (3.4) as a one-step recurrence in a two-dimensional space by setting

$$\begin{pmatrix} X_{i+1} \\ X_i \end{pmatrix} = A \begin{pmatrix} X_i \\ X_{i-1} \end{pmatrix}, \quad i=1,2,\ldots, \quad \text{where} \quad A := \begin{pmatrix} a & c \\ 1 & 0 \end{pmatrix}, \tag{3.5}$$

and looking at the eigenvalues of the companion matrix A. These are exactly the roots of the characteristic polynomial (3.4).

Again as a comparison we apply the above techniques to the stochastic recurrence (2.3). Writing (2.3) in an analogous form to (3.3) as

$$X_{i+1} = (a+b\,\xi_i)X_i \ +\ (c+d\,\xi_{i-1})X_{i-1}, i=1,2,\ldots,$$

it is obvious that the coefficients of this difference equation, or equivalently of the companion matrix when choosing a reformulation of (2.3) analogously to (3.5), depend on the random values ξ_i, ξ_{i-1} and vary from step to step. One then faces the difficulty that a stability investigation necessarily depends on products of random step-dependent companion matrices

$$\prod_{i=1}^{n} \begin{pmatrix} a_i & c_i \\ 1 & 0 \end{pmatrix}, \quad n=1,2,\ldots.$$

These products have to be bounded for all n to ensure stability. Of course, coefficients or companion matrices which vary from step to step also appear in stability investigations of deterministic variable step-size variable coefficient multi-step schemes. The stability analysis of these schemes is sophisticated. It makes use of the fact that the coefficients of the difference equation depend continuously on the step-sizes and the ratios of the stepsizes, which are assumed to be bounded. Due to the random terms in the coefficients of the stochastic difference equation, one can not depend on similar assumptions in the stochastic case.

4 Lyapunov functionals for stochastic difference equations

In this section we present an approach for performing a linear stability analysis of stochastic multi-step methods. With the aid of a Lyapunov-type theorem we obtain sufficient conditions on the parameters in (2.3) to ensure asymptotic mean-square stability of the zero solution of the recurrence equation. Related methods have been used in the case of stochastic delay differential equations in [2]. In [15] a general method is described to construct appropriate Lyapunov functionals and in fact, it is possible to construct different Lyapunov functionals giving different sets of sufficient conditions. We will present several constructions of appropriate Lyapunov functionals and the resulting conditions.

We will subsequently discuss stability of the zero solution of (2.3) and thus of (1.4). Slightly abusing notation, the initial values X_0, X_1 of (1.4) will take the role of the perturbations D_0, D_1 in Definition 2.2. For ease of reading we also omit the superscript on the perturbed solution of the recurrence equation. The following theorem is taken from [15, Theorem 1], where it was stated and proved for the general case of asymptotic stability in the p-th mean, $p > 0$. As the proof is short, we repeat it in our notation for the convenience of the reader.

THEOREM 4.1. *Suppose $X_i \equiv X_i(X_0, X_1)$ is a solution of (2.3) (and thus of (1.4)). Assume that there exist a positive real-valued functional $V(i, X_{i-1}, X_i)$ and positive constants c_1 and c_2, such that*

$$\mathbb{E}\, V(1, X_0, X_1) \leq c_1 \max(\mathbb{E}\,|X_0|^2, \mathbb{E}\,|X_1|^2)\,, \tag{4.1}$$

$$\mathbb{E}\,[V(i+1, X_i, X_{i+1}) - V(i, X_{i-1}, X_i)] \leq -c_2\, \mathbb{E}\,|X_i|^2, \tag{4.2}$$

for all $i \in \mathbb{N}, i \geq 1$. Then the zero solution of (2.3) (and thus of (1.4)) is asymptotically mean-square stable, that is

$$\lim_{i\to\infty} \mathbb{E}\,|X_i|^2 = 0. \tag{4.3}$$

PROOF. From condition (4.2) we obtain

$$\begin{aligned} \mathbb{E}\, V(i+1, X_i, X_{i+1}) \;-\; \mathbb{E}\, V(1, X_0, X_1) \;&=\\ \sum_{j=1}^{i} \mathbb{E}\,[V(j+1, X_j, X_{j+1}) - V(j, X_{j-1}, X_j)] \;&\leq\; -c_2 \sum_{j=1}^{i} \mathbb{E}\,|X_j|^2\,. \end{aligned}$$

Thus

$$\sum_{j=1}^{i} \mathbb{E}\,|X_j|^2 \leq \frac{1}{c_2}\left(\mathbb{E}\, V(1, X_0, X_1) - \mathbb{E}\, V(i+1, X_i, X_{i+1})\right).$$

An application of (4.1) yields

$$\sum_{j=1}^{i} \mathbb{E}\,|X_j|^2 \leq \frac{1}{c_2}\mathbb{E}\,V(1, X_0, X_1) \leq \frac{c_1}{c_2}\max(\mathbb{E}\,|X_0|^2, \mathbb{E}\,|X_1|^2),$$

and therefore

$$\mathbb{E}\,|X_i|^2 \leq \frac{c_1}{c_2}\max(\mathbb{E}\,|X_0|^2, \mathbb{E}\,|X_1|^2). \tag{4.4}$$

Now, for every $\delta_1 > 0$ there exists $\delta = \delta_1 \cdot c_2/c_1$, such that $\mathbb{E}\,|X_i|^2 \leq \delta_1$ if we have $\max(\mathbb{E}\,|X_0|^2, \mathbb{E}\,|X_1|^2) < \delta$. In addition, from (4.4) it follows that

$$\sum_{j=1}^{\infty} \mathbb{E}\,|X_j|^2 \leq \infty\,.$$

Hence $\lim_{j\to\infty} \mathbb{E}\,|X_j|^2 = 0$. Thus, the zero solution of (2.3) (and thus of (1.4)) is asymptotically mean-square stable and the theorem is proved. ☐

Remark 4.1. *Theorem 4.1 can also be formulated for a Lyapunov functional V which additionally depends on random variables $V(i, X_{i-1}, X_i, \xi_{i-1}, \xi_i)$. The proof is identical.*

Lyapunov-type theorems like Theorem 4.1 are strong results. The remaining problem is to find an appropriate Lyapunov function or functional to apply on specific problems and obtain conditions on problem parameters which can be easily checked. In [15] the authors develop a general method to construct Lyapunov functionals for stochastic difference equations and illustrate their method with examples. This method consists of constructing a Lyapunov-functional V as the sum of functionals $\widetilde{V} + \widehat{V}$, where one starts with a first "good guess" $\widetilde{V}$ and finds the second functional $\widehat{V}$ as a correction term. Sometimes several iterations of this process may be necessary. The main point is to obtain the first "good guess" $\widetilde{V}$, for which the authors in [15] have also provided a formal procedure by using an auxiliary simplified difference equation and a Lyapunov functional v for that equation. We follow this procedure in our subsequent analysis. There are several possibilities for the choice of an auxiliary difference equation. One can distinguish between the auxiliary equation being a deterministic or stochastic one-step method (see Subsection 4.1, where we discuss three variants of a first functional $\widetilde{V}$), or a deterministic two-step scheme (see Subsection 4.2).

4.1 The auxiliary difference equation as a one-step method

4.1.1 First guess as $\widetilde{V}(i, X_{i-1}, X_i) := |X_i|^2$

We start with the first guess

$$\widetilde{V}(i, X_{i-1}, X_i) := |X_i|^2, \quad i = 1, 2, \ldots. \tag{4.5}$$

which is a Lyapunov functional for the simplified recursion

$$\widetilde{X}_{i+1} := a\widetilde{X}_i, \quad \widetilde{X}_0 = X_0\,, \tag{4.6}$$

as well as for

$$\widetilde{X}_{i+1} := (a + b\xi_i)\widetilde{X}_i, \quad \widetilde{X}_0 = X_0 . \tag{4.7}$$

This first guess $\widetilde{V}$ satisfies the conditions of Theorem 4.1 for (4.6) or (4.7) if $|a| < 1$ or $|a|^2 + |b|^2 < 1$, respectively. Now, we apply the functional $\widetilde{V}$ to the original recursion (2.3) and check condition (4.2). We compute for $i = 1, 2, \ldots$

$$\begin{aligned}
\mathbb{E}\Delta\widetilde{V}_i &:= \mathbb{E}\big(\widetilde{V}(i+1, X_i, X_{i+1}) - \widetilde{V}(i, X_{i-1}, X_i)\big) \\
&= \mathbb{E}(|X_{i+1}|^2 - |X_i|^2) \\
&= \mathbb{E}\big(|aX_i + cX_{i-1} + bX_i\xi_i + dX_{i-1}\xi_{i-1}|^2 - |X_i|^2\big) \\
&= \mathbb{E}|aX_i + cX_{i-1} + bX_i\xi_i + dX_{i-1}\xi_{i-1}|^2 - \mathbb{E}|X_i|^2 \\
&= \underbrace{\mathbb{E}|aX_i + cX_{i-1}|^2}_{=:Q_1} + \underbrace{\mathbb{E}|bX_i\xi_i + dX_{i-1}\xi_{i-1}|^2}_{=:Q_2} \\
&\qquad + \underbrace{2\Re\{\mathbb{E}[(aX_i + cX_{i-1})\overline{(bX_i\xi_i + dX_{i-1}\xi_{i-1})}]\}}_{=:Q_3} - \mathbb{E}|X_i|^2 \\
&= Q_1 + Q_2 + Q_3 - \mathbb{E}|X_i|^2 .
\end{aligned}$$

Estimating the individual terms we obtain

$$\begin{aligned}
Q_1 &= \mathbb{E}|aX_i + cX_{i-1}|^2 = \mathbb{E}\big[|a|^2|X_i|^2 + |c|^2|X_{i-1}|^2 + 2\Re\{aX_{i-1}\overline{cX_i}\}\big] \\
&\le \mathbb{E}\big[|a|^2|X_i|^2 + |c|^2|X_{i-1}|^2 + 2|aX_{i-1}\overline{cX_i}|\big] \\
&\le \mathbb{E}\big[|a|^2|X_i|^2 + |c|^2|X_{i-1}|^2 + |a||c|(|X_i|^2 + |X_{i-1}|^2)\big] \\
&= (|a| + |c|)(|a|\mathbb{E}|X_i|^2 + |c|\mathbb{E}|X_{i-1}|^2) ,
\end{aligned}$$

$$\begin{aligned}
Q_2 &= \mathbb{E}|bX_i\xi_i + dX_{i-1}\xi_{i-1}|^2 = \mathbb{E}\big[|b|^2|X_i|^2\xi_i^2 + |d|^2|X_{i-1}|^2\xi_i^2\big] \\
&= |b|^2\mathbb{E}|X_i|^2 + |d|^2\mathbb{E}|X_{i-1}|^2 ,
\end{aligned}$$

$$\begin{aligned}
Q_3 &= 2\ \Re\{\mathbb{E}aX_i\overline{dX_{i-1}\xi_{i-1}}\} \\
&\le 2\mathbb{E}|aX_i\overline{dX_{i-1}\xi_{i-1}}| \le |a\bar{d}|(\mathbb{E}|X_i|^2 + \mathbb{E}[|X_{i-1}|^2\xi_{i-1}^2]) \\
&= |a||d|(\mathbb{E}|X_i|^2 + \mathbb{E}|X_{i-1}|^2) .
\end{aligned}$$

Summarizing the terms we have

$$\begin{aligned}
&\mathbb{E}\Delta\widetilde{V}_i = Q_1 + Q_2 + Q_3 - \mathbb{E}|X_i|^2 \\
&\quad \le (|a|+|c|)(|a|\mathbb{E}|X_i|^2 + |c|\mathbb{E}|X_{i-1}|^2) + |b|^2\mathbb{E}|X_i|^2 + |d|^2\mathbb{E}|X_{i-1}|^2 \\
&\qquad\qquad + |a||d|(\mathbb{E}|X_i|^2 + \mathbb{E}|X_{i-1}|^2) - \mathbb{E}|X_i|^2 \\
&\quad = \big((|a|+|c|)|a| + |b|^2 + |a||d| - 1\big)\mathbb{E}|X_i|^2 + \big((|a|+|c|)|c| + |d|^2 + |a||d|\big)\mathbb{E}|X_{i-1}|^2 \\
&\quad =: \mathcal{K}\cdot\mathbb{E}|X_i|^2 + \mathcal{M}\cdot\mathbb{E}|X_{i-1}|^2 .
\end{aligned}$$

In the next step we add a correction $\widehat{V}$ to $\widetilde{V}$ to deal with the term $\mathcal{M}\cdot\mathbb{E}|X_{i-1}|^2$ on the right-hand side of the above inequality. This is done by setting

$$\widehat{V}(i, X_{i-1}, X_i) := \mathcal{M}\cdot|X_{i-1}|^2 , \tag{4.8}$$

since then we have

$$\mathbb{E}\Delta\widehat{V}_i := \mathbb{E}\widehat{V}(i+1, X_i, X_{i+1}) - \mathbb{E}\widehat{V}(i, X_{i-1}, X_i) \;=\; \mathcal{M}\cdot\mathbb{E}|X_i|^2 - \mathcal{M}\cdot\mathbb{E}|X_{i-1}|^2\,.$$

Altogether for $V := \widetilde{V} + \widehat{V}$ one obtains

$$\mathbb{E}\Delta V_i = \mathbb{E}\Delta\widetilde{V}_i + \mathbb{E}\Delta\widehat{V}_i \;\le\; (\mathcal{K}+\mathcal{M})\cdot\mathbb{E}|X_i|^2\,. \tag{4.9}$$

Further, we check the initial condition (4.1) for $V = \widetilde{V} + \widehat{V}$. This condition is always satisfied due to

$$\begin{aligned}\mathbb{E}\,V(1, X_0, X_1) &= \mathbb{E}\,|X_1|^2 + \mathcal{M}\,\mathbb{E}\,|X_0|^2\\ &\le (1+\mathcal{M})\cdot\max(\mathbb{E}\,|X_0|^2\,, \mathbb{E}\,|X_1|^2)\,.\end{aligned}$$

Hence, V is a discrete Lyapunov functional for (2.3), satisfying conditions (4.1,4.2) if $\mathcal{K}+\mathcal{M} < 0$, i.e.

$$(|a|+|c|)^2 + |b|^2 + |d|^2 + 2|a|\,|d| < 1. \tag{4.10}$$ **(Cond.1)**

This condition is sufficient, but in general not necessary to guarantee the asymptotic mean-square stability of (2.3).

4.1.2 First guess as $\widetilde{V}(i, X_{i-1}, X_i) := |X_i + c\,X_{i-1}|^2$

We write

$$\begin{aligned}X_{i+1} &= a\,X_i \;+\; c\,X_{i-1} \;+\; b\,X_i\,\xi_i \;+\; d\,X_{i-1}\,\xi_{i-1}\\ &= (a\;+\;c)X_i - c\,X_i + \;c\,X_{i-1} \;+\; b\,X_i\,\xi_i \;+\; d\,X_{i-1}\,\xi_{i-1},\end{aligned} \tag{4.11}$$

and distinguish

$$\widetilde{X}_{i+1} := (a\;+\;c)\,\widetilde{X}_i, \quad \widetilde{X}_0 = X_0, \tag{4.12}$$

as the auxiliary difference equation, with the Lyapunov function $v(y) = y^2$ (if $a\;+\;c < 1$). According to [15] the first functional $\widetilde{V}$ must be chosen in the form $\widetilde{V}(i, X_{i-1}, X_i) = |X_i + c\,X_{i-1}|^2$. We apply this functional $\widetilde{V}$ to the original recursion (2.3) and check condition (4.2). Using the representation (4.11) we compute, for $i = 1, 2, \ldots$

$$\begin{aligned}\mathbb{E}\Delta\widetilde{V} &= \mathbb{E}\,[\widetilde{V}(i+1, X_i, X_{i+1}) - \widetilde{V}(i, X_{i-1}, X_i)]\\ &= \mathbb{E}\,[|X_{i+1} + cX_i|^2 - |X_i + cX_{i-1}|^2]\\ &= \mathbb{E}\,[|(a+c)X_i \;+\; cX_{i-1} \;+\; bX_i\xi_i \;+\; dX_{i-1}\xi_{i-1}|^2 - |X_i + cX_{i-1}|^2]\\ &= \underbrace{\mathbb{E}\,[|(a+c)X_i + cX_{i-1}|^2 - |X_i + cX_{i-1}|^2]}_{=:Q_4} + \underbrace{\mathbb{E}|\;bX_i\xi_i + dX_{i-1}\xi_{i-1}|^2}_{=:Q_2}\\ &\quad + \underbrace{2\Re\,\{\mathbb{E}[(a+c)X_i + cX_{i-1}]\overline{[\;bX_i\xi_i + dX_{i-1}\xi_{i-1}]}\}}_{=:Q_5}\,.\end{aligned}$$

The term Q_2 is estimated exactly as before. Similarly, we estimate

$$\begin{aligned} Q_4 &= (|a+c|^2-1)\mathbb{E}|X_i|^2 + 2\Re\mathbb{E}[(a+c-1)X_i\overline{cX_{i-1}}] \\ &\le (|a+c|^2-1)\mathbb{E}|X_i|^2 + |a+c-1||c|(\mathbb{E}|X_i|^2+\mathbb{E}|X_{i-1}|^2), \end{aligned}$$

$$Q_5 = 2|\Re\{\mathbb{E}(a+c)X_i\overline{d\xi_{i-1}X_{i-1}}\}| \le |a+c||d|(\mathbb{E}|X_i|^2+\mathbb{E}|X_{i-1}|^2).$$

Summarizing we arrive at

$$\begin{aligned} \mathbb{E}\Delta\widetilde{V} &= Q_4+Q_2+Q_5 \\ &\le (|a+c|^2-1)\mathbb{E}|X_i|^2 + |a+c-1||c|(\mathbb{E}|X_i|^2+\mathbb{E}|X_{i-1}|^2) \\ &\qquad + |b|^2\mathbb{E}|X_i|^2 + |d|^2\mathbb{E}|X_{i-1}|^2 \quad + |a+c||d|(\mathbb{E}|X_i|^2+\mathbb{E}|X_{i-1}|^2) \\ &\le \underbrace{(|a+c|^2-1+|a+c-1||c|+|b|^2+|a+c||d|)}_{=:\mathcal{K}}\ \mathbb{E}\,|X_i|^2 \\ &\quad + \underbrace{(|a+c||d|+|d|^2+|a+c-1||c|)}_{=:\mathcal{M}}\mathbb{E}\,|X_{i-1}|^2. \end{aligned}$$

The correction $\widehat{V}$ has to be taken again in the form (4.8) and then the same estimate (4.9) holds for $V=\widetilde{V}+\widehat{V}$, where now

$$\mathcal{K}+\mathcal{M} = |a+c|^2-1+2|a+c-1||c|+|b|^2+|d|^2+2|a+c||d|.$$

Further, the initial condition (4.1) is satisfied for $V=\widetilde{V}+\widehat{V}$, since

$$\begin{aligned} \mathbb{E}\,V(1,X_0,X_1) &= \mathbb{E}\,[|X_1+c\,X_0|^2+\mathcal{M}|X_0|^2] \\ &\le (2\max(1,|c|^2)+\mathcal{M})\cdot\max(\mathbb{E}\,|X_1|^2,\mathbb{E}\,|X_0|^2)\,. \end{aligned}$$

Thus $V=\widetilde{V}+\widehat{V}$ is a discrete Lyapunov functional for (2.3), satisfying conditions (4.1,4.2) if $\mathcal{K}+\mathcal{M}<0$, i.e., if

$$|a+c|^2+|b|^2+|d|^2+2\,|a+c||d|+2\,|a+c-1||c|<1. \quad \textbf{(Cond.2)} \tag{4.13}$$

This condition is sufficient but not necessary to guarantee that the zero-solution of (2.3) is asymptotically mean-square stable.

4.1.3 First guess as $\widetilde{V}(i,X_{i-1},X_i,\xi_{i-1},\xi_i)=|X_i+cX_{i-1}+d\,\xi_{i-1}X_{i-1}|^2$

Now we write

$$\begin{aligned} X_{i+1} &= a\,X_i\ +\ c\,X_{i-1}\ +\ b\,X_i\,\xi_i\ +\ d\,X_{i-1}\,\xi_{i-1} \\ &= (a+c)X_i-cX_i+cX_{i-1}+(b+d)X_i\xi_i-dX_i\xi_i+dX_{i-1}\xi_{i-1}, \end{aligned} \tag{4.14}$$

and distinguish

$$\widetilde{X}_{i+1} := (a+c+(b+d)\,\xi_i)\,\widetilde{X}_i,\quad \widetilde{X}_0=X_0\,, \tag{4.15}$$

as the auxiliary difference equation, with the Lyapunov function $v(y) = y^2$ (if $(a+c)^2 + (b+d)^2 < 1$). According to [15] the first functional $\widetilde{V}$ must be chosen in the form $\widetilde{V}(i, X_{i-1}, X_i, \xi_{i-1}, \xi_i) = |X_i + c\, X_{i-1} + d\, \xi_{i-1}\, X_{i-1}|^2$. In this case the chosen Lyapunov functional $\widetilde{V}$ also depends on a random value (cf. Remark 4.1). We apply this functional $\widetilde{V}$ to the original recursion (2.3) and check condition (4.2). Using the representation (4.14) we compute, for $i = 1, 2, \dots$

$$
\begin{aligned}
\mathbb{E}\Delta\widetilde{V} &= \mathbb{E}\,[\widetilde{V}(i+1, X_i, X_{i+1}, \xi_i, \xi_{i+1}) - \widetilde{V}(i, X_{i-1}, X_i, \xi_{i-1}, \xi_i)] \\
&= \mathbb{E}\,[|X_{i+1} + (c + d\,\xi_i)X_i|^2 - |X_i + (c + d\,\xi_{-1})X_{i-1}|^2] \\
&= \mathbb{E}\big[|(a+c)X_i + cX_{i-1} + (b+d)X_i\xi_i + dX_{i-1}\xi_{i-1}|^2 - |X_i + (c+d\xi_{i-1})X_{i-1}|^2\big] \\
&= \underbrace{\mathbb{E}\,[|(a+c)X_i + cX_{i-1}|^2 - |X_i + cX_{i-1}|^2]}_{=:Q_4} \\
&\quad + \underbrace{\mathbb{E}|\ (b+d)X_i\xi_i + dX_{i-1}\xi_{i-1}|^2 - |dX_{i-1}\xi_{i-1}|^2}_{=:Q_6} \\
&\quad + \underbrace{2\Re\mathbb{E}[((a+c)X_i + cX_{i-1})\overline{(\ (b+d)X_i\xi_i + dX_{i-1}\xi_{i-1})}]}_{=:Q_7} \\
&\quad - \underbrace{2\Re\mathbb{E}[(X_i + cX_{i-1})\overline{(\ dX_{i-1}\xi_{i-1})}]}_{=:Q_8}.
\end{aligned}
$$

The term Q_4 is estimated exactly as before. Similarly, we estimate

$$
\begin{aligned}
Q_6 &= |b+d|^2\mathbb{E}|X_i|^2 + |d|^2\mathbb{E}|X_i|^2 - |d|^2\mathbb{E}|X_i|^2 = |b+d|^2\mathbb{E}|X_i|^2 \\
Q_7 - Q_8 &= 2\Re\{\mathbb{E}[(a+c)X_i\overline{d\xi_{i-1X_{i-1}}}]\} - 2\Re\{\mathbb{E}[X_i\overline{d\xi_{i-1}X_{i-1}}]\} \\
&= 2\Re\{\mathbb{E}[(a+c-1)X_i\overline{d\xi_{i-1X_{i-1}}}]\} \le |a+c-1||d|(\mathbb{E}|X_i|^2 + \mathbb{E}|X_{i-1}|^2).
\end{aligned}
$$

Together this yields

$$
\begin{aligned}
\mathbb{E}\Delta\widetilde{V} &= Q_4 + Q_6 + Q_7 - Q_8 \\
&\le (|a+c|^2 - 1)\mathbb{E}|X_i|^2 + |a+c-1||c|(\mathbb{E}|X_i|^2 + \mathbb{E}|X_{i-1}|^2) \\
&\quad + |b+d|^2\mathbb{E}|X_i|^2 + |a+c-1||d|(\mathbb{E}|X_i|^2 + \mathbb{E}|X_{i-1}|^2) \\
&= \underbrace{\big(|a+c|^2 - 1 + |b+d|^2 + (|c|+|d|)|a+c-1|\big)}_{=:\mathcal{K}}\mathbb{E}|X_i|^2 \\
&\quad + \underbrace{(|c|+|d|)|a+c-1|}_{=:\mathcal{M}}\mathbb{E}|X_{i-1}|^2.
\end{aligned}
$$

Again we take the correction $\widehat{V}$ in the form (4.8), although with a different constant $\mathcal{M}$, such that (4.9) holds for $V = \widetilde{V} + \widehat{V}$. Finally, we check the initial condition (4.1):

$$
\begin{aligned}
\mathbb{E}\,V(1, X_0, X_1, \xi_0, \xi_1) &= \mathbb{E}\,[|X_1 + (c\ + d\,\xi_0)\,X_0|^2 + \mathcal{M}|X_0|^2] \\
&\le 2\mathbb{E}|X_1|^2 + 2\mathbb{E}|(c\ + d\,\xi_0)\,X_0|^2 + \mathcal{M}\mathbb{E}|X_0|^2 \\
&\le (2\max(1, |c|^2 + |d|^2) + \mathcal{M}) \cdot \max(\mathbb{E}\,|X_1|^2, \mathbb{E}\,|X_0|^2).
\end{aligned}
$$

We conclude that $V = \widetilde{V} + \widehat{V}$ is a discrete Lyapunov functional for (2.3), satisfying conditions (4.1,4.2) if $\mathcal{K} + \mathcal{M} < 0$, i.e., if

$$|a+c|^2 + |b+d|^2 + 2\,(|c|+|d|)|a+c-1| < 1. \qquad (\textbf{Cond.3}) \tag{4.16}$$

This condition is sufficient but not necessary to guarantee that the zero-solution of (2.3) is asymptotically mean-square stable.

4.2 The auxiliary difference equation as a deterministic two-step method

In this section we will consider the auxiliary difference equation in the form of a deterministic two-step scheme written as (3.5). To avoid the subsequent calculations becoming overly technical we assume here that the parameters λ, μ of the considered test equation (1.3) are real-valued. We start with the deterministic part of the equation (2.3)

$$y_{i+1} = ay_i + cy_{i-1}, \tag{4.17}$$

and, upon setting $\mathcal{Y}_i = (y_i, y_{i-1})^T$, rewrite equation (4.17) as a one-step recursion in $\mathbb{R}^2$:

$$\mathcal{Y}_{i+1} = A\mathcal{Y}_i, \qquad \text{where } A = \begin{pmatrix} a & c \\ 1 & 0 \end{pmatrix}. \tag{4.18}$$

The next step is to determine a Lyapunov-function v for the auxiliary problem (4.18). A function $v : \mathbb{R}^2 \to \mathbb{R}^+$ is a Lyapunov-function for (4.18) if its incremental values $\Delta v_i := v(\mathcal{Y}_{i+1}) - v(\mathcal{Y}_i)$ satisfy $\Delta v_i \le -c_0\,\|\mathcal{Y}_i\|^2$ where $\|\cdot\|$ is a norm on $\mathbb{R}^2$ and c_0 is a positive constant . The ansatz $v(\mathcal{Y}) = \mathcal{Y}^T\,Q\,\mathcal{Y}$ with a positive definite matrix Q yields positive values of v and

$$\Delta v_i = Y_{i+1}^T\,Q\,\mathcal{Y}_{i+1} - \mathcal{Y}_i^T\,Q\,\mathcal{Y}_i = \mathcal{Y}_i^T\,[A^TQA - Q]\,\mathcal{Y}_i\,,$$

such that v is a Lyapunov-function if the matrix $A^TQA - Q$ is negative definite. To find a matrix Q with these properties we start from an arbitrary positive definite matrix P and solve the Lyapunov matrix equation $A^TQA - Q = -P$. For simplicity we choose a diagonal matrix $P = \operatorname{diag}(p_{11}, p_{22})$ where the positive parameters p_{11}, p_{22} can be arbitrarily chosen. Then the elements of the matrix $Q = (q_{ij})_{i,j=1,2}$ can be calculated as (supposing that $c \ne -1,\ |a| \ne |1-c|$)

$$\begin{aligned} \begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix} &= p_{ac}\begin{pmatrix} 1-c \\ ac \end{pmatrix}, \qquad \text{where} \quad p_{ac} = \frac{p_{11}+p_{22}}{(1+c)((1-c)^2-a^2)}\,, \\ q_{22} &= p_{22} + c^2 q_{11}\,. \end{aligned}$$

If the conditions

$$|c| < 1 \quad \text{and} \quad |a| < 1-c \tag{4.19}$$

hold, the matrix Q is positive definite with $q_{11}, q_{22}, p_{ac} > 0$. Then we have a Lyapunov function v for the auxiliary problem (4.18) given as $v(\mathcal{Y}) = \mathcal{Y}^T\,Q\,\mathcal{Y}$. Following [15] the first functional $\widetilde{V}$ must be chosen in the form

$$\widetilde{V}(i, X_{i-1}, X_i) = \mathcal{X}_i^T Q \mathcal{X}_i, \quad \text{where} \quad \mathcal{X}_i = (X_i, X_{i-1})^T.$$

After calculating $\mathbb{E}\Delta\widetilde{V}_i = \mathbb{E}[\mathcal{X}_{i+1}^T Q\mathcal{X}_{i+1} - \mathcal{X}_i^T Q\mathcal{X}_i]$, we can determine the correction functional $\widehat{V}$ and thus $V = \widetilde{V} + \widehat{V}$. We obtain

$$\mathbb{E}\Delta\widetilde{V}_i = -p_{11}\mathbb{E}X_i^2 - p_{22}\mathbb{E}X_{i-1}^2 + Q_9 + Q_{10},$$

where

$$\begin{aligned} Q_9 &= q_{11}\mathbb{E}(bX_i\xi_i + dX_{i-1}\xi_{i-1})^2\,, \\ Q_{10} &= 2(q_{12} + q_{11}a)\mathbb{E}X_i(dX_{i-1}\xi_{i-1})\,. \end{aligned}$$

Estimating these terms as

$$\begin{aligned} Q_9 &= p_{ac}(1-c)\,(b^2\mathbb{E}X_i^2 + d^2\mathbb{E}X_{i-1}^2)\,, \\ Q_{10} &\le p_{ac}\,|ad|\,\big(\mathbb{E}X_i^2 + \mathbb{E}X_{i-1}^2\big)\,, \end{aligned}$$

we obtain

$$\mathbb{E}\Delta\widetilde{V}_i \le \mathcal{K}\,\mathbb{E}X_i^2 + \mathcal{M}\,\mathbb{E}X_{i-1}^2\,,$$

with

$$\mathcal{K} = (\gamma_1 - p_{11}), \qquad \gamma_1 = p_{ac}\big(b^2(1-c) + |ad|\big)\,, \tag{4.20}$$

$$\mathcal{M} = (\gamma_2 - p_{22}), \qquad \gamma_2 = p_{ac}\big(d^{\,2}(1-c) + |ad|\big)\,. \tag{4.21}$$

The correction $\widehat{V}$ can be taken again in the form (4.8) with the above value of $\mathcal{M}$, such that (4.9) holds for $V = \widetilde{V} + \widehat{V}$ if $\mathcal{K} + \mathcal{M} < 0$. Finally, we check the initial condition (4.1). We obtain with $\mathcal{X}_1 = (X_1, X_0)^T$

$$\begin{aligned} V(1, X_0, X_1) &= \mathcal{X}_1^T Q\mathcal{X}_1 + \mathcal{M}X_0^2 \\ &= q_{11}X_0^2 + 2q_{12}X_0X_1 + q_{22}X_1^2 + \mathcal{M}X_0^2 \\ &\le q_{11}X_0^2 + |q_{12}|(X_0^2 + X_1^2) + q_{22}X_1^2 + \mathcal{M}X_0^2 \\ &\le (q_{11} + 2|q_{12}| + q_{22} + \gamma_2 - p_{22})\max(X_0^2, X_1^2) \\ &= \big((1+c^2)\,q_{11} + 2|q_{12}| + \gamma_2\big)\max(X_0^2, X_1^2). \end{aligned}$$

Thus condition (4.1) from Theorem 4.1 is satisfied with the positive constant $(1+c^2)\,q_{11} + 2|q_{12}| + \gamma_2$. We conclude that $V = \widetilde{V} + \widehat{V}$ is a discrete Lyapunov functional for (2.3), satisfying conditions (4.1,4.2) if (4.19) and $\mathcal{K} + \mathcal{M} < 0$ hold. The last inequality means

$$\begin{aligned} & & \gamma_1 + \gamma_2 &< p_{11} + p_{22} \\ \Longleftrightarrow & & \frac{(p_{11}+p_{22})}{(1+c)\big((1-c)^2 - a^2\big)}\big(2|ad| + (b^2+d^2)(1-c)\big) &< p_{11} + p_{22} \\ \Longleftrightarrow & & \frac{2|ad| + (b^2+d^2)(1-c)}{(1+c)\big((1-c)^2 - a^2\big)} &< 1\,. \end{aligned}$$

Summarizing, we get the following set of conditions

$$\left.\begin{array}{l} |c| < 1 \quad \text{and} \quad |a| < 1 - c, \\ \dfrac{2|ad| + (b^2+d^2)(1-c)}{(1+c)\big((1-c)^2 - a^2\big)} < 1\,. \end{array}\right\} \qquad \textbf{(Cond.4)} \tag{4.22}$$

This condition is sufficient but not necessary to guarantee that the zero-solution of (2.3) is asymptotically mean-square stable.

5 Regions of guaranteed mean-square absolute stability

The analysis in Section 4 yields the sufficient conditions (4.10), (4.13), (4.16) and (4.22) for asymptotic mean-square stability of the zero solution of the recurrence (2.3) in terms of the parameters a, b, c, d. Using the relations (2.4) and (2.5) these conditions can be expressed in terms of the coefficients $\alpha_2, \alpha_1, \alpha_0, \beta_0, \beta_1, \beta_0$ (and γ_1, γ_0) of the two-step schemes (1.2) and the parameters $z = h\lambda$, $y = h^{1/2}\mu$, $z, y \in \mathbb{C}$, $h > 0$ representing the parameters of the test equation and the applied stepsize. To illustrate these results we discuss the explicit and implicit Adams methods, the Milne-Simpson method and the BDF method. In Table 5.1 we give the values of their coefficients, in Table 5.2 we give the resulting coefficients a, b, c, d for the recurrence (2.3) in terms of $z = h\lambda, y = h^{1/2}\mu$.

Method	α_2	α_1	α_0	β_2	β_1	β_0	γ_1	γ_0
Adams-Bashforth	1	-1	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0
Adams-Moulton	1	-1	0	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	1	0
Milne-Simpson	1	0	-1	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	1	1
BDF2	1	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	1	$-\frac{1}{3}$

Table 5.1: Table of coefficients of two-step schemes

Method	a	c	b	d
Adams-Bashforth	$1 + \frac{3}{2}z$	$-\frac{1}{2}z$	y	0
Adams-Moulton	$\frac{1 + \frac{2}{3}z}{1 - \frac{5}{12}z}$	$\frac{-\frac{1}{12}z}{1 - \frac{5}{12}z}$	$\frac{y}{1 - \frac{5}{12}z}$	0
Milne-Simpson	$\frac{\frac{4}{3}z}{1 - \frac{1}{3}z}$	$\frac{1 + \frac{1}{3}z}{1 - \frac{1}{3}z}$	$\frac{y}{1 - \frac{1}{3}z}$	$\frac{y}{1 - \frac{1}{3}z}$
BDF2	$\frac{\frac{4}{3}}{1 - \frac{2}{3}z}$	$\frac{-\frac{1}{3}}{1 - \frac{2}{3}z}$	$\frac{y}{1 - \frac{2}{3}z}$	$\frac{-\frac{1}{3}y}{1 - \frac{2}{3}z}$

Table 5.2: Table of the parameters a,b,c,d of two-step schemes

We restrict the subsequent discussion to the case of real-valued parameters $\lambda, \mu \in \mathbb{R}$, resp. $z, y \in \mathbb{R}^2$ to be able to visualize stability regions. We will draw these

regions in the (z, y^2)-plane. For given parameters $\lambda\,, \mu$ in equation (1.3), varying the step-size h in the numerical schemes corresponds to moving along a ray that passes through the origin and (λ, μ^2). We compare the regions where the exact solution is asymptotically mean-square stable to those where the sufficient conditions for asymptotic mean-square stability of the schemes are fulfilled.
From (2.2) we have that the exact solution (2.1) is asymptotically mean-square stable if $\lambda < -\frac{1}{2}\mu^2$, or upon multiplying by h and rearranging if $y^2 < -2z$. The boundary of the region of asymptotic stability of the exact solution is thus given by the line $y^2 = -2z$. Any of the inequalities in the examples below are considered as conditions for y^2 in relation to z and the borders of the resulting regions are given in Figures 1 to 4. For the parameters $(z,\, y^2) = h(\lambda,\, \mu^2)$ below these lines the zero solutions of the test equation and the numerical schemes, respectively, are asymptotically mean-square stable. For the test equation, condition (2.2) is sufficient *and necessary*, thus above the corresponding line the zero solution of the test equation is unstable. For the numerical schemes we have only sufficient conditions and therefore we can not make statements for the regions above the corresponding lines.

Example 5.1. *Sufficient conditions for the two-step Adams-Bashforth-Maruyama scheme:*
Inserting the expressions for the Adams-Bashforth-Maruyama scheme

$$a = 1 + \frac{3}{2}z, \quad c = -\frac{1}{2}z, \quad b = y, \quad d = 0$$

from Table 5.2 into the sufficient condition (4.10) yields

$$\begin{aligned}
\textbf{Cond.1} \iff & (|a| + |c|)^2 + |b|^2 + |d|^2 + 2|a|\,|d| &<& \;1\\
\iff & (|1 + \tfrac{3}{2}z| + |\tfrac{1}{2}z|)^2 + y^2 &<& \;1\\
\iff & \boxed{y^2 < 1 - (|1 + \tfrac{3}{2}z| + |\tfrac{1}{2}z|)^2}.
\end{aligned}$$

Because of $d = 0$ for the explicit Adams scheme the sufficient condition (4.16) coincides with (4.13). We compute

$$\begin{aligned}
\textbf{Cond.2,3} \iff & |a + c|^2 + |b|^2 + |d|^2 + 2\,|a + c||d| + 2\,|a + c - 1||c| &<& \;1\\
\iff & (1 + z)^2 + y^2 + 2|1 + z - 1|\,| - \tfrac{1}{2}z| &<& \;1\\
\iff & \boxed{y^2 < 1 - (1 + z)^2 - z^2}.
\end{aligned}$$

Finally we compute for the sufficient condition (4.22)

$$\begin{aligned}
\textbf{Cond.4} \iff & |c| < 1\,,\ |a| < 1 - c\,, \qquad \frac{2|ad| + (b^2 + d^2)(1 - c)}{(1 + c)\big((1 - c)^2 - a^2\big)} < 1\\
\iff & z \in (-1, 0)\,, \qquad y^2(1 + \tfrac{z}{2}) < (1 - \tfrac{z}{2})\big((1 + \tfrac{z}{2})^2 - (1 + \tfrac{3}{2}z)^2\big)\\
\iff & z \in (-1, 0), \qquad \boxed{y^2 < \frac{z^3 - z^2 - 2z}{1 + \frac{z}{2}}}.
\end{aligned}$$

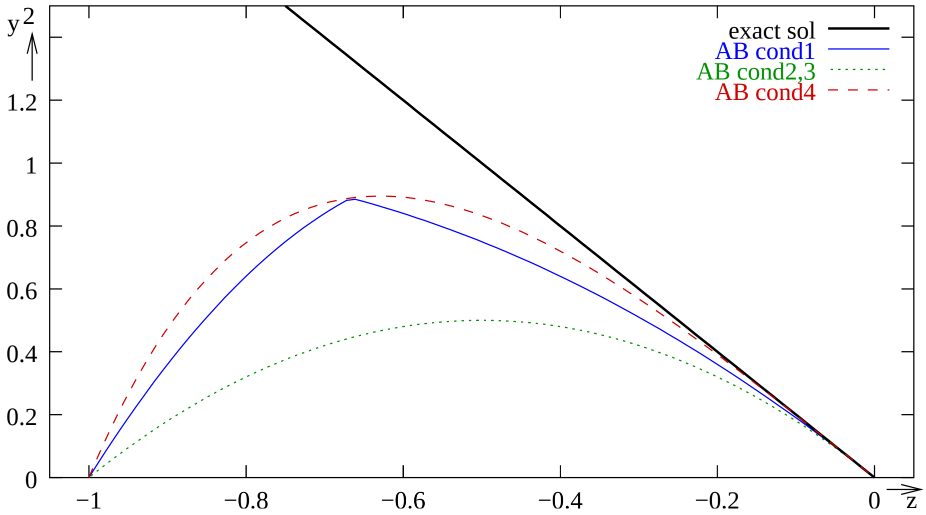

Figure 1: Borders of the range of guaranteed mean-square asymptotic stability for the two-step Adams-Bashforth-Maruyama scheme

We proceed with the results for the other schemes given in Tables 5.1 and 5.2, the calculations are similar.

EXAMPLE 5.2. *Sufficient conditions for the two-step Adams-Moulton-Maruyama scheme:*

One obtains the following inequalities, Figure 2 illustrates the borders of the corresponding regions.

$$\textbf{Cond.1} \iff y^2 < |1 - \tfrac{5}{12}z|^2 - \left(|1 + \tfrac{2}{3}z| + |\tfrac{1}{12}z|\right)^2 ,$$

$$\textbf{Cond.2,3} \iff y^2 < |1 - \tfrac{5}{12}z|^2 - |1 + \tfrac{7}{12}z|^2 - \tfrac{1}{6}z^2 ,$$

$$\textbf{Cond.4} \iff y^2 < \frac{1 - \frac{1}{2}z}{1 - \frac{1}{3}z}(-2z - \tfrac{1}{3}z^2) \quad \text{and } z \in (-6, 0) .$$

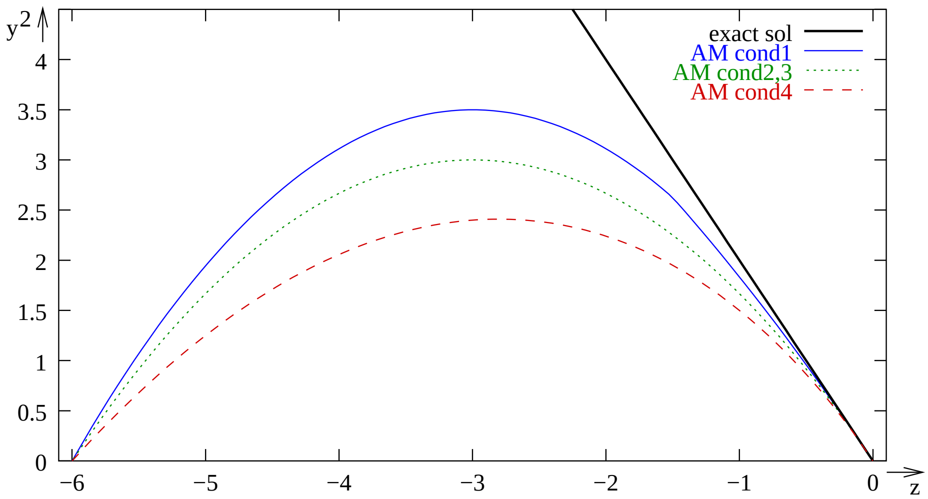

Figure 2: Borders of the range of guaranteed mean-square asymptotic stability for the two-step Adams-Moulton-Maruyama scheme

Example 5.3. *Sufficient conditions for the two-step BDF-Maruyama scheme:* Figure 3 illustrates the regions obtained for the following inequalities.

$$\textbf{Cond.1} \iff y^2 + \tfrac{4}{5}|y| < \tfrac{2}{5}z^2 - \tfrac{6}{5}z - \tfrac{8}{5}\,,$$

$$\textbf{Cond.2} \iff y^2 + \tfrac{3}{5}|y| < \tfrac{2}{5}z^2 - \tfrac{6}{5}z - \tfrac{2}{5}|z|\,,$$

$$\textbf{Cond.3} \iff y^2 + |z|\,|y| < z^2 - 3z - |z|\,,$$

$$\textbf{Cond.4} \iff y^2 + \tfrac{4}{5}\frac{\frac{3}{2}-z}{2-z}|y| < \frac{1-z}{2-z}\left(-\tfrac{8}{5}z + \tfrac{2}{5}z^2\right) \quad \text{and } z \notin [0,4]\,.$$

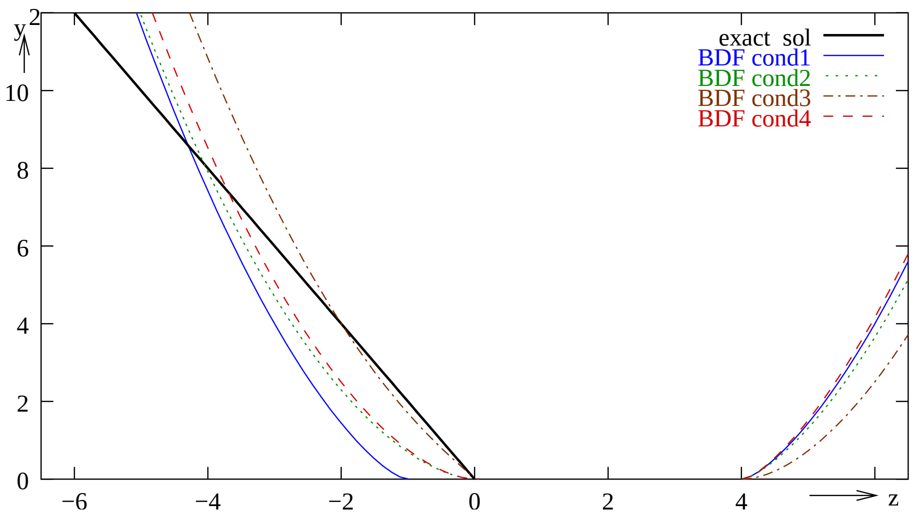

Figure 3: Borders of the range of gauranteed mean-square asymptotic stability for the two-step BDF-Maruyama scheme

Example 5.4. *Sufficient conditions for the Milne-Simpson-Maruyama scheme:* As already in the deterministic case, the range of parameters for which the conditions are satisfied is empty. At the border of the parameter region, i.e. when one takes the inequalities as equations, these are fulfilled for $y = z = 0$.

6 Conclusions and open problems

We have investigated the problem of when a numerical approximation given by a stochastic linear two-step-Maruyama method shares asymptotic properties in the mean-square sense of the exact solution of an SDE. A linear stability analysis has been performed for a linear time-invariant test equation, using Lyapunov-type functionals. We have obtained sufficient conditions for asymptotic mean-square stability of stochastic linear two-step-Maruyama methods, in particular of stochastic counterparts of two-step Adams-Bashforth- and Adams-Moulton-methods, the Milne-Simpson method and the BDF method.
In Figure 4 we give a comparison of the regions where the different schemes are guaranteed to be mean-square asymptotically stable. We also give the stability regions for the stochastic θ-Maruyama methods, based on the results in [12]. In particular, we consider $\theta = 0, \frac{1}{2}, 1$, i. e. the explicit Euler method, the trapezoidal rule and the implicit Euler method, respectively.

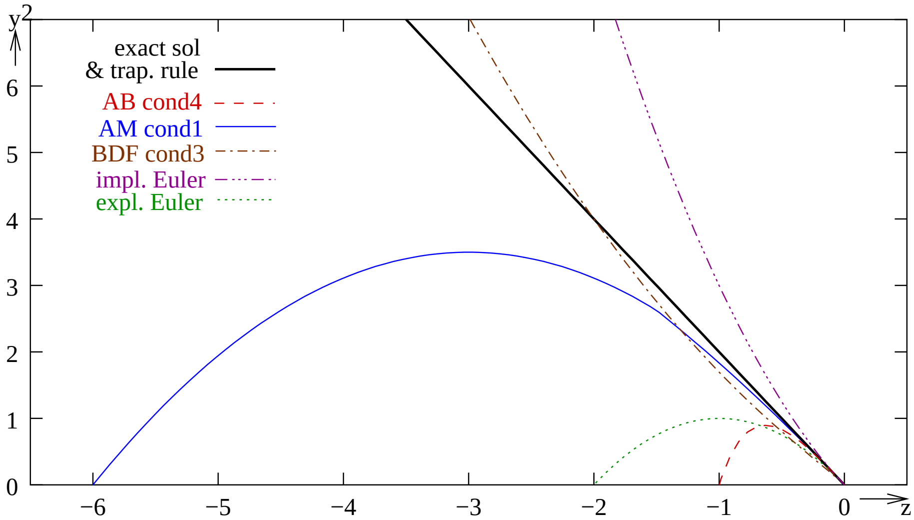

Figure 4: Comparison of the borders of the range of guaranteed mean-square asymptotic stability for several explicit and implicit one- and two-step schemes

It can be seen that the only one of the considered two-step schemes that is asymptotically stable for arbitrarily large negative values of $z = \lambda \cdot h$ is the BDF scheme. However, as in the deterministic setting, this scheme shows a strong damping behaviour in regions where the exact solution is not mean-square asymptotically stable. The regions of mean-square asymptotic stability for the Adams schemes are bounded, such that the test equation with a large negative value of the parameter λ is simulated qualitatively correctly only if the applied step-size h is sufficiently small. The Milne-Simpson scheme can not be recommended. We exemplify this qualitative behaviour with numerical simulations for the test equation (1.3) with parameters $\lambda = -100$, $\mu = 0.01$ in Figure 5. We have plotted the accuracy achieved for the different schemes versus the step-sizes in logarithmic scale. The accuracy is measured as the maximum over $N(h) = \frac{1}{h}$ discrete time-points in the time-interval $[0,1]$ of the mean-square of the difference between exact solution $X(t_\ell)$ and numerical solution X_ℓ for 100 computed paths:

$$\text{err} = \max_{\ell=1,\dots,N(h)} \left(\frac{1}{100} \sum_{j=1}^{100} |X(t_\ell,\omega_j) - X_\ell(\omega_j)|^2 \right)^{1/2}.$$

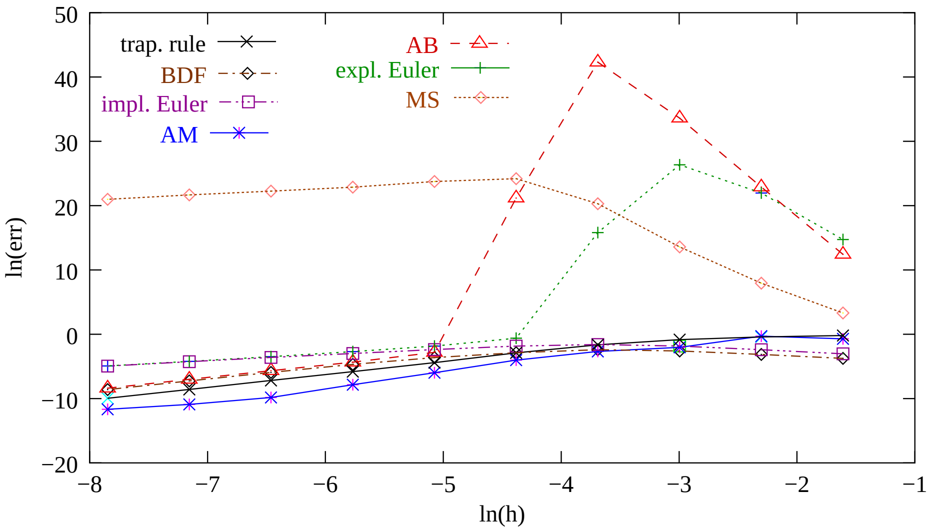

Figure 5: numerical results

Summarizing our findings we observe:
The approach using Lyapunov functionals and the method of construction of Lyapunov functionals has proved to be useful for performing a linear stability analysis of stochastic linear two-step Maruyama methods.
Unfortunately none of the four obtained sufficient conditions (4.10), (4.13), (4.16) and (4.22) has turned out to be "the best" condition for all considered numerical schemes or can be ruled out completely.
In the case of SDEs with small noise (for $\lambda \neq 0$), essentially the experience gained in the area of linear stability analysis of the corresponding deterministic multi-step schemes carries over to the stochastic case.
Further, note that the parameter regions where the solutions of the test equation are asymptotically stable in the p-th mean (for arbitrary $p > 1$) depend on p. The same would be true for the numerical approximations. We conjecture that considering asymptotic stability of the numerical schemes in the p-th mean in comparison to that of the analytic problem would not give a qualitatively different picture, although the conditions are probably harder to deal with com-

putationally.

It would be desirable to obtain *necessary conditions* for mean-square asymptotic stability of the zero solution of the stochastic recurrence (2.3) and thus of the numerical methods (1.2), even when these are not identical to the sufficient conditions.
We are aware that in the area of asymptotic stability analysis of stochastic numerical methods there remain many open problems. We mention here in particular the stability analysis of systems of equations with a single driving Wiener process without commutativity of the coefficient functions, systems of equations with several driving processes, nonlinear systems, and more complex methods, such as Milstein-type schemes.

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