Improved linear multi-step methods for stochastic ordinary differential equations

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Abstract

We consider linear multi-step methods for stochastic ordinary differential equations and study their convergence properties for problems with small noise or additive noise. We present schemes where the drift part is approximated by well-known methods for deterministic ordinary differential equations. Previously, we considered Maruyama-type schemes, where only the increments of the driving Wiener process are used to discretize the diffusion part. Here, we suggest to improve the discretization of the diffusion part by taking into account also mixed classical-stochastic integrals. We show that the relation of the applied step-sizes to the smallness of the noise is essential to decide whether the new methods are worth to be used. Simulation results illustrate the theoretical findings.

1 Introduction

We consider stochastic ordinary differential equations (SODEs) in Itô form

$$X(s)\Big|_0^t = \int_0^t f(s, X(s))\,\mathrm{d}s + \int_0^t G(s, X(s))\,\mathrm{d}W(s)\,,\ X(0) = X_0\,,\ t \in [0, T]\,, \quad (1)$$

where W denotes an m-dimensional Wiener process given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$. The drift and diffusion functions are given as $f : [0,T] \times I\!R^n \to I\!R^n$ and $G = (g_1, \ldots, g_m) : [0,T] \times I\!R^n \to I\!R^{n\times m}$, respectively. The initial value X_0 is assumed to be $\mathcal{F}_0$-measurable, independent of the Wiener process and to possess finite second moments. We assume that there exists a path-wise unique strong solution $X(\cdot)$ of (1).
Often fluctuations, which affect a physical system, are quite small, e.g., if thermal noise has to be included in a physical model. Applications of SODEs with small noise can be found, e.g., in [6, 12]. Following [11] we express the smallness of the noise by introducing a small factor $\epsilon \ll 1$ into the diffusion coefficient such that $G = \epsilon\hat{G}$. Note that the small parameter ϵ is not needed to formulate the suggested numerical schemes. It is used here only to discuss the errors of the numerical schemes which essentially depend on the smallness of the noise.

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Stochastic two-step methods already appear in [10] (for additive noise) and in [7] (see also the references there). In [1] two-step methods for Itô SODEs are analysed. Stochastic versions of Adams methods for order up to five have been implemented and tested for SODEs with additive noise in [6]. Consistency of SLMMs for Stratonovich SODEs has been considered in [2], in addition stochastic Adams methods have been implemented as predictor-corrector schemes and tested. A general convergence theory for stochastic linear multi-step schemes was developed in [3].

In general numerical schemes for SODEs that include only information on the increments of the Wiener process have an asymptotic rate of strong convergence of 1/2 (for additive noise it may be 1). However, when the noise is small, the error behaviour is much better. In fact, the errors are still dominated by the deterministic terms as long as the step-size is large enough. In [3] we analyzed this in detail for schemes we have called stochastic linear multi-step Maruyama methods. For simplicity, in this paper we consider all schemes on equidistant grids with gridpoints $t_\ell = \ell\, h,\ \ell = 0, \ldots, N$ and stepsize $h = T/N$. We denote by X_ℓ the numerical approximation of the exact solution value $X(t_\ell)$ at the time-point t_ℓ and by $I_r^{t,t+h} = \int_t^{t+h} \mathrm{d}W_r(s) = W_r(t+h) - W_r(t)$ the increments of the scalar Wiener Process W_r on subintervals $[t, t+h]$. The abbreviation ϕ_{l-j} is used to denote $\phi(t_{\ell-j}, X_{\ell-j})$ for functions ϕ defined on $[0,T] \times I\!R^n$. Then stochastic linear multi-step Maruyama methods are given by

$$\sum_{j=0}^{k} \alpha_j X_{\ell-j} = h \sum_{j=0}^{k} \beta_j f_{\ell-j} + \sum_{j=1}^{k} \sum_{r=1}^{m} \gamma_j\, g_{r,\ell-j}\ I_r^{t_{\ell-j}, t_{\ell-j+1}}, \quad \ell = k, \ldots, N \quad (2)$$

with parameters α_j, β_j, $j = 0, \ldots, k$ and γ_j, $j = 1, \ldots, k$. For $\beta_0 = 0$, the scheme is explicit, otherwise it is drift-implicit. Note that the discretization of the diffusion part is always explicit. To start the recursion k initial values $X_0, \ldots, X_{k-1}$ are needed. Using the parameters α_j, β_j from deterministic linear multi-step schemes and choosing the parameters γ_j according to $\sum_{j=0}^{k} \alpha_j X_{\ell-j} = \gamma_1(X_\ell - X_{\ell-1}) + \ldots + \gamma_k(X_{\ell-k+1} - X_{\ell-k})$ the global error of the scheme (2) applied to small noise SODEs can be estimated as $\mathcal{O}(\epsilon^2 h^{1/2} + \epsilon h + h^p)$, where p is the order of the deterministic scheme and the coefficient functions f, g_r are assumed to be sufficiently smooth. For $p \geq 2$ one can distinguish three regions of the ϵ-h relations where qualitatively different terms are dominating the global error. For $h \ll \epsilon^2$ the term $\mathcal{O}(\epsilon^2 h^{1/2})$ dominates and one observes the asymptotic order of convergence 1/2. For $\epsilon^{1/(p-1)} \ll h$ the term $\mathcal{O}(h^p)$ dominates and reflects the deterministic order of convergence. For stepsizes between these two extreme cases the term $\mathcal{O}(\epsilon h)$ is dominating the global error. One observes order 1 behaviour with a small error constant that is due to the factor ϵ, such that the errors are still considerably smaller than those for the Euler-Maruyama scheme. The described behaviour has been illustrated by computational results, e.g., in [3, 4].

In this paper we present a careful analysis of the local and global errors of the multi-step Maruyama schemes (2) to gain insight into the terms responsible for the global error term $\mathcal{O}(\epsilon h)$. The necessary theory of mean-square convergence is recaptured in Section 2, while the analysis of the local errors

is done in Section 3. We then aim at improving the methods such that the $\mathcal{O}(\epsilon h)$ term is cancelled out by including suitable terms which involve mixed classical-stochastic integrals in the numerical schemes. The new methods are presented in Section 4. Simplified versions for special cases, such as additive noise and commutative noise SODEs are also discussed. Section 5 contains a brief discussion of the simulation of mixed classical-stochastic integrals. In the final sections we present numerical results illustrating the theoretical findings and draw conclusions.

2 Mean-square convergence of stochastic linear multi-step methods

In the literature on numerical methods for SODEs (see, e.g., [7, 9, 10]) mainly two concepts of convergence are discussed, weak and strong convergence. Weak convergence relates to Monte-Carlo methods and is mostly concerned with statistical properties of the solutions of SODEs. The term *strong* convergence is often used synonymously for the expression *mean-square* convergence, i.e., convergence in the norm $\|\cdot\|_{L_2}$. We denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^n$, by $\|\cdot\|$ the corresponding induced matrix norm and by $\|Z\|_{L_2} := (\mathbb{E}|Z|^2)^{1/2}$ the norm of a vector-valued square-integrable random variable $Z \in L_2(\Omega, \mathbb{R}^n)$. Subsequently we discuss *mean-square convergence* of possibly drift-implicit **s**tochastic **l**inear **m**ulti-step **m**ethods (SLMMs), given in the form

$$\sum_{j=0}^{k} \alpha_j \, X_{\ell-j} = h \sum_{j=0}^{k} \beta_j \, f_{\ell-j} + \sum_{j=1}^{k} \Gamma_{j,\ell-j} \, I^{t_{\ell-j},t_{\ell-j+1}}, \quad \ell = k, \ldots, N\,, \qquad (3)$$

where $I^{t,t+h}$ denotes a collection of multiple stochastic integrals over $[t, t+h]$:

$$I^{t,t+h}_{r_1,r_2,\ldots,r_j} = \int_t^{t+h} \int_t^{s_1} \cdots \int_t^{s_{j-1}} \mathrm{d}W_{r_1}(s_j) \ldots \mathrm{d}W_{r_j}(s_1), \qquad (4)$$

where $r_i \in \{0, 1, \ldots, m\}$ and $\mathrm{d}W_0(s) = \mathrm{d}s$. The Maruyama-type schemes (2) are a special case of (3) with $\Gamma_j(t,x)\ I^{t,t+h} = \sum_{r=1}^{m} \gamma_j g_r(t,x)\ I_r^{t,t+h}$. Again, we emphasize that an explicit discretization of the diffusion part is used and we assume that k initial values $X_0, X_1, \ldots X_{k-1} \in L_2(\Omega, \mathbb{R}^n)$, where X_ℓ is $\mathcal{F}_{t_\ell}$-measuarable for $\ell = 0, \ldots, k-1$, are given (e.g., by appropriate one-step schemes). We aim at mean-square estimates of the global error, i.e., estimates of $\max_{\ell=0,\ldots,N} \|X(t_\ell) - X_\ell\|_{L_2}$. To this end we use the fundamental result of [3] that allows us to estimate the global error from local errors by means of a stability inequality. The local error L_ℓ of the SLMM (3) at time-point t_ℓ is defined as the defect that is obtained when the exact solution values are inserted into the numerical scheme:

$$\begin{aligned} L_\ell &:= \sum_{j=0}^{2} \alpha_j X(t_{\ell-j}) - h \sum_{j=0}^{k} \beta_j \, f(t_{\ell-j}, X(t_{\ell-j})) - \sum_{j=1}^{k} \Gamma_j(t_{\ell-j}, X(t_{\ell-j}))\, I^{t_{\ell-j},t_{\ell-j+1}}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell = k, \ldots, N, \\ L_\ell &:= X(t_\ell) - X_\ell, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell = 0, \ldots, k-1. \end{aligned}$$

As in [3] we represent the local error in the form

$$L_\ell = R_\ell + S_\ell =: R_\ell + S_{1,\ell} + S_{2,\ell-1} + \ldots + S_{k,\ell-k+1}, \quad \ell = k, \ldots, N, \qquad (5)$$

where each $S_{j,\ell}$ is $\mathcal{F}_{t_\ell}$-measurable with $I\!E(S_{j,\ell}|\mathcal{F}_{t_{\ell-1}}) = 0$. The representation (5) is not unique. It is chosen in order to exploit the adaptivity and independence of the stochastic terms arising on disjoint subintervals.
Mean square convergence is implied by local properties of the SLMM by means of numerical stability, sometimes also called zero-stability, in the mean-square sense. Numerical stability estimates the influence of any perturbations of the right-hand side of the discrete scheme on the global solution of that discrete scheme. Taking the local errors as special perturbations and applying the numerical stability estimate to them gives the following convergence theorem [3, Thm 3.2]:

Theorem 2.1 *Let the parameters α_j, $j = 0, \ldots, k$, of the SLMM (3) satisfy Dahlquist's root condition, i.e., all eigenvalues of the characteristic polynomial*

$$\rho(\zeta) = \alpha_0\zeta^k + \alpha_1\zeta^{k-1} + \ldots + \alpha_k$$

have modulus smaller or equal than 1 and those with modulus equal to 1 are simple. Further, assume that the coefficient functions f, Γ_j are globally Lipschitz with respect to their second argument. Then there exist constants $h_0 > 0$ and $S > 0$ such that for all step-sizes $h < h_0$ and all representations (5) of the local error L_ℓ we have the following estimate of the global error

$$\max_{\ell=0,\ldots,N} \|X_\ell - X(t_\ell)\|_{L_2} \le S\Big\{ \max_{\ell=0,..,k-1} \|L_\ell\|_{L_2} + \max_{\ell=k,\ldots,N} \big(\frac{\|R_\ell\|_{L_2}}{h} + \frac{\|S_\ell\|_{L_2}}{h^{1/2}}\big)\Big\}. \qquad (6)$$

Subsequently we assume that the conditions of the preceding theorem are fulfilled. In order to prove mean-square convergence of order γ it is then sufficient to find a representation (5) of the local error L_ℓ such that

$$\|R_\ell\|_{L_2} \le \bar{c} \cdot h^{\gamma+1}, \quad \text{and} \quad \|S_\ell\|_{L_2} \le c \cdot h^{\gamma+\frac{1}{2}}\ , \quad \ell = k, \ldots, N\ . \qquad (7)$$

Together, the conditions (7) imply the estimates

$$\|I\!E(L_\ell|\mathcal{F}_{t_{\ell-k}})\|_{L_2} = \mathcal{O}(h^{\gamma+1}) \quad \text{and} \quad \|L_\ell\|_{L_2} = \mathcal{O}(h^{\gamma+\frac{1}{2}})\ , \quad \ell = k, \ldots, N\ ,$$

(compare Lemma 2.8 in [3]), known as consistency in the mean and in the mean-square. Note that in the case of k-step schemes the conditional mean has to be taken with respect to the σ-algebra $\mathcal{F}_{t_{\ell-k}}$. The analysis of the local errors is based on Itô-Taylor-expansions.

3 Local error analysis for Maruyama-type schemes

We want to establish a suitable representation (5) of the local error L_ℓ of the k-step Maruyama-scheme (2) for the SODE (1) by means of appropriate Itô-Taylor expansions, where we take special care to separate the multiple stochastic integrals over the different subintervals of integration.

For further reference we state the following definitions and results. For a continous function y from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ a general multiple Wiener integral over the subinterval $[t,t+h]\subseteq[0,T]$ is given by

$$I^{t,t+h}_{r_1,r_2,\ldots,r_j}(y)=\int_t^{t+h}\int_t^{s_1}\ldots\int_t^{s_{j-1}} y(s_j,X(s_j))\,\mathrm{d}W_{r_1}(s_j)\ldots\mathrm{d}W_{r_j}(s_1), \qquad (8)$$

where $r_i\in\{0,1,\ldots,m\}$ and $\mathrm{d}W_0(s)=\mathrm{d}s$. According to (4) we write $I^{t,t+h}_{r_1,r_2,\ldots,r_j}$ if $y\equiv 1$. To estimate the multiple integrals (8) we will use the following lemma (cf. Lemmata 2.1 and 2.2 in [10] and in [9, Chap 1]).

Lemma 3.1 *For any function y from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ that satisfies a linear growth condition in the form*

$$|y(t,x)|\le K(1+|x|^2)^{\frac{1}{2}}, \qquad \forall y\in\mathbb{R}^n, t\in[0,T], \qquad (9)$$

and any $t\in[0,T]$, $h>0$, such that $t+h\in[0,T]$, we have that

$$\begin{aligned} \mathbb{E}(I^{t,t+h}_{r_1\ldots r_j}(y)|\mathcal{F}_t) &= 0 \qquad \text{if } r_i\neq 0 \text{ for some } i\in\{1,\ldots,j\}, \quad (10)\\ \|I^{t,t+h}_{r_1,\ldots,r_j}(y)\|_{L_2} &= \mathcal{O}(h^{l_1+l_2/2}), \quad (11)\end{aligned}$$

where l_1 is the number of zero indices and l_2 the number of non-zero indices r_i.

Let $C^{s-1,s}$ denote the class of all functions from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ having continuous partial derivatives up to order $s-1$ and, in addition, continuous partial derivatives of order s with respect to the second variable. We introduce operators Λ_0 and Λ_r, $r=1,\ldots,m$, defined on $C^{1,2}$ and $C^{0,1}$, respectively, by

$$\Lambda_0 y=y'_t+y'_x f+\frac{1}{2}\sum_{r=1}^m\sum_{i,j=1}^n y''_{x_ix_j}g_{ri}g_{rj}\ , \quad \Lambda_r y=y'_x g_r\ ,\ r=1,\ldots,m. \qquad (12)$$

Using these operators and the notation for multiple Wiener integrals (8), the Itô formula for a function y in $C^{1,2}$ and the solution X of (1) reads

$$y(t,X(t))=y(t_0,X(t_0))+I_0^{t_0,t}(\Lambda_0 y)+\sum_{r=1}^m I_r^{t_0,t}(\Lambda_r y), \quad 0\le t_0<t\le T. \qquad (13)$$

Applying the Itô formula especially to the drift and diffusion coefficients f,g_r, which are assumed to be in $C^{1,2}$, and inserting the results into the SODE (1) leads to the first terms of the Itô-Taylor (or Wagner-Platen) expansion of the solution $X(t)$:

$$
\begin{aligned}
X(t) &= X(t_0) + I_0^{t_0,t}(f) + \sum_{r=1}^m I_r^{t_0,t}(g_r) \\
&= X(t_0) + f(t_0, X(t_0)) I_0^{t_0,t} + \sum_{r=1}^m g_r(t_0, X(t_0)) I_r^{t_0,t} \\
&\quad + I_{00}^{t_0,t}(\Lambda_0 f) + \sum_{r=1}^m \big(I_{r0}^{t_0,t}(\Lambda_r f) + I_{0r}^{t_0,t}(\Lambda_0 g_r) \big) + \sum_{r,q=1}^m I_{qr}^{t_0,t}(\Lambda_q g_r)\,.
\end{aligned}
$$

Now we start analyzing the local error L_ℓ of the k-step Maruyama-scheme (2) for the SODE (1). For simplicity we restrict the exposition to $k \le 3$. By rewriting

$$
\begin{aligned}
\sum_{j=0}^3 \alpha_j X(t_{\ell-j}) &= \alpha_0\big(X(t_\ell) - X(t_{\ell-1})\big) + (\alpha_0+\alpha_1)\big(X(t_{\ell-1}) - X(t_{\ell-2})\big) \\
&\quad + (\alpha_0+\alpha_1+\alpha_2)\big(X(t_{\ell-2}) - X(t_{\ell-3})\big) + \big(\sum_{j=0}^3 \alpha_j\big) X(t_{\ell-3}),
\end{aligned}
$$

one immediately observes the consistency conditions

$$
\sum_{j=0}^3 \alpha_j = 0\,, \quad \alpha_0 = \gamma_1\,, \quad \alpha_0+\alpha_1 = \gamma_2\,, \quad \alpha_0+\alpha_1+\alpha_2 = \gamma_3\,, \tag{14}
$$

which we assume to be valid for our subsequent calculations. Then we can express the local error as

$$
\begin{aligned}
L_\ell &= \alpha_0\big(X(t_\ell) - X(t_{\ell-1}) - \textstyle\sum_{r=1}^m g_r(t_{\ell-1}, X(t_{\ell-1})) I_r^{t_{\ell-1},t_\ell}\big) \\
&\quad + (\alpha_0+\alpha_1)\big(X(t_{\ell-1}) - X(t_{\ell-2}) - \textstyle\sum_{r=1}^m g_r(t_{\ell-2}, X(t_{\ell-2})) I_r^{t_{\ell-2},t_{\ell-1}}\big) \\
&\quad + (\alpha_0+\alpha_1+\alpha_2)\big(X(t_{\ell-2}) - X(t_{\ell-3}) - \textstyle\sum_{r=1}^m g_r(t_{\ell-3}, X(t_{\ell-3})) I_r^{t_{\ell-3},t_{\ell-2}}\big) \\
&\quad - h \textstyle\sum_{j=0}^3 \beta_j f(t_{\ell-j}, X(t_{\ell-j})) \\
&= \gamma_1\big(I_0^{t_{\ell-1},t_\ell}(f) + \textstyle\sum_{r=1}^m [I_r^{t_{\ell-1},t_\ell}(g_r) - g_r(t_{\ell-1}, X(t_{\ell-1})) I_r^{t_{\ell-1},t_\ell}]\big) \\
&\quad + \gamma_2\big(I_0^{t_{\ell-2},t_{\ell-1}}(f) + \textstyle\sum_{r=1}^m [I_r^{t_{\ell-2},t_{\ell-1}}(g_r) - g_r(t_{\ell-2}, X(t_{\ell-2})) I_r^{t_{\ell-2},t_{\ell-1}}]\big) \\
&\quad + \gamma_3\big(I_0^{t_{\ell-3},t_{\ell-2}}(f) + \textstyle\sum_{r=1}^m [I_r^{t_{\ell-3},t_{\ell-2}}(g_r) - g_r(t_{\ell-3}, X(t_{\ell-3})) I_r^{t_{\ell-3},t_{\ell-2}}]\big) \\
&\quad - h \textstyle\sum_{j=0}^3 \beta_j f(t_{\ell-j}, X(t_{\ell-j})) \\
&= \gamma_1\big(I_0^{t_{\ell-1},t_\ell}(f) + \textstyle\sum_{r=1}^m I_{0r}^{t_{\ell-1},t_\ell}(\Lambda_0 g_r) + \sum_{q,r=1}^m I_{qr}^{t_{\ell-1},t_\ell}(\Lambda_q g_r)\big) \\
&\quad + \gamma_2\big(I_0^{t_{\ell-2},t_{\ell-1}}(f) + \textstyle\sum_{r=1}^m I_{0r}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 g_r) + \sum_{q,r=1}^m I_{qr}^{t_{\ell-2},t_{\ell-1}}(\Lambda_q g_r)\big) \\
&\quad + \gamma_3\big(I_0^{t_{\ell-3},t_{\ell-2}}(f) + \textstyle\sum_{r=1}^m I_{0r}^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 g_r) + \sum_{q,r=1}^m I_{qr}^{t_{\ell-3},t_{\ell-2}}(\Lambda_q g_r)\big) \\
&\quad - h \textstyle\sum_{j=0}^3 \beta_j f(t_{\ell-j}, X(t_{\ell-j}))\,.
\end{aligned}
$$

The stochastic integrals $I_{0r}^{t_{\ell-j},t_{\ell-j+1}}(\Lambda_0 g_r)$ and $I_{qr}^{t_{\ell-j},t_{\ell-j+1}}(\Lambda_q g_r)$ naturally should be taken as a part of $S_{j,\ell-j+1}$ in the representation (5) of the local error. The

terms that contain values or classical integrals of the drift coefficient need further investigation. To pool together the deterministic parts we use

$$I_0^{t_{\ell-j},t_{\ell-j+1}}(f) = h\, f(t_{\ell-j}, X(t_{\ell-j})) + I_{00}^{t_{\ell-j},t_{\ell-j+1}}(\Lambda_0 f) + \sum_{r=1}^{m} I_{r0}^{t_{\ell-j},t_{\ell-j+1}}(\Lambda_r f)$$

and trace the values of the drift coefficient back to the point $t_{\ell-3}$:

$$\begin{aligned}
f(t_{\ell-2}, X(t_{\ell-2})) &= f(t_{\ell-3}, X(t_{\ell-3})) + I_0^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 f) + \sum_{r=1}^{m} I_r^{t_{\ell-3},t_{\ell-2}}(\Lambda_r f),\\
f(t_{\ell-1}, X(t_{\ell-1})) &= f(t_{\ell-3}, X(t_{\ell-3})) + I_0^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 f) + I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f),\\
&\qquad + \sum_{r=1}^{m} (I_r^{t_{\ell-3},t_{\ell-2}}(\Lambda_r f) + I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f)),\\
f(t_{\ell}, X(t_{\ell})) &= f(t_{\ell-3}, X(t_{\ell-3})) + I_0^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 f) + I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f) + I_0^{t_{\ell-1},t_{\ell}}(\Lambda_0 f)\\
&\qquad + \sum_{r=1}^{m} (I_r^{t_{\ell-3},t_{\ell-2}}(\Lambda_r f) + I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f) + I_r^{t_{\ell-1},t_{\ell}}(\Lambda_r f)).
\end{aligned}$$

Hence, the deterministic part R_ℓ of the representation (5) besides higher order terms always contains the term $h(\sum_{j=1}^{3}\gamma_j - \sum_{j=0}^{3}\beta_j) f((t_{\ell-3}, X(t_{\ell-3}))$ that has to vanish for consistent schemes, thus leading to the consistency condition

$$\big(\sum_{j=0}^{3}(3-j)\alpha_j = \ \big) \qquad \sum_{j=1}^{3}\gamma_j = \sum_{j=0}^{3}\beta_j\,, \tag{15}$$

which we also assume to be fulfilled for the subsequent calculations. Then we arrive at

$$\begin{aligned}
L_\ell \ = \ & \sum_{q,r=1}^{m} \big(\gamma_1 I_{qr}^{t_{\ell-1},t_\ell}(\Lambda_q g_r) + \gamma_2 I_{qr}^{t_{\ell-2},t_{\ell-1}}(\Lambda_q g_r) + \gamma_3 I_{qr}^{t_{\ell-3},t_\ell-2}(\Lambda_q g_r)\big) && (16)\\
& + \sum_{r=1}^{m} \Big(\gamma_1 I_{0r}^{t_{\ell-1},t_\ell}(\Lambda_0 g_r) + \gamma_1 I_{r0}^{t_{\ell-1},t_\ell}(\Lambda_r f) - \beta_0 h I_r^{t_{\ell-1},t_\ell}(\Lambda_r f)\\
& \quad + \gamma_2 I_{0r}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 g_r) + \gamma_2 I_{r0}^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f) + (\gamma_1 - \beta_0 - \beta_1) h I_r^{t_{\ell-2},t_{\ell-1}}(\Lambda_r f)\\
& \quad + \gamma_3 I_{0r}^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 g_r) + \gamma_3 I_{r0}^{t_{\ell-3},t_{\ell-2}}(\Lambda_r f)\\
& \qquad + (\gamma_1 + \gamma_2 - \beta_0 - \beta_1 - \beta_2) h I_r^{t_{\ell-3},t_{\ell-2}}(\Lambda_r f)\Big) && (17)\\
& + \big(\gamma_1 I_{00}^{t_{\ell-1},t_\ell}(\Lambda_0 f) - \beta_0 h I_0^{t_{\ell-1},t_\ell}(\Lambda_0 f)\\
& \quad + \gamma_2 I_{00}^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f) + (\gamma_1 - \beta_0 - \beta_1) h I_0^{t_{\ell-2},t_{\ell-1}}(\Lambda_0 f)\\
& \quad + \gamma_3 I_{00}^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 f) + (\gamma_1 + \gamma_2 - \beta_0 - \beta_1 - \beta_2) h I_0^{t_{\ell-3},t_{\ell-2}}(\Lambda_0 f)\big). && (18)
\end{aligned}$$

With the above representation of the local error we have already separated the terms causing global errors of order $\mathcal{O}(\epsilon^2 h^{1/2})$ and $\mathcal{O}(\epsilon h)$, namely the terms (16) and (17). These terms can be seen as parts of the stochastic terms S_ℓ in the representation (5). By the stability inequality (6) we know that the global

error caused by these terms is only an order $\mathcal{O}(h^{-1/2})$ larger than the local quantities. Using Lemma 3.1 and exploiting the smallness of the noise in the form $g_r = \epsilon \hat{g}_r$ the (so-called Milstein) terms (16) can be estimated as $\mathcal{O}(\epsilon^2 h)$, and the mixed terms in (17) as $\mathcal{O}(\epsilon h^{3/2})$. Without further investigations the remaining terms (18) can be estimated as $\mathcal{O}(h^2)$, thus causing global errors of order $\mathcal{O}(h)$. Supposing that also $\Lambda_0 f$ belongs to $C^{1,2}$ and that the additional consistency condition

$$\sum_{j=0}^{3}(3-j)^2\alpha_j = 2\sum_{j=0}^{3}(3-j)\beta_j \tag{19}$$

is fulfilled, one can further estimate (18) as $\mathcal{O}(h^3+\epsilon h^{5/2})$ and conclude that the induced global errors are of order $\mathcal{O}(h^2+\epsilon h^2)$. The condition (19) guarantees the deterministic order 2.

4 Improved linear multi-step methods

Intending to avoid the global error terms of order $\mathcal{O}(\epsilon h)$ we include the leading parts of (17) in the discretization scheme. This leads us to schemes that include the mixed classical-stochastic integrals $I_{0,r}^{t_{\ell-j},t_{\ell-j+1}}$ and $I_{r,0}^{t_{\ell-j},t_{\ell-j+1}}$ and take the general form

$$\begin{aligned}\sum_{j=0}^{k}\alpha_j X_{\ell-j} = h\sum_{j=0}^{k}\beta_j f_{\ell-j} + \sum_{j=1}^{k}\sum_{r=1}^{m}\Big\{&\gamma_j\, g_{r,\ell-j}\; I_r^{t_{\ell-j},t_{\ell-j+1}} \\ +\gamma_j\,[(g_r)'_t+(g_r)'_x f]_{\ell-j}\, I_{0,r}^{t_{\ell-j},t_{\ell-j+1}} &+ [f'_x g_r]_{\ell-j}\big(\gamma_j\, I_{r,0}^{t_{\ell-j},t_{\ell-j+1}} + \eta_j\, h\, I_r^{t_{\ell-j},t_{\ell-j+1}}\big)\Big\},\\ &\ell = k,\dots,N, \end{aligned} \tag{20}$$

with parameters α_j, β_j, $j=0,\dots,k$ and γ_j, η_j, $j=1,\dots,k$. The parameters in the stochastic part can be computed from those in the deterministic part by

$$\gamma_j = \sum_{i=0}^{j-1}\alpha_i\,,\ j=1,\dots,k\,,\quad \eta_1 = -\beta_0,\quad \eta_{j+1} = \eta_j + \gamma_j - \beta_j\,, j=1,\dots,k-1\,.$$

As an example we give the improved variant of the two-step Adams-Bashforth method. For $\ell = 2,\dots,N$, it takes the form

$$\begin{aligned}X_\ell - X_{\ell-1} = h\,(\frac{3}{2}f_{\ell-1} - \frac{1}{2}f_{\ell-2}) + \sum_{r=1}^{m} g_{r,\ell-1}\, I_r^{t_{\ell-1},t_\ell} \\ +\sum_{r=1}^{m}[(g_r)'_t+(g_r)'_x f]_{\ell-1}\, I_{0r}^{t_{\ell-1},t_\ell} + \sum_{r=1}^{m}[f'_x g_r]_{\ell-1}\, I_{r0}^{t_{\ell-1},t_\ell} - \frac{1}{2}\sum_{r=1}^{m}[f'_x g_r]_{\ell-2}\, h\, I_r^{t_{\ell-2},t_{\ell-1}}.\end{aligned}$$

Due to the identity $I_{0,r} + I_{r,0} = hI_r$ between the mixed integrals the above methods simplify considerably in the case when the coefficients f, g_r are commutative in the sense that $f'_x g_r = (g_r)'_t + (g_r)'_x f$, . The methods then reduce to

$$\sum_{j=0}^{k} \alpha_j X_{\ell-j} = h \sum_{j=0}^{k} \beta_j f_{\ell-j}$$
$$+ \sum_{j=1}^{k} \sum_{r=1}^{m} \Big\{ \gamma_j \; g_{r,\ell-j} \; I_r^{t_{\ell-j},t_{\ell-j+1}} + (\gamma_j + \eta_j) \, [f'_x g_r]_{\ell-j} \; h \, I_r^{t_{\ell-j},t_{\ell-j+1}} \Big\},$$
$$\ell = k, \ldots, N, \quad (21)$$

In case of additive noise, i.e., $g_r(t,x) \equiv g_r(t)$ one has $(g_r)'_x = 0$, hence $\Lambda_0 g_r = (g_r)'_t$ and $\Lambda_q g_r = 0$. The Milstein terms (16) vanish leaving a global error of order $\mathcal{O}(\epsilon h + h^p)$ for k-step Maruyama-schemes with deterministic order p and a global error of order $\mathcal{O}(\epsilon^2 h^{\frac{3}{2}} + \epsilon h^2 + h^p)$ for the improved schemes (20).
In Table 1 we list the parameters for two- and three-step Adams-Bashforth, Adams-Moulton and BDF (backward differentiation formula) schemes. The schemes are scaled such that $\alpha_0 = \gamma_1 = 1$.

Meth.	α_1	α_2	α_3	β_0	β_1	β_2	β_3	γ_2	γ_3	η_1	η_2	η_3
AB2	-1	0	—	0	$\frac{3}{2}$	$-\frac{1}{2}$	—	0	—	0	$-\frac{1}{2}$	—
AB3	-1	0	0	0	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	0	0	0	$-\frac{11}{12}$	$\frac{5}{12}$
AM2	-1	0	—	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	—	0	—	$-\frac{5}{12}$	$-\frac{1}{12}$	—
AM3	-1	0	0	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	0	0	$-\frac{9}{24}$	$-\frac{1}{6}$	$\frac{1}{24}$
BDF2	$-\frac{4}{3}$	$\frac{1}{3}$	—	$\frac{2}{3}$	0	0	—	$-\frac{1}{3}$	—	$-\frac{2}{3}$	$\frac{1}{3}$	—
BDF3	$-\frac{18}{11}$	$\frac{9}{11}$	$-\frac{2}{11}$	$\frac{6}{11}$	0	0	0	$-\frac{7}{11}$	$\frac{2}{11}$	$-\frac{6}{11}$	$\frac{5}{11}$	$-\frac{2}{11}$

Table 1: Coefficients of improved two- and three-step schemes

5 Mixed stochastic integrals

The proposed improved schemes (20) contain the mixed stochastic-classical integrals I_{0r} and I_{r0}, $r = 1, ...m$, on the corresponding subintervals $[t, t+h]$. Unless the coefficients f, g_r are commutative these integrals have to be simulated together with the Wiener increments. This can be done in the following way (cf. [7, 10, 9]): Starting from independent standard normally distributed random variables ξ_r, $\zeta_r \sim N(0,1)$ one computes

$$\begin{aligned} I_r &:= h^{1/2} \xi_r, \\ I_{r0} &:= h^{3/2} (\zeta_r/\sqrt{3} + \xi_r)/2, \\ I_{0r} &:= h I_r - I_{r0}. \end{aligned}$$

For the 2-dimensional system with non-commutative noise in Section 6 we have calculated a reference solution with corresponding trajectories of two Wiener processes using a very small step-size. The Wiener increments for the numerical solutions with the different step-sizes were calculated in the usual way by adding up the correct number of Wiener increments used for the reference solution. Then we have calculated the mixed integrals I_{r0}, $r = 1, ...m$ with the finest step-size of the numerical approximations. To this end we denote the finest step-size of the numerical approximations by h_1 and a Wiener increment calculated for the step-size h_1 by $I_r^{h_1}$. Then $I_r^{h_1}$ is a normal random variable and $I_r^{h_1} \sim N(0, h_1)$, thus $I_r^{h_1}/\sqrt{h_1} \sim N(0, 1)$. Generating another independent standard normally distributed random variable $\zeta_r \sim N(0, 1)$ we form $I_{r0} = h_1^{3/2}(\zeta_r/\sqrt{3} + I_r^{h_1}/\sqrt{h_1})/2$. For all coarser step-sizes we have used the following useful fact, see [8]:

$$I_{r0}^{t_1,t_3} = I_{r0}^{t_1,t_2} + I_{r0}^{t_2,t_3} + I_r^{t_1,t_2}(t_3 - t_2).$$

6 Test results

We implemented several explicit and implicit stochastic linear k-step Maruyama-schemes and improved schemes for $k = 1, 2, 3$ and applied them to several examples of SDEs. Table 1 summarizes the methods we have implemented and tested. For comparisons we also considered the explicit and implicit Euler-Maruyama schemes and the explicit Milstein scheme. To start off the integration with the two- and three-step schemes we needed a second (and third) starting value X_1 (and X_2). If available, we used the exact solution values, thus avoiding introducing additional errors. In computational practice, the starting values could be computed, e.g., by means of the trapezoidal rule.

In our experiments we have investigated the relation between the step-size h and the achieved accuracy. The accuracy is measured as the maximum approximate L_2-norm of the global errors in the considered time-interval $[0, T]$:

$$\max_{\ell=1,\ldots,N} \Big(\frac{1}{M} \sum_{j=1}^{M} |X(t_\ell, \omega_j) - X_\ell(\omega_j)|^2 \Big)^{1/2} \approx \max_{\ell=1,\ldots,N} \|X(t_\ell) - X_\ell\|_{L_2}$$

where N denotes the number of steps and M the number of computed paths. In our computations we used $M = 200$.

The results are presented as figures, where we have plotted the accuracy versus the step-sizes in logarithmic scale with base 10. Then the slope of the resulting lines corresponds to the observed order of the schemes. Lines with slopes 0.5, 1, 2 are provided in some figures to enable comparisons with convergence of these orders.

Our first test example is the simple bilinear scalar SDE with drift and diffusion functions $f(t, x) = -x$, $g_1(t, x) = \epsilon\, x$ and starting value $X_0 = 1$ on the time-interval $[0, 1]$. For this simple example the exact solution is known as the geometric Brownian motion $X(t) = \exp\big((-1 - \frac{1}{2}\epsilon^2)t + \epsilon W(t)\big)$. This example is commutative, hence, we could use the simplified scheme (21). We have chosen the parameter $\epsilon = 0.005$. In Figure 1 we present results for selected schemes

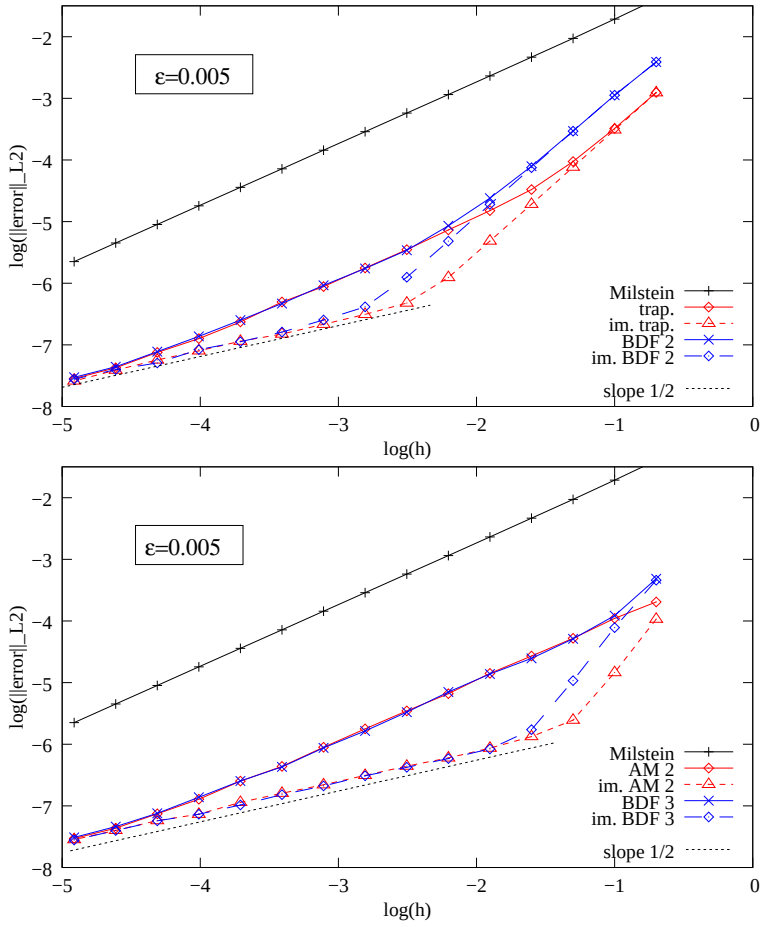

Figure 1: Performance of k-step Maruyama and improved schemes for the geometric Brownian motion

with deterministic order 2 (left) and 3 (right), which reflect the discussion in the previous sections very well. The visual differences between the individual schemes with the same deterministic order are caused by different error constants for the $\mathcal{O}(h^p)$ error terms. The maximum gain in accuracy by the improved methods occurs for those step-sizes, where the $\mathcal{O}(\epsilon^2 h^{1/2})$ and $\mathcal{O}(h^p)$ error terms are of the same magnitude ($p = 2, 3$). For comparison we include results of the Milstein-scheme with mean-square order of convergence 1. The resulting line practically coincided with those for the Euler schemes (which we have not included here) in the considered range of step-sizes. Though the asymptotical order is 1 for the Milstein scheme in contrast to $1/2$ for the other considered schemes, one observes much larger errors for the Milstein scheme for the considered step-sizes.

Our second test example is a linear scalar SDE with additive noise. The drift and diffusion functions are $f(t,x) = \frac{2}{1+t}\,x + (1+t)^2$, $g_1(t,x) = \epsilon\,(1+t)^2$ on the time-interval $[0,1]$. Unfortunately, this example is a bit more specific than we intended. The exact solution is known as $X(t) = (1+t)^2\big(X_0 + t + \epsilon W(t)\big)$ (cf. [7,

(4.44)]). The solution of the noise-free equation ($\epsilon = 0$) is a cubic polynomial, such that the noise-free equation is integrated exactly by any integration scheme with deterministic order 3 or higher. Thus, for k-step schemes with deterministic order $p \geq 3$ the global error term of order $\mathcal{O}(h^p)$ vanishes. Moreover, since f is linear in x, we have $\Lambda_q \Lambda_r f = 0$, such that for the improved schemes also the global error term of order $\mathcal{O}(\epsilon^2 h^{3/2})$ vanishes leaving a global error of order $\mathcal{O}(\epsilon h^2)$ instead of $\mathcal{O}(\epsilon^2 h^{3/2} + \epsilon h^2 + h^3)$ as expected in general. This is illustrated in Figure 2, where we present simulation results for selected schemes with deterministic order 2 (left) and 3 (right) for the initial value $X_0 = 1$ and the parameter $\epsilon = 0.001$. The numerical errors of magnitude $\frac{1}{h} \times$machine-accuracy become visible in the lower corner of the right picture. The previous exam-

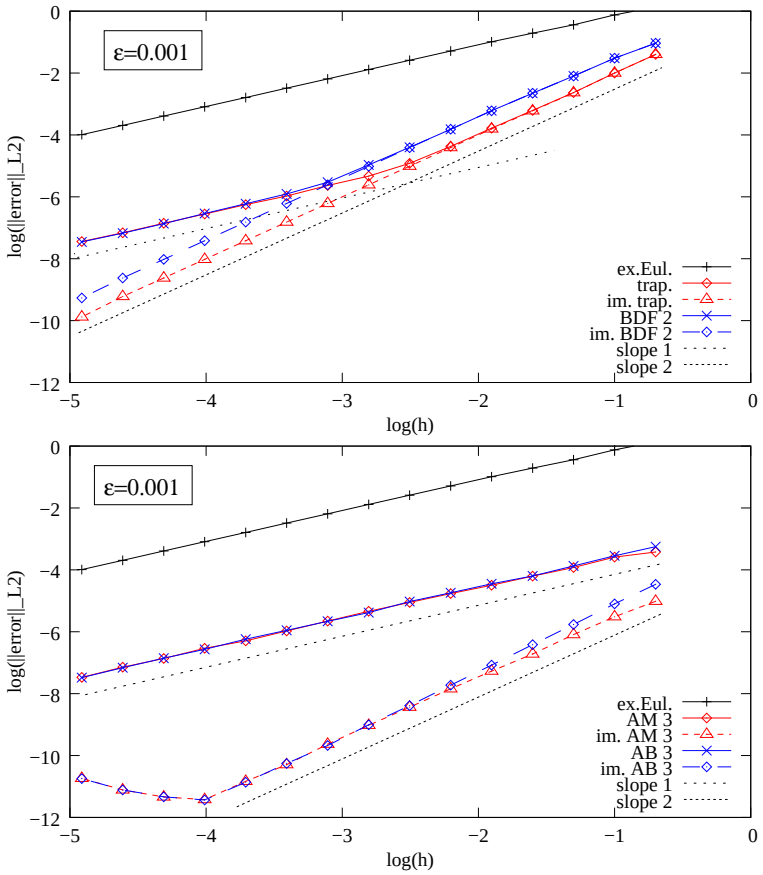

Figure 2: Performance of k-step Maruyama and improved schemes for the additive noise example

ples have exact solutions in the form $X(t) = \phi(t, W(t))$ and the commutativity condition is fulfilled. Then one can use the simplified versions (21) and it is not necessary to simulate the mixed integrals to compute the iterates of the improved schemes. Our third example is a non-commutative two-dimensional linear example with two-dimensional noise, which we have taken from [5] and modified by formulating it in Itô-calculus and introducing a small parameter ϵ

:

$$
\begin{aligned}
X(t) &= \begin{pmatrix}1\\1\end{pmatrix} + \int_0^t F X(s)\,\mathrm{d}s + \int_0^t G_1 X(s)\,\mathrm{d}W_1(s) + \int_0^t G_2 X(s)\,\mathrm{d}W_2(s),\\
F &= \begin{pmatrix}-\frac{9}{10} & 0\\ \frac{1}{4} & -\frac{1}{2}\end{pmatrix} + \frac{\epsilon^2}{2}\begin{pmatrix}(\frac{3}{4})^2+(\frac{9}{10})^2 & 0\\ 0 & (\frac{3}{4})^2+(\frac{9}{10})^2\end{pmatrix}\\
G_1 &= \epsilon\begin{pmatrix}\frac{3}{4} & 0\\ 0 & -\frac{3}{4}\end{pmatrix}, \quad G_2 = \epsilon\begin{pmatrix}0 & \frac{9}{10}\\ \frac{9}{10} & 0\end{pmatrix}, \quad t\in[0,2].
\end{aligned}
$$

Here, the expressions $[F, G_1]$, $[F, G_2]$, $[G_1, G_2]$ are all non-zero, where $[A, B] = AB - BA$ for matrices A, B. Since we do not have an explicit formula for the solution, we have computed a 'reference solution' by means of the trapezoidal rule on a very fine grid by using $2^{18} = 262144$ steps. We have implemented and tested the two-step schemes from Table 1 and have chosen the parameter $\epsilon = 0.01$. The necessary second starting value X_1 has been taken as the corresponding value of the reference solution. The simulation results presented in

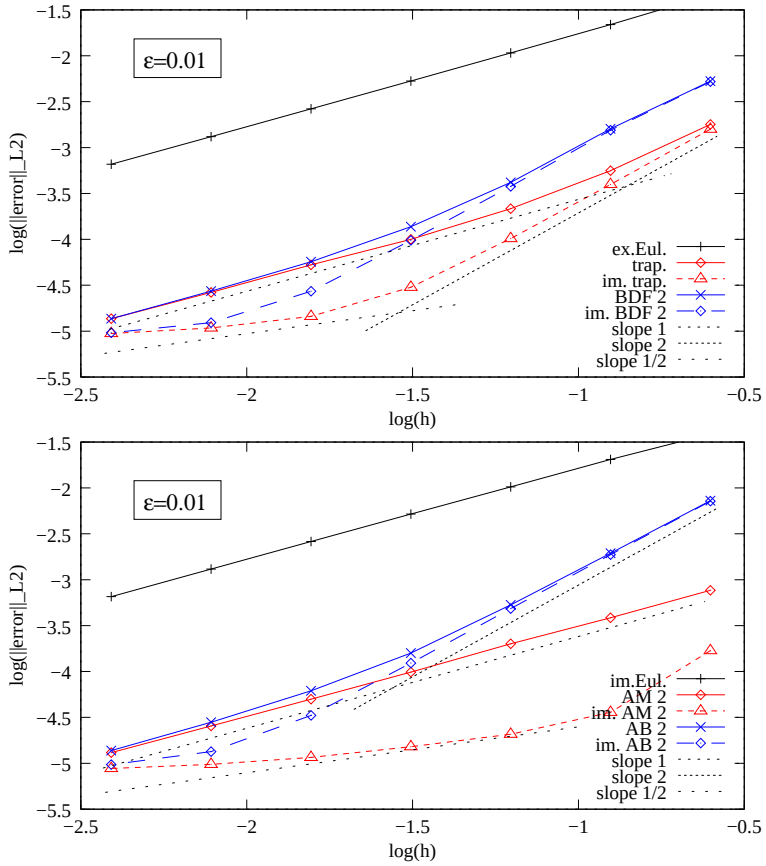

Figure 3: Performance of 2-step Maruyama and improved schemes for the non-commutative example

Figure 3 confirm the error behaviour that was observed already for the geome-

tric Brownian motion though the significance of the observations is restricted by the accuracy of the reference solution.

7 Conclusions

We have analysed the local and global error behaviour of improved stochastic linear multi-step methods for small noise SODEs. We have found that the global error is of the order $\mathcal{O}(\epsilon^2 h^{1/2} + \epsilon h^2 + h^p)$, where p is the deterministic order of the scheme. Compared with the Maruyama-type stochastic linear multi-step methods an error term of order $\mathcal{O}(\epsilon h)$ is avoided. The price to pay for this improvement is the inclusion of the derivatives $f'_x g_r$ and $(g_r)'_t + (g_r)'_x f$ and the additional simulation of mixed classical-stochastic integrals. The latter effort is comparable to that for the simulation of the Wiener increments. The gain in accuracy depends on the relation of the step-size and the smallness of the noise, the maximum gain in accuracy by the improved methods occurs for those step-sizes, where the $\mathcal{O}(\epsilon^2 h^{1/2})$ and $\mathcal{O}(h^p)$ error terms are of the same magnitude. Then the error is reduced by a factor of order $\epsilon^{1/3}$ for $p = 2$ and $\epsilon^{3/5}$ for $p = 3$. In the additive noise case the error is practically reduced to the deterministic error.

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