# Stochastic DAEs in Transient Noise Simulation

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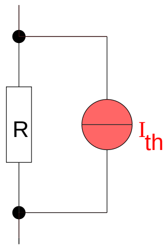
**Abstract.** In this paper we describe how stochastic differential-algebraic equations (SDAEs) arise as a mathematical model for network equations that are influenced by additional sources of Gaussian white noise. We give the necessary analytical theory for the existence and uniqueness of strong solutions, provided that the systems have noise-free constraints and are uniformly of DAE-index 1. We express these conditions in terms of the network-topology for reasons of use within a circuit simulator. In the second part we analyze discretization methods. Due to the differential-algebraic structure, implicit methods will be necessary. By the examples of the drift-implicit Euler and Milstein schemes we show how drift-implicit schemes for SDEs can be adapted to become directly applicable to stochastic DAEs and prove that the convergence properties of these methods known for SDEs are preserved. For illustration we apply the drift-implicit Euler scheme to an oscillator circuit.

## 1 Problem Formulation

The charge-oriented Modified Nodal Analysis (MNA) represents a standard tool in circuit simulation. The equations are generated automatically by combining the network topology, Kirchhoff's Current Law, and the characteristic equations describing the physical behaviour of the network elements. This results in large systems of DAEs, whose special structure was analyzed in a number of papers, e.g. [5,6,16]. Due to decreasing signal to noise ratios in special applications linear noise analysis around the deterministic solution is no longer satisfactory. The noise influences such systems in an essentially nonlinear way, the solution of the deterministic models differs from the mean of the solution of the stochastic models. Transient noise analysis is necessary. Here we deal with models where Gaussian white noise sources are added to the systems. Thermal noise of resistors and shot noise of $pn$-junctions are modelled as external Gaussian white noise sources in parallel to the original element (see Figures 1 and 2), as will be explained next.
Nyquist's theorem (see e.g. [2–4,18]) states that the current through an arbitrary linear resistor having a resistance $R$, maintained in thermal equilibrium at a temperature $T$, can be described as the sum of the noiseless, deterministic current and a current due to a Gaussian white noise process with spectral density $S_{th} := \frac{2kT}{R}$, where $k$ is Boltzmann's constant. Hence, the additional current is modelled as

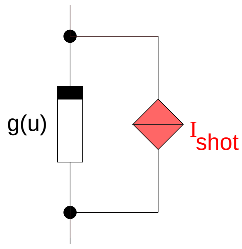
$$I_{th} = \sigma_{th} \cdot \xi(t) = \sqrt{\frac{2kT}{R}} \cdot \xi(t) \ ,$$

where $\xi(t)$ is a standard Gaussian white noise process. In [17,18] a thermodynamical foundation to apply this model to mildly nonlinear resistors and reciprocal networks is given.




thermal noise of a resistor

Figure 1




shot noise of a pn-junction

Figure 2

Shot noise of pn-junctions, caused by the discrete nature of current due to the elementary charge, is also modelled by a Gaussian white noise process, where the spectral density is proportional to the current $I$ through the *pn*-junction: $S_{shot} := q|I|$ , where $q$ is the elementary charge. If the current through the *pn*-junction is described by a characteristic $I = g(u)$ , where $u$ is some voltage, the additional current is modelled by

$$I_{shot} = \sigma_{shot}(u) \cdot \xi(t) = \sqrt{q|g(u)|} \cdot \xi(t) \ ,$$

where $\xi(t)$ is a standard Gaussian white noise process. For a discussion of the model assumptions we refer to [2–4,17,18].

We represent the topology of a network by means of the incidence matrix $(A_C, A_R, A_L, A_V, A_I, A_N)$, with indices referring to branches of capacitances, resistances, inductances, possibly controlled voltage and current sources, and $n_N$ additional noise sources, respectively. Then the charge-oriented MNA system has the following structure (see [5,6] for the deterministic case):

$$A_C q' + f_1(e, j_L, j_V, t) + A_N \mathrm{diag}\,(\sigma(A_N^T e, t))\xi(t) = 0 \qquad (1)$$

$$\phi' - A_L^T e = 0 \qquad (2)$$

$$A_V^T e - v_s(e, j_L, t) = 0 \qquad (3)$$

$$q - q_C(A_C^T e, t) = 0 \qquad (4)$$

$$\phi - \phi_L(j_L, t) = 0 \ , \qquad (5)$$

where $f_1(e, j_L, j_V, t) := A_R g(A_R^T e, t) + A_L j_L + A_V j_V + A_I i_s(e, j_L, j_I, t)$, and $q_C, g, \phi_L, v_s, i_s, \sigma$ are given, noiseless functions. The vector of unknowns describing the system behaviour consists of all node potentials $e$, the branch currents of current-controlled elements (inductances and voltage sources) $j_L, j_V$, and the charges $q$ of capacitances, and the fluxes $\phi$ of inductances. $\xi$ denotes an $n_N$-dimensional vector of independent standard Gaussian white noise processes. In industry-relevant applications one has to deal with a large number of unknowns and noise sources.

The first block of equations (1) denotes a stochastic integral equation :

$$A_C q(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f_1(x(s),s)ds + \int_{t_0}^{t} A_N \mathrm{diag}\,(\sigma)(x(s),s)dw(s) = 0\ ,$$

where the Itô integral is employed and $w$ denotes an $n_N$-dimensional Wiener process (also known as Brownian motion) given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t \geq t_0}$ (see e.g. [7] for the stochastic background). A solution $x = x(t,\omega)$ is now a stochastic process depending on the time $t$ and the chance element $\omega \in \Omega$. The parameter $\omega$ is omitted in the notations above. The solution $x(t) = x(t,\cdot)$ for fixed time $t$ is a vector-valued random variable in $L^2(\Omega)$, a realization $x(\cdot,\omega)$ is called a path.

The equations (1)-(5) form a specially structured **S**tochastic **D**ifferential **A**lgebraic **E**quation (SDAE) of the type

$$Ax(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} G(x(s),s)dw(s) = 0\ , \tag{6}$$

which is the subject of this paper. $A$ is a constant singular matrix, $t$ varies over a compact interval $\mathcal{J}$. The short-hand notation

$$Ax'(t) + f(x(t),t) + G(x(t),t)\xi(t) = 0 \tag{7}$$

emphasizes the relations of (6) to its deterministic counterpart but may be misleading for readers who are less familiar with the stochastic background. Though the notation $x'(t)$ is used in (7), a typical realization $x(\cdot,\omega)$ of the solution is nowhere differentiable.

A process $x(\cdot) = (x(t))_{t \in \mathcal{J}}$ is called a strong solution of (6) if it is adapted to the filtration (i.e., it does not depend on future information), and if, with probability 1, its sample paths are continuous, the integrals in (6) exist and (6) is fulfilled.

## 2 Index 1 SDAEs

Due to the singularity of the matrix $A$ the deterministic part of (6)

$$Ax'(t) + f(x(t),t) = 0\ , \tag{8}$$

where the solution $x$ is now a deterministic function of $t$, forms a DAE. Solutions have to fulfil the constraints of the equation. The solution components belonging to $\ker A$ (we call them the algebraic components) do not occur under the differential operator $d/dt$, and the inherent dynamics live only in a lower-dimensional subspace. The DAE (8) is characterized as an index 1 DAE iff the constraints are locally solvable for the algebraic components. Solving an index 1 DAE involves a coupling of an integration task and a nonlinear equation solving task. If a DAE is of higher index, the constraints are not locally solvable for the algebraic components, and there exist solution components that are determined only by a hidden differentiation step, which may cause serious difficulties in the numerical solution of such problems (see e.g. [1,9]).

We assume here that the deterministic part (8) is globally an index 1 DAE in the sense that the constraints are regularly and globally uniquely solvable for the algebraic variables. The globally unique solvability is stronger than the deterministic index 1 condition, which requires only the non-singularity of the corresponding Jacobian and guarantees only local solvability of the constraints for the algebraic variables. The globally unique solvability holds for the MNA-system (1)-(5) if (see [19])

> there are no loops of capacitances and voltage sources and no cut-sets of inductances or current sources, the capacity, conductance, and inductance matrices are symmetric and uniformly positively definite, and the controlled sources fulfil certain conditions described in [5].

In [13,14] it is shown that special conditions are needed to ensure solution processes that are not directly effected by white noise. Then the SDAEs are called SDAEs without direct noise, otherwise with direct noise. To avoid a solution process that is directly affected by white noise we have to assume that the noise sources do not appear in the constraints. This means that

$$\operatorname{im} G(x,t) \subseteq \operatorname{im} A \quad \forall (x,t) \in \mathbb{R}^n \times \mathcal{J} \ .$$

This is true for (7) if and only if there are always capacitances in parallel to a noise source. This is quite restrictive in the actual noise modelling (see also the example in Section 4). Nevertheless, one can also handle many situations where this condition is violated. Often noisy constraints are only needed for the determination of algebraic solution components that do not interact with the dynamical ones. Future work should be dedicated to classify such situations.

Under these conditions the constraints of the SDAE can be described by the deterministic equation

$$Rf(x(t),t) = 0 \quad ,$$

where $R$ is a projector along $\operatorname{im} A$, i.e., $R^2 = R$, $\ker R = \operatorname{im} A$. Solving the constraints for the algebraic components

$$Rf(u+v,t) = 0 \ , \ Av = 0 \Longleftrightarrow v = \hat{v}(u,t) \ ,$$

inserting the result into the differential equations, and scaling the system by a Pseudo-inverse $A^-$ (with $AA^- = I - R, A^-A$ a projector along $\ker A$) leads to a so-called inherent regular SDE in the differential components $u$:

$$u' + A^- f(u + \hat{v}(u,t),t) + A^- G(u + v(u,t),t)\xi(t) = 0 \tag{9}$$

It can be shown that (9), together with $x(t) = u(t) + \hat{v}(u(t),t)$, is theoretically equivalent to (6). Based on this, the following theorem concerning the existence and uniqueness of solutions of (6) is proved in [19]:

**Theorem 1.** *Let the above conditions be fulfilled for (6), and suppose that* $f$ *and* $G$ *are globally Lipschitz-continuous with respect to* $x$ *, continuous with respect to* $t$*, and that* $Ax^0$ *is* $\mathcal{F}_{t_0}$ *-measurable, independent of the Wiener process* $w$*, and with finite second moments.*
*Then there exists a strong solution* $x(\cdot)$ *of the initial value problem*

$$Ax(t) - Ax^0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} G(x(s), s)dw(s) = 0 \ , \tag{10}$$

*which is pathwise unique. Moreover, the solution* $x(t)$ *is square-integrable.*

## 3 Discretization Schemes for Index 1 SDAEs

Starting with the Euler Maruyama scheme [8] a wide spectrum of numerical methods for SDEs has been developed. However, first decoupling the SDAE numerically and then applying a scheme to the resulting inherent SDE would be an inefficient procedure in general. We aim at numerical methods for SDAEs that should work directly on the given implicit structure, as in the case of deterministic DAEs. Only little previous work has been done in this direction. In [13,14] linear SDAEs are analyzed and the convergence of the drift-implicit (i.e., implicit in the deterministic part, but explicit in the stochastic part) Euler scheme is proved. In [11,12] a scheme with strong order 1 is developed for the specially structured SDAEs arising in transient noise simulation for electronic circuits. Later we will point out its relation to the drift-implicit Milstein scheme.
Our approach applies to nonlinear SDAEs too, and is used here to analyze the drift-implicit Euler scheme, and to derive and analyze a drift-implicit Milstein scheme for SDAEs.

Designing the methods such that the iterates $x_\ell$ have to fulfil the constraints of the SDAE at the current time-point $t_\ell$ is the key idea to adapt known methods for SDEs to (6).

### 3.1 Drift-Implicit Euler Scheme

On the deterministic grid $0 = t_0 < t_1 < \ldots < t_N = T$ the drift-implicit Euler scheme for (6) is given by

$$A\frac{x_\ell - x_{\ell-1}}{h_\ell} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell = 0 \,, \tag{11}$$

where $h_\ell = t_{\ell+1} - t_\ell$ , $\Delta w_\ell = w(t_{\ell+1}) - w(t_\ell)$. Realizations of $\Delta w_\ell$ can be simulated as $N(0, h_\ell I)$-distributed random variables.
The scheme (11) for the SDAE (6) possesses the same convergence properties as the drift-implicit Euler scheme for SDEs. In general, its order of strong convergence is 1/2, i.e.,

$$\varepsilon_\ell := \|x(t_\ell) - x_\ell\|_{L_2} = (E|x(t_\ell) - x_\ell|^2)^{1/2} \leq c \cdot h^{1/2}$$

holds for the mean square norm of the global errors. For additive noise, i.e., $G(x,t) = G(t)$, the order of strong convergence is 1, for small noise, $G(x,t) = \epsilon\tilde{G}(x,t)$, the error is bounded by $O(h+\epsilon^2 h^{1/2})$, (see [10] for related results).

### 3.2 Drift-Implicit Milstein Scheme

We intend to design this method in such a way that it realizes the drift-implicit Milstein scheme for the inherent SDE $u' + \hat{f}(u,t) + \hat{G}(u,t)\xi(t) = 0$ , i.e.,

$$\frac{u_\ell - u_{\ell-1}}{h_\ell} + \hat{f}(u_\ell, t_\ell) + \hat{G}(u_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell - \sum_{j=1}^{m}(\hat{g}_j)_u \hat{G}(u_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell} I_j^\ell = 0\,,$$

where $I_j^\ell = (I_{j,i}^\ell)_{i=1}^m$ , $I_{j,i}^\ell = \int_{t_{\ell-1}}^{t_\ell}\int_{t_\ell}^{s} dw_i(\tau)dw_j(s)$ , $\hat{G} = (\hat{g}_1, \ldots, \hat{g}_m)$ and $\hat{f}(u,t) := A^- f(u + \hat{v}(u,t), t)$ , $\hat{G}(u,t) := A^- G(u + \hat{v}(u,t), t)$ .

The Milstein scheme possesses strong convergence of order 1. It differs from the Euler scheme by an additional correction term for the stochastic part, which includes double stochastic integrals. For additive noise the additional term vanishes and both schemes coincide.

The Milstein scheme for the inherent SDE is realized by

$$A\frac{x_\ell - x_{\ell-1}}{h_\ell} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\frac{\Delta w_\ell}{h_\ell} - \sum_{j=1}^{m}(g_j)_x x_u A^- G(x_{\ell-1}, t_{\ell-1})\frac{I_j^\ell}{h_\ell} = 0\ ,$$

where $G = (g_1, \ldots, g_m)$, which we call the drift-implicit Milstein scheme for (6). The expression $x_u A^-$ in the latter term can be reformulated (cf. [15]) as

$$x_u A^- = (I + \hat{v}_u)A^- = (A + Rf_x)^{-1}(I - R) = (A + hf_x)^{-1}(I - R) + \mathcal{O}(h)\,.$$

Without changing the order of the scheme the partial derivative $x_u$, which is generally not explicitly known, can be substituted reusing the Jacobian of the system. Penski's approach [11,12] results in a similar approximation to the Milstein scheme in a more specialized setting. The higher order 1 of strong convergence of these schemes has to be paid for by the use of a large number of double stochastic integrals and the use of the derivatives of the diffusion coefficients. In an application with a large number of small noise sources one has to pay much for a mostly theoretical gain in accuracy.

## 4 Numerical Example

We have applied the drift-implicit Euler scheme to simulate a ring-oscillator with simple mosfet-models which was used also in [11]. The equivalent circuit diagram is given in Fig. 3.

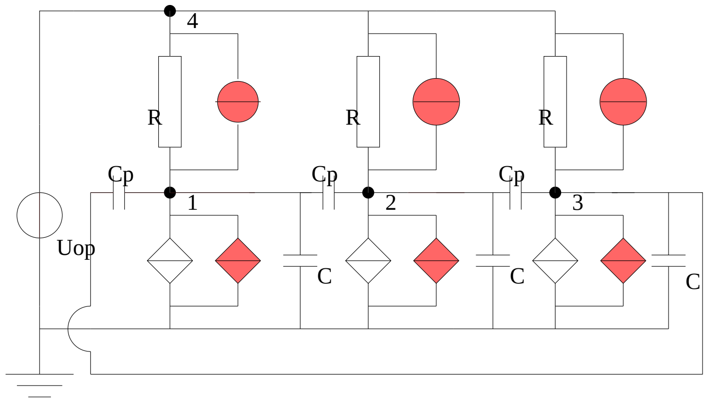



Figure 3: Thermal noise sources in a mosfet ring-oscillator model

The variables in the MNA system are the charges for the six capacities, the four nodal potentials and the current through the voltage source. The system fulfils the assumptions made in Chapter 2 only partly. It is of index 1, but clearly the three thermal resistance noise sources directly affect the current through the voltage source. But, the direct noise occurring in this variable is harmless in the sense that this variable does not influence others. Omitting this critical variable together with the nodal equation for node 4 would lead to a system without direct noise.
In this simple model nearly no differences between the solutions of the noisy and the deterministic problem could be seen. Therefore, we dealt with a system where the diffusion coefficients had been scaled by a factor thousand.
In Fig. 4 we present numerical results obtained with the drift-implicit Euler scheme and fixed stepsize $h = 10^{-11}$. We plotted the nodal potential at node 1. The mean of the computed 100 sample paths differs considerable from the noiseless solution.

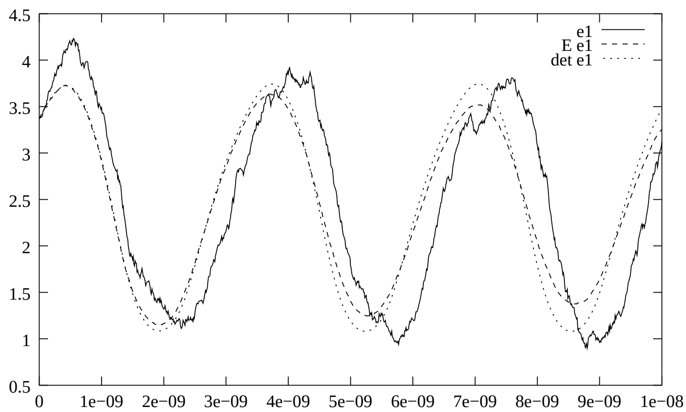



Figure 4: a sample path of the voltage in node 1 (e1) compared to
the mean over 100 sample paths (E e1) and the noiseless voltage (det e1)

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