Stochastic Differential Algebraic Equations in Transient Noise Analysis

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Summary. In this paper we describe how stochastic differential-algebraic equations (SDAEs) arise as a mathematical model for network equations that are influenced by additional sources of Gaussian white noise. We discuss the concepts of weak and strong solutions of SDAEs and give the necessary analytical theory for the existence and uniqueness of strong solutions, provided that the systems have noise-free constraints and are uniformly of DAE-index 1. Further, we analyze discretization methods using the concept of strong convergence. Due to the differential-algebraic structure, implicit methods will be necessary. We present adaptations of known schemes for stochastic differential equations (SDEs) that are implicit in the deterministic and explicit in the stochastic part to SDAEs of index 1, in particular we discuss stochastic analogues to the the drift-implicit Euler and the two-step BDF-scheme.

1 Problem Formulation

The increasing scale of integration, high tact frequencies and low supply voltages cause smaller signal-to-noise-ratios. In several applications the noise influences the system behaviour in an essentially nonlinear way such that linear noise analysis is no longer satisfactory. A possible way out is given by transient noise analysis. Here a circuit model that includes also noisy elements has to be considered and to be simulated in time-domain.

We deal with the thermal noise of resistors as well as the shot noise of semiconductors that are modelled by additional sources of additive or multiplicative Gaussian white noise currents that are shunt in parallel to the ideal, noise-free elements (see Fig. 1). Note, that modeling the internal noise of the elements as external noise sources was originally justified only for linear elements and reciprocal networks.

Nyquist's theorem (see e.g. [B96, DS98, WM98]) states that the current through an arbitrary linear resistor having a resistance R, maintained in thermal equilibrium at a temperature T, can be described as the sum of the deterministic current and a Gaussian white noise process with spectral density

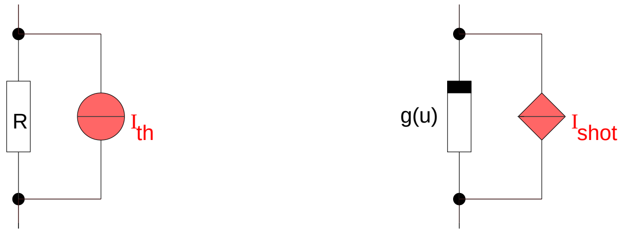

Fig. 1. Thermal noise of a resistor and shot noise of a pn-junction

$S_{th} := \frac{2kT}{R}$, where k is Boltzmann's constant. Hence, the additional current is modelled as

$$I_{th} = \sigma_{th} \cdot \xi(t) = \sqrt{\tfrac{2kT}{R}} \cdot \xi(t) \ ,$$

where $\xi(t)$ is a standard Gaussian white noise process. In [WM98] a thermodynamical foundation to apply this model to mildly nonlinear resistors and reciprocal networks is given.

Shot noise of pn-junctions, caused by the discrete nature of current due to the elementary charge, is also modelled by a Gaussian white noise process. Here the spectral density is proportional to the current I through the pn-junction: $S_{shot} := q|I|$, where q is the elementary charge. If the current through the pn-junction is described by a characteristic $I = g(u)$ in dependence on a voltage u, the additional current is modelled by

$$I_{shot} = \sigma_{shot}(u) \cdot \xi(t) = \sqrt{q|g(u)|} \cdot \xi(t),$$

where $\xi(t)$ is a standard Gaussian white noise process. For a discussion of the model assumptions we refer to [B96, DS98, WM98].

Using the charge-oriented modified nodal analysis (MNA) one formally obtains specially structured differential-algebraic equations with stochastic perturbation terms (see [W03, W02], and for the deterministic case [ET00, GF99]):

$$A\frac{d}{dt}q(x(t)) + f(t, x(t)) + \sum_{r=1}^{m} g^r(t, x(t))\xi^r(t) = 0, \tag{1}$$

where x is the vector of unknowns consisting of the nodal potentials and the branch currents of current-controlled elements (inductances and voltage sources). The vector $q(x)$ consists of the charges of capacitances and the fluxes of inductances. The leading constant matrix A is a singular incidence matrix which is determined by the topology of the network, $f(t, x)$ describes the impact of the static elements, $g^r(t, x)$ denotes the vector of noise intensities for the r-th noise source, and $\xi(t)$ is an m-dimensional vector of independent Gaussian white noise processes. One has to deal with a large number of equations as well as of noise sources. Compared to the other quantities the noise intensities $g^r(t, x)$ are small.

2 Solutions of SDAEs

In established numerical integrations the terms $q(x)$ are treated as extra variables (cf. [GF99]) to guarantee charge-conservation. One can view equation (1) as a compact formulation of the larger system

$$\begin{aligned} A\tfrac{d}{dt}q(t) + f(t,x(t)) + \sum_{r=1}^{m} g^r(t,x(t))\xi^r(t) &= 0, \\ q(t) &= q(x(t)), \end{aligned} \tag{2}$$

with unknowns (q,x). We understand (1) resp. (2) as a stochastic integral equation

$$\begin{aligned} AQ(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(s,X(s))ds + \sum_{r=1}^{m}\int_{t_0}^{t} g^r(s,X(s))dB^r(s) &= 0\ , \\ Q(t) &= q(X(t)), \end{aligned} \tag{3}$$

where the second integral is an Itô-integral, and B denotes an m-dimensional Wiener process (or Brownian motion) given on the probability space $(\Omega,\mathcal{F},P)$ with a filtration $(\mathcal{F}_t)_{t\geq t_0}$. Due to the singularity of the incidence matrix A and the special structure of the system (3), it involves constraints. We call the system (3) a stochastic differential-algebraic equation (SDAE). The solution is a stochastic process $X(t,\omega)$ depending on the time t and on the random sample ω. The value of the solution process at fixed time t is a random variable $X(t,\cdot) = X(t)$ whose argument ω is usually not written. For a fixed sample ω representing a fixed realization of the driving Wiener noise, the function $X(\cdot,\omega)$ is called a realization or a path of the solution. Due to the influence of the Gaussian white noise, typical paths of the solution are nowhere differentiable. A process is called a strong solution of (3) if it is adapted to the filtration (i.e., it does not depend on future information), and if, with probability 1, its sample paths are continuous, the integrals in (3) exist and (3) is fulfilled.

The theory of stochastic differential equations distinguishes between the concepts of weak and strong solutions. The concept of weak solutions is applied if one is interested only in the time-evaluation of the distribution of the solution. One then computes moments of the solution process like expectation and variance. The strong solution concept is applied if one is interested in solution paths.

We illustrate the output of both concepts by simulation results for a noisy MOSFET-ring-oscillator cf. [KRS92, P00]). Only thermal noise in the MOSFETs and in the resistors is considered. The corresponding circuit diagram is given in Fig. 2. To make the differences between the solutions of the noisy and the noise-free model for this simple example more visible, we dealt with a system where the diffusion coefficients had been scaled by a factor of 1000. We plotted values of the nodal potential at node 1 versus the time. The solution of the noise-free system is given by the dashed line. In the left of Fig. 3 we

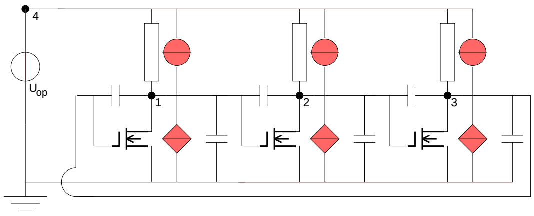

Fig. 2. Figure 2: Thermal noise sources in a MOSFET ring-oscillator model

present quantities obtained from moments of the solution: the mean μ and the boundaries of the confidence interval $[\mu - 3\sigma, \mu + 3\sigma]$, where σ is the estimate for the standard deviation. The mean appears damped and differs considerably from the noiseless, deterministic solution. In the right we present two sample paths (dark solid lines). They indicate that the large deviations from the mean seen in the left picture are mainly due to phase noise. Transient noise

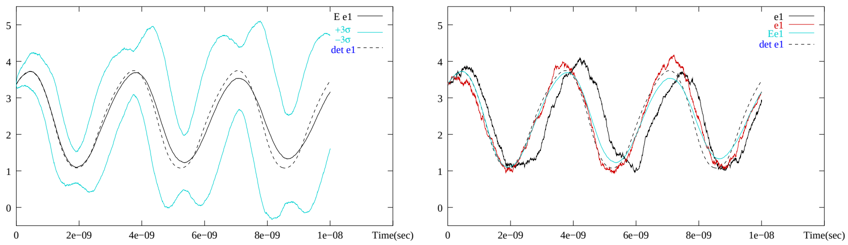

Fig. 3. Statistical parameters and solution paths for the nodal potential at node 1

analysis computes paths of the solution. It allows the computation of moments in a post-processing step. The necessary numerical analysis therefore uses the concept of strong solutions and of strong convergence of approximations.

Combining knowledge from the theory of stochastic differential equations and the theory of differential-algebraic equations the existence and uniqueness of a strong solution of the SDAE (3) is proved in [W03] under the following conditions, which we suppose also in the next chapter: First, assume that the deterministic MNA-system

$$\begin{aligned} A\tfrac{d}{dt}q(t) + f(t, x(t)) &= 0, \\ q(t) &= q(x(t)), \end{aligned} \tag{4}$$

is globally an index 1 DAE in the sense that the constraints are regularly and globally uniquely solvable for the algebraic variables. Second assume that the coefficients f, G, q of the SDAE (3) are globally Lipschitz-continuous with respect to x and continuous with respect to t, and, third, assume that the SDAE

(3) possesses noise-free constraints. This guarantees a solution process that is not directly affected by the white noise process, which is true for the MNA system if there are always capacitances in parallel to a noise source. This is quite restrictive in the actual noise modeling. Nevertheless, one can also handle many situations where this condition is violated. Often noisy constraints are only needed for the determination of algebraic solution components that do not interact with the dynamical ones. Future work should be dedicated to the classification of such situations.

3 Integration schemes for index 1 SDAEs

We present adaptations of known schemes for stochastic differential equations (SDEs) that are implicit in the deterministic and explicit in the stochastic part to the SDAE (3). Designing the methods such that the iterates have to fulfill the constraints of the SDAE at the current time-point is the key idea to adapt known methods for SDEs to (3). This is realized by an implicit Euler or BDF-discretization of the deterministic part.

The noise densities given in Sec. 1 contain small parameters, in fact the square root of Boltzmann's constant $k = 1.3806 \times 10^{-23}$ for thermal noise and of the elementary charge $q_e = 1.602 \times 10^{-19}$ for shot noise. To exploit the smallness of the noise in the analysis of the discretization errors we express the noise densities in the form

$$G(t,x) := \epsilon \tilde{G}(t,x), \quad \epsilon \ll 1. \tag{5}$$

3.1 Drift-Implicit Euler-Maruyama Scheme

On the deterministic grid $0 = t_0 < t_1 < \ldots < t_N = t_{\text{end}}$ the drift-implicit Euler Maruyama scheme for (3) is given by

$$A\frac{q(X_\ell) - q(X_{\ell-1})}{h_\ell} + f(t_\ell, X_\ell) + G(t_{\ell-1}, X_{\ell-1})\frac{\Delta B_\ell}{h_\ell} = 0, \quad \ell = 1, \ldots, N, \tag{6}$$

where $h_\ell = t_\ell - t_{\ell-1}$, $\Delta B_\ell = B(t_\ell) - B(t_{\ell-1})$, and X_ℓ denotes the approximation to $X(t_\ell)$. Realizations of ΔB_ℓ are simulated as $N(0, h_\ell I)$-distributed random variables. The Jacobian of (6) is the same as in the deterministic setting. In general, the Jacobian is solution-dependent and differs from path to path.

The scheme (6) for the SDAE (3) possesses the same convergence properties as the drift-implicit Euler- Maruyama scheme for SDEs (see [W03, W02]). In general, its order of strong convergence is only 1/2, i. e.,

$$\|X(t_\ell) - X_\ell\|_{L_2(\Omega)} := (E|X(t_\ell) - X_\ell|^2)^{1/2} \le c \cdot h^{1/2}, \quad h := \max_{\ell=1,\ldots,N} h_\ell,$$

holds for the mean square norm of the global errors. For additive noise, i. e. $G(t,x) = G(t)$, the order of strong convergence is 1, for small noise (5) the error is bounded by $O(h+\epsilon^2 h^{1/2})$ (see [RW03b], or [MT97] for related results).

The smallness of the noise also allows special estimates of local error terms, which can be used to control the stepsize. The local error for the Euler-Maruyama scheme applied to SDEs with small noise is analyzed in [RW03b]. As long as stepsizes with

$$h_\ell \gg \epsilon^2$$

are used, the dominating local error (per unit step) term of (6) is

$$\eta_\ell := \frac{1}{2}\Big\| A^- \big(f(t_\ell, X_\ell) - f(t_{\ell-1}, X_{\ell-1})\big)\Big\|_{L_2(\Omega)} = \mathcal{O}(h_\ell + \epsilon h_\ell^{1/2}), \qquad (7)$$

where A^- denotes a suitable pseudo-inverse of A. For $\epsilon \to 0$ it approaches the known error estimate in the deterministic setting. If an ensemble of solution paths is computed simultaneously, the estimate η_ℓ can be approximated and may be used to control the local error corresponding to a given tolerance. This results in an adaptive stepsize sequence that is uniform for all solution paths.

3.2 Higher order schemes for small noise SDAEs

Improving the (asymptotic) order of strong convergence of numerical schemes for SDEs or SDAEs would require to include more information on the driving noise process than only the increments of the Wiener process. The so-called Milstein-schemes (see [RW03a, P00]) which possess strong order 1 require derivatives of the diffusion coefficients and double stochastic integrals. In an application with a large number of small noise sources one has to pay much for a merely theoretical gain in accuracy.

When the noise is small, one can believe that the stochastic system, though of a completely different analytical character, has a solution that is somehow ‘close’ to a deterministic one. Then one can hope that for step-sizes that are not asymptotically small the error behaviour is still dominated by the deterministic terms. In addition, one might expect that the stochastic scheme inherits some of the qualitative properties of the deterministic methods. Thus motivated, linear two-step Maruyama methods for SDEs has been analyzed in [BW03, BW04]. Here we present the two-step BDF-Maruyama scheme for the SDAE (3):

$$\begin{aligned} 0 \;=\; & A\frac{q(X_\ell) - \frac{4}{3}q(X_{\ell-1}) + \frac{1}{3}q(X_{\ell-2})}{h} + \frac{2}{3}f(t_\ell, X_\ell) \\ & \qquad + G(t_{\ell-1}, X_{\ell-1})\frac{\Delta B_\ell}{h} - \frac{1}{3}G(t_{\ell-2}, X_{\ell-2})\frac{\Delta B_{\ell-1}}{h} \end{aligned}$$

For small noise the global errors of this scheme are bounded by $\mathcal{O}(h^2 + \epsilon h + \epsilon^2 h^{1/2})$. Below we illustrate this by simulation results for the scalar linear SDE

$$X(t) = 1 + \int_0^t \alpha X_\tau d\tau + \int_0^t \epsilon X_\tau dB_\tau,\ t \in [0,1]$$

with the geometric Brownian motion $X(t) = \exp\big((\alpha - \frac{1}{2}\epsilon^2)t + \epsilon W(t)\big)$ as exact solution. We have chosen the parameters $\alpha = -1,\ \epsilon = 10^{-3}$. In Fig.4 we show the mean-square of the global errors vs. the stepsize for the implicit Euler scheme, the stochastic trapezoidal rule and the two-step BDF-Maruyama-scheme for 100 computed paths in logarithmic scale. The slopes of the lines indicate the observed order of convergence of the schemes. For comparisons a line with slope 1 is given. The error of the Euler scheme shows order 1 behaviour in the considered range of stepsizes. The two schemes with deterministic order 2 show two different regimes. For larger stepsizes the deterministic part of the error of the form $\mathcal{O}(h^2)$ dominates and leads to order 2 behaviour, whereas for smaller stepsizes the error is dominated by a term $\mathcal{O}(\epsilon h)$. In both regimes the errors are considerably smaller than those of the Euler-scheme. The theoretical order 1/2 of the schemes would be observed only for much smaller stepsizes.

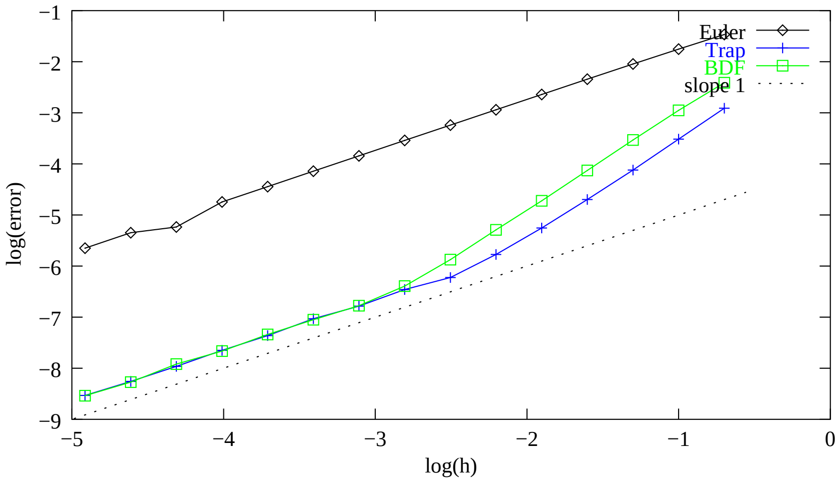

Fig. 4. global error vs. stepsize in logarithmic scale

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