Finiteness Theorems

Emre Can Sertöz

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1 Finiteness of isogeny classes

Set-up 1. Define K to be an algebraic number field and S a finite collection of finite places of K.

Notation. We will use G_K to denote $\mathrm{Gal}(\bar{K}/K)$, the absolute Galois group of K.

Set-up 2. Given a positive integer d, a prime l and assuming Set-up 1, define $\rho : G_K \to \mathrm{GL}_{\mathbb{Q}_l}(V)$ to be a semi-simple representation of dimension d, unramified outside of S.

Set-up 3. Use Set-up 1. Given an integer $g \geq 1$, define A to be an abelian variety over K of dimension g with good reduction outside of S.

Theorem 1.1. *Fix an integer $g \geq 1$. There are only finitely many isogeny classes of abelian varieties A as in Set-up 3.*

Strategy of proof. Choose a prime l. In proving the Tate conjectures we have seen that two abelian varieties A and B are isogenous iff $V_l(A)$ and $V_l(B)$ are isomorphic as G_K representations.

A standard result implies that semi-simple G_K-representations are isomorphic if the corresponding trace functions on G_K coincide. We will prove the theorem by proving the following two statements:

- **Reduction 1.** Find finitely many elements $F_1, \ldots, F_b$ in G_K such that any two representations ρ_1, ρ_2 of dimension $d = 2g$ as in Set-up 2 are isomorphic if the corresponding trace of these finitely many elements are equal, i.e.

$$\mathrm{tr}\, \rho_1(F_i) = \mathrm{tr}\, \rho_2(F_i) \text{ for all } i = 1, \ldots, b.$$

- **Reduction 2.** Show that there is a finite set of values U depending only on Set-up 1 and $g \geq 1$ which contains the values $\mathrm{tr}\, \rho(F_i)$ for all i and all ρ which is the Tate representation $V_l(A)$ of some A as in Set-up 3.

These two facts combined imply that there can be only finitely many G_K-representations arising as $V_l(A)$, which in turn imply that there can only be finitely many isogeny classes.

In the rest of this section we establish these two reductions. □

1.1 Traces and representations - first reduction

Use Set-up 1. Fix an integer $d > 0$ and a prime $l \in \mathbb{Z}$. Let ρ_1, ρ_2 be G_K-representations as in Set-up 2. We will denote by t_i the composition $\mathrm{tr} \circ \rho_i : G_K \to \mathbb{Q}_l$. As mentioned earlier, the two representations ρ_1 and ρ_2 are isomorphic iff $t_1 = t_2$. Let us see how far we can force this result.

Let $\rho = \rho_1 \times \rho_2 : \mathbb{Z}_l[G_K] \to \mathrm{End}(V_1) \times \mathrm{End}(V_2)$ and denote by M the image of ρ. We are interested in the difference of the traces, that is in the function $t = t_2 - t_1$. However, t

factors through M by $(m_1, m_2) \mapsto \operatorname{tr} m_2 - \operatorname{tr} m_1$. If $t : M \to \mathbb{Q}_l$ is identically zero, then the two representations are isomorphic. We now need to find good generators for M to test the value of t.

Since M is finitely generated and $\mathbb{Z}_l$ is local, any set of generators of M/lM lift to generators of M (Nakayama's Lemma). Furthermore, if we follow the quotient we get a map:

$$\bar{\rho} : G_K \to (M/lM)^*$$

into the units of M/lM. Note that the image of $\bar{\rho}$ generates M/lM over $\mathbb{Z}_l$.

Since M is a finitely generated $\mathbb{Z}_l$-module in a $\mathbb{Q}_l$ vector space of dimension $2d^2$, M is a free $\mathbb{Z}_l$-module of rank $\leq 2d^2$. Therefore, M/lM is a $\mathbb{F}_l$-algebra of dimension $\leq 2d^2$.

Because $(M/lM)^*$ is finite, $\bar{\rho}$ factors through a finite quotient of G_K. The Fundamental Theorem of Galois Theory tells us that any finite quotient of G_K is $\operatorname{Gal}(L/K)$ for some finite Galois extension L/K. Therefore we have a commutative diagram:

$$\begin{array}{ccc} G_K & \xrightarrow{\bar{\rho}} & (M/lM)^* \\ \downarrow & \nearrow & \\ \operatorname{Gal}(L/K) & & \end{array}$$

Here L depends on the map ρ and we wish to remove this dependence. To that end observe that

$$\#(M/lM)^* < l^{2d^2}.$$

Consequently, $[L : K] = |\operatorname{Gal}(L/K)| < l^{2d^2}$.

We will now show that L/K is unramified ourside of S, i.e. the inertia group $I_{\mathfrak{p}}$ over any place outside of S is 0. Note that the inertia group over a place is the image of the absolute ramification over that place (upto conjugation). But $\operatorname{Gal}(L/K)$ is isomorphic to the image of $\bar{\rho}$ and we assumed ρ was unramified outside of S, which means the absolute inertia of places outside of S are killed by ρ, and thus by $\bar{\rho}$. Consequently, $\operatorname{Gal}(L/K)$ has no inertia outside of S.

As a consequence of Hermite-Minkowski Theorem, we may choose a *finite* field extension $\tilde{K}/K$ such that $\tilde{K}$ contains *all* finite Galois extensions K'/K of degree $< l^{2d^2}$ unramified outside of S. Then $\bar{\rho}$ factors through as follows:

$$\begin{array}{ccc} G_K & \xrightarrow{\bar{\rho}} & (M/lM)^* \\ \downarrow & \nearrow_{\tilde{\rho}} & \\ \operatorname{Gal}(\tilde{K}/K) & & \end{array}$$

We have thus successfully eliminated the dependence of the finite extension on ρ.

Since the image of $\bar{\rho}$ generates M/lM as a $\mathbb{Z}_l$-module, so does the image of $\tilde{\rho}$. Let $f_1, \ldots, f_b \in \operatorname{Gal}(\tilde{K}/K)$ be elements whose conjugacy classes cover $\operatorname{Gal}(\tilde{K}/K)$. Lift f_i's to G_K and denote them by F_i's. Since the conjugacy classes of f_i's generate M/lM, the conjugacy classes of F_i's generate M by Nakayama. But t is invariant under conjugation, thus we really need to check the finitely many values $t(F_i)$ for $i = 1, \ldots, b$. If all these values are zero then the two representations ρ_1 and ρ_2 are equal.

Notice that F_i's do not depend on the two representations we started with, but only on d and l. Therefore, with d and l as above, if we are given any two representations ρ_1 and ρ_2 as in Set-up 2, then these two representations are isomorphic if and only if $\operatorname{tr} \rho_1(F_i) = \operatorname{tr} \rho_2(F_i)$ for all $i = 1, \ldots, b$.

1.2 The Frobenius element

Our choice of F_i's above will be Frobenius elements. Defining these automorphisms properly is quite a bit of work, which we begin now.

1.2.1 Finite extensions

Let K be a number field and let L/K be a finite Galois extension. Fix a prime $\mathfrak{p}$ of K. Choose a prime $\mathfrak{q}$ of L, lying above $\mathfrak{p}$. Denote the residue field of $\mathfrak{p}$ by $\mathbb{F}(\mathfrak{p}) = \mathcal{O}_K/\mathfrak{p}$, and let $N(\mathfrak{p}) = \#\mathbb{F}(\mathfrak{p})$. Similarly define $\mathbb{F}(\mathfrak{q})$ and $N(\mathfrak{q})$.

Since $\mathcal{O}_L = \operatorname{intcl}_L \mathbb{Z}$, any automorphism of L will restrict to an automorphism of $\mathcal{O}_L$. Define $D(\mathfrak{q}/\mathfrak{p}) = \{\sigma \in \operatorname{Gal}(L/K) \mid \sigma\mathfrak{q} = \mathfrak{q}\}$. Each $\sigma \in D(\mathfrak{q}/\mathfrak{p})$ restricts to an automorphism of $\mathbb{F}(\mathfrak{p}) = \mathcal{O}_L/\mathfrak{q}$ over $\mathbb{F}(\mathfrak{p})$. This gives us a map:

$$D(\mathfrak{q}/\mathfrak{p}) \to \operatorname{Gal}(\mathbb{F}(\mathfrak{q})/\mathbb{F}(\mathfrak{p})).$$

Now we will show that this map is surjective. Write $L \simeq K[x]/(f)$ for some monic and irreducible $f \in \mathcal{O}_K[x]$. Similarly, write $\mathbb{F}(\mathfrak{q}) \simeq \mathbb{F}(\mathfrak{p})[x]/(g)$. It is easy to see that $f \mod \mathfrak{p}$ is g^e for some integer e. Let α be a root of f, note $\alpha \in \mathcal{O}_L$. Let $\bar{\alpha}$ be its image in $\mathbb{F}(\mathfrak{q})$. Any automorphism $\sigma \in \operatorname{Gal}(\mathbb{F}(\mathfrak{q})/\mathbb{F}(\mathfrak{p}))$ is completely determined by $\beta_\sigma := \sigma(\bar{\alpha})$, which is a root of g. Thus we can find a root of f, say γ, which restricts to β_σ. The extension L/K being Galois there is an automorphism of L taking α to γ. This automorphism restricts to σ. Consequently we have an exact sequence:

$$0 \to I(\mathfrak{q}/\mathfrak{p}) \to \operatorname{Gal}(L_\mathfrak{q}/K_\mathfrak{p}) \to \operatorname{Gal}(\mathbb{F}(\mathfrak{q})/\mathbb{F}(\mathfrak{p})) \to 0$$

where the kernel $I(\mathfrak{q}/\mathfrak{p})$ is called an *inertia subgroup*. The inertia group of two primes lying over $\mathfrak{p}$ differ by conjugation. We will say $\mathfrak{p}$ is unramified if any, and hence all, of these inertia groups over $\mathfrak{p}$ vanish.

Notice that there is a canonical choice of a generator for $\operatorname{Gal}(\mathbb{F}(\mathfrak{q})/\mathbb{F}(\mathfrak{p}))$, which we denote by $\operatorname{Frob}(\mathfrak{q}/\mathfrak{p}) : x \mapsto x^{N(\mathfrak{p})}$. One crucial observation is this: If $\mathfrak{p}$ is unramified, then there is a canonical lift of the Frobenius action $\operatorname{Frob}(\mathfrak{q}/\mathfrak{p})$ on the residue fields to an automorphism in $\operatorname{Gal}(L/K)$. We will call this automorphism a Frobenius element, and continue to denote it with the symbol $\operatorname{Frob}(\mathfrak{q}/\mathfrak{p})$. Any other element in the conjugacy class of $\operatorname{Frob}(\mathfrak{q}/\mathfrak{p})$ is of the form $\operatorname{Frob}(\mathfrak{q}'/\mathfrak{p})$ for a prime $\mathfrak{q}'$ lying over $\mathfrak{p}$. Thus, it is convenient to denote this conjugacy class by $\operatorname{Frob}_\mathfrak{p}$.

1.2.2 Absolute Frobenius

Now we wish to construct a Frobenius element in $\operatorname{Gal}(\bar{K}/K)$ corresponding to a prime $\mathfrak{p}$ of $\mathcal{O}_K$. As before, we need actual primes lying over $\mathfrak{p}$ to define such an element. One way to choose primes over $\mathfrak{p}$ is to consider the following

Set-up 4. Let $K \subset \bar{\mathbb{Q}}$ be a number field. Fix an embedding $i : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, where $(p) = \mathfrak{p} \cap \mathbb{Q}$, such that the valuation ideal in $\bar{\mathbb{Q}}_p$ restricts to $\mathfrak{p}$ in K.

This setup is useful because for any field extension $K \subset L \subset \bar{\mathbb{Q}}$, pulling back the valuation of $\bar{\mathbb{Q}}_p$ gives a prime $\mathfrak{q}$ in $\mathcal{O}_L$ lying over $\mathfrak{p}$. Notice that this choice of a prime lying over $\mathfrak{p}$ in each extension of K is done in a consistent manner.

Suppose $K \subset L \subset T$ are a series of finite extensions contained in $\bar{\mathbb{Q}}$, with the chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{r}$ induced from the embedding i. Then we have the following morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & I(\mathfrak{r}/\mathfrak{p}) & \to & D(\mathfrak{r}/\mathfrak{p}) & \to & \operatorname{Gal}(\mathbb{F}(\mathfrak{r})/\mathbb{F}(\mathfrak{p})) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I(\mathfrak{q}/\mathfrak{p}) & \to & D(\mathfrak{q}/\mathfrak{p}) & \to & \operatorname{Gal}(\mathbb{F}(\mathfrak{q})/\mathbb{F}(\mathfrak{p})) & \to & 0. \end{array}$$

Taking the inverse limit of these short exact sequences over all finite extensions of K in $\bar{\mathbb{Q}}$ we get an exact sequence (the inverse limit over a compact set being exact):

$$0 \to I_\mathfrak{p} \to D(\mathfrak{p}) \to \operatorname{Gal}(\bar{\mathbb{F}}(\mathfrak{p})/\mathbb{F}(\mathfrak{p})) \to 0. \; .$$

Here $D(\mathfrak{p})$ is the subgroup of $\operatorname{Gal}(\bar{K}/K)$ fixing the infinite chain of primes above $\mathfrak{p}$.

We will denote by $\mathrm{Frob}_\mathfrak{p}$ the Frobenius automorphism in $\mathrm{Gal}(\bar{\mathbb{F}}(\mathfrak{p})/\mathbb{F}(\mathfrak{p}))$, which is also called the absolute Frobenius. Notice that by construction, any lift of the absolute Frobenius restricts to $\mathrm{Frob}(\mathfrak{q}/\mathfrak{p})$ in $\mathrm{Gal}(L/K)$ (modulo inertia).

1.2.3 Characteristic polynomial of the Frobenius element

Definition 1.2. Use Set-up 4. Let $G_K = \mathrm{Gal}(\bar{K}/K)$. Let V be a k-vector space, for some field k, and consider a representation $\rho : G_K \to \mathrm{GL}_k(V)$. The representation ρ is said to be *unramified* at a prime $\mathfrak{p}$ if $\rho(I_\mathfrak{p}) = 0$. In this case, we can lift the absolute Frobenius element in anyway we like and still the image $\rho(\mathrm{Frob}_\mathfrak{p})$ is well defined. We shall call this the *Frobenius element of the representation ρ over $\mathfrak{p}$.*

Recall that Set-up 4 implies that we have a chosen prime lying over $\mathfrak{p}$ in every finite extension of K. Making a different series of choices, i.e. a different compatible inclusion of K into $\bar{\mathbb{Q}}_p$, would have conjugated the inertia group and any lift of the absolute Frobenius. This means, any construction done with the Frobenius element of the representation that is invariant under conjugation can be defined using just $\mathfrak{p}$. For instance, the characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{p})$ and consequently the trace and determinant are well defined even without the choice of a series of primes lying above $\mathfrak{p}$.

In summary we have:

Fact 1.3. *Let K be a number field, and $\mathfrak{p}$ a prime of $\mathcal{O}_K$. Suppose $\rho : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_k(V)$ is unramified at $\mathfrak{p}$. Then we may unambigously define a polynomial which is the* characteristic polynomial of *$\rho(\mathrm{Frob}_\mathfrak{p})$.*

The primes of $\mathcal{O}_K$ are called finite places. They correspond to inclusions $K \to \bar{\mathbb{Q}}_p$ for primes p. The infinite places are the inclusions $K \to \mathbb{C}$. The corresponding Frobenius element for an infinite place is simply the restriction onto K the conjugation action on $\mathbb{C}$. No lifting is required here. From now on we will freely talk about the Frobenius element Frob_v for a place v of K, whether v is finite or infinite.

1.3 The trace of Frobenius - second reduction

Set-up 5. Evoke Set-up 1. Given a prime $l \in \mathbb{Z}$ and an integer $d \geq 1$, choose $\tilde{K} \subset \bar{K}$ a finite Galois extension of K, containing all Galois extensions K'/K of degree $< l^{2d^2}$ unramified outside of S.

Recall that in Section 1.1, we dealt with finitely many elements $F_1, \ldots, F_b$ whose conjugacy classes covered $\tilde{G} = \mathrm{Gal}(\tilde{K}/K)$. Now we will determine a certain collection of elements having this property, whose traces we can control. Our goal is to prove Reduction 2 of Section 1.

A consequence of the density theorem of Čebotarev is the following:

Theorem 1.4. *Evoke Set-up 1. Let L/K be a finite Galois extension, unramified outside of S. Then there exists a finite set of places T of K such that the conjugacy classes $\{\mathrm{Frob}_v \mid v \in T\}$ cover $\mathrm{Gal}(L/K)$.*

Set-up 6. Use Set-up 5. Find a finite set of places T of K as in Theorem 1.4 for the extension $\tilde{K}/K$ using the set $S_l = S \cup \{\text{places dividing } l\}$.

Notice that if ρ is a representation of $\mathrm{Gal}(\tilde{K}/K)$ unramified at T, then the value $\mathrm{tr}\,\rho(\mathrm{Frob}_v)$ is well defined for all $v \in T$, and so are the characteristic polynomials (see Fact 1.3).

Evoke Set-up 3. Further choose a finite place $v \notin S_l$. Then the $\mathrm{Gal}(\bar{K}/K)$-representation on $V_l(A)$ is unramified at v. Hence, we can define the characteristic polynomial of Frob_v with respect to this action, we will denote it by $P(A, v)$.

Theorem 1.5 (Weil)**.** *The polynomial $P(A, v)$ is independent of l. Furthermore, the coefficients of $P(A, v)$ are integers and all the roots (over $\mathbb{C}$) have absolute value equal to $N(v)^{\frac{1}{2}}$.*

Fix $N \in \mathbb{R}$ and $d \in \mathbb{N}$ to define the set $X = \{f \in \mathbb{C}[x] \mid f = \prod_{i=1}^{d}(x - \lambda_i),\ |\lambda_i| = N\}$. It is clear that the coefficients of the polynomials in X are bounded. Thus, the subset of X consisting of polynomials with integer coefficients must be finite. For each A of dimension g, the polynomial $P(A, v)$ has degree $2g$ and has the bound $N(v)^{\frac{1}{2}}$ on the absolute value of its roots. We have just shown:

Fact 1.6. *Fixing $g \geq 1$, there are only finitely many polynomials that are realized as the characteristic polynomial of* Frob_v *on $V_l(A)$ for* any *A as in Set-up 3 and* any *prime l which is not divisible by v.*

Of course the trace of Frob_v is just the coefficient of x^{2g-1} in $P(A, v)$. Let U_v be the set of finitely many integers which are realized as the coefficient of x^{2g-1} in some $P(A, v)$ (v is fixed). Let $U = \prod_{v \in T} U_v$.

Our first reduction implies that a $\mathrm{Gal}(\bar{K}/K)$-representation $V_l(A)$ is determined by the tuple $(\mathrm{tr}_{V_l(A)} \mathrm{Frob}_v)_{v \in T}$, which lies in the finite set U. This proves Reduction 2 and thus completes the proof of Theorem 1.1.

2 Finiteness of isomorphism classes

The goal of this section is to prove:

Theorem 2.1. *Use Set-up 1. There are only finitely many isomorphism classes of abelian varieties of dimension $g \geq 1$ over K with good reduction outside of S.*

One of the key ingredients we are going to be using is the finiteness of the isogeny classes. Since there are only finitely isogeny classes, we need only show each isogeny class contains finitely many isomorphism classes. Another key ingredient is

Theorem 2.2 (Falting's Height Theorem)**.** *Let K be a number field, $g \geq 1$ an integer and $c > 0$* real *number. Then there are only finitely many isomorphism classes of principally polarized semi-stable abelian varieties of dimension g over K, having height less than c.*

To use Falting's height theorem we need to reduce our problem from that of counting abelian varieties with good reduction outside of S to that of counting abelian varieties that admit a principle polarization and are semi-stable.

Fact 2.3. *Using Zarhin's Trick as explained in [Bru15] we immediately conclude that isomorphism class of abelian varieties are finite if isomorphism classes of abelian varieties with principal polarization is finite.*

Fact 2.4. *Let A be an abelian variety as in Set-up 3. There is a finite field extension K' of K such that all abelian varieties B in the isogeny class $\mathrm{cl}_{K'}(A)$ of A are semi-stable. Here $\mathrm{cl}_{K'}(A)$ denotes the isogeny class of $A \times_K K'$ over K'. See [Fal+92, p. 169] for more clarification.*

In using the last fact we must be careful. In passing from $\mathrm{cl}_K(A)$ to $\mathrm{cl}_{K'}(A)$ we become more generous with the automorphisms that we allow. Therefore, the isomorphism classes 'grow larger' decreasing the number of isomorphism classes. Hence it should be checked that infinitely many isomorphism classes over K can not be identified to one isomorphism class over K'. But I won't go into this here.

Facts 2.3 and 2.4 imply that Theorem 2.1 will follow from the following

Theorem 2.5. *Let K be a number field and $g \geq 1$ an integer. Let A be a semi-stable principally polarized abelian variety over K. The isogeny class* $\mathrm{cl}(A)$ *of A contains only finitely many isomorphism classes of semi-stable principally polarized abelian varieties.*

To achieve this result we need only prove that the Falting's height $h : \mathrm{cl}(A) \to \mathbb{R}_{\geq 0}$ is bounded. This bound will be achieved by the following

Theorem 2.6. *Let K be a number field and $g \geq 1$ an integer. There exists a finite set of primes N such that for any principally polarized abelian variety B over K and any isogeny $\phi : B \to B'$ of degree relatively prime to all elements in N we have*

$$h(B) = h(B').$$

Proof. The proof will be given next week by Barbara. □

To control the degree of isogenies we need the following

Lemma 2.7. *Let N be a set of primes. Suppose two abelian varieties B and B' over K are isogenous and $T_l B \simeq T_l B'$ as* $\mathrm{Gal}(\bar{K}/K)$*-modules for all $l \in N$. Then there exists an isogeny $\phi : B \to B'$ such that the degree of ϕ is prime to all elements in N.*

Proof. See [Fal+92, p. 169]. □

Here is how to use this lemma. Suppose $B \in \mathrm{cl}(A)$ and $\mu : B \to A$ is an isogeny. Then the inclusion $T_l B \hookrightarrow V_l B$ composed with the isomorphism $V_l B \xrightarrow{\sim} V_l A$ induced by ϕ, gives us a $\mathrm{Gal}(\bar{K}/K)$-invariant lattice in $V_l A$. If $B' \in \mathrm{cl}(A)$ has an isogeny to A giving us a lattice in $V_l A$ isomorphic to $T_l B$ then we may conclude, using the lemma above, that B and B' have an isogeny of degree relatively prime to l. This is the idea we are going to pursue now. To that end we need a standard result about lattices in $V_l A$.

Lemma 2.8. *Let U_l be the set of isomorphism classes of* $\mathrm{Gal}(\bar{K}/K)$*-invariant lattices in $V_l A$. Then U_l is finite.*

Proof of Theorem 2.5. Let N be the finite set of primes given by Theorem 2.6 for the abelian variety A. Define

$$U = \prod_{l \in N} U_l$$

where U_l is the finite set defined in Lemma 2.8. Any isogeny $A' \to A$ gives us a lattice $\lambda_{A',l} \in U_l$ as discussed above. Therefore we get a tuple $\lambda_{A'} \in U$. Despite the notation here, this tuple depends on the isogeny and not just on A'. Let $\tilde{U}$ be the subset of U consisting of tuples that are realized by an isogeny. Since $\tilde{U}$ is a finite set, we can find finitely many isogenies

$$\mu_i : A_i \to A \quad i = 1, \dots, n$$

such that $\tilde{U} = \{\lambda_{A_1}, \dots, \lambda_{A_n}\}$.

Consequently, any isogeny $B \to A$ must satisfy $\lambda_B = \lambda_{A_i}$ for some i. We may now apply Lemma 2.7 to conclude that there exists an isogeny $\phi_i : B \to A_i$ of degree relatively prime to all primes in N.

We must further insist that *whenever possible* we choose A_i to be semi-stable and principally polarizable. Thus, for any semi-stable and principally polarizable B having $\lambda_B = \lambda_{A_i}$ the conditions of Theorem 2.6 are satisfied and therefore $h(B) = h(A_i)$.

In particular, the height function on semi-stable principally polarizable elements of $\mathrm{cl}(A)$ is bounded by $\max_i h(A_i)$. Using Falting's Height Theorem completes the proof. □

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