The Tate Conjecture from Finiteness

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These are notes for a talk given at the IRTG College Seminar on Faltings' proof of the Mordell conjecture in Berlin on May 6, 2015. The main references are [2], [3] and [4].

1 Introduction & Motivation

Fix a number field K with absolute Galois group $G := \mathrm{Gal}(\overline{K}/K)$. For an abelian variety A over K, recall that we defined the Tate module

$$T_\ell A := \varprojlim_n A[\ell^n](\overline{K}) = \{(a_n)_{n\geq 1} \in A[\ell^n](\overline{K}) \mid \forall n \geq 2 : \ell a_n = a_{n-1}\}.$$

For convenience, we also define $V_\ell A := T_\ell A \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$.

Theorem 1 (Finiteness of isogeny classes)**.** *Let A be an abelian variety over K. Up to isomorphism, there are only finitely many abelian varieties over K isogenous to A.*

This finiteness statement has the following six corollaries:

(1) **Semisimplicity Theorem:** If A is an abelian variety over K and ℓ is a prime, then the action of G on $V_\ell A$ is semisimple.

(2) **Tate conjecture:** Let A, B be abelian varieties over K. Then the natural injection

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell[G]}(T_\ell A, T_\ell B)$$

is an isomorphism.

(3) **Isogeny theorem:** Let A, B be abelian varieties over K. Then A and B are isogenous if and only if $V_\ell A \cong V_\ell B$ as $\mathbb{Q}_\ell[G]$-modules.

(4) **Finiteness II:** Fix an integer g and a finite set S of primes of K. Up to isomorphism, there are only finitely many abelian varieties over K of dimension g with good reduction outside S.

(5) **Shafarevich conjecture:** Fix $g \in \mathbb{Z}$ and a finite set S of primes of K. Up to isomorphism, there are only finitely many smooth projective curves over K of genus g with good reduction outside S.

(6) **Mordell conjecture:** Let C be a smooth projective curve over K of genus ≥ 2. Then $C(K)$ is finite.

The aim of this talk is to focus on the Tate conjecture and to explain how Theorem 1 implies (1) and (2). This was done by Tate himself in [7]. It is easy to see that (2) implies (3):

Lemma 2. *The Tate conjecture (2) implies the isogeny theorem (3).*

Proof. Since $\mathbb{Q}_\ell$ is flat over $\mathbb{Z}_\ell$, the space $\operatorname{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$ is naturally included in $\operatorname{Hom}_{\mathbb{Q}_\ell[G]}(V_\ell A, V_\ell B)$.

If $f \in \operatorname{Hom}(A,B)$ is an isogeny, then there is another isogeny $g : B \to A$ such that $f \circ g = [n]_B$ and $g \circ f = [n]_A$ for some integer $n \geq 1$. So f becomes invertible in $\operatorname{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$ and thus gives a $\mathbb{Q}_\ell[G]$-isomorphism of $V_\ell A$ with $V_\ell B$.

Conversely, if $\varphi : V_\ell A \to V_\ell B$ is a $\mathbb{Q}_\ell[G]$-isomorphism, then $\ell^n\varphi \in \operatorname{Hom}_{\mathbb{Z}_\ell[G]}(T_\ell A, T_\ell B)$ for some $n \geq 1$ with $\det(\ell^n\varphi) \neq 0$. By the Tate conjecture, this comes from an element in $\operatorname{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$ and can therefore be approximated by elements of $\operatorname{Hom}(A,B)$ with non-zero determinant. This gives the desired isogeny. □

Remark. This is not quite the strategy Faltings used in his 1983 paper [1] to prove Mordell's conjecture. He first developped a weaker finiteness result (see the end of Section 3), which sufficed to prove semisimplicity and the Tate conjecture. From that he deduced the finiteness of the isogeny classess, Finiteness II and the Shafaravich conjecture. Together with the work of Parshin [6], this proved the Mordell conjecture.

2 Prerequisites

Let K be a perfect field with absolute Galois group $G := \operatorname{Gal}(\overline{K}/K)$.

Lemma 3. *Let $f : A \to B$ be an isogeny of abelian varieties over K and $\ell \neq \operatorname{char} K$ be a prime. Then f induces an injective $\mathbb{Z}_\ell[G]$-homomorphism $T_\ell f : T_\ell A \to T_\ell B$.*

Proof. The existence of $T_\ell f$ follows from the natural inclusion

$$\operatorname{Hom}(A,B)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \hookrightarrow \operatorname{Hom}_{\mathbb{Z}_\ell[G]}(T_\ell A, T_\ell B).$$

Let $(a_n)_{n\geq 1}$ be an element of

$$\ker T_\ell f = \varprojlim_n K_n = \{(a_n)_{n\geq 1} \in K_n \mid \forall n \geq 2 : \ell a_n = a_{n-1}\}.$$

For $n \geq 1$, denote by K_n the kernel of the restriction of f to $A[\ell^n](\overline{K})$. Then the chain $K_1 \subseteq K_2 \subseteq \ldots$ stabilizes at some point because $\ker f$ is finite. So if n is large enough, we can assume that $a_{2n} \in K_{2n}$ lies in fact in K_n. In particular $0 = \ell^n a_{2n} = \ell^{n-1}a_{2n-1} = \cdots = \ell a_{n+1} = a_n$, as $a_{2n} \in K_n \subseteq A[\ell^n](\overline{K})$. Since $a_{n-m} = \ell^m a_n = 0$ for $0 \leq m < n$, all elements in the sequece $(a_n)_{n\geq 1}$ are 0. □

Remark. Moreover, one can show that the cokernel of such a $T_\ell f$ is finite.

The following is the key to the proof of the semisimplicity theorem and the Tate conjecture:

Lemma 4. *Suppose that Theorem 1 holds. Let A be an abelian variety over K and $W \subseteq V_\ell A$ be a G-stable subspace. Then there is an $u \in \operatorname{End}(A)\otimes\mathbb{Q}_\ell$ such that $u(V_\ell A) = W$.*

Proof.

A family of isogenies. Consider for $n \geq 1$ the natural inclusion

$$G_n := (W \cap T_\ell A)/\ell^n (W \cap T_\ell A) \hookrightarrow T_\ell A/\ell^n T_\ell A \cong A[\ell^n](\overline{K}).$$

These groups are G-stable and thus the quotients $A_n := A/G_n$ exist. They are abelian varieties over K and isogenous to A via the canonical quotient maps $p_n : A \to A_n$. Since G_n is contained in the ℓ^n-torsion points, the multiplication-by-ℓ^n map on A factors through A_n. In particular, there are isogenies $f_n : A_n \to A$ such that $f_n \circ p_n = [\ell^n]_A$. The kernel of f_n is a quotient of $A[\ell^n]$, hence the degree of f_n is a power of the prime ℓ.

Induced maps on the Tate module. The f_n induce maps $T_\ell f_n : T_\ell A_n \to T_\ell A$, which are injective by Lemma 3. Hence they induce an identification of their image $U_n := T_\ell f_n(T_\ell A_n) \subseteq T_\ell A$ with $T_\ell A_n$. We claim that $U_n = W \cap T_\ell A + \ell^n T_\ell A$. Since multiplication by ℓ^n on $T_\ell A$ is given as the composition $T_\ell f_n \circ T_\ell p_n$, clearly $\ell^n T_\ell A \subseteq \operatorname{im}(T_\ell f_n)$. Thus it suffices to show that the image of $T_\ell f_n$ in $T_\ell A/\ell^n T_\ell A \cong A[\ell^n](\overline{K})$ is G_n. If $x \in A_n[\ell^n](\overline{K})$, then there is an element $y \in A(\overline{K})$ such that $p_n(y) = x$, $\ell^n y \in G_n$ and thus $f_n(x) = \ell^n y \in G_n$. Conversely, since G_n is ℓ-divisible, every $g \in G_n$ can be written as $\ell^n \cdot y$ for some $y \in A(\overline{K})$.

Using finiteness. By Theorem 1, we know that there are only finitely many abelian varieties isogenous to A. Hence there must be an infinite subset $I \subseteq \mathbb{N}$ such that $A_i \cong A_j$ for all $m, n \in I$. Let m denote the smallest integer in I. For all $n \in I$, define $u_n \in \operatorname{End}(A) \otimes \mathbb{Q}_\ell$ as the composition

$$u_n : A \xrightarrow{f_m^{-1}} A_m \cong A_n \xrightarrow{f_n} A,$$

remembering that isogenies are invertible after tensoring with $\mathbb{Q}_\ell$. On the level of Tate modules,

$$V_\ell u_n : V_\ell A \xrightarrow{V_\ell f_m^{-1}} V_\ell A_m \cong V_\ell A_n \xrightarrow{V_\ell f_n} V_\ell A$$

satisfies $V_\ell u_n(U_m) \cong U_n \subseteq U_m$ and gives hence an element $v_n \in \operatorname{End}(U_m)$.

Using compactness. Since $T_\ell A_m$ is a free $\mathbb{Z}_\ell$-module of rank $2 \dim A_m$ (Gregor's talk), $\operatorname{End}(T_\ell A_m)$ is a free $\mathbb{Z}_\ell$-module of rank $4(\dim A_m)^2$. The p-adic integers are compact in the ℓ-adic topology, and therefore $\operatorname{End}(U_m) \cong \operatorname{End}(T_\ell A_m)$ is compact as well. As a consequence, we can assume that the sequence $(v_n)_{n \in I}$ converges to some element $v \in \operatorname{End}(U_m)$. As $\operatorname{End}(A) \otimes \mathbb{Q}_\ell$ is closed in $\operatorname{End}(V_\ell A)$, and any v_n comes from some $u_n \in \operatorname{End}(A) \otimes \mathbb{Q}_\ell$, also v comes from some $u \in \operatorname{End}(A) \otimes \mathbb{Q}_\ell$.

We have that $W \cap T_\ell A = \bigcap_{n \geq 1} U_n$. As $v_n(U_m) = U_n$ for all $n \in I$, any $x \in W \cap T_\ell(A)$ is a limit of $v_n(y_n)$ for certain $y_n \in U_m$. Since $U_m \cong T_\ell A_m \cong \mathbb{Z}_\ell^{2 \dim A_m}$ is compact, we can pass to an accumulation point y of the sequence (y_n) and choose a subsequence converging to y. Then

$$x = \lim_n v_n(y_n) = \lim_n v(y_n) = v(y),$$

which shows that $W \cap T_\ell A \subseteq v(U_m)$. Tensoring with $\mathbb{Q}_\ell$ gives $W \subseteq u(V_\ell A)$. Conversely,

$$u(V_\ell A) = \lim_n u_n(V_\ell A) \subseteq \bigcap_{n \geq 1} (U_n \otimes \mathbb{Q}_\ell) = W.$$

Thus $u(V_\ell A) = W$ as desired. □

3 Semisimplicity and Tate conjecture

Definition. Let R be a not necessarily commutative ring and M an R-module.

- M is called *semisimple* if every R-submodule of M is a direct summand of M, or equivalently, if M decomposes as a direct sum of simple modules.
- An R-algebra is *(left-/right-)semisimple* if it is semisimple as a (left-/right-)module over itself.

Remark. For R-algebras, left semisimplicity is equivalent to right semisimplicity, thus we may simply speak of semisimplicity.

Lemma 5. *Let A be an abelian variety over K. Then* $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$ *is a semisimple* $\mathbb{Q}_\ell$*-algebra.*

Proof. A is isogenous to a product $A_1^{n_1} \times \cdots \times A_r^{n_r}$ of simple abelian varieties by Poincaré's reducibility theorem. Then the $\mathrm{End}(A_i) \otimes \mathbb{Q}_\ell$ are skew fields: If $\alpha \in \mathrm{End}(A_i)$, then the connected component of the identity in $\ker \alpha$ is an abelian subvariety of A. As A_i is simple, this means that $\ker \alpha$ is either finite or all of A_i, so α is either an isogeny or 0. In the first case, there is another isogeny $\beta : A_i \to A_i$ such that $\beta\alpha = \alpha\beta = [\deg \alpha]$. Hence after tensoring with $\mathbb{Q}_\ell$, α becomes invertible. Now

$$\mathrm{End}(A) \otimes \mathbb{Q}_\ell \cong \bigoplus_{1 \le i,j \le r} \mathrm{Hom}(A_i^{n_i}, A_j^{n_j}) \otimes \mathbb{Q}_\ell \cong \bigoplus_{i=1}^{r} \mathrm{End}(A_i^{n_i}) \otimes \mathbb{Q}_\ell,$$

as there are no non-zero homomorphisms between non-isogenous simple abelian varieties. Finally one can identify $\mathrm{End}(A_i^{n_i}) \otimes \mathbb{Q}_\ell$ with the (simple) algebra of $n_i \times n_i$-matrices over the skew field $\mathrm{End}(A_i) \otimes \mathbb{Q}_\ell$. □

Theorem 6. *Suppose that Theorem 1 holds. Let A be an abelian variety over K, ℓ a prime. Then $V_\ell A$ is a semisimple* $\mathbb{Q}_\ell[G]$*-module.*

Proof. Let $W \subseteq V_\ell A$ be a G-stable subspace and consider the set

$$I := \{u \in \mathrm{End}(A) \otimes \mathbb{Q}_\ell \mid u(V_\ell A) \subseteq W\}.$$

This is obviously a right ideal, and since $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$ is semisimple by Lemma 5, I is generated by an idempotent e: We can write $\mathrm{End}(A) \otimes \mathbb{Q}_\ell = I \oplus J$ by semisimplicity and find elements $e \in I, f \in J$ such that $\mathrm{id} = e + f$. Now $e = e^2 + fe$, so $e - e^2 = fe \in I \cap J = \{0\}$, and e is idempotent. Similarly, for any $u \in I$, $u - ue = uf \in I \cap J = \{0\}$, so e is indeed a generator for I.

By Lemma 4, there is an $u \in I$ such that $u(V_\ell A) = W$, hence $e(V_\ell A) = W$. As e is idempotent, we can decompose $V_\ell A$ as

$$V_\ell A = e(V_\ell A) \oplus (\mathrm{id} - e)(V_\ell A) = W \oplus W'.$$

Since $(\mathrm{id} - e)$ is G-equivariant, W' is G-stable as well. So any G-stable subspace W is a direct summand of $V_\ell A$, which implies that $V_\ell A$ is semisimple as $\mathbb{Q}_\ell[G]$-module. □

Theorem 7. *Suppose that Theorem 1 holds. Let A be an abelian variety over a field K and let ℓ be a prime not equal to* $\mathrm{char}\, K$*. The natural map*

$$\mathrm{End}(A) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell[G]}(V_\ell A)$$

is an isomorphism.

Remarks. (1) If A, B are two abelian varieties over K, then there is a commutative diagram ($\mathbb{Q}_\ell$'s omitted)

$$\begin{array}{ccc}
\mathrm{End}(A \times B) & = & \mathrm{End}(A) \times \mathrm{End}(B) \times \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, A) \\
\downarrow & & \downarrow \\
\mathrm{End}_G(T_\ell A \times B) & = & \mathrm{End}_G(T_\ell A) \times \mathrm{End}_G(T_\ell B) \times \mathrm{Hom}_G(T_\ell A, T_\ell B) \times \mathrm{Hom}_G(T_\ell B, T_\ell A),
\end{array}$$

where the vertical maps are the natural injections.

(2) The cokernel C of the "integral" map $\mathrm{Hom}(A, B) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell[G]}(T_\ell A, T_\ell B)$ is a torsion-free $\mathbb{Z}_\ell$-module by Gregor's talk. Hence $C = 0$ if and only if $C \otimes \mathbb{Q}_\ell = 0$. Hence Theorem 7 directly implies the Tate conjecture.

Proof. Let $\alpha \in \mathrm{End}(V_\ell A)$ be G-equivariant, i. e. $\alpha(gx) = g\alpha(x)$ for any $x \in V_\ell(A)$. Hence the graph Γ of α is a G-stable subspace of $V_\ell A \times V_\ell A = V_\ell(A \times A)$. By Lemma 4, there is an endomorphism $u \in \mathrm{End}(A \times A) \otimes \mathbb{Q}_\ell$ whose image is Γ. We can view u as a 2×2-matrix over $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$. Pick an element c of the centralizer C of $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$ in $\mathrm{End}(V_\ell A)$, i. e.

$$c \in C = \{c \in \mathrm{End}(V_\ell A) \mid \forall \gamma \in \mathrm{End}(A) \otimes \mathbb{Q}_\ell : \gamma \circ c = c \circ \gamma\}.$$

Then the diagonal matrix $\mathrm{diag}(c, c)$ commutes with u and therefore

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} (\Gamma) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \circ u(V_\ell A \times V_\ell A) = u \circ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} (V_\ell A \times V_\ell A) \subseteq u(V_\ell A \times V_\ell A) = \Gamma.$$

So if $(x, \alpha(x)) \in \Gamma$, then $(c(x), c(\alpha(x))) \in \Gamma$. Since Γ is the graph of α, also $(c(x), \alpha(c(x))) \in \Gamma$. But this implies $\alpha \circ c = c \circ \alpha$ and hence α lies in the centralizer of C in $\mathrm{End}(V_\ell A)$. Now since $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$ is semisimple by Lemma 5, the centralizer of its centralizer is again equal to $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$ (algebra fact, the so-called double centralizer theorem). Consequently, $\alpha \in \mathrm{End}(A) \otimes \mathbb{Q}_l$, which shows the surjectivity of the natural injection $\mathrm{End}(A) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell[G]}(V_\ell A)$. ☐

Remark. The same proof strategy works over number fields K when replacing the hypothesis 1 by the following weaker assumption: Given g and c, there are only finitely many isomorphism classes of abelian varieties of dimension g with semi-stable reduction over K such that their modular height is at most c. Of course, one needs to show that A and all the A_n have semistable reduction and that the height of the A_n is independent of n for large enough n.

4 The Tate conjecture over finite fields

Theorem 8. *Let K be a finite field and g an integer. Then there are only finitely many isomorphism classes of abelian varieties of dimension g over K.*

Remark. This clearly implies the Tate conjecture for finite fields, since the A_n in the proof of Lemma 4 do all have the same dimension.

Proof. See also Gregor's notes. The strategy is as follows:

- Show that $(A \times A^\vee)^4$ is principally polarized (Zarhin's trick).

- Show that there are only finitely many isomorphism classes of principally polarized abelian varieties of dimension $8 \dim A$ over K (Theorem 10).
- Show that any abelian variety has only finitely many direct factors up to isomorphism. (Theorem 11). □

Proposition 9. *Let A be an abelian variety over a field K. Then $(A \times A^\vee)^4$ is principally polarized.*

Proof. Omitted, see Gregor's notes or [4, Theorem I.13.12]. □

Theorem 10. *Let K be a finite field and fix integers g, d. Up to isomorphism, there are only finitely many abelian varieties A over K of dimension g having a polarization of degree d^2.*

Remark. If $\phi_{\mathcal{L}} : A \to A^\vee$ is a polarization, then $\deg \phi_{\mathcal{L}} = \chi(\mathcal{L})^2$, hence is always a square number (see [5, p. 150]).

Sketch of proof. The proof uses lots of results of Mumford [5, pp. 150-165]. Let $\phi_{\mathcal{L}}$ be a polarization of degree $d^2 = \chi(\mathcal{L})^2$. Then $\mathcal{L}^{\otimes 3}$ is very ample. Since $H^i(A, \mathcal{L}^{\otimes 3n}) = 0$ for all $n \geq 1$ and $i \neq 0$, we can therefore embed A into some projective space with Hilbert polynomial

$$P(n) = h^0(A, \mathcal{L}^{\otimes 3n}) = \chi(A, \mathcal{L}^{\otimes 3n}) = 3^g d \cdot n^g.$$

Note that P depends only on g and d. Hence A gives a K-valued point of the Hilbert scheme of subschemes of $\mathbb{P}^{3^g d - 1}$ with Hilbert polynomial P. This Hilbert scheme is of finite type over $\mathbb{Z}$ and therefore has only finitely many K-rational points. □

Theorem 11. *Up to isomorphism, an abelian variety over an arbitrary field has only finitely many direct factors.*

Sketch of proof. Let A be an abelian variety over a field K and suppose that $A \cong B \times C$, where B, C are abelian subvarieties of A. Define the map

$$e : A \cong B \times C \to B \times C \cong A, \quad (b, c) \mapsto (b, 0).$$

This is an idempotent endomorphism of A, $B \cong \ker(\mathrm{id} - e)$ and $C \cong \ker(e)$. Conversely, any idempotent e gives rise to a direct factor, as $A \cong \ker(\mathrm{id} - e) \times \ker(e)$. Hence we get a surjective map

$$\{\text{idempotents in } \operatorname{End}(A)\} \to \{\text{isomorphism classes of direct factors of } A\}, \quad e \mapsto [\ker(\mathrm{id} - e)].$$

Note that if $u \in \operatorname{End}(A)^*$ is a unit, then $u^{-1}eu$ is idempotent as well and $\ker(\mathrm{id} - u^{-1}eu) \cong \ker(\mathrm{id} - e)$. So the above map factors through units of $\operatorname{End}(A)$, and we end up with a surjective map

$$\{\text{idempotents in } \operatorname{End}(A)\}/\operatorname{End}(A)^* \to \{\text{isomorphism classes of direct factors of } A\}.$$

Since $\operatorname{End}(A) \otimes \mathbb{Q}$ is semisimple by Lemma 5, this implies that the left-hand side is finite (see [4, Proposition I.15.4]). □

There is a funny corollary:

Corollary 12. *Let A and B be abelian varieties over a finite field $K = \mathbb{F}_q$. Then A and B are isogenous if and only if their Hasse-Weil zeta functions concide, i.e. if and only if for all $n \geq 1$, $\#A(\mathbb{F}_q) = \#B(\mathbb{F}_q)$.*

Sketch of proof. Consider the geometric Frobenius $F_A \in \operatorname{End}(A)$, being the identity on the underlying topological space and defined as $f \mapsto f^q$ on sections of $\mathcal{O}_A$. This is a purely inseparable isogeny of degree $q^{\dim A}$. Let f_A denote the characteristic polynomial of $V_\ell(F_A)$ on $V_\ell A$.

Now it can be shown that $V_\ell A \cong V_\ell B$ for some $\ell \neq \operatorname{char} K$ as $\mathbb{Q}_\ell[G]$-modules if and only if $f_A = f_B$. As abelian varieties satisfy the Weil conjectures, the Hasse-Weil zeta function of A is given by

$$Z(A,T) = \prod_{i=0}^{2\dim A} \det(\operatorname{id} - T \cdot F_A \mid H^i(A, \mathbb{Q}_\ell))^{(-1)^{j+1}}.$$

Since $H^i(A, \mathbb{Q}_\ell) \cong (V_\ell A)^\vee$, this implies $Z(A,T) = Z(B,T)$ if and only if $f_A = f_B$. □

Remark. The analogous statement holds over number fields K using the Hasse-Weil L-function

$$L(A,s) = \prod_{v \text{ place of } K} \det(1 - N(v)^{-s} \cdot F_v \mid (T_\ell A)^{I_v}), \quad \operatorname{Re}(s) > \frac{3}{2},$$

where $I_v \subseteq G$ is an inertia subgroup at v, $F_v \in G/I_v$ is a Frobenius element and $N(v)$ is the cardinality of the residue field at v.

5 The T^r conjecture

Let X be a smooth projective geometrically irreducible variety over a field K. Let $\overline{X} := X \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$. Pick a prime $\ell \neq \operatorname{char} K$. In étale cohomology, there are cycle class map

$$c^r : Z^r(\overline{X}) \to H^{2r}(\overline{X}, \mathbb{Q}_\ell(r)), \quad r = 1, \dots, \dim X,$$

associating to every algebraic cycle of codimension r an ℓ-adic étale cohomology class. Modding out by the kernel, i. e. by ℓ-adic homological equivalence, and tensoring with $\mathbb{Q}_\ell$ yields $\mathbb{Q}_\ell$-linear maps

$$c^r \otimes 1 : A^r(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2r}(\overline{X}, \mathbb{Q}_\ell(r)),$$

where $A^r(\overline{X}) = Z^r(\overline{X}) / \ker c^r$. Denote by $A^r(X)$ the image of $Z^r(X)$ in $A^r(\overline{X})$. The absolute Galois group $G := \operatorname{Gal}(\overline{K}/K)$ acts on the étale cohomology of $\overline{X}$, and we obtain well-defined maps

$$C^r : A^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2r}(\overline{X}, \mathbb{Q}_\ell(r))^G.$$

Conjecture ($T^r(X)$)**.** Let $r \in \{1, \dots, \dim X\}$ and suppose that K is finitely generated over its prime field. Then C^r is bijective.

The Tate conjecture implies T^1:

Proposition 13. *Let A be an abelian variety over K. Then the Tate conjecture as in Theorem 7 holds for A if and only if $T^1(A)$ and $T^1(A^\vee)$ hold.*

Sketch of proof. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{End}(A) \otimes \mathbb{Q}_\ell \arrow[r, "(2)"] \arrow[ddd, "(1)"'] & \mathrm{Pic}^0(A \times A^\vee) \arrow[d, "(3)"'] \\
& H^2(A \times A^\vee, \mathbb{Q}_\ell(1)) \arrow[d, "(4)"'] \\
& H^1(A, \mathbb{Q}_\ell) \otimes H^1(A^\vee, \mathbb{Q}_\ell(1)) \arrow[d, "(5)"'] \\
\mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell A, V_\ell B) \arrow[r, equal] & (V_\ell A)^\vee \otimes V_\ell B
\end{tikzcd}$$

with the following maps

(1) the natural inclusion as in Theorem 7,

(2) $u \mapsto (u \times \mathrm{id})^* \mathcal{P}$, where $\mathcal{P}$ denotes the Poincaré bundle on $A \times A^\vee$,

(3) the cycle class map for $A \times A^\vee$,

(4) Künneth formula for étale cohomology,

(5) Natural identification $H^1(A, \mathbb{Q}_\ell) = (V_\ell A)^\vee$ plus using that the Weil pairing

$$H^1(A, \mathbb{Q}_\ell) \times H^1(A^\vee, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-1)$$

gives an isomorphism $H^1(A^\vee, \mathbb{Q}_\ell(1)) \cong H^1(A, \mathbb{Q}_\ell)^\vee \cong V_\ell A$.

All maps are G-equivariant and (5) is an isomorphism. Hence $(1)^G$ is an isomorphism if and only if $(4)^G \circ (3) \circ (2)$ is an isomorphism. But the first statement is Theorem 7 and the last statement is $T^1(A \times A^\vee)$, which is in turn equivalent to ($T^1(A)$ and $T^1(A^\vee)$) [7, Theorem 3]. $\square$

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