The Tate conjecture

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1. Introduction

In this lecture we fill in part of the proof the Tate conjecture. Recall our goal:

Claim 1.1. *Let K be a number field with absolute Galois group $\Gamma = \mathrm{Gal}(\bar{K}/K)$, and A/K an abelian variety. Let $W \subseteq V_l A$ be a $\mathbb{Q}_l[\Gamma]$-submodule. Write $G_n = (W \cap T_l A)/l^n(W \cap T_l A) \hookrightarrow A[l^n](\bar{K})$ and $A_n = A/G_n$. Then the set $\{A_n : n \in \mathbb{N}\}$ falls into finitely many isomorphism classes.*

Niels explained how the Tate conjecture would follow from this statement, and Daniele showed that one can reduce to the case where A has semistable reduction over $\mathcal{O}_K$. Thus it suffices to prove the following.

Claim 1.2. *The Faltings height*

$$h\colon \big\{\text{semistable abelian varieties over } K\big\} \to \mathbb{R}$$

satisfies

(1) for all $g \in \mathbb{N}$, $c \in \mathbb{R}$ there are (up to isomorphism) but finitely many semistable abelian varieties over K of dimension g and height at most c, and

(2) h is bounded on $\{A_n : n \in \mathbb{N}\}$.

As explained by Daniele, (1) follows from the analogous statement for principally polarized abelian varieties. That boils down to counting points on moduli spaces and will be dealt with later. Today we prove (2).

Sketch of the proof. For simplicity let $K = \mathbb{Q}$. Let $\mathcal{A}$, $\mathcal{A}_n$ be the connected Néron models of A, A_n over $\mathbb{Z}$. Define $\mathcal{G}_n = \ker(\mathcal{A} \to \mathcal{A}_n)$. Let's assume $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ is an l-divisible group (false in general). Then we claim that $h(A_n) = h(A)$ for all $n \in \mathbb{N}$. By the isogeny formula, this means $h/2 = d$, where h is the height of $G/\mathbb{Q}$ and d the dimension of $\mathcal{G}/\mathbb{Z}_l$.

To prove this, let $\chi\colon \Gamma \to \mathbb{Z}_l^\times$ be the determinant character of $T_l G$. We compute $\chi = \chi_0^d$, where χ_0 is the l-cyclotomic character. Choose a Frobenius $F \in \Gamma$ at a suitable prime p. Then $|\chi(F)| = p^{h/2}$ by the Weil conjectures, and $\chi_0(F) = p$.

2. Reduction to l-divisible groups

We work in the following setting. Let K be a number field, A/K an abelian variety, $f_n\colon A \to A_n$ isogenies with kernels G_n such that $G = (G_n)_{n\in\mathbb{N}}$ is an l-divisible group of height h. Let $\mathcal{A}$, $\mathcal{A}_n$ be the connected Néron models of A, A_n over $R = \mathcal{O}_K$. Define $\mathcal{G}_n = \ker(\mathcal{A} \to \mathcal{A}_n)$. Recall that it is quasi-finite. In this section we resolve the issue that $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ is not necessarily an l-divisible group.

Let $v \mid l$ be a place of K. We write $R_{(v)}$ for the localization of R at v and R_v for the completion at v. As $\mathcal{G}_n/R_v$ is quasi-finite over an henselian local ring, there is a decomposition

$$\mathcal{G}_n = \tilde{\mathcal{G}}_n \sqcup \mathcal{H}_n$$

with $\tilde{\mathcal{G}}_n/R_v$ finite and $\mathcal{H}_n$ having empty special fiber. By functoriality, $\tilde{\mathcal{G}}_n$ is a closed subgroup of $\mathcal{G}_n$. Of course, there is no reason for $\tilde{\mathcal{G}} = (\tilde{\mathcal{G}}_n)_{n\in\mathbb{N}}$ to be l-divisible.

Lemma 2.1. *We may assume $\tilde{\mathcal{G}}/R_v$ is l-divisible.*

Proof. Observe that $\mathcal{G}$ is l-divisible on the generic fiber, being a base change of G. In particular $\mathcal{G}_n$ is finite of order l^{nh} on the generic fiber. Then by induction to n one sees that $\tilde{\mathcal{G}}_n$ is finite of order l^{nh_n} on the generic fiber with $h_n \leq h$ non-increasing. There exists $N \in \mathbb{N}$ after which h_n is constant. We see $\tilde{\mathcal{G}}(\bar{K}_v) = F(\bar{K}_v) \oplus C$ for some l-divisible group F/K_v of height h_N and some finite abelian group $C \subseteq \tilde{\mathcal{G}}_N(\bar{K}_v)$. Now $(\tilde{\mathcal{G}}_{N+n}/\tilde{\mathcal{G}}_N)_{n\in\mathbb{N}}$ on the generic fiber is $(F_{N+n}/F_N)_{n\in\mathbb{N}}$, hence l-divisible. Replacing A by A_N and A_n by A_{N+n} we may assume $N = 0$, i.e. that $\tilde{\mathcal{G}}$ is l-divisible on the generic fiber. (This substitution is known as *Tate's trick*.)

Consider the maps $\varphi_n\colon \tilde{\mathcal{G}}_{n+2}/\tilde{\mathcal{G}}_{n+1} \to \tilde{\mathcal{G}}_{n+1}/\tilde{\mathcal{G}}_n$ given by multiplication by l. It is an isomorphism on the generic fiber. Writing $\tilde{\mathcal{G}}_{n+1}/\tilde{\mathcal{G}}_n = \operatorname{Spec} E_n$, this means that the maps $\varphi_n^*\colon E_n \to E_{n+1}$ are isomorphisms after $- \otimes_{R_v} K_v$. Therefore $(E_n)_{n\in\mathbb{N}}$ is an increasing system of orders in the finite-dimensional K_v-algebra $E = E_0 \otimes_{R_v} K_v$. It must stabilize at some $N \in \mathbb{N}$, meaning that φ_n is an isomorphism for all $n \geq N$. Applying Tate's trick once again, we may assume $N = 0$. By a simple diagram chase we see that then $\tilde{\mathcal{G}}/R_v$ is l-divisible. ■

Theorem 2.2. *Assume $\tilde{\mathcal{G}}/R_v$ is l-divisible for all places $v \mid l$. Then $h(A) = h(A_n)$ for all $n \in \mathbb{N}$.*

We prove this in next section. Note that without the assumption, the proof of lemma 2.1 shows that $h(A_m) = h(A_n)$ for all m, n large enough.

To finish this section, we compute $\#s^*\Omega^1_{\mathcal{G}_n/R}$. Daniele showed $s^*\Omega^1_{\mathcal{G}_n/R} = \bigoplus_{v|l} s^*\Omega^1_{\mathcal{G}_n/R_v}$. Now, $s^*\Omega^1_{\mathcal{G}_n/R_v} = s^*\Omega^1_{\tilde{\mathcal{G}}_n/R_v}$ since taking differentials commutes with taking completions; and the completions of $\mathcal{G}_n$ and $\tilde{\mathcal{G}}_n$ coincide since they have the same special fiber. Then apply proposition 2.4 below to $\tilde{\mathcal{G}}/R_v$ to obtain

$$\#s^*\Omega^1_{\mathcal{G}_n/R} = \prod_{v|l} \#(R_v/l^n R_v)^{d_v} = \prod_{v|l} l^{n[K_v:\mathbb{Q}_l]d_v} \tag{2.3}$$

where d_v is the dimension of the l-divisible group $\tilde{\mathcal{G}}/R_v$.

Proposition 2.4. *Let R be a noetherian complete local ring with residue characteristic $l > 0$, and G/R an l-divisible group of dimension d. Then $s^*\Omega^1_{G_n/R} = (R/l^n R)^d$.*

Proof. We may assume G is connected as the étale part does not contribute to differentials. Then G corresponds to some d-dimensional divisible formal Lie group $\operatorname{Spf}(L)/R$. Write $L = R[\![x_1, \ldots, x_d]\!]$ and let $\psi\colon L \to L$ correspond to multiplication by l on $\operatorname{Spf}(L)$. Recall that we have $G_n = \operatorname{Spec}(L/\langle \psi^n(x_i) : i = 1, \ldots, d\rangle)$. Now we have

$$\hat{\Omega}^1_{L/R} = \bigoplus_{i=1}^{d} L\mathrm{d}x_i, \qquad \hat{\Omega}^1_{G_n/R} = \bigoplus_{i=1}^{d} \frac{L}{\langle \psi^n(x_i), \frac{\partial}{\partial x_i}\psi^n(x_1), \ldots, \frac{\partial}{\partial x_i}\psi^n(x_d)\rangle} \mathrm{d}x_i.$$

The proposition follows from taking $- \otimes_L R$ and observing that $\psi^n(x_i) = l^n x_i +$ higher order terms. ■

3. Representation theory

As promised we prove theorem 2.2 using the isogeny formula. From (2.3) we find

$$\begin{aligned} h(A_n) - h(A) &= \tfrac{1}{2}\log\deg f_n - \frac{1}{[K:\mathbb{Q}]}\log \#s^*\Omega^1_{\mathcal{G}_n/R} \\ &= \tfrac{1}{2}\log l^{nh} - \frac{1}{[K:\mathbb{Q}]}\log\prod_{v\mid l} l^{n[K_v:\mathbb{Q}_l]d_v} \\ &= n\log(l)\Big(\tfrac{1}{2}h - \frac{1}{[K:\mathbb{Q}]}\sum_{v\mid l}[K_v:\mathbb{Q}_l]d_v\Big). \end{aligned}$$

Theorem 3.1. *$h[K:\mathbb{Q}]/2 = \sum_{v\mid l}[K_v:\mathbb{Q}_l]d_v$.*

Let $\Gamma = \mathrm{Gal}(\bar{K}/K)$. We consider its rank h representation $U = T_l G$. Let $\chi\colon \Gamma \to \mathbb{Z}_l^\times$ be the determinant character of U.

As we need to apply class field theory results, we move the entire setting from K to $\mathbb{Q}$. Let $\Gamma' = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $U' = \mathrm{Ind}_\Gamma^{\Gamma'} U = \mathbb{Z}_l[\Gamma'] \otimes_{\mathbb{Z}_l[\Gamma]} U$. It is a rank $h[K:\mathbb{Q}]$ representation of Γ'. Let $\chi'\colon \Gamma' \to \mathbb{Z}_l^\times$ be its determinant character.

Finally, let $\chi_0\colon \Gamma' \to \mathbb{Z}_l^\times$ be the l-cyclotomic character, and $\varepsilon\colon \Gamma' \to \{\pm 1\} \subset \mathbb{Z}_l^\times$ the sign of the action of Γ' on Γ'/Γ.

Lemma 3.2. *$\chi' = \varepsilon^h \cdot \chi_0^{\sum_{v\mid l}[K_v:\mathbb{Q}_l]d_v}$.*

Proof. Let $\tau\colon \Gamma'^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$ be the transfer map. From general representation theory we have $\chi' = \varepsilon^h \cdot (\chi \circ \tau)$. (Note that characters factor over the abelian quotients, so this is well-defined.)

Let v be a finite place of K. We want to compute χ at v, i.e. compute its restriction to $D_v \subset \Gamma$. Recall that here the *decomposition group* at v is

$$D_v = \mathrm{Gal}(\bar{K}_v/K_v) = \mathrm{Aut}(\bar{R}_v/R_v) \hookrightarrow \mathrm{Aut}(\bar{R}_{(v)}/R_{(v)}) = \Gamma.$$

The *inertia subgroup* is

$$I_v = \ker\big(D_v \to \mathrm{Aut}(\bar{k}_v/k_v)\big)$$

where k_v is the residue field of R_v. A character of Γ is *unramified* at v if it is trivial on I_v.

For $v \nmid l$, I_v acts unipotently (i.e. by linear transformations whose eigenvalues are all 1) on $T_l A$ as A has semistable reduction. Then I_v acts unipotently on $U \subseteq T_l A$. As unipotent transformations have determinant 1, χ is unramified at v.

For $v \mid l$, one can show that $\chi_{G_{K_v}}$ differs from $\chi_{\tilde{\mathcal{G}}_{K_v}}$ by an unramified character, as I_v acts trivially on $T_l G_{K_v}/T_l\tilde{\mathcal{G}}_{K_v} \subseteq T_l A_{K_v}/T_l\tilde{\mathcal{A}}_{K_v}$. This follows from the orthogonality theorem for the Weil pairing $T_l A_{K_v} \times T_l A_{K_v}^\vee \to \mathbb{Z}_l(1)$. Antareep has explained that $\chi_{\tilde{\mathcal{G}}_{K_v}} = \chi_0^{d_v}$.

Combining the above, we find that χ' and $\varepsilon^h \cdot \chi_0^{\sum_{v\mid l}[K_v:\mathbb{Q}_l]d_v}$ have the same ramification at all primes p, more precisely: their quotient is nowhere ramified. By class field theory they must be equal. ∎

Let $B = \mathrm{Res}_{K/\mathbb{Q}} A$ be the Weil restriction of A to $\mathbb{Q}$. It is an abelian variety over $\mathbb{Q}$ of dimension $[K:\mathbb{Q}]\dim A$. Let $p \neq l$ be a prime where B has good reduction. Let $F \in \Gamma'$ be a Frobenius at p, i.e. a lift along the surjection $D'_p \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ of the p-Frobenius.

Lemma 3.3. $|\chi'(F)| = p^{h[K:\mathbb{Q}]/2}$.

Proof. Consider

$$U' = \mathrm{Ind}_{\Gamma}^{\Gamma'} U \hookrightarrow \mathrm{Ind}_{\Gamma}^{\Gamma'} T_l A = T_l B.$$

By (the known part of) the Weil conjectures, the eigenvalues of F acting on $T_l B$ are algebraic with absolute value $p^{1/2}$. Then the same is true for the subrepresentation U', and we just observe that this representation has rank $h[K : \mathbb{Q}]$. ■

On the other hand, $\chi_0(F) = p$. So also $|\chi'(F)| = p^{\sum_{v|l}[K_v:\mathbb{Q}_l]d_v}$, and theorem 3.1 follows.