Higher Analysis I and Linear Functional Analysis
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Remarks/mistakes to

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Contents

0 Introduction

Linear Algebra	Analysis
• matrices • eigenvalues • eigenvectors in finite-dimensional spaces	• real and complex valued functions • differentiation, integration • sequences • Limits and Estimates

Table 1: Comparison between linear algebra and analysis

The aim of functional analysis is to solve equations (even linear) where the unknown is a function u. Examples:

ODE $u'(x) = A(x)u(x) + f(x)$ with $u \in \mathcal{C}^1([y_0, y_1], \mathbb{R}^m)$

PDE $\Omega = B_R(0) \subset \mathbb{R}^d$, $\Delta u = f(x)$ in Ω and $u|_{\partial\Omega} = 0$ on $\partial\Omega$. Given $f \in \mathcal{C}^0(\overline{\Omega})$, does there exist $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, such that the equations hold.

Basic objects:

- Function spaces (more general: infinite-dimensional vector spaces)
- Linear operators on or between such spaces

1 Linear spaces

linear space = vector space
$(X, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.1 Banach spaces

Definition 1.1.1
If $(X, \mathbb{K})$ is a vector space, then a function $\|\cdot\| : X \to \mathbb{R}$ is called a norm *on X if for $x, y \in X$ and $\alpha \in \mathbb{K}$*

(i) $x \neq 0 \implies \|x\| > 0$ *(positivity)*

(ii) $\|\alpha x\| = |\alpha|\|x\|$ *(scaling)*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *(triangle estimate)*

The function $\|\cdot\|$ is called seminorm *on X, if only (ii) and (iii) hold.*

Example 1.1.2

(i) $X = \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ or $\|x\|_\infty = \max\{|x_i| \mid i = 1, \ldots, n\}$ or $\|x\|_{\text{Mielke}} = \|x\|_2 + \|x\|_{\pi^2/7}$

(ii) $X = \mathcal{C}([a, b], \mathbb{K})$ with $u \in X$. Then we have the norms $\|u\|_\infty = \sup\{|u(t)| \mid t \in [a, b]\}$ or $\|u\|_1 = \int_a^b |u(t)|\, dt$.

(iii) $X = \mathcal{C}^1([a, b], \mathbb{K})$. $|||u||| = \int_a^b |u'(t)|\, dt$ is a seminorm. Positivity fails as $u \equiv 1$, the $u' \equiv 0$ and hence $|||u||| = 0$.

Proposition 1.1.3
If $(X, \mathbb{K}, \|\cdot\|)$ is a seminormed vector space, we obtain a normed space by factoring with respect to the equivalence relation $\sim$ defined by

$$x \sim y \overset{def}{\iff} \|x - y\| = 0$$

This means the following:
We define equivalence classes $[x]_\sim = \{y \in X \mid x \sim y\} \subset X$ and $X_\sim = \{[x]_\sim \mid x \in X\}$ is again a vector space over $\mathbb{K}$. We can also define a norm *$\|[x]_\sim\| = \|x\|$. Of course, one needs to show that this definition is well-posed, meaning "independence of the chosen representative".*

Proof: Exercise 1.

Definition 1.1.4
(i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called a Cauchy sequence, *if $\forall \varepsilon > 0 \exists n_0 \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.*

(ii) *The point* $x \in X$ *is called* limit *of the sequence* $(x_n)_{n\in\mathbb{N}}$ *if* $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 : \|x_n - x\| < \varepsilon$.

(iii) *A normed vector space* $(X, \mathbb{K}, \|\cdot\|)$ *is called* complete *if for every Cauchy sequence there exists a limit.*

(iv) *Complete normed vector spaces are called* Banach spaces*.*

Well known fact: Limits of Cauchy sequences are unique! We will write

$$x = \lim_{n\to\infty} x_n \text{ in } X \quad \text{or} \quad x = X\text{-}\lim_{x\to\infty} x_n \quad \text{or} \quad x_n \xrightarrow{X} x \quad \text{or} \quad \|x_n - x\| \to 0$$

Example 1.1.5 (i) Finite dimensional case:

$(\mathbb{Q}^n, \mathbb{Q}, \|\cdot\|_2)$ normed space, but not complete

$(\mathbb{R}^n, \mathbb{R}, \|\cdot\|_p)$ Banach space

$(\mathbb{C}^n, \mathbb{C}, \|\cdot\|_p)$, $(\mathbb{C}^n, \mathbb{R}, \|\cdot\|_p)$ Banach space

(ii) $X = \mathcal{C}([a,b], \mathbb{K})$ with $\|u\|_\infty = \sup\{|u(t)| \mid t \in [a,b]\}$ Cauchy sequence in the $\|\cdot\|_\infty$-norm means uniform convergence. Hence, a continuous limit function exists. $(\mathcal{C}([a,b], \mathbb{K}, \|\cdot\|_\infty)$ is a Banach space.

(iii) $X = \mathcal{C}([a,b])$, $\|u\|_1 = \int_a^b |u(t)|\, dt$ is not complete! Consider

$$u_n(t) = \begin{cases} c_n(t-a)^n & \text{for } t \in \left[a, \frac{a+b}{2}\right] \\ 1 & \text{for } t \in \left[\frac{a+b}{2}, b\right] \end{cases} \quad \text{where} \quad u_n \in \mathcal{C}([a,b]).$$

Claim: (i) $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (ii) u_n does not converge, i.e. there is no limit.

(i)

$$\begin{aligned} \|u_n - u_m\|_1 &= \int_a^b |u_n(t) - u_m(t)|\, dt \\ &= \int_a^{\frac{a+b}{2}} |u_n(t) - u_m(t)|\, dt \\ &\underset{m>n}{\overset{\text{wlog}}{=}} \int_a^{\frac{a+b}{2}} \underbrace{u_n(t) - u_m(t)}_{\geq 0}\, dt = \left(\frac{1}{n} - \frac{1}{m}\right)\frac{a+b}{2} \to 0 \quad \text{for} \quad n,m \to \infty \end{aligned}$$

(ii) To produce a contradiction, assume there exists a limit $\tilde{u} \in \mathcal{C}([a,b])$ such that $\|u_n - \tilde{u}\|_1 \to 0$.

$$\begin{aligned} \int_{\frac{a+b}{2}}^b |u_n(t) - \tilde{u}(t)|\, dt &= \int_{\frac{a+b}{2}}^b |u_n(t) - \tilde{u}(t)|\, dt \leq \|u_n - \tilde{u}\|_1 \to 0 \\ \implies \int_{\frac{a+b}{2}}^b |1 - \tilde{u}(t)|\, dt &= 0 \implies \tilde{u}(t) = 1 \quad \text{for} \quad t \in \left[\tfrac{a+b}{2}, b\right] \\ \text{Similarly } \int_a^{\frac{a+b}{2}} |0 - \tilde{u}(t)|\, dt &\leq \underbrace{\int_a^{\frac{a+b}{2}} |0 - u_n(t)|\, dt}_{\to 0} + \underbrace{\int_a^{\frac{a+b}{2}} |u_n - \tilde{u}(t)|\, dt}_{\to 0} \\ \implies \tilde{u}(t) &= 0 \quad \text{for} \quad t \in \left[a, \tfrac{a+b}{2}\right] \end{aligned}$$

This is a contradiction and therefore no limit exists.

(iv) Consider the sequence space $X = \{(a_k)_{k\in\mathbb{N}} \mid \exists k_0 \in \mathbb{N} \forall k \geq k_0 : a_k = 0\}$ with $\|(a_k)_{k\in\mathbb{N}}\|_\infty = \max\{|a_k| \mid k \in \mathbb{N}\}$. $(X, \|\cdot\|_\infty)$ is a normed space. This space is not complete. Take the sequence $(a^{(n)})_{n\in\mathbb{N}}$ with $a^{(n)} \in X$ and $a_k^{(n)} = \begin{cases} \frac{1}{k} & \text{for } k \leq n \\ 0 & \text{otherwise} \end{cases}$. Then (for $n < m$) $\|a^{(n)} - a^{(m)}\|_\infty = \frac{1}{n+1} \to 0$ for $n, m \to \infty$.

There is no limit! Assume $\tilde{a} \in X$ is the limit. Fix $k \in \mathbb{N}$, then $(a_k^{(n)})_{n\in\mathbb{N}}$ is a CS in $\mathbb{K}$:

$$|a_k^{(n)} - a_k^{(m)}|_{\mathbb{K}} \leq \|a^{(n)} - a^{(m)}\|_\infty \to 0$$

As $a_k^{(n)} \to \frac{1}{k}$ for $n \to \infty$ the limit $\tilde{a}$ must satisfy $\tilde{a}_k = \frac{1}{k} \implies \tilde{a} \notin X$.

Definition 1.1.6
Assume $(X, \|\cdot\|)$ is a Banach space. We call a subset $A \subset X$ open, *if $\forall x \in A \exists \varepsilon > 0 : B_\varepsilon^{\|\cdot\|}(x) \subset A$.*

A is called closed *if $X \setminus A$ is open. For Banach spaces we have A closed $\iff$ ((x_n) CS in $A \implies \lim_{n\to\infty} x_n \in A$).*

Lemma 1.1.7
$(X, \|\cdot\|)$ is a Banach space and $U \subset X$ is a closed subspace, then $(U, \|\cdot\||_U)$ is again a Banach space.

Proof: (i) $(U, \|\cdot\|)$ is a vector space.

(ii) $(U, \|\cdot\|)$ is complete as U is closed. □

Every normed vector space can be completed in a "natural and unique" way to a Banach space. Examples to complete $\mathbb{Q}$ to $\mathbb{R}$:

(i) Cauchy sequence construction

(ii) Dedekind cuts: (A, B) is a Dedekind cut of $\mathbb{Q}$, if $A, B \subset \mathbb{Q}$.

$\forall a \in A \forall b \in B : a < b$

$A \cup B = \mathbb{Q}$.

(iii) Infinite fractions ($\pi = 3.1415805\ldots$)

(iv) nested intervals

There are many completions, but essentially all of them are the same.

Definition 1.1.8
If $(X, \|\cdot\|)$ is a normed vector space, we call $(Y, \|\!|\cdot|\!\|)$ a completion *of $(X, \|\cdot\|)$ if there exists a mapping $J : X \to Y$ such that the following holds:*

(i) J is linear

(ii) J is norm preserving, i.e. $\|\!|Jx|\!\| = \|x\|$ for all $x \in X$.

(iii) $(Y, \|\!|\cdot|\!\|)$ is Banach space.

(iv) JX is dense in Y, i.e. Y is the closure of JX.

Theorem 1.1.9
(i) For every normed vector space exists a completion.

(ii) For any two completions $(Y_1, \|\cdot\|_1)$ and $(Y_2, \|\cdot\|_2)$ for a normed vector space $(X, \|\cdot\|)$ there exists a linear, bijective and norm preserving mapping $L_1 : Y_1 \to Y_2$.

Point (ii) allows us to talk about the *completion.*

Proof: (ii)

We define $\tilde{L}_1 : J_1X \to J_2X, y = J_1x \mapsto J_2x = J_2J_1^{-1}y$. $\tilde{L}_1 : J_1X \to Y_2$ is defined as $J_2J_1^{-1}$. Moreover $\tilde{L}_1$ is norm preserving, namely

$$\begin{tikzcd} Y_1 & X \arrow[l, "J_1"'] \arrow[r, "J_2"] \arrow[dl, "1-1"] \arrow[dr, "1-1"'] & Y_2 \\ J_1X & & J_2X \end{tikzcd}$$

$$\|\tilde{L}_1 y_1\|_2 \overset{\text{def } \tilde{L}_1}{=} \|J_2J_1^{-1}y_1\|_2 \underset{\text{preserving}}{\overset{J_2\text{norm}}{=}} \|J_1^{-1}y_1\|_X \underset{\text{preserving}}{\overset{J_1\text{norm}}{=}} \|y_1\|_1$$

Since J_1X is dense in Y_1 for every $y \in Y$ exists a Cauchy sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in J_1X$ and $y_n \xrightarrow{Y_1} y$. Define

$$L_1 y = \begin{cases} \tilde{L}_1 y & \text{if } y \in J_1X \\ \lim\limits_{n\to\infty} \tilde{L}_1 y_n & \text{for } y_n \in J_1X \text{ and } y_n \to y, y \notin J_1X \end{cases}$$

L_1 is well-defined, linear and norm preserving, as

$$\|\tilde{L}_1 y_n - \tilde{L}_1 y_m\|_2 = \|\tilde{L}_1(y_n - y_m)\|_2 = \|y_n - y_m\|_1 \to 0$$

(i) Construction of a completion via Cauchy sequences. Define Z = linear space of CS.

$$Z = \{(x_n)_{n\in\mathbb{N}} \mid \|x_n - x_m\| \to 0 \text{ for } n, m \to \infty\} \text{ with seminorm } \|(x_n)_{n\in\mathbb{N}}\|_Z = \lim_{n\to\infty} \|x_n\|_X \quad \text{(exists)}$$

Note that for CS $(x_n)_{n\in\mathbb{N}}$ the norms $(\|x_n\|_X)_{n\in\mathbb{N}}$ form a CS in the complete reals (use $|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\|$). It is easy to check that $\|\cdot\|_Z$ is a seminorm on Z. We define the equivalence relation

$$\begin{aligned}(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} &\overset{\text{def}}{\iff} \|(x_n)_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}}\|_Z = 0\\ &\iff x_n - y_n \to 0 \quad \text{in } X.\end{aligned}$$

$Y = Z_{/\sim} = \{[(x_n)_{n\in\mathbb{N}}]_\sim \mid (x_n)_{n\in\mathbb{N}} \in Z\}$ with $\|[(x_n)_n]\|_Y = \|(x_n)\|_Z = \lim_{n\to\infty} \|x_n\|_X$. Hence, we found a normed vector space $(Y, \|\cdot\|_Y)$, which is a candidate for the completion of $(X, \|\cdot\|)$. We define $J : X \to Y$ via $Jx = [(x, x, x, \dots)]_\sim$ such that (i) and (ii) hold.
ad (iv): Given $y \in Y$ and $\varepsilon > 0$. Find $x \in X$, such that $\|Jx - y\|_Y < \varepsilon$. $y = [(\tilde{x}_n)_{n\in\mathbb{N}}]_\sim$, $(\tilde{x}_n)_{n\in\mathbb{N}}$ CS in X. Hence $\exists n_0 \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$. Choose $x = \tilde{x}_{n_0}$, then $\|Jx - y\|_Y = \lim_{n\to\infty} \|x_{n_0} - x_n\| \leq \varepsilon$.
ad (iii): To prove: Completeness of $(Y, \|\cdot\|)$. CS in Y: $y^{(n)} = [(x_k^{(n)})_{k\in\mathbb{N}}]_\sim$, $\|y^{(n)} - y^{(m)}\|_Y \to 0$. For $k \in \mathbb{N}$ consider $N(k)$ such that $\|x_{n_1}^{(k)} - x_{n_2}^{(k)}\|_X \leq \frac{1}{k}$ for all $n_1, n_2 \geq N(K)$. Now define $\tilde{y} \in Y$ as potential limit via $\tilde{y} = [(\tilde{x}_k)_{k\in\mathbb{N}}]_\sim$ with $\tilde{x}_k = x_{N(k)}^{(k)}$. To show, that $\tilde{y}$ is a CS we use

$$\begin{aligned}\|\tilde{x}_n - \tilde{x}_m\| &\overset{\text{Def}}{=} \|x_{N(n)}^{(n)} - x_{N(m)}^{(m)}\|\\ &\leq \|x_{N(n)}^{(n)} - x_k^{(n)}\| + \|x_k^{(n)} - x_k^{(m)}\| + \|x_k^{(m)} - x_{N(m)}^{(m)}\| \quad \text{for every } k\\ &\overset{\text{Def of } N}{\leq} \frac{1}{n} + \|x_k^{(n)} - x_k^{(m)}\| + \frac{1}{m} \quad \text{for } k \geq \max\{N(m), N(n)\}\end{aligned}$$

For $k \to \infty$ we can use $\lim_{k\to\infty} \left\|x_k^{(n)} - x_k^{(m)}\right\| = \left\|\left|x^{(n)} - x^{(m)}\right\|\right|$. Together, we find

$$\|\tilde{x}_n - \tilde{x}_m\| \leq \frac{1}{n} + \left\|\left|x^{(n)} - x^{(m)}\right\|\right| + \frac{1}{m} \to 0 \quad \text{for } n, m \to \infty$$

We have to show convergence, i.e. $\|y^{(n)} - y^k\|_Y \to 0$ for $n \to \infty$. By definition, we have $\|y^{(n)} - y^k\|_Y = \lim_{k\to\infty} \|x_k^{(n)} - \tilde{x}_k\|$. To show: $\lim_{n\to\infty} \left(\lim_{k\to\infty} \|x_k^{(n)} - \tilde{x}_k\|\right) = 0$.

$$\begin{aligned}\|x_k^{(n)} - \tilde{x}_k\| &\overset{\text{Def}}{=} \|x_k^{(n)} - x_{N(k)}^{(k)}\|\\ &\leq \|x_k^{(n)} - x_{N(k)}^{(n)}\| + \|x_{N(n)}^{(n)} - x_{N(k)}^{(k)}\|\\ &\leq \frac{1}{n} + \frac{1}{n} + \|y^{(k)} - y^{(n)}\|_Y + \frac{1}{k}\\ \text{For } k \geq N(n) \text{ we have} \quad \|x_k^{(n)} - \tilde{x}_k\| &\leq \frac{2}{n} + \|y^{(k)} - y^{(n)}\|_Y + \frac{1}{k}\\ \text{Hence} \quad \lim_{k\to\infty} \|x_k^{(n)} - \tilde{x}_k\| &\leq \frac{2}{n} + \lim_{k\to\infty} \|y^{(k)} - y^{(n)}\|_Y\end{aligned}$$

Since $(y^{(n)})_{n\in\mathbb{N}}$ is a CS, we obtain $\lim_{n\to\infty}(\lim_{k\to\infty} \|x_k^{(n)} - \tilde{x}_k\|) \leq 0 \implies y^{(n)} \to \tilde{y}$ in Y. □

Remark In practice, the abstract completion is not very useful. One needs a more explicit description of *one of the completions*. The simplest and most common situation is the following:

(i) Given a normed vector space $(X, \|\cdot\|)$.

(ii) Find a Banach space $(B, \|\cdot\|_B)$ and a norm-preserving linear mapping $E : X \to B$ (E for embedding).

(iii) Define $Y = \overline{(EX)}^B = \text{closure}_B(EX)$ and $\|y\|_Y = \|y\|_B$

(iv) Then, $(Y, \|\cdot\|_Y)$ is a completion of $(X, \|\cdot\|)$ (simply by def. of "completion")

Example 1.1.10 $X = \{(a_k)_{k\in\mathbb{N}} \mid \exists K \forall k \geq K : a_k = 0\}$. $\|(a_k)\|_\infty = \max\{|a_k| \mid k \in \mathbb{N}\}$, which is a non-complete normed space. Let B be the sequence space with $X \subset B$ such that $E : X \to B; (a_k) \mapsto (a_k)$. With $B = \ell_\infty =$ set of all bounded sequences with the usual norm $\|(a_k)_{k\in\mathbb{N}}\|_\infty = \sup\{|a_k| \mid k \in \mathbb{N}\}$. $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space. (We omit the straightforward proof). Obviously, we have the desired norm-preserving embedding. We define $Y = \overline{X}^{\ell_\infty}$ and claim $Y = c_0 = \{(a_k)_{k\in\mathbb{N}} \in \ell_\infty \mid a_k \to 0 \text{ for } k \to \infty\}$.

Proof:

"$Y \subset c_0$" $X \subset c_0 \implies Y = \overline{X}^{\ell_\infty} \subset \overline{c}_0^{\ell_\infty}$ Hence, it suffices to show that c_0 is closed in B. Take $(a^{(n)})_{n\in\mathbb{N}} \subset c_0$ and $a^{(n)} \to b$ in ℓ_∞. We have to show $b \in c_0$, i.e. $\forall \varepsilon > 0 \exists K \forall k \geq K.\ |b_k| \leq \varepsilon$. Choose n such that $\|a^{(n)} - b\|_\infty \leq \frac{\varepsilon}{2}$ and K such that $|a_k^{(n)}| \leq \frac{\varepsilon}{2}$ for $k \geq K$. That implies $|b_k| \leq |a_k^{(n)} - b_k| + |a_k^{(n)}| \leq \|a^{(n)} - b\|_\infty + \frac{\varepsilon}{2} \leq \varepsilon$.

"$Y \supset c_0$" Given $a \in c_0$ and $\varepsilon > 0$, we have to find $\tilde{a} \in X$ such that $\|\tilde{a} - a\|_\infty \leq \varepsilon$. As $a \in c_0$ we have $|a_k| \leq \varepsilon$ for $k \geq K = K(a)$. Define $\tilde{a}_k = \begin{cases} a_k & \text{for } k \leq K \\ 0 & \text{for } k > K \end{cases}$ the $\tilde{a} \in X$ and $\|\tilde{a} - a\| = \sup_{k>K} |a_k| \leq \varepsilon$. □

Limits of CS exist in Banach spaces. Therefore we can solve equations by approximation procedures.

Theorem 1.1.11 (Banach fixed-point theorem)
(also contraction mapping theorem*)*
Every contraction on a Banach space has a unique fixed point. More precisely, given a Banach space $(X, \|\cdot\|)$ and $F : X \to X$ with $\exists \rho \in [0,1[\forall x, \tilde{x} \in X : \|F(x) - F(\hat{x})\| \leq \rho \|x - \hat{x}\|$, there exists a unique $y \in X$, such that $F(y) = y$. Moreover, for each initial value $x^0 \in X$ the Banach iterations $x^{k+1} = F(x^k)$ for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ converge to y such that

$$\|y - x^k\| \leq \frac{\rho^{k-1}}{1-\rho} \|F(x^0) - x^0\|$$

1.2 Hilbert spaces

There are many essentially different Banach spaces. The Hilbert spaces are a small subclass. The new features are inner products also called scalar product. Hilbert spaces are possibly infinite-dimensional generalizations of the Euclidean spaces $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$.

Definition 1.2.1
$(X, \mathbb{K})$ vector space. A mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{K}$ is called a scalar product *(or* inner product*), if it satisfies the following conditions.*

(i) $\forall \lambda_1, \lambda_2 \in \mathbb{K} \forall x_1, x_2, y \in X : \langle \lambda_1 x_1 + \lambda_2 x_2, y\rangle = \lambda_1 \langle x_1, y\rangle + \lambda_2 \langle x_2, y\rangle$ (linearity)

(ii) $\forall x, y \in X : \langle y, x\rangle = \overline{\langle x, y\rangle}$ (symmetry)

(iii) $\langle x, x\rangle > 0$ for $x \neq 0$ (positivity)

Example 1.2.2 (i) Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$ with $\langle a, b\rangle_{\text{Euclid}} = \sum_{k=1}^n a_k \overline{b}_k$

(ii) $(\mathbb{C}^n, \langle\cdot,\cdot\rangle_H)$ where $H = \overline{H}^\top =: H^* > 0$ (H hermitian and positive definite) with $\langle a, b\rangle_H = \langle Ha, b\rangle_{\text{Euclid}}$

(iii) $X = C^0([a,b], \mathbb{C}^n)$ $H \in C^0([a,b], \mathbb{C}^{n\times n}_{\text{hermitian}})$ and $H(t) > 0$ for all $t \in]a,b[$. We define

$$\langle f, g\rangle := \int_a^b \langle H(t)f(t), g(t)\rangle_{\text{Euclid}}\, dt$$

This is a scalar product. (i) and (ii) are straightforward. For (iii), note that $\langle f, f\rangle = \int_a^b \langle H(t)f(t), f(t)\rangle_{\text{Euclid}}\, dt$. If $\langle f, f\rangle = 0$, then $\langle H(t)f(t), f(t)\rangle = 0$ for all $t \in [a,b]$. Therefore $f(t) = 0$ for $t \in]a,b[$, since $H(t) > 0$. Since f is continuous, we have $f \equiv 0$.

Each scalar product defines a norm via

$$\|x\|_{\langle\cdot,\cdot\rangle} = \sqrt{\langle x, x\rangle}$$

Obviously, $\|\cdot\|_{\langle\cdot,\cdot\rangle}$ satisfies "scaling" and "positivity". To obtain the "triangle estimate" we need the *Cauchy-Schwarz estimate*

Theorem 1.2.3
Cauchy-Schwarz estimate

$$|\langle x, y\rangle| \leq \|x\|\|y\| \quad \textit{for all} \quad x, y \in X$$

Proof:

Case 1 $x = 0$ or $y = 0$, $0 \leq 0$ holds.

Case 2 $x \neq 0$ and $y \neq 0$. For all $\mu \in \mathbb{K}$ we have

$$\begin{aligned} 0 &\leq \|x+\mu y\|^2 = \langle x+\mu y, x+\mu y\rangle \\ &\leq \langle x,x\rangle + \mu\langle y,x\rangle + \overline{\mu}\langle x,y\rangle + |\mu|^2\langle y,y\rangle \\ &\leq \|x\|^2 + 2\,\mathrm{Re}(\mu\langle y,x\rangle) + |\mu|^2\|y\|^2 \end{aligned}$$

Choose $\mu \in \mathbb{K}$ s.t. $\mu\frac{\|x\|}{\|y\|} = e^{i\theta}$ and $\mathrm{Re}(\mu\langle y,x\rangle) = -|\mu\langle y,x\rangle|$

$$\begin{aligned} &\leq \|x\|^2 - 2|\mu\langle x,y\rangle| + \|\mu\|^2\|y\|^2 \\ &= \frac{\|x\|}{\|y\|}(\|x\|\|y\| - 2|\langle x,y\rangle| + \|x\|\|y\|) \implies CSE \end{aligned}$$

□

We have shown that inner-product spaces $(X, \langle\cdot,\cdot\rangle)$ lead to normed vector spaces $(X, \|\cdot\|_{\langle\cdot,\cdot\rangle})$. Now we can ask about completeness of $(X, \|\cdot\|_{\langle\cdot,\cdot\rangle})$.

Definition 1.2.4
An inner product space $(X, \langle\cdot,\cdot\rangle)$ is also called pre-Hilbert-space.

Theorem 1.2.5
Consider a pre-Hilbert-space $(X, \langle\cdot,\cdot\rangle)$. Then, the normed space $(X, \|\cdot\|_{\langle\cdot,\cdot\rangle})$ has a Banach space completion $(Y, \|\cdot\|_Y)$ and this space is called a Hilbert space.

Proof: Given $(X, \langle\cdot,\cdot\rangle)$ we obtain $(X, \|\cdot\|_{\langle\cdot,\cdot\rangle})$ and consider any completion $(Y, \|\cdot\|_Y)$. We define a scalar product on Y as follows:

$$E : X \to Y \quad \text{is norm preserving and has a dense range } E(X) \subset Y$$

For $y, \tilde{y} \in Y$ we define

$$\langle y, \tilde{y}\rangle_Y \overset{\text{Def}}{=} \lim_{n\to\infty} \langle x_n, \tilde{x}_n\rangle_X \quad \text{where} \quad Ex_n \to y \quad \text{and} \quad E\tilde{x}_n \to \tilde{y}$$

(i) Well-definedness of $\langle\cdot,\cdot\rangle_Y : Y \times Y \to \mathbb{K}$

$\forall y, \tilde{y} \exists$ CS $(x_n), (\tilde{x}_n)$ such that $Ex_n \to y, E\tilde{x}_n \to \tilde{y}$. Then $\lim_{n\to\infty}\langle x_n, \tilde{x}_n\rangle$ exists. We use continuity of the scalar product on $(X, \|\cdot\|_{\langle\cdot,\cdot\rangle})$. We use the Cauchy-Schwarz estimate

$$\begin{aligned} |\langle x,\tilde{x}\rangle - \langle\bar{x},\hat{x}\rangle| &\leq \overbrace{|\langle x,\tilde{x}\rangle - \langle\bar{x},\tilde{x}\rangle|}^{\langle x-\bar{x},\tilde{x}\rangle} + \overbrace{|\langle\bar{x},\tilde{x}\rangle - \langle\bar{x},\hat{x}\rangle|}^{\langle\bar{x},\tilde{x}-\hat{x}\rangle} \\ &\overset{\text{CSE}}{\leq} \|x-\bar{x}\|\|\tilde{x}\| + \|\tilde{x}-\hat{x}\|\|\bar{x}\| \end{aligned}$$

Hence we find

$$|\langle x_n,\tilde{x}_n\rangle - \langle x_m,\tilde{x}_m\rangle| \leq \underbrace{\|x_n - x_m\|}_{\to 0}\underbrace{\|\tilde{x}_n\|}_{\leq C} + \underbrace{\|\tilde{x}_n - \tilde{x}_m\|}_{\to 0}\underbrace{\|x_m\|}_{\leq C}$$

Taking any other sequences $(z_n), (\tilde{z}_n)$ in X such that $Ez_n \to y$ and $E\tilde{z}_n \to \tilde{y}$, we obtain the same limit. To see this estimate $|\langle x_n,\tilde{x}_n\rangle - \langle z_n,\tilde{z}_n\rangle|$ as above using

$$\|x_n - z_n\|_X \underset{\text{preserving}}{\overset{E \text{ norm}}{=}} \|Ex_n - Ez_n\|_Y \to \|y-y\|_Y = 0$$

(ii) $\langle\cdot,\cdot\rangle_Y$ is a scalar product.

linearity clear as "lim" and $\langle\cdot,\cdot\rangle_X$ are linear

symmetry clear

positivity $0 = \langle y,y\rangle = \lim_{n\to\infty}\langle x_n,x_n\rangle$ implies $x_n \to 0$ in $X \implies Ex_n \to 0 = y$.

(iii) $\langle\cdot,\cdot\rangle_Y$ induces the norm $\|\cdot\|_Y$. (this was defined first). In Banach space completion we have $\|y\|_Y = \lim_{n\to\infty}\|x_n\|_X$ for $(x_n)_{n\in\mathbb{N}}\subset X$ with $Ex_n\to y$. This agrees with the def. of $\langle\cdot,\cdot\rangle_Y$ as

$$\|y\|^2_{\langle\cdot,\cdot\rangle_Y} \overset{\text{def}}{=} \langle y,y\rangle_Y = \lim_{n\to\infty}\langle x_n,x_n\rangle_X \underset{\text{Hilbert space}}{\overset{X\text{ pre-}}{=}} \lim_{n\to\infty}\|x_n\|_X^2 \overset{\text{def}}{=} \|y\|_Y^2$$

□

Example 1.2.6 (i) $(\mathbb{R}^n,\langle\cdot,\cdot\rangle_{\text{Euclid}})$ is a Hilbert space (as well as $(\mathbb{C}^n,\langle\cdot,\cdot\rangle_{\text{Euclid}})$).

(ii) The first of *the two important Hilbert spaces* is $\ell_2 = \left\{a=(a_n)_{\mathbb{N}} \mid \sum_{n=1}^\infty |a_n|^2<\infty\right\}$ with $\langle a,b\rangle = \sum_{n=1}^\infty a_n\overline{b}_n$. Claim: This is an infinite dimensional Hilbert space

Proof:
- ℓ_2 vector space is trivial.
- $\langle\cdot,\cdot\rangle:\ell_2\times\ell_2\to\mathbb{K}$ is well defined. ($\sum_{n=1}^\infty a_n\overline{b}_n$ converges absolutely)

$$\sum_{n=1}^N |a_n\overline{b}_n| \overset{\text{CSE}}{\le} \left(\sum_{n=1}^N |a_n|^2\right)^{1/2}\left(\sum_{n=1}^N |b_n|^2\right)^{1/2} \le \left(\sum_{n=1}^\infty |a_n|^2\right)^{1/2}\left(\sum_{n=1}^\infty |b_n|^2\right)^{1/2} < \infty$$

- $\langle\cdot,\cdot\rangle$ satisfies all axioms of the inner product.
- $\|a\|_{\ell_2} = (\langle a,a\rangle)^{1/2} = (\sum_{n=1}^\infty |a_n|^2)^{1/2}$
- Completeness is less trivial. Assume $(a_n)_{n\in\mathbb{N}}$ is a CS in $(\ell_2,\|\cdot\|_{\ell_2})$. To show

$$\exists\tilde{a}\in\ell_2 : \|a^{(n)}-\tilde{a}\|_{\ell_2}\to 0$$

take the component k: $(a_k^{(n)})_{n\in\mathbb{N}}$ is a CS in $\mathbb{K}$, $|a_k^{(n)}-a_k^{(m)}|\le\|a^{(n)}-a^{(m)}\|_{\ell_2}$. Define $\tilde{a}_k=\lim_{n\to\infty}a_k^{(n)}$. We still need to show

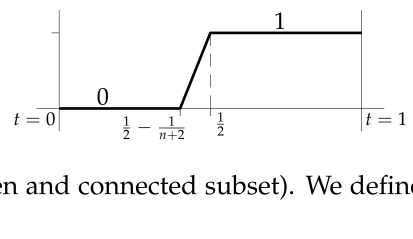
(i) $\tilde{a}\in\ell_2$

(ii) $a^{(n)}\to\tilde{a}$ in ℓ_2

ad (i): $\sum_{k=1}^K|\tilde{a}_k|^2 = \lim_{n\to\infty}\sum_{k=1}^K|a_k^{(n)}|^2 \le \lim_{n\to\infty}\|a^{(n)}\|^2_{\ell_2} =: C<\infty$. With $K\to\infty$ we find $\|\tilde{a}\|^2_{\ell_2}\le C$.

ad (ii) We argue similarly $\sum_{k=1}^K|a_k^{(n)}-a_k^{(l)}|^2 \le \|a^{(n)}-a^{(l)}\|^2_{\ell_2}\le\varepsilon$ for $n,l\ge M_\varepsilon$. With $l\to\infty$ we obtain $\sum_{k=1}^K|a_k^{(n)}-\tilde{a}_k|^2\le\varepsilon$ for $n\ge M_\varepsilon$. since M_ε is independent of $K\in\mathbb{N}$ we can take the limit $K\to\infty$ which yields $\sum_{k=1}^\infty|a_k^{(n)}-\tilde{a}_k|^2\le\varepsilon$ for $n\ge M_\varepsilon$. □

(iii) $X=\mathcal{C}^0([0,1])$, $\langle f,g\rangle_{L^2}=\int_0^1 f(t)\overline{g(t)}\,dt$ non-complete pre-Hilbert space. Consider the CS f_n defined via
Show
- $\|f_n-f_m\|_{L^2}\to 0$
- but there is no limit function



(iv) The *second* of the two important Hilbert spaces. $\Omega\subset\mathbb{R}^d$ "domain" (open and connected subset). We define a pre-Hilbert space as follows

$$X=\mathcal{C}_c(\Omega)\overset{\text{def}}{=}\{u:\Omega\to\mathbb{R}\mid u\text{ continuous, }\operatorname{supp}(u)\Subset\Omega\}$$

$\operatorname{supp}(u)=\overline{\{x\in\Omega\mid u(x)\ne 0\}}$ (*support*) and $\Subset$ means compact contained, so $A\Subset B$ if there exists a compact set G such that $A\subset G$ and $B\subset G$. We define an inner product via

$$\langle u,v\rangle_{L^2(\Omega)} = \int_\Omega u(x)\overline{v(x)}\,dx \qquad \left(=\int_{\operatorname{supp}(u)\cap\operatorname{supp}(v)} u(x)\overline{v(x)}\,dx\right)$$

$(\mathcal{C}_c(\Omega),\langle\cdot,\cdot\rangle)$ is a pre-Hilbert space.

Definition 1.2.7
$L^2(\Omega)$ is (one of) the completion(s) of this pre-Hilbert space.

What is a useful version of this completion? The answer will be given in Section 1.3. It is the classical Lebesgue space $L^2(\Omega)$ of square integrable functions. For instance, $\Omega =]0,1[$, the function $u : x \mapsto x^{-1/3}$ lies in $L^2(]0,1[)$ since $\int_0^1 |u(x)|^2\,dx = \int_0^1 x^{-2/3}\,dx = 3 < \infty$.
Question: Given a norm, is there an associated scalar product?

Proposition 1.2.8
A norm is induced by a scalar product if and only if it satisfies the parallelogram identity*:*

$$x, y \in X : \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof: (Sketch)

- Scalar product implies parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$
$$\overset{\text{linearity}}{=} \|x\|^2 + \cancel{\langle y,x\rangle} + \cancel{\langle x,y\rangle} + \|y\|^2 + \|x\|^2 - \cancel{\langle y,x\rangle} - \cancel{\langle x,y\rangle} + \|y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

- Parallelogram identity implies scalar product: Assume $\mathbb{K} = \mathbb{R}$. $S(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ (check: if $\|\cdot\| = \|\cdot\|_{\langle\cdot,\cdot\rangle}$, then $S = \langle\cdot,\cdot\rangle$). One can prove that S really is a scalar product. One needs continuity arguments ($\to$ analysis). $\square$

Example 1.2.9 $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ is a Banach space but not a Hilbert space: Find $f, g \in \mathcal{C}([0,1])$ such that $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 \neq 2\|f\|_\infty^2 + 2\|g\|_\infty^2$. Take $f(t) = t$, $g(t) = 1-t$, $(f-g)(t) = 2t-1$. Then you have

$$\|f\|_\infty = 1 = \|g\|_\infty = \|f+g\|_\infty = \|f-g\|_\infty$$
$$1^2 + 1^2 \neq 2\cdot 1^2 + 2\cdot 1^2 = 4$$

In Hilbert spaces we can define angles between two vectors:

Definition 1.2.10 (Angle)
If $\mathbb{K} = \mathbb{R}$, we have $\langle x,y\rangle = \rho\|x\|\|y\|$ with $\rho \in [-1,1]$.

$$\sphericalangle(x,y) = \arccos\rho \in [0,\pi]$$

If $\mathbb{K} = \mathbb{C}$, then $\mathbb{C} \ni \langle x,y\rangle = z\|x\|\|y\|$ with $z \in \mathbb{C}$ and $|z| \le 1$. We define

$$\sphericalangle_{\mathbb{C}}(x,y) = \arccos|z| \in [0,\pi]$$

Definition 1.2.11
x is orthogonal *to y (we write $x \perp y$) if $\langle x,y\rangle = 0$. For orthogonal vectors we have Pythagoras' relation*

$$\langle x,y\rangle = 0 \implies \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Definition 1.2.12
$(X, \langle\cdot,\cdot\rangle)$ Hilbert space. J index set. $S = \{x_j \mid j \in J\} \subset X$.

- *S is called an* orthogonal system *(OGS) if for all $j_1, j_2 \in J : j_1 \neq j_2 \implies \langle x_{j_1}, x_{j_2}\rangle = 0$.*
- *We call an OGS an* orthonormal system *(ONS), if additionally $\|x_j\| = 1$ for all $j \in J$.*
- *S is called an* orthonormal basis *(Hamel basis) if every $x \in X$ is a* finite *linear combination of elements in S, i.e. $X = \operatorname{span} S$.*
- *S is called a complete ONS (cONS), if it is an ONS and* $\operatorname{span} S$ *is dense in X. (In that case S is also called a* Hilbert basis*, as every $x \in X$ can be written as a converging infinite linear combination, see below).*

Example 1.2.13 (i) $X = \ell_2$, $e_n = (0,\dots,1,0,\dots)$. $\|e_n\|_{\ell_2} = 1$, $\langle e_n, e_m\rangle_{\ell_2} = \delta_{nm}$. S is an ONS. Since $\operatorname{span} S = \{(a_j)_{j\in\mathbb{N}} \in \ell_2 \mid \exists n \in \mathbb{N}\forall m > n : a_m = 0\}$. We know from earlier, that $\operatorname{span} S$ is dense. Hence, S is a complete ONS.

(ii) $X = \ell_2$ and $S_{\text{even}} = S_e = \{e_{2n} \mid n \in \mathbb{N}\}$. Then, S_e is an ONS, but it is not complete. Claim: e_1 is not in the closure of S_e. We use $\langle e_1, e_{2n}\rangle = 0$. For $y \in \operatorname{span} S_e$, we conclude $\langle e_1, y\rangle = \left\langle e_1, \sum_{n=1}^{N} \alpha_{2n} e_{2n}\right\rangle = \sum_{n=1}^{N} \overline{\alpha_{2n}} \langle e_1, e_{2n}\rangle = 0$. $\forall y \in \operatorname{span} S_e : \|e_1 - y\|^2 = \|e_1 + (-y)\|^2 \overset{\text{Pyt.}}{=} \|e_1\|^2 + \|-y\|^2 \geq 1$. No sequence in $\operatorname{span} S_e$ can approach e_1.

(iii) For S in (i) we see that S is not an algebraic basis, since $\operatorname{span} S \subsetneq X = \ell_2$. For instance $a = (\frac{1}{n})_{n\in\mathbb{N}} \in \ell_2 \setminus \operatorname{span} S$.

We now want to prove that a cONS allows us to write every $x \in X$ in a unique way as a converging infinite linear combination of the elements in S. We consider a Hilbert space $(X, \langle \cdot, \cdot \rangle)$, a countable index set J (e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^d$) and an ONS $S = \{e_j \mid j \in J\}$. We want to write $x \in X$ in this basis.

$$x \mapsto Q_S x := \sum_{j\in J} \langle x, e_j\rangle e_j$$

The next result shows that this definition is well posed and has all the desired properties.

Theorem 1.2.14
The mapping $Q_S : X \to X$ is well posed and linear. Moreover, Q_S is a projection, i.e. $Q_S(Q_S) = Q_S$ and we have Bessel's estimate

$$\|Q_S x\|^2 = \sum_{j\in J} |\langle x, e_j\rangle|^2 \leq \|x\|^2$$

If S is also complete, then $Q_S = id_X$, i.e.

$$\forall x \in X : x = Q_S x = \sum_{j\in J} \langle x, e_j\rangle e_j$$

and we have Parseval's identity

$$\|x\|^2 = \sum_{j\in J} |\langle x, e_j\rangle|^2$$

This explains the notion "Hilbert basis", which is sometimes used for cONS.

Proof:

Step 1 Assume $J = \mathbb{N}$ (one particular counting) We want to define

$$Q_S x \overset{??}{:=} \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j = \lim_{N\to\infty} \overbrace{\sum_{j=1}^{N} \underbrace{\langle x, e_j\rangle}_{\alpha_j} e_j}^{=:x_N}$$

To show $(x_N)_{N\in\mathbb{N}}$ is a CS. Then $Q_S x := \lim_{N\to\infty} x_N$, $y_N := x - x_N$.

$$\implies \forall j \in \{1, \dots, N\} : \langle y_N, e_j\rangle = \langle x, e_j\rangle - \alpha_j = 0$$

$$x = \sum_{j=1}^{N} \alpha_j e_j + y \quad \text{orthog. decomp.}$$

$$\|x\|^2 = \sum_{j=1}^{N} |\alpha_j|^2 + \|y\| \quad \text{Pythagoras}$$

Hence for all $N : \sum_{j=1}^{N} |\alpha_j|^2 = \|x\|^2 - \|y_N\|^2 \leq \|x\|^2$. For $N \to \infty$ we obtain Bessel's estimate

$$\sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2 \leq \|x\|^2$$

We now show that (x_N) is a CS. For $M > N$ we have

$$\|x_N - x_M\|^2 = \left\| \sum_{j=N+1}^{M} \alpha_j e_j \right\|^2 \overset{\text{Pyt.}}{=} \sum_{j=N+1}^{M} |\alpha_j|^2 \xrightarrow{N\to\infty} 0$$

Therefore we have a CS and $Q_S x := \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j$ is well defined. (Linearity of Q_S is simple).

Step 2 Consider any other numbering of the index set J. Renumbering $k_{\mathbb{N}} \to \mathbb{N}$ bijective. Claim:

$$\lim_{N\to\infty} \sum_{n=1}^{N} \left\langle x, e_{k(n)} e_{k(n)} \right\rangle = Q_S x$$

We define $\hat{x}_N$ as above and $M_N = \max\{m \mid \{1,\dots,m\} \subset \{k(1),\dots,k(N)\}\}$. Since k is surjective, we have $M_N \to \infty$ for $N \to \infty$ and hence $x_{M_N} \overset{\text{as before}}{=} \sum_{j=1}^{M_N} \left\langle x, e_j \right\rangle e_j \to Q_S x$. Now

$$\|\hat{x}_N - x_{M_N}\|^2 \overset{\text{ONS}}{=} \sum_{\substack{n\in\{1,\dots,N\}\\ k(n)>M_N}} |\alpha_{k(n)}|^2 \le \sum_{m>M_N} |\alpha_m|^2 \xrightarrow{N\to\infty} 0$$

Hence $\hat{x}_N - x_{M_N} \to 0$ and using $x_{M_N} \to Q_S x$ we conclude $\hat{x}_N \to Q_S x$ as well. Hence we can define $\sum_{j\in J} \left\langle x, e_j \right\rangle e_j$ without referring to any particular ordering in J.

Step 3 Assume that S is complete. We have to show that $Q_S x = x$. For x given, define $y = x - Q_S x$. By definition of Q_S we have $\left\langle y, e_j \right\rangle = 0$ for $j \in J$. Thus: $\|x\|^2 = \|Q_S x\|^2 + \|y\|^2$. Since span S is dense, there exists $(s_m)_{m\in\mathbb{N}}$ with $\|y - s_m\| \to 0$ and $s_m \in \operatorname{span} S$. Note $\langle y, s_m \rangle = 0$ as $\left\langle y, e_j \right\rangle = 0$. $\|y\|^2 = \langle y, y \rangle = \langle y, \lim_{m\to\infty} s_m \rangle = \lim_{m\to\infty} \langle y, s_m \rangle = 0$. Hence $y = 0 \implies x = Q_S x$. □

Proposition 1.2.15 (Ex. 9)
Assume that $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space and S is an ONS and J countable (e.g $\mathbb{N}$). Then the following statements are equivalent:

(i) S is complete.

(ii) If x satisfies $\left\langle x, e_j \right\rangle = 0$ for all $j \in J$, then $x = 0$.

(iii) Parseval's identity holds

(iv) $\forall x \in X : \sum_{j\in J} \left\langle x, e_j \right\rangle e_j = x$.

The most important application are the

Fourier series

$X = L^2(]0, 2\pi[)$.

real case	complex case
$J = \mathbb{Z}$	$J = \mathbb{Z}$
$j \ge 1\ e_j(t) = \frac{1}{\sqrt{\pi}} \cos(jt)$	$\tilde{e}_j(t) = \frac{1}{\sqrt{2\pi}} \exp(ijt)$
$j = 0\ e_0(t) = \frac{1}{\sqrt{2\pi}} 1$	
$j < 0\ e_j(t) = \frac{1}{\sqrt{\pi}} \sin(jt)$	

In both cases we have an ONS. e.g.

$$\langle \tilde{e}_n, \tilde{e}_m \rangle = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{int} \frac{1}{\sqrt{2\pi}} e^{-imt}\, dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)t}\, dt = \frac{1}{2\pi} \delta_{n-m} 2\pi = \delta_{n-m}$$

Claim: These ONS are complete. Classical analysis shows: If $f \in \mathcal{C}^2([0, 2\pi])$ then $\underbrace{\sum_{j=-N}^{N} \left\langle f, \tilde{e}_j \right\rangle e_j}_{\in \operatorname{span} \tilde{S}} \to f$ uniformly on $[0, 2\pi]$. Hence, span $\tilde{S}$ is dense in $\mathcal{C}^2([0, 2\pi])$, in $\mathcal{C}^0([0, 2\pi])$, in $L^2(]0, 2\pi[)$ and therefore it is complete.

Remark $f : [0, 2\pi] \to \mathbb{R}$ piecewise $\mathcal{C}^1$, $f|_{]t_k, t_{k+1}[}$ has an extension $f_k \in \mathcal{C}([t_k, t_{k+1}])$. Then for all $t \in [0, 2\pi]$ we have $f_N(t) = \sum_{j=-N}^{N} \left\langle f, e_j \right\rangle e_j(t) \to \tilde{f}(t)$ with $\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \in]0, 2\pi[\setminus \{t_1, \dots, t_k\} \\ \frac{1}{2}(f(t_k - 0) + f_k(t + 0) & \text{for } t = t_k \\ \frac{1}{2}(f(0+0) + f(2\pi - 0) & \text{for } t = 0 \text{ or } t = 2\pi \end{cases}$ each f_N is

analytic. We have pointwise convergence, but no uniform convergence.
Gibbs' phenomenon: "overshooting"

$$f(t) = \begin{cases} 1 & \text{for } t \in [\frac{\pi}{2}, \frac{3\pi}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \|f_N\|_\infty \to 1.09$$

But nevertheless $\|f_N - f\|_{L^2} \to 0$ as we have a cONS. In $L^2(\Omega)$ we have $\|f - \tilde{f}\|_{L^2} = 0$.

Does every Hilbert space has a complete ONS? Yes, there are many.

Definition 1.2.16
A Banach space $(X, \|\cdot\|)$ is called separable, *if there exists a countable set that is dense in X.*

Example 1.2.17 (i) $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidean norm are separable, since $\mathbb{Q}^n$ and $(\mathbb{Q} + i\mathbb{Q})^n$ are dense and countable.

(ii) $\ell_2 = \{(a)_{j\in\mathbb{N}} \mid \sum |a_j|^2 < \infty\}$. $D = \{\sum_{i=N}^N q_i e_i \mid N \in \mathbb{N}, q_i \in \mathbb{Q}\}$ is countable and dense.

(iii) $(\mathcal{C}[a,b], \|\cdot\|_\infty)$ is separable. $D =$ polynomials with rational coefficients.

(iv) $X = B([0,1])$ space of bdd. functions $\|f\|_\infty = \sup\{|f(t)| \mid t \in [0,1]\}$. There exists an uncountable set U such that $\forall f, g \in U : \|f - g\|_\infty = 0$ or 2. Hence X cannot be separable. For $\xi \in [0,1]$ set $f_\xi(t) = \delta_{\xi t}$ and $U = \{f_\xi \mid \xi \in [0,1]\}$. $\|f_{\xi_1} - f_{\xi_2}\|_\infty = 1$ if $\xi_1 \neq \xi_2$.

(v) Without proof $(\mathcal{C}(]0,1[), \|\cdot\|_\infty)$ is a non-separable Banach space. $f(t) = \sin(\frac{1}{t}) \in \mathcal{C}(]0,1[)$.

Theorem 1.2.18
For every separable Hilbert space, there exists a countable cONS.

Proof: We use the *Gram-Schmidt algorithm*. $D = \{x_j \mid j \in \mathbb{N}\}$ is our dense and countable set. By induction we construct a ONS $\{e_k \mid k \in \mathbb{N}\}$ as follows.

$$\begin{aligned} &e_1 = \frac{1}{\|x_1\|} x_1 \implies V_1 = \operatorname{span}\{e_1\} \\ &V_k := \operatorname{span}\{x_1, \dots, x_k\} \quad \text{and} \quad m_k = \dim V_k \leq k \\ &V_k = \operatorname{span}\{e_1, \dots, e_k\} \end{aligned}$$

If $m_{k+1} > m_k$, then $e_{m_{k+1}}$ is obtained by Gram-Schmidt.
If $m_k \to m^* < \infty$ after finally many steps, then we have to show that $V_{m^*} = X$, i.e. X is finite dimensional.
If $m_k \to \infty$, we obtain an infinite countable ONS.
In both cases $S = \{e_j \mid j \in J\}$ where $J = \{1, \dots, m^*\}$ or $J = \mathbb{N}$.
Claim: S is complete. We use that D is dense. $\forall j : D \ni x_j \in \operatorname{span} S$. In fact, $x_j \in V_j \subset \operatorname{span} S$. Hence, using $D \subset \operatorname{span} S$ we have shown density of $\operatorname{span} S$ in X $\square$

Example 1.2.19 Using Gram-Schmidt, we generate new cONS. $L^2([-1,1])$ completion of $(\mathcal{C}([-1,1], \langle\cdot,\cdot\rangle_{L^2})$. Possible set for starting are polynomials $q_1(t) = 1$, $q_n(t) = t^{n-1} \implies Q = \{q_n \mid n \in \mathbb{N}\}$. We know that $\operatorname{span} Q$ is dense in $L^2(]-1,1[)$. Applying Gram-Schmidt to the sequence, we obtain a cONS $S = \{p_n \mid n \in \mathbb{N}\}$ with $\langle p_n, p_m\rangle_{L^2} = \int_{-1}^1 p_n(t) p_m(t)\, dt = \delta_{n,m}$. Up to a normalisation constant, we obtain the *Legendre polynomials*

$$\begin{aligned} &p_1(t) = c_1 \cdot 1 \qquad p_2(t) = c_2 t \qquad p_3(t) = c_3(t^2 - 3) \qquad p_4(t) = c_4(5t^3 - 3t) \\ &p_5(t) = c_5(35t^4 - 30t^2 + 3) \end{aligned}$$

Theorem 1.2.20 (structure of separable Hilbert spaces)
Every separable Hilbert space is norm-isomorphic to either $\mathbb{K}^n_{Euclid}$ or ℓ_2.

We say that two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are called norm-isomorphic, *if there exist, a linear bijective mapping $A : X \to Y$ such that $\|Ax\|_Y = \|x\|_X$ for all x. For Hilbert spaces even the scaler products are preserved*

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

Proof: Wlog assume the case of a infinite cONS. $(X, \langle\cdot,\cdot\rangle)$ has some cONS $S_X = \{e_j^X \mid j \in \mathbb{N}\}$. $(\ell_2, \langle\cdot,\cdot\rangle_X)$ has the natural cONS $S_{\ell_2} = \{\tilde{e}_j \mid j \in \mathbb{N}\}$ with $\tilde{e}_j = (0, \dots, 0, 1, 0, \dots)$. We define

$$A : \begin{cases} X \to \ell_2 \\ x \mapsto \sum_{j=1}^\infty \left\langle x, e_j^X \right\rangle \tilde{e}_j \end{cases}$$

and find $\|Ax\|_{\ell_2} \overset{\text{Parseval}}{\underset{\text{in } \ell_2}{=}} \sum_{j=1}^{\infty} \left\langle x, e_j^X \right\rangle \overset{\text{Pars.}}{\underset{\text{in } X}{=}} \|x\|_X^2$ because $x \in X \implies \|x\|^2 = \sum | \left\langle x, e_j^X \right\rangle |^2 < \infty$. A is injective, since $Ax = 0 \implies 0 = \|Ax\| = \|x\|$. A is surjective, because

$$B : \begin{cases} \ell_2 \to X \\ (a_j) \mapsto \sum_{j=1}^{\infty} a_j e_j^X \end{cases}$$

is a norm preserving inverse. $A = B^{-1}$, $B = A^{-1}$. □

Wavelets

invented by Ingrid Daubechies. Wavelets are *localized* versions of trigonometric functions.

(i) Generate the first generation by translation.

(ii) Second generation. refine by factor 2.

$$W = \{\underbrace{2^{d\sigma(j)}\Psi\left(\frac{x - x(j)}{2^{\sigma(j)}}\right)}_{W_j} \mid j \in J\} \qquad d = \text{ dimension}$$

(iii) Choose Ψ suitable (art of I.Daubechies) gives span $W = L^2(\Omega)$.

$$\text{b/w picture} \qquad u : \Omega \to [0,1] \quad u \in L^2(\Omega)$$

$u = \sum_{j \in J} \alpha_j W_j$ complete expansion where $\alpha_j = \langle u, V_j \rangle$. compression factor $\varepsilon \in]0,1[$.

$$u_{\text{jpg}} = \sum_{\substack{j \in J \\ |\alpha_j| \leq \varepsilon \|u\|_{L^2}}} \alpha_j W_j$$

This is a non-linear mapping.

1.3 The Spaces L^p

Elements of Measure Theory

Definition 1.3.1
Let X be a set and $\Sigma \subseteq \mathcal{P}(X)$ a system of subsets, Σ is called a σ-algebra *if*

(i) $\emptyset \in \Sigma$

(ii) $E \in \Sigma \implies X \setminus E \in \Sigma$

(iii) $(E_m)_{m \in \mathbb{N}}$ *sequence with* $E_m \in \Sigma \implies \bigcup_{m \geq 1} E_m \in \Sigma$

Definition 1.3.2
Let X be a set and $\Sigma \in \mathcal{P}(X)$ a σ-algebra. Then (X, Σ) is called a measurable space.

Lemma 1.3.3
Let $P \subseteq \mathcal{P}(X)$ be a system of subsets. Then there exists a unique *σ-algebra denoted $\Sigma(P)$ such that $P \subseteq \Sigma(P)$.*

Definition 1.3.4
Let $(X, \mathcal{O})$ topological space, (e.g. (X, d) metric space, $\mathcal{O}$ open subset wrt d). $\mathcal{B}(X) := \Sigma(\mathcal{O})$ is called the σ-algebra of Borel sets.

Definition 1.3.5
Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces, $f : X_1 \to X_2$ is called measurable *, if $f^{-1}(E) \in \Sigma_1$ for all $E \in \Sigma_2$.*

Lemma 1.3.6
Let $P_2 \subseteq \mathcal{P}(X_2)$ and $\Sigma_2 = \Sigma(P_2)$, $f : X_1 \to X_2$. Then $\forall E \in P_2 : f^{-1}(E) \in \Sigma_1 \implies f$ is measurable.

Corollary 1.3.7
Let (X, Σ) be a measurable space and $f : (X, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the following statements are equivalent:

(i) f *measurable*

(ii) *preimages of all intervals* (a, ∞) *are measurable*

(iii) *preimages of all closed intervals are measurable.*

Definition 1.3.8
$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $\mathcal{B}(\overline{\mathbb{R}}) := \{A \in \mathcal{P}(\mathbb{R}) \mid A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$

Definition 1.3.9
Let (X, Σ) *be a measurable space. A* measure *on* (X, Σ) *is a function* $\mu : \Sigma \to [0, \infty)$ *such that*

(i) $\mu(\emptyset) = 0$

(ii) *Let* $(E_m)_{m \in \mathbb{N}}$ *be a seq. of pairwise disjoint sets* $E_m \in \Sigma$*. Then* $\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$

Definition 1.3.10
Let μ *be a measure on* (X, Σ)*. Then* (X, Σ, μ) *is called a* measure space*.*

Lemma 1.3.11
On $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$*, there exists a* unique measure *such that* $\lambda((a, b)) = (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$ *for all* $a = (a_1, \dots, a_n)$*,* $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ *with* $a_j \leq b_j$ *for all* j*.* λ *is called the* Lebesgue measure*.*

Definition 1.3.12
(X, Σ, μ) *measure space.* $N \subseteq X$ *is called a* μ-zeroset *if* $N \subseteq E \in \Sigma$ *such that* $\mu(E) = 0$*.*

Definition 1.3.13
Let $S(x)$ *be a statement depending on* $x \in X$*. We say,* S *holds* μ-almost-everywhere *(a.e.) if* $\exists X_0 \in \Sigma$ *such that*

(i) $\forall x \in X_0 : S(x)$

(ii) $\mu(X \setminus X_0) = 0$.

Lemma 1.3.14
Let (X, Σ, μ) *be a measure space, such that every zeroset is in* Σ*. Let* $f_n : (X, \Sigma, \mu) \to (\overline{\mathbb{R}}, \mathcal{B}(\mathbb{R}))$ *be a sequence of measurable functions, s.t.* $f_n \to f : X \to \overline{\mathbb{R}}$ *a.e. (i.e.* $f_n(x) \to f(x)$ *a.e.). Then* f *is measurable.*

Definition 1.3.15
(X, Σ, μ) *measure space,* $f : X \to \overline{\mathbb{R}}$*,* $E \in \Sigma$*, the Lebesgue integral wrt* μ *is denoted as* $\int_E f \, d\mu$*.*

Definition 1.3.16
$f : X \to \overline{\mathbb{R}}$ *is called* integrable *if*

(i) f *measurable*

(ii) $\int_X |f| \, dx < \infty$.

Convergence Theorems

Here $X = \Omega \subseteq \mathbb{R}^n$ domain, i.e. open and connected. $\Sigma = \mathcal{B}(\Omega, \mu = \lambda)$. The following theorems hold in a more general case than this.

Theorem 1.3.17 (Beppo-Levi, monotone convergence)
Let $f_n : \Omega \to \overline{\mathbb{R}}$ *be a sequence such that*

(i) f_n *measurable*

(ii) $f_n \leq f_{n+1}$ *a.e.* $E \in \Sigma$

Then f_n *converges pointwise a.e. to a measurable function* f *and* $\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n$*.*

Theorem 1.3.18 (Fatou, one-sided boundedness)

(i) $f_n : \Omega \to \overline{\mathbb{R}}$*,* $E \in \Sigma$

(ii) $\exists$ *integrable* g *such that* $f_n \geq g$ *a.e. (e.g.* $g \equiv 0$*)*

Then

$$\int_E (\liminf_{n\to\infty} f_n) \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu$$

Theorem 1.3.19 (Lebesgue, dominated convergence)

(i) $f_n : \Omega \to \overline{\mathbb{R}}$ *measurable*

(ii) $f_n \to f$ *pointwise a.e.*

(iii) $\exists$ *measurable* $g : |f_n| \leq g$ *a.e.*

Then for $E \in \Sigma$, g *is integrable on* E *implies* f, f_n *is integrable on* E *and* $\lim_{n\to\infty} \int_E f_n = \int_E (\lim_{n\to\infty} f_n)$.

The Spaces L^p

Definition 1.3.20
Let $p \in [1,\infty)$, $\mathcal{L}^p(\Omega) := \{f : \Omega \to \mathbb{R} \mid f \text{ measurable and } |f|^p \text{ integrable } \}$ *with* $\|f\|_p := (\int_\Omega |f|^p \, dx)^{\frac{1}{p}}$

Lemma 1.3.21
$0 = \|f - g\|_p \iff f = g$ *a.e., therefore* $\|\cdot\|_p$ *is not a norm. But*

Lemma 1.3.22
$(\mathcal{L}^p(\Omega), \|\cdot\|_p)$ *is a half-normed space.*

Proof: triangle estimate by Minkowski inequality, other properties are clear.

Lemma 1.3.23 (Hölder inequality)
$p, q, \in (1,\infty)$ *s.t.* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f \in \mathcal{L}^p$, $q \in \mathcal{L}^q$ *implies* $f \cdot g \in \mathcal{L}^1$ *and* $\|f \cdot g\|_1 \leq \|f\|_p \|g\|_q$.

Proof: Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ with $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(x)|}{\|g\|_q}$.

$$\int \frac{|f(x)||g(x)|}{\|f\|_p \|g\|_q} \leq \int \ldots = \frac{1}{p} \frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q} = 1$$

$\square$

Lemma 1.3.24 (Minkowski-inequality)
$f, g \in \mathcal{L}^p \implies f + g \in \mathcal{L}^p$ *and* $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof: (for $p \neq 1$)

$$|f+g|^p \leq (|f| + |g|)^p \leq 2^{p-1}(|f|^p + |g|^p) \implies f + g \in \mathcal{L}^p$$

$$|f+g|^p \leq |f||f+g|^{p-1} + |g||f+g|^{p-1} \overset{\text{integrate}}{\implies} \|f+g\|_p^p = \int |f+g|^p \, dx \leq \||f||f+g|^{p-1}\|_1 + \||g|f+g|^{p-1}\|_1$$

$$\overset{\text{Hölder}}{\leq} \|f\|_p \||f+g|^{p-1}\|_q + \|g\|_p \||f+g|^{p-1}\|_q \quad \text{with} \quad q = \frac{p}{p-1}$$

$$= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}$$

Definition 1.3.25 (L^p)
Let $f, g \in \mathcal{L}^p(\Omega)$, $f \sim g :\iff f = g$ *a.e.* $\iff \|f - g\|_p = 0$ *and* $L^p(\Omega) := \mathcal{L}^p(\Omega)_{/\sim}$ *equivalence classes* $\{g \in \mathcal{L}^p(\Omega) \mid g = f \text{ a.e. } \} = [f] \in L^p(\Omega)$.

Lemma 1.3.26
$\sim$ *is an equivalence relation and* $(L^p, \|\cdot\|_p)$ *is a normed vector space (Proof Ex. 1).*

Example 1.3.27 $\Omega = (0,1)$, $f, g : \Omega \to \mathbb{R}$, $f(x) := 1$, $g(x) := \begin{cases} 0 & x \in \mathbb{Q} \cap (0,1) \\ 1 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1) \end{cases}$
$f \sim g$ since $\lambda(\mathbb{Q} \cap (0,1)) = 0$. Notation; $f \in L^p$ although not quite correct.

Example 1.3.28 Let $f : \Omega \to \mathbb{R}$, $\operatorname{supp} f := \overline{\{x \in \Omega \mid f(x) \neq 0\}}$ the support of f. $\mathcal{C}_c^0(\Omega) = \{f \in \mathcal{C}^0(\Omega) \mid \operatorname{supp} f$ is compact$\}$. $f \in \mathcal{C}_c^0(\Omega) \implies [f] \in L^p(\Omega)$.

Properties of L^p

The aim is $(L^p(\Omega), \|\cdot\|_p)$ is a completion of $(\mathcal{C}_c^0(\Omega), \|\cdot\|_p)$.

Theorem 1.3.29 (Riesz-Fischer)
$(L^p(\Omega), \|\cdot\|_p)$ *is complete.*

Proof: Let $(f_k)_{k\in\mathbb{N}}$ be a CS in $L^p(\Omega)$. Choose a subsequence $(k_i)_{i\in\mathbb{N}}$ such that $\|f_k - f_l\|_p \le 2^{-i}\forall k,l \ge k_i$. in the following denote this subsequence again by $(f_k)_{k_i}$. Then

$$\sum_{k\in\mathbb{N}} \|f_{k+1} - f_k\|_p \le \sum_{k\in\mathbb{N}} 2^{-k} < \infty$$

Now define $g_l := \sum_{k=1}^{l} |f_{k+1} - f_k|$. Now using the Theorem of Fatou (1.3.18) and the Minkowski inequality (1.3.24):

$$\int \lim_{l\to\infty} g_l^p \, d\mu \le \liminf_{l\to\infty} \int g_l^p \, d\mu = (\liminf \|g_l\|_p)^p \le (\sum_{k\in\mathbb{N}} \|f_{k+1} - f_k\|_p)^p < \infty$$

$$\implies \lim_{l\to\infty} g_l(x) < \infty \text{ a.e.} \implies f_k(x) \text{ is a CS a.e.}$$

Therefore $\exists$limit $f(x) = \lim_{l\to\infty} f_k(x)$ a.e. By Lemma 14 $\implies$ f is measurable and, by Fatou

$$\int (f - f_l)^p \, d\mu \le \liminf_{k\to\infty} \int |f_k - f_l|^p \, d\mu = (\liminf_{k\to\infty} \|f_k - f_l\|_p)^p \xrightarrow{l\to\infty} 0$$

$$\implies \|f - f_l\| \to 0$$

$\square$

Theorem 1.3.30
$L^p(\Omega)$ is complete for $1 \le p < \infty$.

Corollary 1.3.31
Let $f_m \to f$ in $L^p(\Omega)$, there exists a subsequence $(m_k)_{k\in\mathbb{N}}$ such that $f_{m_k}(x) \to f(x)$ a.e.

Theorem 1.3.32
$\mathcal{C}_c^0(\Omega)$ is dense in $(L^p(\Omega), \|\cdot\|_p)$ for $p \in [1,\infty)$.

Remark This is true for $\mathcal{C}_c^\infty(\Omega)$ as well.

Proof: **Step 1** Define s *simple* if $s = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$ with A_i measurable and $\chi_{A_i}(x) = 1$ if $x \in A_i$ and 0 otherwise. Let $f \in L^p$ and then $f^+, f^- \ge 0$ with $f = f^+ - f^-$. To choose s_m^+ (and similarly s_m^-) decompose $[0,\infty)$ into the disjoint intervals

$$I_j^m = [\tfrac{j-1}{M}, \tfrac{j}{M}) \quad \text{for} \quad j = 1, ..., M^2 \quad \text{and} \quad I_{M^2+1} = [M, \infty) \quad \text{where} \quad M = 2^m$$

Now define

$$A_j^m = \{x \in \Omega | f(x) \in I_j^m\} \quad \text{and} \quad s_m^+ = \sum_{j=1}^{M^2+1} \frac{j-1}{M} \chi_{A_j^m}$$

Then

$$|f - s_m|^p = |f^+ - s_m^+|^p + |f^- - s_m^-|^p \le |f^+|^p + |f^-|^p = |f|^p \quad \text{a.e.}$$

The theorem of dominated convergence (1.3.19) implies $\|f - s_m\| \to 0$.

Step 2 Let $\lambda(A) < \infty$. We want to show, that

$$\forall \varepsilon > 0 \exists f_\varepsilon \in \mathcal{C}_c^0(\Omega) \quad \text{such that} \quad \|f_\varepsilon - \chi_A\|_p < \varepsilon$$

From the theory of the Lebesgue measure we know that there exists an open set $\mathcal{O}_\varepsilon$ such that $A \subset \mathcal{O}_\varepsilon$ and $\lambda(\mathcal{O}_\varepsilon \setminus A) < \varepsilon$. Hence, we have $\|\chi_A - \chi_{\mathcal{O}_\varepsilon}\|_p < \varepsilon^{\frac{1}{p}}$.

Thus, we have reduced the task to considering open sets $\mathcal{O} = A$. We consider a decomposition of $\mathbb{R}^d$ into small semi-open cubes $Q_n^N = \frac{1}{2^N}(n + [0,1)^d)$, where $n \in \mathbb{Z}^d$ is the integer lattice and 2^{-N} is the side-length of the cubes. For a given $N \in \mathbb{N}$ consider $A_N \subset A$ as the union of all cubes Q_n^N that are contained in A. Then, $A_N \subset A_{N+1}$ by construction (subdivision of the cubes). Moreover, $A = \sup_{N\in\mathbb{N}} A_N$, since A is open.

Thus, we conclude $\lambda(A \setminus A_M) \to 0$ for $N \to \infty$. And by the same argument going from general A to open $A = \mathcal{O}$, it now suffices to consider a set A_M instead of a general A. Since A_M is a finite union of cubes $A_M = \bigcup_{j=1}^{K} Q_{n_j}^M$ we only need to find $f_{j,\varepsilon} \in \mathcal{C}_c(\Omega)$ such that $\|f_{j,\varepsilon} - \chi_{Q_{n_j}^M}\|_p < \frac{\varepsilon}{K}$. For this simply define

$f_{j,\varepsilon}(x)$ as 0 outside of $Q^M_{n_j}$ and as $\min\{\mathrm{dist}(x, \partial Q^M_{n_j})/\delta, 1\}$ for $x \in Q^M_{n_j}$. Choosing δ sufficiently small, we have the estimate. Moreover, using the triangle estimate, the sum $f_\varepsilon = \sum_{j=1}^K f_{j,\varepsilon}$ satisfies $\|f_\varepsilon - \chi_{A_m}\|_p < \varepsilon$.

$\square$

Definition 1.3.33 ($L^\infty(\Omega)$)
For $f : \Omega \to \mathbb{R}$, define the essential supremum

$$\|f\|_\infty := \mathrm{esssupp}\{|f(x)| \mid x \in \Omega\} := \inf_{\lambda(N)=0} \sup_{x \in \Omega \setminus N} |f(x)| = \inf\{R \geq 0 \mid \lambda(S_R) = 0\}$$

with $S_R := \{x \in \Omega \mid |f(x)| > R\}$.

Definition 1.3.34
$\mathcal{L}^\infty(\Omega) : \{f : \Omega \to \mathbb{R} \mid f$ *measurable and* $\|f\|_\infty < \infty\}$.

Lemma 1.3.35
$(\mathcal{L}^\infty(\Omega), \|\cdot\|_\infty)$ *is a half-normed vector space.*

Definition 1.3.36
$L^\infty(\Omega) := \mathcal{L}^\infty(\Omega)_{/\sim}$ *where* $f \sim g :\iff \|f - g\|_\infty = 0$.

Lemma 1.3.37
$(L^\infty(\Omega), \|\cdot\|_\infty)$ *is complete.*

Proof: Let $[f_n] \in L^\infty$ be a CS. That implies f_n is a CS in $\mathcal{L}^\infty(\Omega)$.

$$\Theta := \bigcup_{n=1}^\infty \{x \in \Omega \mid |f_n(x)| > \|f_n\|_\infty\} \cup \bigcup_{m,n=1}^\infty \{x \in \Omega \mid |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\} \implies \lambda(\Theta) = 0$$
$$\implies \forall x \in \Omega \setminus \Theta : |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty$$

$\implies f_n$ converges pointwise uniformly on $\Omega \setminus \Theta$ to $f := \lim_{n\to\infty}(f \cdot \chi_{\Omega\setminus\Theta}) \in \mathcal{L}^\infty(\Omega) \implies [f_n] \to [f]$ in L^∞.
$\square$

Lemma 1.3.38
$\mathcal{C}^0_c(\Omega)$ not *dense in* $L^\infty(\Omega)$.

Addition to earlier proof: Covering of an open set by a *disjoint* union of open balls.

Definition 1.3.39
If $\Omega \subset \mathbb{R}^d$ is an open set, we call $V = \{A_j \mid j \in J\}$ a Vitali covering *of Ω if*

(i) $\forall j \in J : A_j$ *is open*

(ii) $\forall x \in \Omega \forall \varepsilon > 0 \exists j \in J : x \in A_j \subset (B_\varepsilon(x) \cap \Omega)$.

It is easy to obtain Vitali coverings as follows. Take your favorite open bounded set F with $0 \in F$. $\{A_{x,\varepsilon} \mid x \in \Omega, \varepsilon > 0 : A_{x,\varepsilon} = x + \varepsilon F \subset \Omega\}$.

Theorem 1.3.40 (Covering Theorem, Vitali 1908, Lebesgue)
If $\Omega \in \mathbb{R}^d$ is open and $V = \{A_j \mid j \in J\}$ is a Vitali covering, then there exists a zeroset $N \subset \Omega$ and a countable subset $\{A_{jn} \mid n \in \mathbb{N}\}$ of V such that the following holds:

(i) $n \neq m \implies A_{jn} \cap A_{jm} = \emptyset$ *(disjointness)*

(ii) $\bigcup_{n\in\mathbb{N}} A_{jn} = \Omega \setminus N$

back to the proof: $\mathcal{O}_\varepsilon$ cover with balls $F = B_1(0)$. $\mathcal{O}_\varepsilon \setminus N = \bigcup_{i=1}^\infty B_{r_i}(x_i)$ disjoint union. $\chi_{\mathcal{O}_\varepsilon \setminus N}$ in $L^1(\Omega)$ since $\chi_{\mathcal{O}_\varepsilon}(x) = \chi_{\mathcal{O}_\varepsilon \setminus N}(x)$ a.e. in Ω.

$\mathcal{C}^0_c(\Omega)$ is dense in $L^p(\Omega)$ for all $p \in [1, \infty[$, not for $p = \infty$.

1.4 Sobolev spaces

These spaces contain L^p functions whose derivatives are also in L^p. They are fundamental for Mathematical physics (quantum mechanics), partial differential equations (financial mathematics, finite elements,...). To take a weak definition of derivatives that can be generalized to L^p functions. We use the framework of *distributions*.

Definition 1.4.1
Given $\Omega \subset \mathbb{R}^d$ open. A linear mapping $T : \mathcal{C}_c^\infty(\Omega) \to \mathbb{K}$ is called a distribution, *if for all open $D \Subset \Omega$ there exists $k \in \mathbb{N}_0$ and $C > 0$ such that*

$$|T(\varphi)| \le C\|\varphi\|_{\mathcal{C}^k(\Omega)} \quad \textit{for all } \varphi \in \mathcal{C}_c^\infty(\Omega) \textit{ with } \operatorname{supp}(\varphi) \subset D.$$

Where $A \Subset B \overset{Def}{\iff} \exists K$ compact $\exists \mathcal{O}$ set such that $A \subset K \subset \mathcal{O} \subset B$.

Example 1.4.2

(i) *Delta-distribution* (*Dirac-distribution*) For $x_* \in \Omega$ define

$$T(\varphi) = \varphi(x_*) \quad \text{choose } C = 1 \text{ and } k = 0$$

(ii) $T(\varphi) = \partial_{x_1}^{17}\varphi(x_*) + \Delta\varphi(\tilde{x})$, $C = d + 1$, $k = 17$

(iii) $\Omega = \mathbb{R}$, $f(x) = e^{x^2}$

$$T_f(\varphi) := \int_{\mathbb{R}} f(x)\varphi(x)\,dx = \int\limits_{\operatorname{supp}(\varphi)} f(x)\varphi(x)\,dx \quad \text{well defined}$$

For $\mathcal{D} \subset]-l, l[$ we find $|T_f(\varphi)| \le 2le^{l^2}\|\varphi\|_{\mathcal{C}^0(\Omega)}$ if $\operatorname{supp}(\varphi) \subset D$.

(iv) For $f \in L^p(\Omega)$ we define a distribution T_f via

$$T_f(\varphi) \overset{\text{def}}{=} \int_\Omega f(x)\varphi(x)\,dx = \int_{\mathcal{D}} f(x)\varphi(x)\,dx \quad \text{where} \quad \varphi \in \mathcal{C}_c^0(\Omega) \quad \text{with } \operatorname{supp}(\varphi) \subset \mathcal{D} \Subset \Omega$$

$$\implies |T_f(\Omega)| \overset{\text{Hölder}}{\le} \|f\|_{L^p(\mathcal{D})} \underbrace{\|\varphi\|_{L^{p'}(\mathcal{D})}}_{\text{finite}} \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$\le \|f\|_{L^p(\Omega)} \underbrace{\left(\int_{\mathcal{D}} 1^{p'}\,dx\right)^{1/p'}}_{\mu(D)} \|\varphi\|_{\mathcal{C}^0}$$

If $\mu(\Omega) < \infty$, then $\mu(\mathcal{D})$ can be estimated by $\mu(\Omega) < \infty$ uniformly.

(v) $\Omega = \mathbb{R}^3 \quad T_{S^2}(\varphi) = \dfrac{1}{4\pi}\displaystyle\int_{|x|=1} \varphi(x)\,da$

Definition 1.4.3 (multi dimensional derivatives)

$$D^\alpha\varphi = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}}\varphi \quad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \quad |\alpha| = \alpha_1 + \cdots \alpha_d$$

$$\|\varphi\|_{\mathcal{C}^k(\Omega)} := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \|D^\alpha\varphi\|_{\mathcal{C}^0(\Omega)} \quad \textit{where} \quad \mathcal{C}^0(\Omega) \quad \textit{has } \|\cdot\|_\infty$$

We motivate the definition of derivatives for distributions as follows: Take $u \in \mathcal{C}^k(\Omega)$ and consider the induced distribution T_u then we want $D^\alpha(T_u) = T_{D^\alpha u}$. (Note that $u \mapsto T_u$ is an embedding of $\mathcal{C}^k(\Omega)$ into the set of distributions.) This means

$$D^\alpha(T_u)(\varphi) \overset{\text{should be}}{=} (T_{D^\alpha u})(\varphi) = \int_\Omega D^\alpha u(x)\varphi(x)\,dx = (-1)^{|\alpha|}\int_{\mathcal{D}} D^\alpha u(x)\varphi(x)\,dx = (-1)^{|\alpha|}T_u(D^\alpha\varphi)$$

Definition 1.4.4
For any distribution $T : \mathcal{C}_c^\infty(\Omega) \to \mathbb{K}$ and any multi-index $\alpha \in \mathbb{N}_0$ we define

$$(D^\alpha T) : \begin{cases} \mathcal{C}_c^\infty(\Omega) \to \mathbb{K} \\ \varphi \mapsto (-1)^{|\alpha|} T(D^\alpha \varphi) \end{cases}$$

Remark (Consequences) This is compatible to classical derivatives since we have

(i) $D^\alpha(T_u) = T_{D^\alpha u}$

(ii) Linearity

(iii) $D^\alpha(D^\beta T) = D^{\alpha+\beta} T$

Example 1.4.5 $\Omega = \mathbb{R}$, $u(x) = |x|$. We consider $T_u : \varphi \mapsto \int_{\mathbb{R}} |x| \varphi(x)\, dx$.

$$\begin{aligned}
(DT_u)(\varphi) &\overset{\text{Def}}{=} -T_u(\varphi') = -\int_{\mathbb{R}} |x| \varphi'(x)\, dx \\
&= +\int_{-\infty}^0 (+x)\varphi'(x)\, dx - \int_0^\infty x\varphi'(x)\, dx \\
&= [x\varphi(x)]_{-\infty}^0 - \int_{-\infty}^0 1\varphi(x)\, dx - [x\varphi(x)]_0^\infty + \int_0^\infty 1\varphi(x)\, dx \\
&= \int_{\mathbb{R}} v(x)\varphi(x)\, dx \quad \text{with} \quad v(x) = \begin{cases} 1 & \text{for } x > 0 \\ \gamma & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases} \quad \text{with arbitrary } \gamma \text{, e.g. } \gamma = 42.
\end{aligned}$$

$DT_u = T_v$. We call (any such) v the distributional derivative of u.

If a distribution $\tilde{T}$ has an integral representation in the sense that $\tilde{T} = T_f : \varphi \mapsto \int_\Omega f\varphi\, dx$, then f is unique (in the sense of a.e. equality).

Proposition 1.4.6 (Uniqueness of integral representation)
If $f, g \in L^1(\Omega)$ and $T_f = T_g$ in the sense of distributions, then $f = g$ in $L^1(\Omega)$, i.e. $f = g$ a.e. in Ω.

Proof: Define $h = f - g$, then $T_h = T_{f-g} = T_f - T_g = 0$. We have to proof that for $h \in L^1(\Omega)$ we have

$$\forall \varphi \in \mathcal{C}_c^\infty \quad \int_\Omega h(x)\varphi(x)\, dx = 0 \implies h = 0$$

We know that $\mathcal{C}_c^0(\Omega)$ is dense in $L^1(\Omega)$. Hence we can approximate h by $h_\varepsilon \in \mathcal{C}_c^0(\Omega)$ with $\|h - h_\varepsilon\|$. Moreover, $\mathcal{C}_c^\infty(\Omega)$ is dense in $\mathcal{C}_c^0(\Omega)$ (see Exercise sheet 6). Hence, for $\psi \in \mathcal{C}_c^0(\Omega)$ we find $\varphi_n \in \mathcal{C}_c^\infty(\Omega)$ with $\|\psi - \varphi_n\|_{\mathcal{C}^0} \to 0$. Hence

$$\begin{aligned}
\int_\Omega h\psi\, dx &= \underbrace{\int_\Omega h(\psi - \varphi_n)\, dx}_{|\cdot| \le \|h\|_{L^1} \|\psi - \varphi_n\|_{\mathcal{C}^0} \to 0} + \underbrace{\int_\Omega h\varphi_n\, dx}_{=0} \\
\implies \int_\Omega h\psi\, dx &= 0 \quad \text{for all } \psi \in \mathcal{C}_c^0(\Omega)
\end{aligned}$$

We consider T_{h_ε} and obtain

$$\begin{aligned}
|T_{h_\varepsilon}(\psi)| = \left| \int_\Omega h_\varepsilon \psi\, dx \right| &= \left| \int_\Omega (h_\varepsilon - h)\psi\, dx + \underbrace{\int_\Omega h\psi\, dx}_{=0} \right| \\
&\le \|h_\varepsilon - h\|_{L^1} \|\psi\|_{\mathcal{C}^0} \overset{(**)}{\le} \varepsilon \|\psi\|_{\mathcal{C}^0}
\end{aligned}$$

We now choose a test function $\psi = \psi_k^\varepsilon$. Idea: Use $\psi^\varepsilon = \operatorname{sign}(h_\varepsilon)$ (not continuous). Then

$$\begin{aligned}
T_{h_\varepsilon}(\psi^\varepsilon) = \int h_\varepsilon \psi^\varepsilon\, dx &= \int |h_\varepsilon|\, dx = \|h_\varepsilon\|_{L^1} \\
&\overset{**}{\le} \varepsilon \underbrace{\|\psi^\varepsilon\|_{L^\infty}}_{=1} \le \varepsilon
\end{aligned}$$

To be precise, choose approximations of $\operatorname{sign}(h_\varepsilon)$ via $\psi_k^\varepsilon(x) = \operatorname{sign}(h_\varepsilon(x))|h_\varepsilon(x)|^{1/k}$ and therefore $\psi_k^\varepsilon = M_k \circ h_\varepsilon \in \mathcal{C}_c^0(\Omega)$.

$$\begin{aligned} T_{h_\varepsilon}(\psi_k^\varepsilon) &= \int_\Omega |h_\varepsilon|^{1+1/k}\,dx \xrightarrow[\text{Domin. convergence}]{k\to\infty} \int_\Omega |h_\varepsilon|\,dx - \|h_\varepsilon\|_{L^1} \\ \implies \|h_\varepsilon\|_{L^1} &= \lim_{k\to\infty} T_{h_\varepsilon}(\psi_k^\varepsilon) \overset{**}{\leq} \limsup_{k\to\infty} \varepsilon \|\psi_k^\varepsilon\|_{\mathcal{C}^0} \\ &= \varepsilon \lim_{k\to\infty} \|\psi_k^\varepsilon\|_{\mathcal{C}^0} = \varepsilon \end{aligned}$$

Since $\max\{|\psi_k^\varepsilon(x)| \mid x \in \Omega\} = \|h_\varepsilon\|_\infty^{1/k} \to 1$. Hence $\|h_\varepsilon\|_{L^1} \leq \varepsilon$ and thus

$$\|h\|_{L^1} \leq \|h - h_\varepsilon\|_{L^1} + \|h_\varepsilon\|_{L^1} < 2\varepsilon \to 0 \implies h = 0 \quad \text{in} \quad L^1(\Omega)$$

□

We say that $f \in L^p(\Omega)$ has an L^p-derivative $D^\alpha f$ if there exists $g \in L^p(\Omega)$ such that

$$D^\alpha(T_f) = T_g$$

Then we write $D^\alpha f = g$ and call $D^\alpha f$ *the* distributional derivative.

Theorem 1.4.7 (Gauß' theorem, Divergence Theorem)
$v : \Omega \to \mathbb{R}^d$ vector field.

$$\int_\Omega \underbrace{\operatorname{div} v}_{\sum_{i=1}^d \partial_{x_i} v_i}\; dx = \int_{\partial\Omega} \underbrace{v \cdot \nu}_{\sum_{i=1}^d v_i \cdot \nu_i}\; da \quad \textit{with } \nu \textit{ the outer normal vector}$$

The fundamental version is: $f \in \mathcal{C}^1(\overline{\Omega})$

$$\int_\Omega \partial_{x_i} f(x)\,dx = \int_{\partial\Omega} f(\underbrace{\nu \cdot e_i}_{\nu_i})\,da$$

Applying to products leads to integration by parts

$$\int_\Omega \partial_{x_i}(u\varphi)\,dx = \int_\Omega u\partial_{x_i}\varphi\,dx + \int_\Omega (\partial_{x_i} u)\varphi\,dx = \int_{\partial\Omega} u\varphi\nu_i\,da$$

By induction

$$\int_\Omega (D^\alpha u)\varphi\,dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi\,dx + \int_{\partial\Omega} \dots\,da$$

Definition 1.4.8 (Sobolev spaces)
Let Ω be a domain (open and connected) and $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}_0^d \textit{ with } |\alpha| \leq k : D^\alpha f \in L^p(\Omega) \right\}$$

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}^p \right)^{1/p}$$

Definition 1.4.9
Sobolev spaces are Banach spaces. For $p = 2$ we have Hilbert spaces

$$H^k(\Omega) := W^{k,2}(\Omega) \quad \textit{with} \quad \langle f, g \rangle_{H^k(\Omega)} \overset{def}{=} \sum_{|\alpha| < k} \int_\Omega D^\alpha f \overline{D^\alpha g}\,dx$$

Proof:

Step 1 We have a normed vector space. That is straight forward.

Step 2 : Completeness. (interchanging of two limit procedures: converging of CS and differentiation). Take a CS $(f_n)_{n\in\mathbb{N}}$ in $W^{k,p}(\Omega)$, then for all α with $|\alpha| \le k$ the sequence $(D^\alpha f_n)_{n\in\mathbb{N}}$ is a CS in $L^p(\Omega)$.

$$\|D^\alpha f_n - D^\alpha f_m\|_p \le \|f_n - f_m\|_{W^{k,p}} \to 0$$

Since $L^p(\Omega)$ is complete, we find $g_\alpha \in L^p(\Omega)$ such that $\|D^\alpha f_n - g_\alpha\|_p \to 0$. Of course, we have $f = g_0$ is the limit if we forget all derivatives. We expect $D^\alpha f = g_\alpha$.
Claim: For all α with $|\alpha| \le k : D^\alpha f = g_\alpha$ (in the sense of distributions). If the claim holds, then

$$\|f_n - f\|_{W^{k,p}}^p = \sum \|D^\alpha f_n - D^\alpha f\|_p^p \overset{\text{Claim}}{=} \sum \|D^\alpha f_n - g_\alpha\|_p^p \to 0$$

Using the definition of distributional derivative, we have to show

$$D^\alpha(T_f)(\varphi) = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi \, dx \overset{!!}{=} T_{g_\alpha}(\varphi) = \int_\Omega g_\alpha \varphi \, dx$$

We use continuity of the integral $h \mapsto \int_\Omega h\varphi \, dx$ for $\varphi \in \mathcal{C}_c^\infty(\Omega)$, namely

$$\left| \int_\Omega h_1 \varphi \, dx - \int_\Omega h_2 \varphi \, dx \right| \overset{\text{Hölder}}{\le} \|h_1 - h_2\|_{L^p} \underbrace{\|\varphi\|_{L^{p'}}}_{\text{finite}}$$

$$\begin{aligned} D^\alpha(T_f)(\varphi) = (-1)^{|\alpha|} \int f \underbrace{D^\alpha \varphi}_{\in \mathcal{C}_c^\infty(\Omega)} \, dx &\overset{\text{continuity}}{=} \lim_{n\to\infty} f_n D^\alpha \varphi \, dx \\ &\underset{\substack{\varphi\in\mathcal{C}_c^\infty(\Omega) \\ f_n \in W^{k,p}(\Omega)}}{\overset{\text{int. by parts}}{=}} \lim_{n\to\infty} \int_\Omega (D^\alpha f_n)\varphi \, dx \\ &\underset{D^\alpha f_n \to g_\alpha}{\overset{\text{continuity}}{=}} \int_\Omega g_\alpha \varphi \, dx = T_{g_\alpha}(\varphi) \end{aligned}$$

This is the claim $D^\alpha f = g\alpha$, i.e. the distributional derivative $D^\alpha T_f$ has the L^p-representation T_{g_α}. □

Lemma 1.4.10
$\Omega = B_1(0) \subset \mathbb{R}^d$, then

$$u(x) = \begin{cases} \frac{1}{|x|^\alpha} & \text{for } 0 < |x| < 1 \\ \gamma & \text{for } x = 0 \end{cases} \qquad \text{for } \gamma \text{ arbitrary}$$

lies in $W^{k,p}(\Omega)$ if $\alpha + k - \frac{p}{d} < 0$. For later we introduce

$$\dot{W}^{k,p}(\Omega) = W_0^{k,p}(\Omega) = H_0^k(\Omega) = \mathring{W}^{k,2}(\Omega)$$

$W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$. Hence, it is again a Banach space.

Topological vector spaces

Let M be a set. A subset $\mathcal{T}$ of $\mathcal{P}(M)$ (the power set of M) is called topology on M, if

(i) $\emptyset, M \in \mathcal{T}$

(ii) arbitrary unions of sets in $\mathcal{T}$ again are in $\mathcal{T}$.

(iii) finite intersections of sets in $\mathcal{T}$ lie in $\mathcal{T}$.

The sets in $\mathcal{T}$ are called open sets. The complements $M \setminus T$ for $T \in \mathcal{T}$ are the closed sets. The pair $(M, \mathcal{T})$ is called a topological space. If $(M, \mathcal{T})$ and $(N, \mathcal{S})$ are topological spaces, a mapping $f : M \to N$ is called continuous, if preimages of open sets (in N) are open (in M):

$$\forall S \in \mathcal{S} : f^{-1}(S) = \{m \in M \mid f(m) \in S\} \in \mathcal{T}$$

Definition 1.4.11
Let (X, K) be a vector space and $\mathcal{T}$ a topology on X. We call $(X, \mathbb{K}, \mathcal{T})$ a topological vector space, *if the vector-space operations are continuous.*

$$+ : \begin{cases} X \times X \to X \\ (u, v) \mapsto u + v \end{cases} \qquad \cdot : \begin{cases} \mathbb{K} \times X \to X \\ (\lambda, u) \mapsto \lambda u \end{cases}$$

Here $(\mathbb{K}, \mathcal{T}_{Euclid})$ is the field with the usual Euclidean topology.

Example 1.4.12 (i) $X = \mathbb{K}^n$, $\mathcal{T} = \mathcal{T}_{\text{Euclid}}$.

(ii) Metric spaces (M, d), $d : M \times M \to [0, \infty[$ satisfies

M1 positivity $d(x, y) = 0 \iff x = y$

M2 symmetry $d(x, y) = d(y, x)$

M3 triangle estimate $d(x, z) \leq d(x, y) + d(y, z)$.

Metric topology $\mathcal{T}_d$ on M is defined via open ball $B_r(x) = \{y \in M \mid d(x, y) < r\}$. $\mathcal{T}_d$= smallest topology containing all $B_r(x)$.

(iii) A normed vector space is a topological vector space in the above sense, if we take the topology induced by norm balls $B_r(x) = \{y \in X \mid \|x - y\| < r\}$.

(iv) On vector spaces, we call a metric translation invariant if $\forall x, y, v \in X : d(x + v, y + v) = d(x, y)$. This implies $d(x, y) = d(x - y, 0)$. For example on $\mathbb{R}^n$, take $d(x, y) = \sqrt{|x - y|_{\text{Euclid}}}$. If additionally d has the scaling property $d(\lambda x, \lambda y) = |\lambda| d(x, y)$, then $x \mapsto d(x, 0)$ is a norm.

We have Hilbert spaces $\subset$ Banach Spaces $\subset$ topological vector spaces.

Remark In many cases, the topology on a vector space is generated by a family (or system) of seminorms $(X, \mathbb{K}, +, \cdot)$ vector space. $\{p_\alpha \mid \alpha \in A\}$ with A index set, $p_\alpha : X \to [0, \infty[$ a seminorm.

Definition 1.4.13
A family of seminorms is called Hausdorff, *if we have the following implication*

$$p_\alpha(x) = 0 \quad \textit{for all} \quad \alpha \in A \implies x = 0$$

More general, a topological space $(M, \mathcal{T})$ is called Hausdorff, if any two points can be separated by open sets, in quantifiers:

$$\forall x, y \in M \quad \textit{with} \quad x \neq y \exists \mathcal{O}_x, \mathcal{O}_y \in \mathcal{T} : x \in \mathcal{O}_x, y \in \mathcal{O}_y \quad \textit{and} \quad \mathcal{O}_x \cap \mathcal{O}_y = \emptyset$$

Definition 1.4.14
Given a family of seminorms $SN = \{p_\alpha \mid \alpha \in A\}$ on X, we define a topology *on X as follows: We define the open sets*

$$\mathcal{O}_{\alpha,r}(x) = \{y \in X \mid p_\alpha(x - y) < r\}$$

Now, $\mathcal{T}_{SN}$ is defined to be the smallest topology containing all $\mathcal{O}_{\alpha,r}(x)$ for $x \in X, \alpha \in A, r > 0$. Since finite intersections lie in $\mathcal{T}_{SN}$, we define

$$\mathcal{O}_{\alpha_1,\ldots,\alpha_n,r}(x) = \{y \in X \mid p_{\alpha_j}(x - y) < r \text{ for } j = 1, \ldots, n\}$$

which is still open. It can be shown (exercise) that a general subset $M \subset X$ is open if and only if

$$\forall x \in M \exists \alpha_1, \ldots \alpha_n \in A \exists \varepsilon > 0 : \mathcal{O}_{\alpha_1,\ldots,\alpha_n}(x) \subset M$$

For every vector space X and every family of seminorms SN, the topology $\mathcal{T}_{SN}$ makes $(X, \mathcal{T}_{SN})$ into a topological vector space. If SN is Hausdorff, then T_{SN} is a Hausdorff topology .

Example 1.4.15

(i) $X = \mathbb{K}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$) with seminorms $p_j(u) = |u_j|$, $u = (u_1, \ldots, u_n) \in \mathbb{K}^n$, index set $A = \{1, \ldots, n\}$. SN is Hausdorff.

(ii) $X = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ arbitrary}\}$, index set $A = \mathbb{R}$, seminorm $p_\alpha(f) = |f(\alpha)|$. Then $SN = \{p_\alpha \mid \alpha \in A\}$ is Hausdorff, as $0 = p_\alpha(f) = |f(\alpha)|$ for all $\alpha \in \mathbb{R}$ implies $f \equiv 0$.

(iii) $X = \mathcal{C}(]0,1[)$ (no boundedness assumption). Take $f_*(t) = \exp(\frac{1}{t+1}) \implies f_* \in X$. Typical question: $f_n(t) = \sum_{k=0}^n t^k \xrightarrow{n\to\infty} \frac{1}{1-t} \in X$ in what sense? We want to find a topology for uniform convergence on compact subintervals. $A = \mathbb{N}$, $p_k(f) = \|f|_{[-1+\frac{1}{k},1-\frac{1}{k}]}\|_{L^\infty([-1+\frac{1}{k},1-\frac{1}{k}])}$. This defines a Hausdorff family of seminorms. $p_k(f) = 0$ for all k then

$$\forall k: \quad f(t) = 0 \quad \text{for} \quad t \in [-1+\tfrac{1}{k}, 1-\tfrac{1}{k}] \implies f = 0 \text{ on }]-1,1[$$

Note: for every $K \Subset]-1,1[$ there exists $k \in A = \mathbb{N}$ such that $K \subset [-1+\frac{1}{k}, 1-\frac{1}{k}]$.

(iv) $X = \mathcal{C}^\infty(\mathbb{R})$, $A = \mathbb{N}^2$, $p_{n,k}(f) = \|f^{(n)}|_{[-k,k]}\|_{L^\infty([-k,k])}$.

Definition 1.4.16 (of convergence and Cauchy sequences)
Consider a topological vector space $(X,\mathcal{T})$ and a sequence $(x_n)_{n\in\mathbb{N}}$. We say that $(x_n)_{n\in\mathbb{N}}$ converges to x in $(X,\mathcal{T})$ if

$$\forall \mathcal{O} \in \mathcal{T} \quad \textit{with} \quad x \in \mathcal{O} \exists n \in \mathbb{N} \forall k \geq n : x_k \in \mathcal{O}$$

We write $x_n \xrightarrow{\mathcal{T}} x$. The sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence, *if*

$$\forall \mathcal{O} \in \mathcal{T} \quad \textit{with} \quad 0 \in \mathcal{O} \exists n \forall k,m \geq n : x_k - x_m \in \mathcal{O}$$

Lemma 1.4.17 (Convergence in case of seminorms)
Assume that the topology $\mathcal{T}$ on X is induced by a Hausdorff family of seminorms. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,\mathcal{T}_{SN})$ if and only if

$$\forall \alpha \in A : p_\alpha(x_n - x_m) \xrightarrow{n,m\to\infty} 0$$

Convergence to x is equivalent to

$$\forall \alpha \in A : p_\alpha(x_n - x) \xrightarrow{n\to\infty} 0$$

The limit point (if it exists) is unique.

Proof:

Step 1 CS $(x_n)_{n\in\mathbb{N}} \implies (p_\alpha(x_n))$ CS. Choose $\mathcal{O} = \mathcal{O}_{\alpha,\varepsilon}(0) = \{y \in X \mid p_\alpha(x-y) < \varepsilon\}$ then $x_k - x_m \in \mathcal{O}_{\alpha,\varepsilon}(0) \iff p_\alpha(x_k - x_m) < \varepsilon$.

Step 2 $p_\alpha(x_k - x_m) \to 0 \implies (x_n)_{n\in\mathbb{N}}$ in $(X,\mathcal{T})$. Take any $\mathcal{O} \in \mathcal{T}$ with $0 \in \mathcal{O}$. Then

$$\exists \beta_1,\dots,\beta_l \in A \exists \varepsilon > 0 : 0 \in \underbrace{\mathcal{O}_{\beta_1,\dots,\beta_l,\varepsilon}(0)}_{\{y \in X \mid p_{\beta_j}(y-0)<\varepsilon \text{ for } j=1,\dots,l} \subset \mathcal{O}$$

$\forall j \in \{1,\dots,l\} \exists n_j \forall k,m \geq n_j : p_{\beta_j}(x_n - x_m) < \varepsilon$. For $k,m \geq \max\{n_1,\dots,n_j\}$ we find $x_k - x_m \in \mathcal{O}_{\beta_1,\dots,\beta_l,\varepsilon(0)} \subset \mathcal{O}$.

Step 3 Characterization of convergence works similarly.

Step 4 Uniqueness of limit. Assume $x_n \xrightarrow{\mathcal{T}} x$ and $x_n \xrightarrow{\mathcal{T}} \tilde{x}$. We find for $\alpha \in A$

$$p_\alpha(x - \tilde{x}) \overset{\Delta\text{-est.}}{\leq} p_\alpha(x - x_n) + p_\alpha(x_n - \tilde{x}_n) \xrightarrow{n\to\infty} 0$$

Hence, $p_\alpha(x - \tilde{x}) = 0$ for all $\alpha \in A$. Since SN is Hausdorff we conclude $x = \tilde{x}$. □

With the above examples, we have

Example 1.4.18

(i) $X = \mathbb{K}^n$, $p_j(u) = |u_j|$. SN convergence $\iff$ coordinatewise convergence. $\|u\| = \sum_{j=1}^\infty p_j(u)$ is a norm.

(ii) $X = \{f : \mathbb{R} \to \mathbb{R}\}$, $p_\alpha(f) = |f(\alpha)|$. $p_\alpha(f_n - f) \to 0 \iff f_n(\alpha) \to \alpha$. Thus, convergence means pointwise convergence. $\forall \alpha \in A : p_\alpha(f_n - f) \to 0$ means here $\forall t \in \mathbb{R} : f_n(t) \to f(t)$ (simple pointwise convergence).

(iii) $X = \mathcal{C}(]-1,1[)$, $p_n(f) = \|f\|_{\mathcal{C}^0([-1+\frac{1}{n},1-\frac{1}{n}])}$. $\forall n \in \mathbb{N} : f_k \to f$ uniformly on $[-1+\frac{1}{n}, 1-\frac{1}{n}]$. Uniform convergence n compact subsets.

(iv) $X = \mathcal{C}^\infty(\mathbb{R})$. $p_{n,k}(f) = \|f^{(n)}\|_{\mathcal{C}^0([-k,k])}$ implies uniform converges of all derivatives on every compact set. If $f_m(t) = \sum_{l=0}^m \frac{1}{l!} t^l$, then $f_l \xrightarrow{\mathcal{T}} f : t \mapsto \exp(t)$.

If a family of seminorms is finite and Hausdorff, then $\|x\| = \sum_{\alpha\in A} p_\alpha(x)$. is a norm generating the same topology, i.e. $\mathcal{T}_{SN} = \mathcal{T}_{\|\cdot\|}$. If the index set is countable, one can still define a translation invariant metric d_{SN} as follows

$$d_{SN}(x,y) = \sum_{j=1}^{\infty} \frac{1}{2j} \frac{p_{\alpha_j}(x-y)}{1+p_{\alpha_j}(x-y)} \qquad d_{SN} : X \times X \to [0,1[$$

is well defined and satisfies

(i) $d_{SN}(x,y) = d_{SN}(x-y,0)$ (translation invariance)

(ii) $d_{SN}(x,y) = 0 \implies x = y$ (positivity). Proof:$\sum = 0 \implies \frac{1}{2^j}\frac{p_{\alpha_j(x-y)}}{1+p_{\alpha_j}(x-y)} = 0$ for all $j \implies \forall j : p_{\alpha_j}(x-y) = 0 \implies SN$ Hausdorff $x-y$.

(iii) triangle estimate

$$d_{SN}(x,z) \le d_{SN}(x,y) + d_{SN}(y,z) \iff d_{SN}(\underbrace{x-z}_{x_1+x_2},0) \le d_{SN}(\underbrace{x-y}_{x_1},0) + d_{SN}(\underbrace{y-z}_{=x_2},0)$$

$$\begin{aligned} d_{SN}(x_1+x_2,0) &= \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{p_{\alpha_j}(x_1+x_2)}{1+p_{\alpha_j}(x_1+x_2)} \underset{\text{monotonicity}}{\overset{p_\alpha \text{ seminorm}}{\le}} \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{p_{\alpha_j}(x_1)+p_{\alpha_j}(x_2)}{1+p_{\alpha_j}(x_1)+p_{\alpha_j}(x_2)} \\ &=\le \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{p_{\alpha_j}(x_1)}{1+p_{\alpha_j}(x_1)+\cancel{p_{\alpha_j}(x_2)}} + \sum \frac{1}{2^j} \frac{p_{\alpha_j}(x_2)}{1+\cancel{p_{\alpha_j}(x_1)}+p_{\alpha_j}(x_2)} \\ &= d_{SN}(x_1,0) + d_{SN}(x_2,0) \end{aligned}$$

Theorem 1.4.19
Take a vector space X and a countable Hausdorff family $SN = \{p_k \mid k \in \mathbb{N}\}$ of seminorms. Define the metric

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1+p_k(x-y)}$$

then the topologies $\mathcal{T}_{SN}$ and $\mathcal{T}_d$ coincide, i.e. $\mathcal{T}_{SN} = \mathcal{T}_d \subset \mathcal{P}(X)$

Remark We can define more general metrics via $d(x,y) = \sum_{k=1}^{\infty} a_k g(p_k(x-y))$ where $a_k > 0$, $\sum_{k=1}^{\infty} a_k < \infty$, $g : [0,\infty[\to [0,1]$ continuous, concave and $g(0) = 0$.

Proof: We have to prove:

(i) $\forall \mathcal{O}_d \in \mathcal{T}_d : \mathcal{O}_d \in \mathcal{T}_{SN}$ this means $\forall x \in \mathcal{O}_d \exists n \in \mathbb{N} \exists \varepsilon > 0 : \mathcal{O}_{1,\dots,n,\varepsilon}(x) \subset \mathcal{O}_d$.

(ii) $\forall \mathcal{O}_{SN} \in \mathcal{T}_{SN} \forall x \in \mathcal{O}_{SN} \exists \varepsilon > 0 : B^d_\varepsilon(x) \subset \mathcal{O}_{SN}$.

ad (i) Given $\mathcal{O}_d \in \mathcal{T}_d$ and $x \in \mathcal{O}_d$. By definition of $\mathcal{T}_d$, there exists a $\delta > 0$ such that $B_\delta(x) \subset \mathcal{O}_d$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N+1}} < \frac{\delta}{4}$ and define $\mathcal{O}_{1,\dots,N,\delta/2}(x) \in \mathcal{O}_{SN}$.

Claim: $x \in \mathcal{O}_{1,\dots,N,\delta/2}(x) \subset B^d_\delta(x) \subset \mathcal{O}_d$.

Take $y \in \mathcal{O}_{1,\dots,N,\delta/2}(x)$, then

$$\begin{aligned} d(x,y) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1+p_k(x-y)} \le \sum_{k=1}^{N} \frac{1}{2^k} \frac{\frac{\delta}{2}}{1+\frac{\delta}{2}} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \cdot 1 < \frac{\delta}{2} + \frac{1}{2^N} \le \delta \\ &\implies y \in B^d_\delta(x) \end{aligned}$$

ad (ii) Given $\mathcal{O}_{SN} \in \mathcal{T}_{SN}$ and $x \in \mathcal{O}_{SN}$, there exists $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $x \in \mathcal{O}_{1,\dots,N,\varepsilon}(x) \subset \mathcal{O}_{SN}$ (by definition of $\mathcal{T}_{SN}$). Choose $\delta \le \frac{\varepsilon}{2^{N+1}}$ and $\delta < \frac{1}{2}$ and define $B^d_\delta(x) \in \mathcal{T}_d$.

Claim: $x \in B^d_\delta(x) \subseteq \mathcal{O}_{1,\dots,N,\varepsilon}(x) \in \mathcal{O}_{SN}$. For $y \in B^d_\delta(x)$ we have $\sum_{k=1}^{\infty} \frac{1}{2^k}\frac{p_k(x-y)}{1+p_k(x-y)} < \delta$. That implies

$$\frac{p_k(x-y)}{1+p_k(x-y)} \le \delta 2^k \le 2^k \frac{\varepsilon}{2^{N+1}} \le \frac{\varepsilon}{2} \implies p_k(x-y) < \varepsilon \quad \text{for } k = 1,\dots N$$

□

Definition 1.4.20
For a topological space $(M,\mathcal{T})$ the topology is called metrizable *if there exists a metric d on M, such that $\mathcal{T} = \mathcal{T}_d$.*

2 Linear operators

2.1 Continuous linear operators

Definition 2.1.1
If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological vector spaces, then an operator (map, mapping) T is called continuous *if preimages of open sets are open.*

$$\forall S \in \mathcal{S} : \{x \in X \mid T(x) \in S\} \in \mathcal{T}$$

Theorem 2.1.2
If we have two normed spaces $(X, \mathcal{T}_{\|\cdot\|})$ and $(Y, \mathcal{S}_{|||\cdot|||})$, then a linear *operator $T : X \to Y$ is continuous if and only if there exists a $C \geq 0$ such that*

$$|||Tx||| \leq C\|x\| \quad \textit{for all } x \in X$$

Remark A linear operator T satisfying $|||Tx||| \leq C\|x\|$ is called *linear bounded* or simply *bounded*. A *bounded linear operator* means continuous linear operator between normed spaces.

Proof:

"$\Rightarrow$" T is continuous. We have to find C. Take $S = B_1^Y(0) = \{y \in Y \mid |||y||| < 1\} \in \mathcal{S}_{|||\cdot|||}$. The preimage $T^{-1}(S) := \{x \in X \mid Tx \in S\}$ is open by continuity of T. Since $0 \in T^{-1}(S)$ (note $|||T0_X||| = |||0_X||| = 0 < 1$) there exists $\varepsilon > 0$ such that $B_\varepsilon^X(0_X) \subset T^{-1}(S) \implies \forall x \in X$ with $\|x\| < \varepsilon : |||Tx||| < 1$. By linearity (scaling) we have $|||Tx||| \leq \frac{1}{\varepsilon}\|x\|$.

"$\Leftarrow$" Boundedness implies continuity. To show: preimages of open sets are open. For $x \in T^{-1}(S)$ consider $Tx \in S$ open. Hence, $\exists \varepsilon > 0 : B_\varepsilon^Y(Tx) \subset S$. Consider $B_{\varepsilon/C}^X(x) \subset X$, then for $z \in B_{\varepsilon/C}^X(x)$ we have

$$|||Tz - Tx||| = |||T(x - z)||| \leq C\|x - z\| < \varepsilon$$

Hence

$$T_z \in B_\varepsilon^Y(Tx) \subset S \implies z \in T^{-1}(S) \text{ for all } z \in B_{\varepsilon/C}^X(x) \implies B_{\varepsilon/C} \subset T^{-1}(S) \implies T^{-1}(S) \text{ open}$$

□

Remark (Notation) For two topological vector spaces X and Y we write $\mathcal{L}(X, Y)$ for the vector space of all continuous linear operators. In the case of normed spaces, we introduce the *operator norm*

$$\|T\|_{\mathcal{L}(X,Y)} = \sup\left\{\frac{|||Tx|||}{\|x\|} \mid x \in X \setminus \{0\}\right\}$$

Proposition 2.1.3
If $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ are normed spaces, then $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is again a normed space. Moreover, if Y is a Banach space, then $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is a Banach space.

Proof:

Step 1 $\mathcal{L}(X, Y)$ is a vector space.

Step 2 $\|\cdot\|_{\mathcal{L}(X,Y)}$ is a norm on $\mathcal{L}(X, Y)$. Positivity, scaling and triangle estimate are trivial.

Step 3 Completeness (exercise).

□

Remark (Notation) Simplifying notation: $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $X' = \mathcal{L}(X, \mathbb{K})$. For $T_1, T_2 \in \mathcal{L}(X)$ we can write the composition $(T_1 \circ T_2)(x) = T_1(T_2(x))$ as T_1T_2x. We have $T_1T_2 \in \mathcal{L}(X)$ since the composition of continuous functions is continuous. The operator norm can be estimated via

$$\begin{aligned}
\|T_1T_2x\| &\leq \|T_1\|_{\mathcal{L}(X)} \\
\|T_1T_2x\| &\leq \|T_1\|_{\mathcal{L}(X)}\|T_2\|_{\mathcal{L}(X)}\|x\| \\
\|T_1T_2\|_{\mathcal{L}(X)} &\leq \|T_1\|_{\mathcal{L}(X)}\|T_2\|_{\mathcal{L}(X)}
\end{aligned}$$

A Banach space $(Y, \|\cdot\|)$ with an additional multiplication satisfying $\|y_1 \odot y_2\| \leq \|y_1\|\|y_2\|$ is called a *Banach algebra*.

Example 2.1.4

(i) Multiplication operator: $X \in L^p(\Omega)$, $Y \in L^q(\Omega)$ with $1 \leq q \leq p$. Take a function $m \in L^r(\Omega)$ with $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$.

$$T_m : \begin{cases} L^p(\Omega) \to L^q(\Omega) \\ u \mapsto mu \end{cases}$$

For m fixed, this is a linear operator, as

$$\|T_m u\|_{L^q} = \|mu\|_{L^q} \overset{\text{Hölder}}{\leq} \|m\|_{L^r} \|u\|_{L^p}$$

Hence, $T_m \in \mathcal{L}(L^p, L^q)$ with $\|T_m\|_{\mathcal{L}(L^p, L^q)} \leq \|m\|_{L^r}$.

(ii) Differential operator: $\Omega \subset \mathbb{R}^d$ domain. $X = W^{1,p}(\Omega)$, $Y = (L^p(\Omega))^d$.

$$T : \begin{cases} W^{1,p}(\Omega) \to L^p(\Omega)^d \\ u \mapsto \nabla u \end{cases} \qquad \text{with} \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}\right)$$

$$\|Tu\|_{L^p} = \|\nabla u\|_{L^p} \leq 1 \|u\|_{W^{1,p}(\Omega)} \quad \text{where} \quad \|u\|_{W^{1,p}} = \left(\sum_{i=1}^d \|\partial_{x_i} u\|_{L^p}^p\right)^{\frac{1}{p}}$$

$$\nabla \in \mathcal{L}(W^{1,p}, L^p)$$

$$D^\alpha \in \mathcal{L}(W^{k+|\alpha|,p}(\Omega), W^{k,p}(\Omega)) \quad \text{where} \quad \alpha \in \mathbb{N}_0^d$$

(iii) Integral operator: $\Omega \subset \mathbb{R}^d$ domain, integral Kernel $K : \Omega \times \Omega \to \mathbb{R}$, $X = Y = C^0_{\text{bdd}}(\overline{\Omega})$, $\|\cdot\|_X = \|\cdot\|_{L^\infty}$

$$(T_K u)(x) = v(x) = \int_\Omega K(x,y) u(y)\, dy$$

$$\begin{aligned} \|v\|_\infty &= \sup_{x \in \Omega} \left| \int_\Omega K(x,y) u(y)\, dy \right| \\ &\leq \sup_{x \in \Omega} \int |K(x,y)| \underbrace{|u(y)|}_{\leq \|u\|_{L^\infty}} \, dy \\ &\leq \underbrace{\sup_{x \in \Omega} \|K(x,\cdot)\|_{L^1} \|u\|_{L^\infty}}_{\geq \|T_K\|_{L(C^0,C^0)} \|u\|_{L^\infty}} \end{aligned}$$

Take $\Omega = \mathbb{R}$, $K(x,y) = \frac{1}{2} e^{-|x-y|}$, $\|K(x,\cdot)\|_{L^1} = 1$, $v(x) = \int_\mathbb{R} \frac{1}{2} e^{-|x-y|} u(x)\, dx$

Two topological vector spaces $(X, \mathcal{T})$, and $(Y, \mathcal{S})$, $L : X \to Y$, is continuous if preimages of open sets are open. With $\mathcal{T} = \mathcal{T}_{\|\cdot\|_X}$ and $\mathcal{S} = \mathcal{T}_{\|\cdot\|_Y}$ we have

$$L \in \mathcal{L}(X,Y) \iff \exists C \geq 0 \forall x \in X : \|Lx\|_Y \leq C \|x\|_X$$

if $\mathcal{T}$ and $\mathcal{S}$ are induced by families of seminorms

$$\{p^X_\alpha \mid \alpha \in I\} \quad \text{for } \mathcal{T} \text{ and} \quad \{p^Y_\beta \mid \beta \in J\} \quad \text{for } \mathcal{S}$$

Theorem 2.1.5
If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are induced by the above families of seminorms and both Hausdorff, then a linear mapping $L : X \to Y$ is continuous if and only if

$$\forall \beta \in J \exists \alpha_1, \ldots, \alpha_{N(\beta)} \in I \exists C(\beta) \forall x \in X : p_\beta(Lx) \leq C(\beta) \left(p_{\alpha_1}(x) + \cdots + p_{\alpha_{N(\beta)}}(x)\right)$$

(Normed spaces are special cases with $\#I = |I| = 1 = \#J$.)

Proof: See exercise 27.

As an application we consider the space of *Schwartz functions*

$$S(\mathbb{R}^d) := \{f \in \mathcal{C}^\infty(\mathbb{R}^\infty) \mid \forall n,k \in \mathbb{N} : p_{n,k}(f) < \infty\}$$

where $p_{n,k}(f) = \sup\{(1+|x|^2)^{n/2}|D^\alpha f(x)| \mid x \in \mathbb{R}^d, \alpha \in \mathbb{N}_0^d, |\alpha| = k\}$. $\{p_{n,k} \mid (n,k) \in \mathbb{N}_0^2\}$ is a family of seminorm. Exercise 28 shows that $(S(\mathbb{R}^d), \mathcal{T}_{SN})$ is a complete topological vector space.

There are many classical linear operators that turn out to be continuous:

(i) Multiplication by polynomials (by functions that can be estimated by polynomials)

(ii) $f \mapsto D^\alpha f$ is continuous

(iii) Fourier transform

$$\mathcal{F} : \begin{cases} S(\mathbb{R}^d) \to S(\mathbb{R}^d) \\ u \mapsto \mathcal{F}u \end{cases} \qquad (\mathcal{F}u)(\xi) = c_d \int_{x\in\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\, dx$$

With $c_d = 1$ in statistics and $c_d = \frac{1}{(2\pi)^{d/2}}$ in functional analysis.

Proposition 2.1.6
$\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is continuous.

Proof:

Step 1 integrand for each ξ is bounded by

$$|e^{ix\xi}u(x)| = |u(x)| \leq \frac{1}{(1+|x|)^{d+1}} p_{d+1,0}(u) \implies |\mathcal{F}u(\xi)| \leq \underbrace{\int \frac{1}{(1+|x|)^{d+1}}\, dx}_{=:\tilde{c}_d}\, p_{d+1,0}(u)$$

$p_{0,0}(\mathcal{F}u) \leq \tilde{c}_d p_{d+1,0}(u)$.

Step 2 Assuming that all calculations can be made rigorous by interchanging limits (i.e. dominated convergence of Lebesgue (1.3.19))

$$D^\alpha(\mathcal{F}u)(\xi) = c_d \int_{x\in\mathbb{R}^d} D_\xi^\alpha(e^{-ix\xi})u(x) = c_d \int_{x\in\mathbb{R}^d} (-ix)^\alpha e^{-ix\xi} u(x)\ x$$

$$(-ix)^\alpha \overset{\text{def}}{=} (-i)^{|\alpha|} \prod_{j=1}^d x_j^{\alpha_j}$$

We found the formula $\quad D^\alpha(\mathcal{F}u) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha u(x))$

$$\text{Hence} \quad \sup|D^\alpha \mathcal{F}u| = \sup|\mathcal{F}(x^\alpha u(x))| \overset{\text{Step 1}}{\leq} \tilde{c}_d p_{d+1,0}(x^\alpha u(x)) \leq \tilde{c}_d p_{d+1+|\alpha|,0}(u)$$

Result:

$$p_{0,k}(\mathcal{F}u) \leq \tilde{c}_d p_{d+1+k,0}(u)$$

Step 3

$$\begin{aligned} \xi^\alpha \mathcal{F}(u)(\xi) &= c_d \int_{\mathbb{R}^d} \underbrace{\xi^\alpha e^{-x\xi}}_{(-i)^{=|\alpha|}D_x^\alpha(e^{-ix\xi})} u(x)\, dx \\ &= c_d(-i)^{|\alpha|} \int_{\mathbb{R}^d} D_x^\alpha(e^{-ix\xi})u(x)\, dx \\ &\underset{\text{integration by parts}}{\overset{\text{make rigorous}}{=}} c_d \int e^{-ix\xi}(D^\alpha u)(x)\, dx \end{aligned}$$

$$\xi^\alpha \mathcal{F}(u)(\xi) = \mathcal{F}(\mathcal{D}^\alpha u)$$

$$p_{L,0}(\mathcal{F}u) \overset{\text{Step 1}}{\leq} \tilde{c}_d \sum_{|\alpha|=L} p_{d+1,0}(D^\alpha u) \leq \hat{c}_d p_{d+1,L}(u)$$

Step 4 For all $n,k \in \mathbb{N}_0 \exists \tilde{c}_{n,k} \forall u \in S(\mathbb{R}^d) : p_{n,k}(\mathcal{F}u) \leq \tilde{c}_{n,k} p_{d+1+k,n}(u)$.

Message: Fourier transform interchanges smoothness against decay. □

2.2 The (four) principles of functional analysis

We want to use specific properties of Banach spaces.

(i) Open mapping principle.

(ii) Principle of the continuous inverse

(iii) Closed graph theorem.

(iv) Theorem/principle of Banach Steinhaus.

(i)-(iv) rely on Baire category theorem.

Theorem 2.2.1 (René-Louis Baire, 1874-1932)
Let (X, d) be a complete metric space. Assume that X is the countable union of closed sets, then at least one of these closed sets have nonempty interior. In formulas: Let $A_1, A_2, \dots$ closed in X and $X = \bigcup_{n=1}^{\infty} A_n$. Then $\exists x \in X \exists r > 0 \exists n : B_r(x) \subset A_n$.
The equivalent formulation via negation and complements says:
If the sets $\mathcal{O}_n$ are open and dense (i.e. $A_n = X \setminus \mathcal{O}_n$ is closed and has empty interior), then the intersection $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ is still dense.

Proof: We consider open and dense sets and have to show

$$\forall y \in X \forall \varepsilon > 0 \exists x \in B_\varepsilon(y) \forall n \in \mathbb{N} : x \in \mathcal{O}_n$$

We start from a given y and $\varepsilon > 0$ and construct a sequence x_n such that $x_n \to x$. $B_r(z) = \{x \in X \mid d(z, x) < r\}$. $\mathcal{O}_1$ is open and dense, hence $B_{\varepsilon/2} \cap \mathcal{O}_1$ is still open and nonempty (by density) .

$$\implies \exists x_1 \in X \exists \varepsilon_1 > 0 : B_{3\varepsilon_1}(x_1) \subset B_{\varepsilon/2}(y) \cap \mathcal{O}_1$$

By induction we find x_{k+1} and $\varepsilon_{k+1} > 0$ such that $B_{3\varepsilon_{k+1}}(x_{k+1}) \subset B_{\varepsilon_k} \cap \mathcal{O}_{k+1}$ by density of $\mathcal{O}_{k+1}$ and openness and $\varepsilon_{k+1} \leq \varepsilon_k/3$. For the centers and $n > k$ we find the estimate

$$\begin{aligned} d(x_k, x_n) &\leq d(x_k, x_{k+1}) + \cdots + d(x_{n-1}, x_n) \leq \varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_{n-1} \\ &\leq \varepsilon_k + \varepsilon_k/3 + \varepsilon_k/9 + \cdots \leq \frac{3}{2}\varepsilon_k \leq \frac{3}{2}\frac{1}{3^{k-1}}\varepsilon_1 \to 0 \end{aligned}$$

We have a Cauchy sequence and completeness gives a limit $x \in X$ such that $d(x_n, x) \to 0$. Moreover

$$d(x_k, x) = \lim_{n\to\infty} d(x_k, x_n) \leq \frac{3}{2}\varepsilon_k \implies x \in \overline{B_{3/2\varepsilon_k}(x_k)} \subset B_{3\varepsilon_k}(x_k) \overset{\text{construction}}{\subset} B_{\varepsilon_{k-1}}(x_{k-1}) \cap \mathcal{O}_k$$

Hence $x \in \bigcap_{k=1} \mathcal{O}_k$ and $x \in \overline{B_{\varepsilon/2}}(y) \subset B_\varepsilon(y)$. □

Definition 2.2.2
A mapping $T : X \to Y$ between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ is called open, *if the image of open sets is open.*

$$\forall \mathcal{O} \in \mathcal{T} : T(\mathcal{O}) \subset \mathcal{S}$$

Example 2.2.3 $X = Y = \mathbb{R}$ with Euclidean topology. $T(x) = x^2$ is not, since $T(]-1, 1[) = [0, 1[$ is not open.

Definition 2.2.4
We call $L : X \to Y$ open *if $L\mathcal{O}$ is open in Y for open $\mathcal{O} \subset X$. For normed spaces and linear maps we have an easy characterization:*

$$L \in \mathcal{L}(X, Y) \text{ open} \iff LB_1^X(0_X) \supset B_\varepsilon^Y(0_Y) \quad \text{for some } \varepsilon > 0$$

Proof: "⇒" Note $0_Y \in LB_1^X(0_X)$ since $L0_X = 0_Y$. Since $LB_1^X(0_X)$ is open, we find $\varepsilon > 0$ as desired.

"⇐" Take any $\mathcal{O} \subset X$ open and $y \in L\mathcal{O}$. Choose $x \in \mathcal{O}$ such that $Lx = y$ and $\delta > 0$ such that $B_\delta(x) \subset \mathcal{O}$. By linearity

$$LB_\delta(x) = L(x + B_\delta(0_X)) = Lx + LB_\delta^X(0_X) = y + LB_\delta^X(0_X) \overset{\text{assumption}}{\supset} y + B_{\varepsilon\delta}^Y(0_Y) = B_{\varepsilon\delta}^Y(y)$$

Since $y \in B_{\varepsilon\delta}^Y(y) \subset LB_\delta^X(x) \subset L\mathcal{O}$ we have shown that $L\mathcal{O}$ is open. □

Theorem 2.2.5 (Open mapping theorem)
If X and Y are Banach spaces and $L \in \mathcal{L}(X,Y)$ is surjective, then L is open.

Proof:

Step 1 Claim: $\exists c > 0 : A = \overline{LB_1^X(0_X)} \supset B_{2c}^Y(0_Y)$. A is symmetric and convex, i.e.

(i) $y \in A \iff -y \in A \quad (B_1^X(0)$ is symmetric and L linear)

(ii) $y_1, y_2 \in A \implies \frac{1}{2}(y_1 + y_2) \in A \quad (B_1^X(0)$ is convex and L is linear)

Since L is surjective and $X = \bigcup_{n=1}^\infty B_n(0)$ (countability, balls) we have

$$Y \overset{\text{surj}}{=} LX = L\left(\bigcup_{n=1}^\infty B_n^X(0)\right) = \bigcup_{n=1}^\infty LB_n^X(0)$$
$$= \bigcup_{n=1}^\infty A_n \quad \text{with} \quad A_n = \overline{LB_n^X(0)}$$

Baire's theorem provides an n_0 and a ball $B_{\varepsilon_0}(y_0) \subset A_{n_0}$ with $\varepsilon_0 > 0$. By scaling and linearity of L we know $A_n = nA_1 := \{ny \mid y \in A_1\}$. Hence, again by scaling and linearity, $A = A_1 \supset B_{\varepsilon_0/n_0}(\frac{1}{n_0}y_0)$. By symmetry we also know $-B_{\varepsilon_0/n_0}(\frac{1}{n_0}y_0) = B_{\varepsilon_0/n_0}(-\frac{1}{n_0}y_0)$ lies in A. By convexity we also know that

$$\frac{1}{2}B_{\varepsilon_0/n_0}(\frac{1}{n_0}y_0) + \frac{1}{2}B_{\varepsilon_0/n_0}(-\frac{1}{n_0}y_0) \subset A := \{\frac{1}{2}y_+ + \frac{1}{2}y_- \mid y_\pm \in B_{\varepsilon_0/n_0}(\pm\frac{1}{n_0}y_0)\}$$
$$= B_{\varepsilon_0/n_0}(0_x) \quad \text{after some calculation}$$

Claim holds for $c = \frac{\varepsilon_0}{2n_0}$.

Step 2 Claim: $LB_1^X(0) \supset B_c^Y(0)$ (no closure but half the radius). For every $y \in B_c^Y(0)$ we have to find a preimage $x \in B_1^X(0)$, i.e. $Lx = y$ and $\|x\| < 1$. We do this by approximation:

Step 2.1 Choose $z_1 \in B_{\frac{1}{2}}^X(0) : \{\|y - Lz - 1\| < \varepsilon_1 = \frac{c}{2}\}$. The existence of such a z_1 follows by Step 1. (Note that $2y \in A$. Hence $2y$ can be approximated by Lx_k with $\|x_k\| < 1$, e.g. $\|2y - Lx_1\| < 2\varepsilon$. Then let $z_1 = \frac{1}{2}x_1$.) We define $y_1 = y - Tz_1 \in B_{\varepsilon_1}^Y(0_Y)$.

Step 2.k By induction (Step 2.k) we let $\varepsilon_{k+1} = \varepsilon_k/2$ and choose $z_k \in B_{1/2^k}^X(0_X)$ s.t. $\|y_{k-1} - Lz_k\| < \varepsilon_k$. $y_k = y_{k-1} - Lz_k$.

We define $x_n = \sum_{k=1}^n z_k$ and see that x_n is a Cauchy sequence, since $\sum_{k=1}^\infty \|z_k\| < \sum_{k=1}^\infty \frac{1}{2^k} = 1 \implies x \in B_1^X(0)$ is the limit. We still have to check that $Lx = y$. We know $y_{k+1} = y_k - Lz_{k+1}$ by construction. Summing over $k = 0, \ldots, n-1$ gives

$$0 \leftarrow \underbrace{y_n}_{\in B_{\varepsilon_k}(0)} = y - \overbrace{L}^{\text{continuous}} \underbrace{\left(\sum_{k=1}^n z_k\right)}_{x_n \to x} \quad \text{Taking the limit we have } 0 = y - Lx$$

□

Example 2.2.6 Let $X = Y = \ell^2(\mathbb{N}) = \{(a_n)_{n\in\mathbb{N}} \mid \sum a_n^2 < \infty\}$ a Hilbert space.

$$La = (\tfrac{1}{n}a_n)_{n\in\mathbb{N}} \qquad L \in \mathcal{L}(X,Y) \quad \text{since} \quad \|L\|_{\ell^2 \to \ell^2} = 1$$

L is injective but not surjective. L is not open since $LB_1(0)$ does not contain a ball. Assume $B_\varepsilon(0) \subset LB_1(0)$. Take $a = \frac{\varepsilon}{2}c_\beta(\frac{1}{n^\beta})_{n\in\mathbb{N}}$ and $\beta \in]\frac{1}{2}, \frac{2}{3}[$ where $\frac{1}{c_\beta^2} = \sum_{\in\mathbb{N}} \frac{1}{n^{2\beta}} < \infty \implies a \in B_\varepsilon(0)$. But any $b \in \ell^2$ with $Lb = a$ has to satisfy $\frac{1}{n}b_n = a_n \implies b_n = \frac{\varepsilon}{2}c_\beta n^{1-\beta}$. This is a contradiction to $b \in \ell^2$.

Theorem 2.2.7 (Inverse Mapping Theorem)
If X and Y are Banach spaces and $L \in \mathcal{L}(X,Y)$ is bijective, then its inverse is continuous, i.e. $L^{-1} \in \mathcal{L}(Y,X)$.

This is a simple corollary of the open mapping theorem, since the inverse mapping (assuming it exists) of an open mapping is continuous.

Definition 2.2.8
If $f : X \to Y$ is a mapping, then $G(f) = \{(x, f(x)) \in X \times Y \mid x \in X\}$ is called the graph *of f. For a linear mapping $L : X \to Y$ the graph $G(L)$ is a linear subspace of $X \times Y$. A graph $G(f)$ is called* closed *, if $G(f)$ is closed in $X \times Y$. For normed spaces this means for $(x_n, y_n) \in G(f)$ and $x_n \to x, y_n \to y$ we have $(x, y) \in G(f)$.*

Obviously, continuous maps have closed graphs, since $y_n = f(x_n)$ implies $y = f(x)$. However, there are many interesting operators with closed graphs that are not continuous. Take $f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = \frac{\pi^2}{17}$, that has a closed graph.

Theorem 2.2.9 (Closed graph theorem)
If X and Y are Banach spaces, and $L : X \to Y$ is linear, then we have the equivalence

$$L \text{ continuous} \iff G(L) \text{ is closed}$$

Proof: "$\Rightarrow$" is trivial.

"$\Leftarrow$" $Z = G(L) \subset X \times Y =$ Banach space with $|||(x,y)||| = \|x\|_X + \|y\|_Y$. Hence, $(Z, |||\cdot|||)$ is a Banach space. We define $P_X : X \times Y \to X$ as $(x, y) \mapsto x$. Then, the restriction P_X to Z is bijective and continuous $P_X(x, Lx) = x$.

$$\|P_X(x, Lx)\|_X = \|x\|_X \leq 1 \cdot |||(x, Lx)|||$$

By the inverse mapping theorem (2.2.7) $P_X^{-1} : X \to Z$ is continuous. Moreover, $P_Y : X \times Y \to Y, (x, y) \mapsto y$ lies in $\mathcal{L}(X \times Y, Y)$ and hence $P_Y|_Z \in \mathcal{L}(Z, Y)$. The composition $L = P_Y P_X^{-1}$ is continuous. □

Theorem 2.2.10 (Theorem of Banach-Steinhaus)
Let X and Y be Banach spaces and $\mathcal{M} \subset \mathcal{L}(X, Y)$ (family of bounded linear operators) with

$$\forall x \in X : \sup\{\|Tx\|_Y \mid T \in \mathcal{M}\} < \infty.$$

Then there exists a constant $c > 0$ such that $\|T\|_{X \to Y} \leq c$ for all $T \in \mathcal{M}$. In words: Pointwise boundedness of an operator family implies uniform boundedness. This explains the name Uniform Boundedness Principle *(of Banach and Steinhaus).*

Proof: We define the set

$$A_n = \{x \in X \mid \forall T \in \mathcal{M} : \|Tx\|_Y \leq n\}$$

This set is closed, as all $T \in \mathcal{M}$ are continuous ($\|Tx_k\| \leq n$ and $x_k \to x \implies \|Tx\| \leq n$). By the pointwise boundedness we know $X = \bigcup_{n=1}^{\infty} A_n$. Since X is a Banach space, Blaire's theorem gives an $n_0 \in \mathbb{N}$ and a ball $B_\varepsilon(x) \subset A_{n_0}$ with $\varepsilon > 0$. This means

$$\forall z \in B_1^X(0) \forall T \in \mathcal{M} : \|T(\underbrace{x + \varepsilon z}_{\in B_\varepsilon(x)})\| \leq n_0$$

By scaling and linearity we obtain for $\tilde{x} \neq 0$

$$\begin{aligned}\|T\tilde{x}\| &= \frac{\|\tilde{x}\|}{\varepsilon} \|T(x + \varepsilon \frac{\tilde{x}}{\|\tilde{x}\|}) - Tx\| \\ &\overset{\Delta\text{-est.}}{\leq} \frac{\|\tilde{x}\|}{\varepsilon} \left(\underbrace{\| T(x + \frac{\tilde{x}}{\|\tilde{x}\|}) \|}_{\leq n_0} + \underbrace{\|Tx\|}_{\leq c_x} \right)\end{aligned}$$

Hence $\|T\tilde{x}\| \leq \underbrace{\frac{1}{\varepsilon}(n_0 + c_X)}_{:=c} \|\tilde{x}\|$. □

Corollary 2.2.11
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $(T_n)_{n \in \mathbb{N}}$ in $\mathcal{L}(X, Y)$ be a pointwise convergent sequence, i.e. $\forall x \in X : (T_n x)_{n \in \mathbb{N}}$ converges in $(Y, \|\cdot\|_Y)$. Let $T : X \to Y$, $Tx := \lim T_n x \; \forall x$. Then

(i) $(T_n)_{n \in \mathbb{N}}$ is uniformly bounded, i.e. $\exists c > 0 : \|T_n\|_{X \to Y} \leq c$.

(ii) $T \in \mathcal{L}(X, Y)$ and $\|T\|_{X \to Y} \leq \liminf_{n \to \infty} \|T_n\|_{X \to Y}$.

Proof:

Ad (i): $\forall x \in X : (T_n x)_{n \in \mathbb{N}}$ converges in Y and that implies $(T_n x)_{n \in \mathbb{N}}$ is bounded. Hence $(T_n)_{n \in \mathbb{N}}$ (or $\{T_n \mid n \in \mathbb{N}\}$) is bounded pointwise and by Banach-Steinhaus $(T_n)_{n \in \mathbb{N}}$ uniformly bounded.

Ad (ii): T is linear, because every T_n and "lim" are linear.

$$\begin{aligned}\forall x \in X : \|Tx\|_Y = \lim_{n\to\infty} \|T_n x\|_Y = \liminf_{n\to\infty} \|T_n x\|_Y \leq \liminf_{n\to\infty}(\|T_n\|_{X\to Y} \cdot \|x\|_X) \\ = (\liminf_{n\to\infty} \|T_n\|_{X\to Y}) \cdot \|x\|_X \leq c\|x\|_X\end{aligned}$$

$$\implies T \in \mathcal{L}(X,Y) \quad \text{and} \quad \|T\|_{X\to Y} \leq \liminf_{n\to\infty} \|T_n\|_{X\to Y}$$

□

Proposition 2.2.12
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(X,Y)$. Then the following conditions are equivalent:

(i) $(T_n)_{n\in\mathbb{N}}$ converges pointwise, i.e. $\forall x \in X : (T_n x)_{n\in\mathbb{N}}$ converges in Y.

(ii) $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded and there is a dense subset $S \subset X$ s.t. $\forall x \in S : (T_n x)_{n\in\mathbb{N}}$ converges in Y.

If these conditions are fulfilled, then $T : X \to Y$, defined by $Tx := \lim_{n\to\infty} T_n x$ is linear and bounded and $\|T\|_{X\to Y} \leq \liminf_{n\to\infty} \|T_n\|_{X\to Y}$.

Proof: (i)⇒(ii) and the additional statements are consequences of the previous corollary. It remains to show (ii)⇒(i).

If (ii) is fulfilled, there is a $c > 0$ s.t. $\|T_n\|_{X\to Y} \leq c$. Let $x \in X$, let $\varepsilon > 0$. As S is dense in X, there is a $z \in S$ s.t. $\|x - z\|_X < \frac{\varepsilon}{3c}$. $z \in S \implies (T_n z)_{n\in\mathbb{N}}$ converges in Y, hence is a Cauchy sequence. Hence $\exists n_0 \in \mathbb{N} \forall m, n \geq n_0 : \|T_m z - T_n z\|_Y < \frac{\varepsilon}{3}$.

$$\begin{aligned}\forall m, n \geq n_0 : \|T_m x - T_x\|_Y &\leq \|T_m x - T_m z\|_Y + \|T_m z - T_n z\|_Y + \|T_m z - T_n x\|_Y \\ &\leq \|T_m\|_{X\to Y}\|x - z\|_X + \|T_m z - T_n z\|_Y + \|T_m\|_{X\to Y} \cdot \|z - x\|_X \\ &< c \cdot \frac{\varepsilon}{3c} + \frac{\varepsilon}{3} + c\frac{\varepsilon}{3c} = \varepsilon\end{aligned}$$

Hence $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y and as Y is a Banach space, we know that $(T_n)_{n\in\mathbb{N}}$ converges in Y.
□

2.3 Projections

In linear algebra, one considers direct sum decompositions $X = Y \oplus Z$ of a vector space X into two linear subspaces Y and Z, and the associated projections

$$P_Y : X \to X \qquad P_Z : X \to X$$

where P_Y is defined as $\forall x \in X \exists!(y,z) \in Y \times Z : x = y + z$, then $P_Y(x) := y$.

$$X = Y \oplus Z \iff X = Y + Z := \{y + z \mid y \in Y, z \in Z\} \quad \text{and} \quad Y \cap Z = \{0\}$$

In functional analysis, we are interested in decompositions which respect the topological structure of X, e.g. P_Y and P_Z should be continuous, or Y and Z are closed subspaces of X.

Important question: When does a closed subspace Y has a closed complement Z (i.e. $X = Y \oplus Z$).

Definition 2.3.1
Let M be a set. Then a map $P : M \to M$ is called a projection *if*

$$P \circ P = P \quad \textit{i.e.} \quad P^2 = P \iff \forall x \in M : P(P(x)) = P(x) \iff P_{|P(M)} = id_{P(M)}$$

Remark (Linear projections) Let X be a vector space and $P : X \to X$ a linear projection.

(i) $id_X - P$ is also a linear projection. Linearity is clear and

$$\begin{aligned}\forall x \in X : (id_X - P)((id_X - P))(x) &= (id_X - P)(x - P(x)) = x - P(x) - P(x - P(x)) \\ &= x - P(x) - P(x) + P(P(x)) \\ &= x - P(x) - P(x) + P(x) = x - P(x) = (id_X - P)(x)\end{aligned}$$

$$\implies (id_X - P) \circ (id_X - P) = id_X - P$$

(ii)

$$R(P) := \mathrm{Im}(P) = P(X) = \{P(x) \mid x \in X\} \quad \text{is the } \textit{range} \text{ of } P$$
$$\text{and} \quad N(P) := \ker(P) = P^{-1}(0) = \{x \in X \mid P(x) = 0\}$$
$$\text{with} \quad y \in R(P) = \mathrm{Im}(P) \iff P(y) = y \iff y - P(y) = 0 \iff (id_X - P)(y) = 0$$
$$\iff y \in N(id_X - P) = \ker(id_X - P)$$

Hence $R(P) = N(id_X - P)$ and $N(P) = R(id_X - P)$.

(iii) Claim: $X = R(P) \oplus N(P) = \mathrm{Im}(P) \oplus \ker(P)$.

- $x \in R(P) \cap N(P) \implies x \overset{P_{|R(P)}=id_{R(P)}}{=} P(x) \overset{x \in \ker P}{=} 0$. That implies $R(P) \cap N(P) = \{0\}$.
- Let $x \in X$. That means $P(x) \in R(P)$, $x - P(x) = (id_X - P)(x) \in R(id_X - P) = N(P)$ or directly:
$$P(x - P(x)) = P(x) - P(P(x)) = P(x) - P(x) = 0$$
Hence $x = P(x) + (x - P(x)) \implies X = R(P) + N(P)$.

Hence $X = R(P) \oplus N(P)$ and P is the projection onto $R(P)$ with respect to this decomposition.

Remark If $(X, \|\cdot\|_X)$ is a normed space and $P : X \to X$ is a linear bounded projection, then

$$\forall x \in X : \|Px\|_X = \|P(P(x))\|_X \leq \|P\|_{X\to X} \cdot \|Px\|_X \leq \underbrace{(\|P\|_{X\to X})^2}_{c} \|x\|_X$$
$$\implies \|P\|_{X\to X} \leq c = (\|P\|_{X\to X})^2 \implies \|P\|_{X\to X} = 0 \quad \text{or} \quad \|P\|_{X\to X} \geq 1$$

Theorem 2.3.2 (continuous projections)
Let $(X, \|\cdot\|_X)$ be a Banach space and $P : X \to X$ a linear projection. Then

$$P \in \mathcal{L}(X, X) \iff R(P) \text{ and } N(P) \text{ are closed subspaces of } X.$$

Proof: "$\Rightarrow$" $P \in \mathcal{L}(X, X)$ implies $N(P) = P^{-1}(0) = P^{-1}(\{0\})$ is the preimage of a closed subset, hence itself closed. Alternatively: $(x_n)_{n\in\mathbb{N}}$ sequence in $N(P)$, $x_n \to x \in X$. Then $P(x) = \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} 0 = 0$. Moreover, $P \in \mathcal{L}(X, X)$ implies $id_X - P \in \mathcal{L}(X, X)$ continuous linear projection. Hence $R(P) = N(id_X - P)$ is closed. $(x_n)_{n\in\mathbb{N}}$ sequence in R(P):

$$x_n \to x \in X \implies x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} P(x_n) = P(\lim_{n\to\infty} x_n) = P(x) \implies x \in R(P)$$

"$\Leftarrow$" $(X, \|\cdot\|_X)$ Banach implies by closed graph theorem (2.2.9), that is suffices to show that the graph $\Gamma(P) = \{(x, Px) \mid x \in X\}$ is closed in $X \times X$. Let $(x_n, Px_n) \xrightarrow{n\to\infty} (x, y)$ in $X \times X$. We have to show, that $(x, y) \in \Gamma(P)$, i.e. $y = Px$.

- $(Px_x)_{\in\mathbb{N}}$ converges to y in X and $Px_n \in R(P) \forall n \in \mathbb{N} \overset{R(P) \text{ closed}}{\implies} y \in R(P)$. Hence $y = P(y)$.
- $(x_n - Px_n)_{n\in\mathbb{N}}$ converges to $x - y$ in X and $x_n - Px_n \in N(P) \forall n \in \mathbb{N} \overset{N(P) \text{ closed}}{\implies} x - y \in N(P)$. $0 = P(x - y) = Px - Py = Px - y \implies y = Px$.

$\implies \Gamma(P)$ is closed and, by closed graph theorem, $P \in \mathcal{L}(X, X)$. □

Example 2.3.3 (i) $X = L^p((-a, a))$, $a > 0$, $p \in [1, \infty]$. $P : X \to X$, $f \mapsto Pf$, where

$$(Pf)(t) := \frac{1}{2}(f(t) + \underbrace{f(-t)}_{=:g(t)}) = \frac{1}{2}(f(t) + g(t)) \quad \text{for all} \quad t \in (-a, a)$$

Hence $Pf(-t) = Pf(t)$ for all $t \in (-a, a)$. P is linear and $R(P) = \{h \in X \mid h(t) = h(-t) \forall t \in (-a, a)\}$.

"$\subseteq$" is clear and $\forall h$ with $h(t) = h(-t)$, $(Pf)(t) = \frac{1}{2}(h(t) + h(-t)) = \frac{1}{2}(h(t) + h(-t)) = h(t)$. Hence $Ph = h$. That implies $R(P) = \{\text{even fct.} \in X\}$ and $P_{|R(P)} = id_{R(P)}$. Therefore P is a projection.

$$\forall f \in X : \|Pf\|_{L^p} = \|\tfrac{1}{2}f + \tfrac{1}{2}g\|_{L^p} \leq \tfrac{1}{2}\|f\|_{L^p} + \tfrac{1}{2}\|g\|_{L^p} = \|f\|_{L^p}$$
$$\implies P \text{ continuous and } \|P\|_{L^p\to L^p} \leq 1 \overset{P\neq 0}{\implies} \|P\|_{L^p\to L^p} = 1$$
$$N(P) = \{f \in X \mid f(t) = -f(-t) \forall t \in (-a, a)\} \text{ is the set of odd fcts.}$$
$$((id_X - P)f)(t) = \tfrac{1}{2}(f(t) - f(-t))$$

(ii) $(X, \langle\cdot,\cdot\rangle)$ inner product space, $f, g \in X : \langle f, g\rangle \neq 0$. $P : X \to X, x \mapsto \frac{\langle x,g\rangle}{\langle f,g\rangle} f$ linear.

- projection: $\forall x \in X : P(Px) = P\left(\frac{\langle x,g\rangle}{\langle f,g\rangle} f\right) = \frac{\left\langle \frac{\langle x,g\rangle}{\langle f,g\rangle} f, g\right\rangle}{\langle f,g\rangle} f = \frac{\langle x,g\rangle}{\langle f,g\rangle}\frac{\langle f,g\rangle}{\langle f,g\rangle} f = \frac{\langle x,g\rangle}{\langle f,g\rangle} f = P_X$
- $\forall x \in X : \|Px\| = \left|\frac{\langle x,g\rangle}{\langle f,g\rangle}\right| \cdot \|f\| \overset{\text{CSE}}{\leq} \frac{\|x\|\cdot\|g\|}{|\langle f,g\rangle|}\|f\| \implies P$ is continuous and $\|P\|_{X\to X} \leq \frac{\|f\|\|g\|}{|\langle f,g\rangle|}$. In fact, the following inequality holds: $\|Pg\| = \frac{|\langle g,g\rangle|}{|\langle f,g\rangle|}\|f\| = \frac{\|f\|\|g\|}{|\langle f,g\rangle|}\cdot\|g\| \implies \|P\|_{X\to X} = \frac{\|f\|\|g\|}{|\langle f,g\rangle|}$

$R(P) = \operatorname{span}\{f\}$, $N(P) = \{x \in X \mid \langle x,g\rangle = 0\} = \{g\}^\perp$.

Projections in Hilbert spaces can be described in more detail in Hilbert spaces because of the additional geometric structure given by a scalar product. We can even investigate non-linear continuous projections such as

$$P : X \to X, x \mapsto \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1 \end{cases}$$

Theorem 2.3.4 (Orthogonal projection)
Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\emptyset \neq A \subseteq X$ a nonempty convex and closed subset. Then

(i) For every $x \in X$ there is a unique point $Px \in A$ s.t. $\|x - Px\| = \inf_{z\in A}\|x - z\|$. The map $P_A := P : X \to X, x \mapsto Px$ is then called orthogonal projection *onto A.*

(ii) For $x \in X$, $Px \in A$ is the unique point satisfying $\operatorname{Re}(\langle x - Px, z - Px\rangle) \leq 0$ *for all $z \in A$.*

(iii) For $x, y \in X : 0 \leq \operatorname{Re}(\langle Px - Py, x - y\rangle)$ and $\|Px - Py\| \leq \|x - y\|$. In particular, P is Lipschitz-continuous.

Proof: A closed, X Hilbert space. Then A is a complete metric space.

(i) **Existence** Let $x \in X \implies \exists (y_n)_{n\in\mathbb{N}}$ in A s.t. $\|x - y_n\| \xrightarrow{n\to\infty} \inf_{z\in A}\|x - z\| =: d$. Then

$$\forall m, n \in \mathbb{N} : \frac{y_m + y_n}{2} \in A \quad \text{by convexity}$$

$$\begin{aligned} 4d^2 &\leq 4\left\|x - \frac{y_m + y_n}{2}\right\|^2 = \|(x - y_m) + (x - y_n)\|^2 \\ &\overset{\text{Parallelogram}}{=} 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - \|y_n - y_m\|^2 \qquad (*) \\ \|y_m - y_n\|^2 &\leq 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4d^2 \end{aligned}$$

If $\varepsilon > 0$, then choose $n_0 < \mathbb{N}$ s.t. $\|x - y_n\|^2 - d^2 < \frac{\varepsilon}{4}$

$$< 2\frac{\varepsilon}{4} + 2\frac{\varepsilon}{4} = \varepsilon \quad \forall m, n \geq n_0$$

Hence $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A and that implies $Px = \lim_{n\to\infty} y_n$ exists, $Px \in A$. Therefore $\|x - Px\| = \lim_{n\to\infty}\|x - y_n\| = d \implies$ existence proved.

Uniqueness If $y \in A$ also satisfies $\|x - y\| = d$, then $\frac{y+Px}{2} \in A$.

$$\begin{aligned} 4d^2 &\overset{(*)}{\leq} 2\|x - Px\|^2 + 2\|x - y\|^2 - \|Px - y\|^2 \\ &= 2d^2 + 2d^2 - \|Px - y\|^2 \implies \|Px - y\|^2 \leq 0 \implies y = Px \end{aligned}$$

Hence P is well defined.

Projection with $R(P) = A : R(P) \subset A$ and $\forall x \in A : \inf_{z\in A}\|x - z\| = \|x - x\| = 0$, so $Px = x$. Hence $A \subseteq R(P) \implies A = R(P)$ and $P_{|A} = id_A \implies P$ is a projection.

(ii) • Let $y \in A$ satisfy $\operatorname{Re}(\langle x - y, x - z\rangle) \leq 0$ for all $z \in A$. Then

$$\forall z \in A : \|x - z\|^2 = \|x - y + y - z\|^2 = \|x - y\|^2 + 2\underbrace{\operatorname{Re}(\langle x - y, y - z\rangle)}_{\geq 0} + \underbrace{\|y - z\|^2}_{\geq 0} \geq \|x - y\|^2$$

$$\implies \|x - y\| = \inf_{z\in A}\|x - z\| \implies y = Px.$$

- $z \in A, \lambda \in (0,1) \implies Px + \lambda(z - Px) \in A$.

$$\begin{aligned}\|x-Px\|^2 &\leq \|x-(Px-\lambda(z-Px))\|^2 = \|(x-Px)+\lambda(z-Px)\|^2\\ &= \|x-Px\|^2 - 2\,\mathrm{Re}(\langle x-Px, \lambda(z-Px)\rangle)\\ &\implies 2\lambda\cdot\mathrm{Re}(\langle x-Px, z-Px\rangle) \leq \lambda^2\|z-Px\|^2\\ &\implies \mathrm{Re}(\langle x-Px, z-Px\rangle) \leq \lambda\|z-Px\|^2 \quad \text{for all } \lambda\in(0,1)\\ &\overset{\lambda\to 0}{\implies} \mathrm{Re}(\langle x-Px, z-Px\rangle) \leq 0\end{aligned}$$

- $\forall x, y \in X$

$$\begin{aligned}\mathrm{Re}(\langle x-Px, Py-Px\rangle) &\leq 0 \quad \text{by (ii) as } Py\in A\\ \mathrm{Re}(\langle y-Py, Px-Py\rangle) &\leq 0\\ \implies -\mathrm{Re}(\langle Px, Py-Px\rangle) &\leq -\mathrm{Re}(\langle x, Py-Px\rangle)\\ \mathrm{Re}(\langle Py, Py-Px\rangle) &\leq \mathrm{Re}(\langle y, Py-Px\rangle)\\ \overset{\text{Add both inequalities}}{\implies} \mathrm{Re}(\langle Py-Px, Py-Px\rangle) &\leq \mathrm{Re}(\langle y-x, Py-Px\rangle)\\ \implies 0 \leq \|Py-Px\|^2 = \mathrm{Re}(\langle Py-Px, Py-Px\rangle) &\leq \mathrm{Re}(\langle y-x, Py-Px\rangle)\end{aligned}$$

$$\begin{aligned}\|Py-Px\|^2 \leq \mathrm{Re}(\langle y-x, Py-Px\rangle) \leq |\langle y-x, Py-Px\rangle| &\overset{\text{CSE}}{\leq} \|y-x\|\cdot\|Py-Px\|\\ \implies \|Py-Px\| \leq \|y-x\| \quad &\text{if} \quad \|Py-Px\|\neq 0\end{aligned}$$

If $\|Py-Px\| = 0$, then this is trivially fulfilled. $\square$

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space, $M \subseteq X$ a subset. We put $M^\perp := \{x\in X \mid \langle x,y\rangle = 0 \forall y\in M\}$.

Theorem 2.3.5 (orthonormal projections onto linear subspaces)
Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $V \subseteq X$ a closed linear subspace ($\implies V\neq\emptyset, V$ convex). Let $P = P_V$ be the projection from the previous theorem. Then

(i) $\forall x\in X: Px\in V$ is the unique point s.t. $x - Px\in V^\perp$.

(ii) P is linear, $N(P) = \ker(P) = V^\perp$, $R(P) = V$ and $X = V\oplus V^\perp$ is the corresponding direct sum decomposition.

Proof: (i)

Uniqueness: Let $y\in V$ be s.t. $x-y\in V^\perp$. Hence to show: $y = Px$.

$$\begin{aligned}\forall v\in V: \|x-v\|^2 = \|\underbrace{(x-y)}_{\in V^\perp} + \underbrace{(y-v)}_{\in V}\|^2 &\overset{\text{as above}}{\geq} \|x-y\|^2\\ \implies \|x-y\| = \inf_{v\in V}\|x-v\| &\implies y = Px\end{aligned}$$

$x - Px\in V^\top$**:** $\forall v\in V: z = v + Px\in V \implies \mathrm{Re}(\langle x-Px, v\rangle) = \mathrm{Re}(\langle x-Px, z-Px\rangle)\leq 0$. This also holds for $-v\in V$ instead of v. Hence $\mathrm{Re}(\langle x-Px, v\rangle) = 0$ for all $v\in V$.

If $\mathbb{K} = \mathbb{R}$, then $\langle x-Px, v\rangle = 0$ for all $v\in V$ implies $x - Px\in V^\perp$.

If $\mathbb{K} = \mathbb{C}$, then $\forall v\in V: iv\in V \implies 0 = \mathrm{Re}(\langle x-Px, iv\rangle) = \mathrm{Re}(-i, \langle x-Px, v\rangle) = \mathrm{Im}(\langle x-Px, v\rangle)$. Therefore $\langle x-Px, v\rangle = 0$ for all $v\in V$ and hence $x-Px\in V^\perp$. $\square$

(ii) $x\in X \overset{(1)}{\implies} x-Px\in V^\perp, Px\in V \implies x = \underbrace{(x-Px)}_{\in V^\perp} + \underbrace{Px}_{\in V} \implies X = V + V^\perp$. Is this a direct decomposition?

$$\begin{aligned}&v\in V\cap V^\perp \implies \|v\|^2 = \langle v,v\rangle = 0 \implies v = 0\\ &\implies X = V\oplus V^\perp \quad \text{and } P \text{ is the projection onto } V \text{ w.r.t. this decomposition}\\ &\implies P \quad \text{linear and } N(P) = R(id_X - P) = V^\perp\end{aligned}$$

Corollary 2.3.6
If $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space, then every closed linear subspace V has a closed complement (namely $V^\perp$).

2.4 The Riesz Representation Theorem

General considerations: If $(X, \|\cdot\|)$ is a normed space, then $X' := \mathcal{L}(X, \mathbb{K})$ is called the *dual space* to X. $(X', \|\cdot\|_{X'})$, where $\|\cdot\|_{X'} := \|\cdot\|_{X\to\mathbb{K}}$ is a Banach space.

Lemma 2.4.1
If $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then

$$\forall y \in X: \quad \begin{cases} J_y : X \to \mathbb{K} \\ x \mapsto \langle x, y\rangle \end{cases} \quad \text{is an element of } X'$$

The map $J : X \to X'$, $y \mapsto J_y = J(y)$ is norm-preserving and antilinear, i.e.

$$\forall y, z \in X : J(y+z) = J(y) + J(z) \qquad \forall \lambda \in \mathbb{K}\forall y \in X : J(\lambda y) = \overline{\lambda}\cdot J(y)$$

Proof: $\forall y \in X : J_y$ linear is clear.

$$\begin{aligned}\forall x \in X : |J_y(x)| = |\langle x, y\rangle| &\overset{\text{CSE}}{\leq} \|x\|\cdot\|y\| \implies J_y \in X', \|J_y\|_{X'} \leq \|y\|_X \\ y = 0 \implies J_y = 0 &\implies \|J_y\|_{X'} = 0 = \|y\|_X \\ y \neq 0 \implies |J_y(y)| = |\langle y, y\rangle| = \|y\|^2 &\implies \|J_y\|_{X'} = \|y\|_X\end{aligned}$$

Hence J is norm-preserving. J is antilinear by properties of $\langle\cdot,\cdot\rangle$. $\square$

Theorem 2.4.2 (Riesz Representation Theorem)
Let $H = (X, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then $J : X \to X'$ is a norm-preserving, antilinear bijection, i.e.

$$\forall \varphi \in X' \exists! y \in X : \varphi(x) = \langle x, y\rangle \quad \textit{for all } x \in X$$

Every element $l \in H'$ can be uniquely represented via the scalar product $\langle\cdot,\cdot\rangle_H$.

Proof: J injective, norm-preserving, antilinear by the lemma. Remains surjectivity: Let $\varphi = 0$, then take $y = 0$. If $\varphi \neq 0$: $\ker(\varphi) = \varphi^{-1}(0) \subsetneq X$ is a closed proper subspace. Hence $X = \ker(\varphi) \oplus (\ker\varphi)^\perp \implies (\ker\varphi)^\perp \neq \{0\}$. Hence $\exists z \in (\ker\varphi)^\perp \setminus \{0\}$.

$$\begin{aligned}\forall x \in X : &\varphi\left(x - \frac{\varphi(x)}{\varphi(z)}z\right) = \varphi(x) - \frac{\varphi(x)}{\varphi(z)}\varphi(z) = 0 \\ &\implies x - \frac{\varphi(x)}{\varphi(z)}z \in \ker\varphi \implies \left\langle x - \frac{\varphi(x)}{\varphi(z)}, z\right\rangle = 0 \\ &\implies \langle x, z\rangle = \left\langle \frac{\varphi(x)}{\varphi(z)}z, z\right\rangle = \frac{\varphi(x)}{\varphi(z)}\|z\|^2 \\ &\implies \varphi(x) = \left\langle x, \frac{\overline{\varphi(z)}}{\|z\|^2}z\right\rangle\end{aligned}$$

So we can put $y = \frac{\overline{\varphi(z)}}{\|z\|^2}z$. $\square$

We also have an existence and uniqueness result. It provides existence and uniqueness of solutions for certain partial differential equations. However, there are many other representations of H'. In fact, often other representations are more important.

Example 2.4.3 (i) For $a \in \mathbb{R}$ we define the Hilbert space

$$L^2_a(\mathbb{R}^d) = \{u : \mathbb{R}^d \to \mathbb{R} \mid u \text{ measurable}, \rho^a u \in L^2(\mathbb{R}^d)\}$$

where the weight ρ is given via $\rho(x) = \sqrt{1+|x|^2}$, $\|u\|_{L^2_a} := \left(\int_{\mathbb{R}} |\rho^a u|^2\,dx\right)^{1/2}$. Note, that $K_a : L^2(\mathbb{R}^d) \to L^2_a(\mathbb{R}^d), v \mapsto \rho^{-a}v$ is a linear norm-preserving mapping. The scalar product $\langle u, v\rangle = \int_{\mathbb{R}^d} \rho^{2a}uv\,dx$. What is the dual space $L^2_a(\mathbb{R}^d)'$? Of course $(L^2_a)' = \mathcal{L}(L^2_a, \mathbb{R})$. But we really want to know representations.

Remark (Representations) (i) Riesz: $(L^2_a)' \cong_i L^2_a \quad \forall l \in L^2_a \exists! v \in L^2_a \forall u \in L^2_a : l(u) = \int_{\mathbb{R}^d} \rho^{2a}vu\,dx$.

(ii) $(L_a^2)' \cong_{ii} L_{-a}^2 \quad \forall l \in L_a^2 \exists! w \in L_{-a}^2 \forall u \in L_a^2 : l(u) = \int_{\mathbb{R}^d} \rho^0 wu \, dx$. Simply choose $w = \rho^{2a} v$.

(iii) Generalization: $(L_a^2)' \cong_{iii} L_b^2$ for any b by choosing $c \in \mathbb{R}$ in the representation $l(u) = \int_{\mathbb{R}^d} \rho^c zu \, dx$.

There are many representations, only two are special:

- Riesz representation using the H scalar product
- use classical representation for distributions. $u \in L^1(\Omega) = Tu(\varphi) = \int_\Omega u\varphi \, dx$.

Example 2.4.4 (ii) Sobolev spaces $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ Hilbert spaces. $H_0^k(\Omega) = \overline{\mathcal{C}_c^\infty(\Omega)}^{\|\cdot\|_{H^k}}$. Representations of $(H_0^k(\Omega))' = \mathcal{L}(H_0^k(\Omega), \mathbb{R})$.

(a) Riesz representation: $\forall l \exists! v \in H_0^k(\Omega) \forall u \in H_0^k(\Omega) : l(u) = \langle u, v\rangle_{H^k} = \int_\Omega \sum_{|\alpha|\le k} D^\alpha u D^\alpha v \, dx$

(b) Representation by distributions: $H^{-k}(\Omega) \stackrel{\text{def}}{=} \{T : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R} \mid \exists C \forall \varphi \in \mathcal{C}_c^\infty(\Omega) : |T(\varphi)| \le C\|\varphi\|_{H^k}\}$. $\|T\|_{H^{-k}} = \sup_{q\neq 0} \frac{|T(\varphi)|}{\|\varphi\|_{H^k}}$. This gives a representation of $H_0^k(\Omega)' = \mathcal{L}(H_0^k(\Omega), \mathbb{R})$ via the distributional representation. Recall that $\mathcal{C}_c^\infty(\Omega)$ is dense in $H_0^k(\Omega)$, hence every T in $H^{-k}(\Omega)$ has a unique continuous extension $T : H_0^k(\Omega) \to \mathbb{R}$.

(iii) $H_0^1(]-1,1[)$ ($\Omega \subset \mathbb{R}^1$ one dimension).

$$\delta_0 : \begin{cases} \mathcal{C}_c^\infty(\Omega) \to \mathbb{R} \\ \varphi \mapsto \varphi(0) \end{cases}$$

Claim: $\delta_0 \in H^{-1}(]-1,1[)$ to show $\exists C > 0 \forall \varphi \in \mathcal{C}_c^\infty(]-1,1[)$. $|\delta_0(\varphi)| \le C\|\varphi\|_{H^1}$

$$\begin{aligned} \varphi(0) &= \int_{-1}^0 \varphi'(x)\, dx \quad \text{where we use } \varphi(-1) = 0 \\ |\varphi(0)| &\le \int_{-1}^0 1 \cdot |\varphi'(x)|\, dx \le \left(\int_{-1}^0 1^2\, dx\right)^{1/2} \left(\int_{-1}^0 |\varphi'(x)|^2\, dx\right)^{1/2} \\ &\le 1 \cdot \left(\int_{-1}^{+1} |\varphi'(x)|^2\, dx\right)^{1/2} \le 1 \cdot \|\varphi\|_{H^1} \end{aligned}$$

"$\delta_0 \in H_0^1(\Omega)'$" if we use the distributional representation. "$\delta_0 \notin H_0^1(\Omega)'$" if we use Riesz representation.

Riesz representation theorem can be generalized as follows

Theorem 2.4.5 (Lax-Milgram Lemma, 1957)
Let H be a Hilbert space of $\mathbb{K}$ and $a : H \times H \to \mathbb{K}$ sesquilinear form such that there exist constants $0 < c \le C < \infty$ with

(i) Continuity $|a(x,y)| \le C\|x\|\|y\|$ for all $x, y \in H$.

(ii) Coercivity $|a(x,x)| \ge c\|x\|^2$ for all $x \in H$.

Then for all $l \in H'$ there exists a unique u in H such that $a(x,u) = l(x)$ for all x. Moreover the mapping $K : H' \to H, l \mapsto u$ is antilinear and invertible satisfying

$$\|K\|_{H'\to H} \le \frac{1}{c} \quad , \qquad \|K^{-1}\|_{H\to H'} \le C$$

Remark Taking $a(x,y) = \langle x, y\rangle_H$ the theorem is reduced to the Riesz representation and $c = C = 1$.

Note that a is not assumed to be symmetric, i.e. $a(x,y) \neq \overline{a(y,x)}$ is allowed.

Proof: (similar to Riesz' theorem)

Step 1 For $u \in H$ fixed, the mapping $l_u : x \mapsto a(x,u)$ satisfies $|a(x,u)| \stackrel{(i)}{\le} C\|u\|\|x\|$. Hence $l_u \in H'$ and Riesz representation yields a unique $v = Au$ such that $a(x,u) = l_u(x) = \langle x, v\rangle$ for all $x \in H$. The mapping $A : H \to H, u \mapsto v$ is linear and

$$\|Au\| = \|v\| = \sup_{x\neq 0} \frac{|\langle v, x\rangle|}{\|x\|} = \sup_{x\neq 0} \frac{|a(x,u)|}{\|x\|} \stackrel{(i)}{\le} C\|u\|$$

Hence $A \in \mathcal{L}(H,H) =: \mathcal{L}(H)$.

Step 2 *Aim:* Show that A is invertible. We have

$$c\|x\|^2 \leq |a(x,x)| = |\langle x, Ax\rangle| \overset{\text{CSE}}{\leq} \|x\|\|Ax\|$$
$$(*) \qquad \forall x \in X : \|Ax\| \geq c\|x\|$$

This implies $N(A) = \ker A = \{0\}$ and hence A is injective. We need to show that A is also surjective. For the range $R(A) = \{Ax \mid x \in H\}$ is in fact closed, since $y_k \in R(A)$ and $y_k \to y$. Then $y_k = Ax_k$ for some x_k.

$$\|x_k - x_m\| \overset{(*)}{\leq} \frac{1}{c}\|A(x_k - x_m)\| = \frac{1}{c}\|y_k - y_m\| \to 0$$

Using $A \in \mathcal{L}(H)$ we conclude $x_k \to x \implies y_k = Ax_k \to Ax = y$. Hence $y \in R(A)$. Thus $R(A)$ is closed. In order to produce a contradiction we assume $R(A) \subsetneq H$. Take $P \in \mathcal{L}(H)$ to be the uniquely defined orthogonal projection with $R(P) = R(A)$. Choose $x_0 \in H \setminus R(A)$ then $x_1 = x_0 - Px_0$ satisfies $x_1 \neq 0$ and $Px_1 = 0$. Hence x_1 is in the orthogonal complement of $R(A)$, i.e. $\langle x_1, y\rangle = 0$ for all $y = Ax$ with $x \in H$. However

$$c\|x_1\|^2 \overset{(ii)}{\leq} |a(x_1,x_1)| = |\langle x_1, Ax_1\rangle| = 0 \implies x_1 = 0 \quad \text{contradiction!}$$

We conclude $R(A) = H$. Hence $A : H \to H$ is bijective. By $(*)$ we know $\|A^{-1}u\| \leq \frac{1}{c}\|u\|$

Step 3 $K : H' \to H$ is the composition of $J : H' \to H$ Riesz mapping and A^{-1}, i.e. $K = A^{-1}J$ and hence all properties of K follow. □

2.5 Elliptic partial differential equations

ellipticity means coercivity of a quadratic form.

$$\text{Laplace equation} \quad (*) \begin{cases} -\Delta u + u &= f \quad \text{in } \Omega \qquad \text{PDE} \\ \qquad\quad u &= 0 \quad \text{on } \partial\Omega \qquad \text{BC (boundary condition)} \end{cases}$$

PDE for the unknown $u : \Omega \to \mathbb{R}$.

classical solution	$u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$	PDE and BC are satisfied everywhere
strong solution	$u \in H^2(\Omega) = W^{2,2}(\Omega), f \in L^2(\Omega)$	PDE is satisfied in $L^2(\Omega)$, BC are imposed by $u \in H^1_0(\Omega)$

For the *weak solution* $u : \Omega \to \mathbb{R}$ we only ask for $u \in H^1_0(\Omega)$. The second derivatives are taken in the distributional sense, namely in $H^{-1}(\Omega)$. We test the equation $(*)$ by a test function $\varphi \in \mathcal{C}^\infty_c(\Omega)$.

$$\int_\Omega f\varphi\, dx = \int_\Omega (-\Delta u + u)\varphi\, dx = \int_\Omega (-\operatorname{div}(\underbrace{\nabla u}_{L^2(\Omega)}) + u)\varphi\, dx \underset{\text{Integr. by parts}}{\overset{\substack{\text{Div. Theorem} \\ \text{Gauß}}}{=}} \int_\Omega (\nabla u \cdot \nabla\varphi + u\varphi)\, dx = \langle u, \varphi\rangle_{H^1}$$

where $-\Delta u + u = f$ in $H^{-1}(\Omega)$ is defined as

$$\int_\Omega f\varphi\, dx = \int_\Omega \nabla u \cdot \nabla\varphi + u\varphi\, dx \quad \text{for all} \quad \varphi \in H^1_0(\Omega)$$

Here we use density of $\mathcal{C}^\infty_c(\Omega)$ in $H^1_0(\Omega)$. The weak form $(*)$ is the following problem. Given $f \in L^2(\Omega)$ find $u \in H^1_0(\Omega)$ such that

$$\langle u, \varphi\rangle_{H^1_0(\Omega)} = \int_\Omega f\varphi\, dx \quad \text{for all} \quad \varphi \in H^1_0(\Omega)$$

The concept of weak solutions is compatible with the concept of strong or classical solutions. If a weak solution is smoother than needed for a weak solution, then it is also a strong solution.

Proof: Assume $u \in H^2(\Omega)$ is a weak solution. Then, we have to show that u is a strong solution. We know

$$\int_\Omega f\varphi\, dx = \int_\Omega \nabla u \cdot \nabla\varphi + u\varphi\, dx \quad \text{for all } \varphi \in \mathcal{C}^\infty_c(\Omega)$$

□

For distributional derivatives we know $D^\alpha(T_u) = T_{D^\alpha u}$ if $u \in H^2(\Omega)$ and $|\alpha| \le 2$.

$$\begin{aligned} &\implies \int_\Omega f\varphi\, x = \int_\Omega (\underbrace{-\Delta u + u}_{\in L^2(\Omega)})\varphi\, dx \\ &\implies 0 = \int_\Omega (\underbrace{-\Delta u + u - f}_{\in L^2(\Omega)})\varphi\, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega) \overset{\text{dense}}{\implies} \in L^2(\Omega) \\ &\implies -\Delta u + u = f \quad \text{in } L^2(\Omega) \end{aligned}$$

Proposition 2.5.1
For each $f \in L^2(\Omega)$ the problem $(*)$ has a unique weak solution.

Proof: $l : \varphi \mapsto \int_\Omega f\varphi\, dx$ is a continuous linear mapping on $H_0^1(\Omega)$.

$$|l(\varphi)| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le \|f\|_{L^2}\|\varphi\|_{H^1}$$

Riesz representation theorem (2.4.2) gives a unique $u \in H_0^1(\Omega)$ s.t.

$$\langle u, \varphi\rangle_{H^1} = \int_\Omega \nabla u \cdot \nabla\varphi + u\varphi\, dx = l(\varphi) = \int_\Omega f\varphi\, dx \quad \text{for all } \varphi \in H_0^1(\Omega)$$

□

Example 2.5.2 $\Omega =]-1,1[$, $-u'' + u = \delta_0 \in H^{-1}(]-1,1[)$, $u \in H_0^1(\Omega)$. There is no strong solution but there is a unique weak solution:

$$\int_{-1}^1 u'\varphi' + u\varphi\, dx = \delta_0(\varphi) = \varphi(0)$$

The solution is given by $u(x) = \begin{cases} \alpha\sinh(x+1) \\ \alpha\sinh(1-x) \end{cases}$. $u'(0^-) - u'(0^+) = +1 \implies \alpha = \frac{1}{\cosh(1)}$.

We look at more general linear PDEs in the form

$$(**) \qquad \begin{cases} -\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u + c(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

A function $u \in H_0^1(\Omega)$ is called a weak solution of $(**)$ if it satisfies

$$\int_\Omega (A(x)\nabla u)\cdot\nabla\varphi + (b(x)\nabla u + c(x)u)\varphi\, dx = \int_\Omega f\varphi\, dx \quad \text{for all } \varphi \in H_0^1(\Omega)$$

We define the bilinear form

$$a(u,v) = \int_\Omega (A(x)\nabla u)\cdot\nabla v + (b\cdot\nabla u)v + cuv\, dx$$

$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ must be a bilinear form with (i) boundedness $\implies$ continuity, (ii) coercitity.

$$A \in L^\infty(\Omega, \mathbb{R}^{d\times d}) \qquad b \in L^\infty(\Omega, \mathbb{R}^d) \qquad c \in L^\infty(\Omega)$$

Then

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |A||\nabla u||\nabla v| + |b||\nabla u||v| + |c||u||v|\, dx \\ &\le \|A\|_\infty\|\nabla\|_{L^2}\|\nabla v\|_{L^2} + \|b\|_\infty\|\nabla u\|_{L^2}\|v\|_2 + \|c\|_\infty\|u\|_2\|v\|_2 \\ &\le \underbrace{(\|A\|_\infty + \|b\|_\infty + \|c\|_\infty)}_{=:C}\|u\|_{H^1}\|v\|_{H^1} \end{aligned}$$

For coercivity, we have to bound $a(u,u)$ from below.

$$a(u,u) = \int_\Omega \nabla u \cdot A(x)\nabla u + ub(x)\cdot\nabla u + cu^2\, dx$$

We assume

(i) $\exists \alpha_{\text{ell}} > 0 \forall x \in \Omega \forall \xi \in \mathbb{R}^d : \xi \cdot A(x)\xi \geq \alpha_{\text{ell}}|\xi|^2$

(ii) $c(x) \geq \gamma > 0$ for all $x \in \Omega$

(iii) $|b(x)| \leq (\alpha_{\text{ell}}\gamma)^{1/2}$ for all $x \in \Omega$.

Hence we find

$$\begin{aligned} a(u,u) &\geq \int_\Omega \alpha_{\text{ell}}|\nabla u|^2 - |b||\nabla u|u + \gamma u^2\, dx \qquad xy \leq \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 \\ |b||\nabla u|u &\leq \sqrt{\alpha_{\text{ell}}\gamma}|\nabla u||u| \leq \tfrac{1}{2}\alpha_{\text{ell}}|\nabla u|^2 + \tfrac{1}{2}\gamma u^2 \\ a(u,u) &\geq \int_\Omega \alpha_{\text{ell}}|\nabla u|^2 - \tfrac{1}{2}\alpha_{\text{ell}}|\nabla u|^2 - \tfrac{\gamma}{2}u^2 + \gamma u^2\, dx \\ &= \int_\Omega \tfrac{\alpha_{\text{ell}}}{2}|\nabla u|^2 + \tfrac{\gamma}{2}u^2\, dx \geq \frac{\min\{\alpha_{\text{ell}}, \gamma\}}{2}\|u\|_{H^1}^2 \end{aligned}$$

Theorem 2.5.3 (Existence and uniqueness for "simple elliptic" PDEs)
If A, b and c satisfy the above assumptions, then $(**)$ has a unique weak solution $u \in H_0^1(\Omega)$ for every $f \in H^{-1}(\Omega)$.

Remark In many applications c is not uniformly positive. In fact, quite often one has $c \equiv 0$ and $b \equiv 0$. In fact, using that $u = 0$ on $\partial\Omega$ sometimes we can estimate $\int u^2\, dx$ in terms of $\int_\Omega |\nabla u|^2\, dx$.

Theorem 2.5.4 (Poincaré inequality)
Assume that Ω is bounded, then there exists a constant C_Ω such that

$$\forall n \in H_0^1(\Omega) : \int_\Omega u^2\, dx \leq C_\Omega \int_\Omega |\nabla u|^2\, dx \quad \textit{Poincaré's estimate}$$

Corollary 2.5.5
If Ω satisfies the Poincaré estimate, then the two norms $\|u\|_{H^1}$ and $\|\nabla u\|_{L^2}$ are equivalent, i.e.

$$\exists c_\Omega > 0 : c_\Omega\|u\|_{H^1} \leq \|\nabla u\|_{L^2} \leq 1 \cdot \|u\|_{H^1}$$

for all $u \in H_0^1(\Omega)$

Proof:

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (1 + C_\Omega)\|\nabla u\|_{L^2}^2 \implies c_\Omega = (1 + C_\Omega)^{-1/2}$$

Corollary 2.5.6
If Ω satisfies the Poincaré estimate, then $-\operatorname{div}(A(x)\nabla u) = f$ in Ω, $u = 0$ on $\partial\Omega$ has a unique weak solution $u \in H_0^1(\Omega)$ (A and f are as above).

Proof: $a(u,v) = \int_\Omega \nabla v \cdot A(x)\nabla u\, x$ bounded bilinear form on $H_0^1(\Omega)$.

$$a(u,u) = \int_\Omega \nabla u \cdot A\nabla u\, dx \geq \int_\Omega \alpha_{\text{ell}}|\nabla u|^2\, dx = \alpha_{\text{ell}}\|\nabla u\|_{L^2}^2 \geq \alpha_{\text{ell}}c_\Omega\|u\|_{H_0^1} \implies \text{coercivity}$$

□

Proof: (of Poincaré inequality) It suffices to show the estimate on the dense set Ω is bounded implies $\exists a, b \in \mathbb{R}$ such that $\Omega \subset]a,b[\times\mathbb{R}^{d-1} = \Sigma$. Every $\varphi \in C_c^\infty(\Omega)$ can be excluded by 0 to a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ where $\varphi_{|\partial\Sigma} = 0$. For $x \in \Sigma$ we have

$$\begin{aligned} \varphi(x) &= \int_a^{x_1} \partial_{x_1}\varphi(\xi_1, x_2, \ldots, x_d)\, d\xi \quad \text{since } \varphi(a, x_2, \ldots, x_d) = 0 \\ |\varphi(x)|^2 &\leq \left|\int_\Omega 1 \cdot \cdots\, d\xi\right|^2 \leq \underbrace{\left(\int_a^{x_1} 1^2\, d\xi\right)}_{\leq b-a} \left(\int_1^{x_1} |\partial_{x_1}\varphi|^2\, d\xi\right) \end{aligned}$$

$$\begin{aligned} \int_a^b |\varphi(x_1, x_2, \ldots, x_d)|^2\, dx_1 &\leq (b-a) \int_a^b \int_a^{x_1} |\partial_{x_1}\varphi(\xi_1, x_2, \ldots, x_d)\, d\xi\, dx_1 \\ &\leq (b-a) \int_a^b \int_a^b |\partial_{x_1}\varphi|^2\, d\xi\, dx_1 \\ &= (b-a)^2 \int_a^b |\partial_{x_1}\varphi(x)|^2\, dx_1 \end{aligned}$$

Now integrate over $(x_2, \dots, x_d) \in \mathbb{R}^{d-1}$ to obtain

$$\begin{aligned} \int_\Omega |\varphi|^2 \, dx = \int_\Sigma \varphi^2 \, dx &= \int_{\mathbb{R}^{d-1}} \int_a^b \varphi^2 \, dx_1 \, d(x_2, \dots, x_d) \\ &\leq \int_{\mathbb{R}^{d-1}} (b-a)^2 \int_a^b |\partial_{x_1} \varphi|^2 \, dx_1 \, d(x_2, \dots, x_d) \\ &= (b-a)^2 \|\partial_{x_1}\varphi\|_{L^2}^2 \leq (b-a) \|\nabla \varphi\|_{L^2}^2 \end{aligned}$$

3 Dual spaces and weak topologies

$(X, \|\cdot\|)$ normed space

$$X' := \mathcal{L}(X, \mathbb{K}) \qquad \text{with } \mathcal{L} = \text{linear and continuous}$$

is called the *topological dual* (the algebraic dual contains all linear functionals, also the non-continuous ones). X' is always a Banach space if it is equipped with the operator norm

$$\|\ell\|_{X'} = \sup \left\{ \frac{|\ell(x)|}{\|x\|} \mid x \in X \setminus \{0\} \right\}$$

The elements in X' are simply called linear functionals on X.
Duality questions:

- Does X' characterize X uniquely?
- Can we use X' to study properties of X?
- Linear functionals define hyperplanes $H = \{x \in X \mid \ell(x) + \alpha = 0 \text{ for } \ell \in X' \text{ and } \alpha \in \mathbb{R}\}$. ($\hookrightarrow$ Geometry, $\hookrightarrow$ separation into two parts.)
- Is there for each $x \in X$ at least one $\ell \in X'$ such that $\ell(x) \neq 0$?
- Can we extend continuous linear mappings from subspaces to the whole space?

Example
- $X = \ell_\infty =$ bounded sequences. $Y \subset X$, $Y = \{(a_n) \mid a_n \text{ is periodic }\}$. (a_n) periodic iff $\exists N : a_{n+N} = a_n$ for all $n \in \mathbb{N}$. $\ell : Y \to \mathbb{R}$, $(a_n)_{n\in\mathbb{N}} \mapsto \frac{1}{N}\sum_{n=1}^N a_n$. What is a continuous extension of this "average" to all sequences?
- $X = L^\infty([0,1]), Y = \mathcal{C}^0([0,1]) \subset X$. $\ell : Y \to \mathbb{R}, f \mapsto f(\frac{1}{2})$.

3.1 Hahn-Banach theorem

Another name is the *extension principle of Hahn and Banach*. It states the existence of a suitable extension from a subspace to the full space. The general version relies on the *Axiom of Choice* (existence result). An equivalent "axiom" is *Zorn's Lemma* (transfinite induction, well-ordering theorem).

Lemma 3.1.1 (Zorn's Lemma)
Let M be a partially ordered set with order $\leq$ such that every chain (=totally ordered subset) has an upper bound. Then there exists at least one maximum element.

Definition 3.1.2
- $(M, \leq)$ *is a* partially ordered set *if*
  - *(P.i)* $\forall m : m \leq m$
  - *(P.ii)* $\forall m_1, m_2, m_3 : (m_1 \leq m_2 \wedge m_2 \leq m_3) \implies m_1 \leq m_3$
  - *(P.iii)* $\forall m_1, m_2 : (m_1 \leq m_2 \wedge m_2 \leq m_1) \implies m_1 = m_2$
- m *is called* upper bound *of $S \subset M$ if $\forall s \in S : s \leq m$.*
- $C \subset M$ *is called a* chain *if $\forall m_1, m_2 \in C$: either $m_1 \leq m_2$ or $m_2 \leq m_1$*
- m_* *is called a* maximal element *if there does not exist any $m \in M$ such that $m \neq m_*$ and $m_* \leq m$.*

We apply this to sets of linear mappings from subspaces Y of X into $\mathbb{R}$. The ordering will be by "extension"

$$\ell_1 : Y_1 \to \mathbb{R}, \ell_2 : Y_2 \to \mathbb{R} \quad \text{we say} \quad \ell_1 \leq \ell_2 \quad \text{if} \quad Y_1 \subset Y_2 \quad \text{and} \quad \ell_2|_{Y_1} = \ell_1$$

This means ℓ_2 is an *extension* of ℓ_1. A maximal element in terms of extension is a mapping that cannot be extended further.

Theorem 3.1.3 (of Hahn (1926) and Banach(1929))

- *Let X be a real vector space (no norm needed).*
- *A sublinear function $p : X \to \mathbb{R}$, i.e.*
  (i) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0, x \in X$
  (ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.
- *Y subspace and $f : Y \to \mathbb{R}$ is linear with bound $f(y) \leq p(y)$ for all $y \in Y$*

Then, there exists a linear extension $\ell : X \to \mathbb{R}$ (i.e. $\ell(y) = f(y)$ for $y \in Y$) such that

$$\ell(x) \leq p(x) \quad \textit{for all} \quad x \in X$$

Proof: We apply Zorn's Lemma using the following set

$$M := \{G : Y_G \to \mathbb{R} \mid Y \subset Y_G, G_{|Y} = f, Y_G \text{ subspace of } X, G(y) \leq p(y) \text{ for all } y \in Y_G, G \text{ linear}\}$$

We use the partial order defined via extension. We have to show that each chain has an upper bound. Let $C = \{G_\alpha : Y_\alpha \to \mathbb{R} \mid \alpha \in \mathcal{A}\}$. We define $\tilde{Y} = \bigcup_{\alpha \in \mathcal{A}} Y_\alpha$ and $\tilde{G} : \tilde{Y} \to \mathbb{R}, \tilde{G}(y) = G_\alpha(y)$ if $y \in Y_\alpha$. This is well-defined, since for every $\beta \in \mathcal{A}$ with $y \in Y_\beta$ we have $G_\alpha(y) = G_\beta(y)$ since either G_α is an extension of G_β or vice versa. By construction $\tilde{G}$ is an extension for all $G_\alpha, \alpha \in \mathcal{A}$. Hence $\tilde{G}$ is an upper bound.

Hence Zorn's lemma provides a maximal element $\ell : Y_\ell \to \mathbb{R}$, which means that ℓ cannot be extended non-trivially. This is certainly the case if $Y_\ell = X$, and this is the last point we have to show. We argue by contradiction. If $Y_\ell \neq X$, we construct a non-trivial extension.

Assume $Y_\ell \neq X$, i.e. $z \in X \setminus Y_\ell$. We look for a linear extension on $\hat{Y} = Y_\ell \oplus \operatorname{span}\{z\} \supsetneq Y_\ell$ in the form

$$\hat{\ell}(y + \lambda z) = \ell(y) + c\lambda \quad \text{for a suitable } c \in \mathbb{R} \text{ to be chosen}$$

$\hat{Y}$ is a subspace of X and $\hat{\ell}$ is linear. We have to find $c \in \mathbb{R}$ such that $\hat{\ell}(y + \lambda z) \leq p(y + \lambda z)$ for all $y \in Y_\ell, \lambda \in \mathbb{R}$. We choose c such that

$$S_* = \sup\{\ell(y) - p(y - z) \mid y \in Y_\ell\} \leq c \leq I_* := \inf\{p(\tilde{y} + z) - \ell(\tilde{y}) \mid \tilde{y} \in Y\}$$

such a c exists if and only if $S_* \leq I_*$

$$\begin{aligned}
\ell(y) + \ell(\tilde{y}) &\overset{\ell \text{ linear}}{=} \ell(y + \tilde{y}) \overset{\ell \in M}{\leq} p(y + \tilde{y}) \qquad (0 = z - z) \\
&= p(y - z + \tilde{y} + z) \\
&\overset{p \text{ sub-linear}}{\leq} \quad p(y - z) + p(\tilde{y} + z) \quad \text{for all} \quad y, \hat{y} \in Y_\ell \\
\text{Hence} \quad \ell(y) - p(y - z) &\leq p(\tilde{y} + z) - \ell(\tilde{y}) \quad \text{for all} \quad y, \tilde{y} \in Y_\ell \\
&\Longrightarrow S_* \leq I_*
\end{aligned}$$

We now show $\hat{\ell}(y + \lambda z) = \ell(y) + \lambda c \leq p(y + \lambda z)$. For $\lambda = 0$ this is clear as $\ell \in M$. For $\lambda > 0$

$$p(y + \lambda z) - \hat{\ell}(y + \lambda z) = \lambda(p(\frac{1}{\lambda}y + z) - \ell(\frac{1}{\lambda}y) - c) \geq \lambda(I_* - c) \geq 0$$

The case $\lambda < 0$ is analogous using S_*. □

A Banach space version reads as follows

Proposition 3.1.4

$(X, \|\cdot\|)$ Banach space, $Y \subset X$ subspace $f : Y \to \mathbb{R}$ is continuous (i.e. $|f(y)| \leq C\|y\|$). Then, there exists an $\ell \in X' = \mathcal{L}(X, \mathbb{R})$ such that $\ell_{|Y} = f$ and

$$\|\ell\|_{X'} = \sup\left\{\frac{|f(y)|}{\|y\|} \mid y \in Y \setminus \{0\}\right\} =: N$$

(Short: There always exists an extension with the same norm.)

Proof: We define $p : X \to \mathbb{R}$, $x \mapsto N\|x\|$ which is sub-linear and

$$|f(y)| \leq N\|y\| = p(y) \quad \text{for all} \quad y \in Y$$

Hahn-Banach gives the extension ℓ with $\ell(x) \leq p(x)$ for all x. Hence $\ell(x) \leq p(x) = N\|x\|$ and

$$\ell(x) = -\ell(-x) \geq -p(-x) = -N\|x\|$$

Hence $|\ell(x)| \leq N\|x\|$. □

Corollary 3.1.5
X Banach and $x \in X \setminus \{0\}$. Then there exists $\ell \in X'$ such that $\ell(x) = \|x\|$ and $\|\ell\|_{X'} = 1$.

Proof: Choose $Y = \operatorname{span}\{x\}$ and $f(\alpha x) = \alpha\|x\|$. Then, $f : Y \to \mathbb{R}$ is linear and $\sup\left\{\frac{|f(y)|}{\|y\|} \mid y \in Y \setminus \{0\}\right\} = 1$. By the above proposition (version of Hahn-Banach) there exists an extension ℓ with norm 1. □

For mappings between Banach spaces we define the *adjoint mapping* as to the right with $T \in \mathcal{L}(X, Y)$. We define the adjoint $T' \in \mathcal{L}(Y', X')$ as follows :

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, squiggly] & Y \arrow[d, squiggly, "\text{dualization}"] \\ X' & Y' \arrow[l, "T'"] \end{tikzcd}$$

$$(T'y')(x) \overset{\text{def}}{=} y'(Tx) \quad \text{for} \quad y' \in Y', x \in X$$

Lemma 3.1.6
If X and Y are Banach spaces, then for each $T \in \mathcal{L}(X, Y)$ we have $\|T'\|_{\mathcal{L}(Y',X')} = \|T\|_{\mathcal{L}(X,Y)}$

Proof: For all $y' \in Y'$ we have

$$\begin{aligned} \|T'y'\|_{X'} &\underset{\text{of norm in } X'}{\overset{\text{by def}}{=}} \sup_{x \neq 0} \frac{|(T'y')(x)|}{\|x\|} \overset{\text{def of } T'}{=} \sup_{x \neq 0} \frac{|y'(Tx)|}{\|x\|} \\ &\underset{\text{of norm in } Y'}{\overset{\text{by def}}{\leq}} \sup_{x \neq 0} \frac{\|y'\|_{Y'}\|Tx\|_Y}{\|x\|} \leq \sup_{x \neq 0} \frac{\|y'\|_{Y'}\|T\|_{\mathcal{L}(X,Y)}\cancel{\|x\|_X}}{\cancel{\|x\|}} \\ &= \|T\|_{L(X,Y)}\|y'\|_{Y'} \implies T' \in \mathcal{L}(Y', X') \quad \text{and} \quad \|T'\|_{\mathcal{L}(Y',X')} \leq \|T\|_{\mathcal{L}(X,Y)} \end{aligned}$$

For the opposite inequality choose x_n with $\|x_n\| = 1$ and $\|Tx_n\|_Y \to \|T\|_{\mathcal{L}(X,Y)}$. The corollary of the HBP provides an element $y'_n \in Y'$ such that $\|y'_n\|_{Y'} = 1$ and $y'_n(Tx_n) = \|Tx_n\|_Y$ We can estimate $\|T'\|_{\mathcal{L}(Y',X')}$ from below by inserting one element in the corresponding $\sup \frac{\|T'y'\|}{\|y'\|}$

$$\|T'\|_{\mathcal{L}(Y',X')} \overset{\|y'_n\|=1}{\geq} \|T'y'_n\|_{X'} \overset{\|x_n\|=1}{\geq} |(T'y'_n)(x_n)| \overset{\text{def of } T'}{=} |y'_n(Tx_n)| \overset{\text{by constr.}}{=} \|Tx_n\|_Y \xrightarrow{\text{by constr.}} \|T\|_{\mathcal{L}(X,Y)}$$

Hence $\|T'\|_{\mathcal{L}(Y',X')} \geq \|T\|_{\mathcal{L}(X,Y)'}$ □

The geometric versions of the HBP are the so-called separation theorems. We want to separate disjoint convex sets by hyperplanes.

Warning: A, B convex and disjoint is not sufficient for existence of a separation hyperplane (only valid in finite dimensions).

Theorem 3.1.7 (Geometric form of *HBP*)
$(X, \|\cdot\|)$ normed space. A, B are convex subsets. $A \cap B = \emptyset$, A is open. Then there exists an $\ell \in X'$ and a constant $c \in \mathbb{R}$ such that

$$\ell(a) < c \quad \textit{for all} \quad a \in A \quad \textit{and} \quad \ell(b) \geq c \quad \textit{for all} \quad b \in B.$$

I.e. the hyperplane $H = \{x \in X \mid \ell(x) = c\}$ separates A and B.

Proof:

Step 1: We assume that $B = \{x_0\}$. Wlog we assume $0 \in A$ (otherwise do a translation ($\implies$ affine)). Since A is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset A$. We define the Minkowsky functional for the set A:

$$p_A(x) \overset{\text{def}}{=} \inf\{r > 0 \mid \tfrac{1}{r}x \in A\}$$

Claim: p_A is sub-linear. $p_A(\alpha x) = \alpha p_A(x)$ follows from the definition. $p_A(x+y) \leq p_A(x) + p_A(y)$ by convexity of A.

For $x_0 \in B$ we obtain $p_A(x_0) \geq 1$, since $x_0 \notin A$. We define a linear functional $f : \operatorname{span}\{x_0\} \to \mathbb{R}$ via $f(\alpha x_0) = \alpha p_A(x_0)$. This f is dominated by p_A on $\operatorname{span}\{x_0\}$.

$$f(\alpha x_0) = \alpha p_A(x_0) = \begin{cases} p_A(\alpha x_0) & \text{if } \alpha \geq 0 \\ \underbrace{-p_A(-\alpha x_0)}_{\leq 0 \leq p_A(\alpha x_0} & \text{if } \alpha < 0 \end{cases}$$

Hence the HBP can be applied and we find $\ell : X \to \mathbb{R}$ linear such that

$$\ell(x) \leq p_A(x) \quad \text{for all} \quad x \in X$$

Since $B_\varepsilon(0) \subset A$ we have $p_A(x) \leq 1$ for all $x \in B_\varepsilon(0)$. By scaling we find $p_A(x) \leq \frac{1}{\varepsilon}\|x\|$ for all $x \in X$. Hence, $\ell(x) \leq \frac{1}{\varepsilon}\|x\|$ and $\ell(x) = -\ell(-x) \geq -\frac{1}{\varepsilon}\|x\|$ implies $\|\ell\|_{X'} \leq \frac{1}{\varepsilon}$, i.e. $\ell \in X'$.

We still have to show that ℓ separates A and B. For $a \in A$ we have

$$\ell(a) \overset{\substack{<\text{still needed}\\ \text{by constr. of } \ell}}{\leq} p_A(a) \overset{\text{def. Mink.}}{\leq} 1 \leq p_A(x_0) \quad \text{as } x_0 \notin A.$$

Step 2: Now B is general again. To transfer to the first step we define

$$C \overset{\text{def}}{=} A - B = \{a - b \mid a \in A, b \in B\}$$

Claim: $0 \notin C$, C is open and convex.

(a) $0 \notin C$ since $A \cap B = \emptyset$.

(b) $C = \bigcup_{b \in B}(A - b)$ open as arbitrary union of open sets.

(c) straight forward

By Step 1 we can separate C from $\{0\}$ by a functional ℓ, i.e. $\forall x \in C : \ell(x) \leq \gamma \leq \ell(0) = 0$. Hence

$$\forall a \in A \forall b \in B : \ell(a - b) \leq 0$$

Define $c = \inf\{\ell(b) \mid b \in B\}$, then $\forall a \in A : \ell(a) \leq c$. We still need to show the *strict* estimate. We use again that A is open. Assume $\exists a_* \in A : \ell(a_*) = c$. Then $\exists \delta > 0 : B_\delta(a_*) \subset A$ (since A open). Hence, $\forall a \in B_\delta(a_*) : \ell(a) \leq c = \ell(a_*) \implies \ell(a - a_*) \leq 0$ for all $a \in B_\delta(a_*) \implies \ell = 0$ which is a contradiction to our assumption that ℓ is non-trivial. Hence, we have strict inequality. □

3.2 Reflexivity and the bidual X''

$X' = \mathcal{L}(X, \mathbb{K})$ is a Banach space. The *bidual* is the dual of the dual $X'' = (X')'$ (topological duals). We can embed the space X in a natural way into X'':

$$J_{\text{BD}} : \begin{cases} X \to X'' \\ x \mapsto \begin{cases} X' \to \mathbb{K} \\ \ell \mapsto \ell(x) \end{cases} \end{cases}$$

J_{BD} is called the *evaluation mapping*. We have to show $J_{\text{BD}}(x) \in X''$

$$|J_{\text{BD}}(x)(\ell)| \overset{\text{def. of } J}{=} |\ell(x)| \leq \|x\|_X \|\ell\|_{X'} \implies J_{\text{BD}}(x) \in X'' \quad \text{and} \quad \|J_{\text{BD}}(x)\|_{X''} \leq \|x\|_X$$

Obviously, $J_{\text{BD}} \in \mathcal{L}(X, X'')$ and moreover it is norm-preserving, i.e. $\|J_{\text{BD}}(x)\|_{X''} = \|x\|_X$. By Corollary of HBP for each $x \in X \exists \ell_x$ such that $\|\ell_x\|_{X'} = 1$ and $\ell_x(x) = \|x\|$. Now

$$\|J_{\text{BD}}(x)\|_{X''} \geq |J_{\text{BD}}(x)(\ell_x)| \overset{\text{def } J}{=} |\ell_x(x)| \overset{\text{by constr.}}{=} \|x\|$$

X'' contains a "copy of X''". In fact, $J_{\text{BD}}(X)$ is a closed subspace of X''.

Definition 3.2.1
A Banach space X is called reflexive, *if $J_{BD} : X \to X''$ is surjective (i.e. it is a norm-preserving isomorphism).*

Example 3.2.2

(i) All finite-dimensional Banach spaces are reflexive. $(\mathbb{R}^n)' = \mathbb{R}^n = (\mathbb{R}^n)''$.

(ii) All Hilbert spaces are reflexive, see below.

(iii) c_0, ℓ_1, ℓ_∞ are non-reflexive Banach spaces.

Exercise 37: $(c_0)' \cong \ell_1$, $(\ell_1)' \cong \ell_\infty$, $c_0 \subsetneq \ell_\infty = (c_0)''$.

Theorem 3.2.3
All Hilbert spaces are reflexive.

Proof: Idea: We use Riesz representation theorem: $H' \cong H$. Twice $(H')' \cong H' \cong H$.
For a given $x'' \in X''$ we have to construct $\hat{x} \in X$ such that $J_{\mathrm{BD}}(\hat{x}) = x''$. Denote by $I_R : H \to H'$ the Riesz isom. and define $\tilde{x}'(y) = x''(I_R(y))$ for $y \in H$. Now set $\hat{x} = I_R^{-1}(\tilde{x}')$. Then $x''(I_R(y)) = \hat{x}'(y) = I_R(y)(\hat{x})$ for all $y \in H$. I_R is surjective, hence $x''(\ell') = \ell'(\hat{x})$ for all $\ell' \in H'$. Hence, $x'' = J_{\mathrm{BD}}(\hat{x})$. □

Theorem 3.2.4

(i) X, Y Banach spaces, $T \in \mathcal{L}(X, Y)$ topol. isomorphism ($T^{-1} \in \mathcal{L}(Y, X)$). Then, X reflexive $\iff$ Y reflexive.

(ii) If X is reflexive, then every closed subspace $Z \subset X$ is reflexive as a Banach space with the same norm.

(iii) X is reflexive if and only if X'' is reflexive.

Proof: iii) is an exercise on sheet 11.

i) It suffices to show "$\Rightarrow$", the other direction follows by exchanging X and Y. We know that T and J_X have continuous inverses. We simply check the identity

$$J_Y = T'' J_X T^{-1}$$

$$\begin{tikzcd} X \arrow[r, shift left, "T"] \arrow[d, squiggly] \arrow[dd, bend right, "J_X"'] & Y \arrow[l, shift left, "T^{-1}"] \arrow[d, squiggly] \arrow[dd, bend left, "J_Y"] \\ X' \arrow[d, squiggly] & Y' \arrow[l, "T'"'] \arrow[d, squiggly] \\ X'' \arrow[r, "T''=(T')'"] & Y'' \end{tikzcd}$$

(straight forward by definitions). An exercise gave that T' has a continuous inverse, namely $(T^{-1})'$. Iterating gives T'' has a continuous inverse, namely $(T^{-1})''$. Hence, $J_Y^{-1} = T J_X^{-1} (T'')^{-1}$. In particular J_Y is surjective.

ii) X reflexive, $Z \subset X$ closed subspace. To show: $J_Z : Z \to Z''$ is surjective. This means $\forall z'' \exists z \in Z : J_Z(z) = z''$ for all $z' : z''(z') = z'(z)$. We have to bring in X''. For $x' \in X'$ we consider $x'_{|Z} \in Z'$. Namely

$$\|x'_{|Z}\|_{Z'} = \sup\{|x'(z)| \mid \|z\| = 1\} \leq \sup\{|x'(x)| \mid \|x\| = 1\} = \|x'\|_{X'}$$

Hence Z and X have the same norm.

For $z'' \in Z''$ given we define

$$x'' : \begin{cases} X' \to \mathbb{K} \\ x' \mapsto z''(x'_{|Z}) \end{cases}$$

We have $x' \in X''$ since

$$\begin{aligned} \|x''\|_{X''} &= \sup\{|x''(x')| \mid \|x'\|_{X'} = 1\} = \sup\{|z''(x'_{|Z})| \mid \|x'\|_{X'} = 1\} \\ &\leq \sup\{\|z''\|_{Z''} \underbrace{\|x'_{|Z}\|_{Z'}}_{\leq 1} \mid \|x'\|_{X'} = 1\} \leq \|z''\|_{Z''} \end{aligned}$$

Since X is reflexive, there exists $\hat{x} \in X$ with $J_X(\hat{x}) = x''$.

Claim 1 : $\hat{x} \in Z$. Take any $x' \in X'$ with $x'_{|Z} = 0$, then

$$x'(\hat{x}) \overset{\text{def of } \hat{x}}{=} J_X(\hat{x})(x') = x''(x') \overset{\text{constr. of } x''}{=} z''(x'_{|Z}) = 0$$

Now assume $\hat{x} \notin Z$ to produce a contradiction. Choose $\tilde{x}'$ such that $\tilde{x}'_{|Z} = 0$ and choose $\tilde{x}'(\hat{x}) = 1$ (usage of Hahn-Banach) (see literature for $\mathbb{C}$-version). This contradicts the above statement for general x' with $x'_{|Y} = 0$.

Claim 2: $J_Z(\hat{x}) = z''$. Take any $z' \in Z'$. We can extend it to $x' \in X'$, i.e. $x'_{|Z} = z'$. Then

$$z'(\hat{x}) = x'(\hat{x}) \overset{\text{def of } \hat{x}}{=} x''(x') \overset{\text{def } x''}{=} z''(x'_{|Z}) \overset{\text{def } x'}{=} z''(z') \quad \text{for all } z' \quad \implies z'' = J_Z(\hat{x})$$

iii) as exercise. □

Example 3.2.5 (i) Finite-dim. spaces are always reflexive.

(ii) Hilbert spaces are reflexive.

(iii) Later: ℓ^p sequences, $L^p(\Omega)$ Lebesgue spaces and $W^{k,p}(\Omega)$ Sobolev spaces are reflexive for $1 < p < \infty$

(iv) $\mathcal{C}^0(\Omega), L^1(\Omega), L^\infty(\Omega), c_0, \ell^1, \ell^\infty$ are *not* reflexive

Lemma 3.2.6
If X is reflexive, then for all $x' \in X'$ there exists $x \in X$ such that $\|x\| = 1$ and $x'(x) = \|x'\|_{X'}$ (i.e. $\sup = \max$, there exists a maximum).

Proof: Using the corollary of Hahn-Banach, there exists $x'' \in X''$ with $\|x''\|_{X''} = 1$ and $x''(x') = \|x'\|_{X'}$. Since X is reflexive, there exists $x \in X$ such that $J_X(x) = x''$. Moreover

$$\|x\|_X = \|x''\|_{X''} = 1 \quad \text{and} \quad x'(x) = x''(x') = \|x'\|_{X'}$$

□

Proof: ($\mathcal{C}^0(\Omega), L^1(\Omega), L^\infty(\Omega)$ are not reflexive) (i) Choose $X = \mathcal{C}^0(\Omega)$, $\Omega \subset \mathbb{R}^d$ domain. Define a norm-preserving embedding of $L^1(\Omega)$ in $\mathcal{C}^0(\overline{\Omega})'$.

$$L^1(\Omega) \ni f \mapsto x'_f : \begin{cases} \mathcal{C}^0(\Omega) \to \mathbb{R} \\ u \mapsto \int_\Omega f u \, dx \end{cases}$$

We have $|x'_f(u)| \leq \int |f| \|u\|_{L^\infty} \, dx \leq \|f\|_{L^1} \|u\|_X$. To show $\|x'_f\|_{X'} = \|f\|_{L^1}$ we choose $u_\varepsilon \in \mathcal{C}^0(\Omega)$ with $\|u_\varepsilon\|_{\mathcal{C}^0} \leq 1$ and $u_\varepsilon(x) \to \operatorname{sign} f(x)$ (pointwise) for $\varepsilon \to 0$.

$$\text{for example} \quad \mathcal{C}_c^\infty \ni f_\varepsilon \text{ in } L^1 \qquad u_\varepsilon \frac{f_\varepsilon}{\sqrt{\varepsilon^2 + f_\varepsilon^2}}$$

$$x'_f(u_\varepsilon) = \int_\Omega f u_\varepsilon \, dx \xrightarrow[\text{convergence}]{\text{Dom.}} \int f \operatorname{sign}(f) \, dx = \|f\|_{L^1}$$

Now take an $f \in L^1(\Omega)$ such that $f(x) = 1$ for Ω_+ and $f(x) = -1$ for Ω_-. with $\Omega = \Omega_+ \cup \Omega_-, \Omega_+ \cap \Omega_- = \emptyset$ and $\mu(\Omega_+) > 0, \mu(\Omega_-) > 0$.

Claim: $\forall u \in X = \mathcal{C}^0(\Omega)$ with $\|u\| = 1$ we have $|x'_f(u)| < \|f\|_{L^1} - \int_\Omega |f(x)| \, dx$. We have $x'_f(u) = \int f u \, dx$ with $|u(x)| \leq 1$ for all $x \in \Omega$.

$$\|f\|_{L^1} - x'_f(u) = \int \underbrace{|f(x)|}_{+1} - f(x) \underbrace{u(x)}_{|\cdot| \leq 1} \, dx \overset{?}{=} 0$$

$$\begin{aligned} &\implies f(x)u(x) = +1 \quad \text{a.e. in } \Omega \\ &\implies u(x) = \pm 1 \quad \text{a.e. in } \Omega_\pm \qquad \text{contradicts continuity} \\ &\implies X = \mathcal{C}^0(\Omega) \quad \text{cannot be reflexive} \end{aligned}$$

(ii) $\mathcal{C}^0(\Omega)$ is a closed subspace of $L^\infty(\Omega)$. Hence $L^\infty(\Omega)$ is not reflexive.

(iii) Exercise: $L^\infty \cong L^1(\Omega)' \implies L^1(\Omega)$ is not reflexive. Otherwise $L^\infty(\Omega)$ would be reflexive. □

3.3 Uniformly convex Banach spaces

Definition 3.3.1
$(X, \|\cdot\|)$ *is called* uniformly convex *if*

$$\forall \varepsilon > 0 \exists \delta > 0 : x, y \in \overline{B_1(0)} \quad and \quad \|x-y\| > \varepsilon \implies \|\tfrac{1}{2}(x+y)\| < 1-\delta$$

Theorem 3.3.2 (Milman and Pattis)
Uniformly convex Banach spaces are reflexive.

Proof: (idea) (for full proof, see standard literature)

$$\begin{array}{ccc} X \text{ reflexive} & \Longrightarrow & \begin{array}{c}\text{Attainment } \sup = \max \\ \forall x' \exists x : \|x\| = 1 \wedge x'(x) = \|x'\|\end{array} \\ \text{M\&P}\Uparrow & \nearrow & \\ X \text{ uniformly convex} & & \end{array}$$

Lemma 3.3.3
If X is uniformly convex Banach space, then

$$\forall x' \in X' \exists x \in X : \|x\| = 1 \quad and \quad x'(x) = \|x'\|_{X'}$$

Moreover, if $x' \neq 0$, then for all sequences $(x_n)_{n\in\mathbb{N}}$ with $\|x_n\| = 1$ and $x'(x_n) \to \|x'\|_{X'}$ we have $x_n \to x_*$ with $x'(x_*) = \|x'\|_{X'}$ and $\|x_*\| = 1$. In particular this $x = x_*$ is unique. (strict convexity gives uniqueness)

Proof: For $x' = 0$ the result is trivial (take any x with $\|x\| = 1$). Consider $x' \neq 0$ and any sequence $(x_n)_{n\in\mathbb{N}}$

$$\|x_n\|_X = 1 \qquad x'(x_n) \to \sup\{x'(x) \mid x \in X, \|x\| \leq 1\} = \|x'\|_{X'}$$

$$\|\underbrace{\tfrac{1}{2}(x_n + x_m)}_{\in B_1(0)}\| \geq \frac{1}{\|x'\|}|\underbrace{x'(\tfrac{1}{2}(x_n + x_m))}_{\to\|x'\|}| \to 1 \quad \text{for} \quad n, m \to \infty$$

$$\forall \varepsilon > 0 \exists n_0(\delta(\varepsilon)) \forall n, m \geq n_0(\delta) : \|x_n - x_m\| \leq \varepsilon$$

$$\implies (x_n)_{n\in\mathbb{N}} \quad \text{is a CS and} \quad x_n \to x_*$$

$$\implies 1 = \|x_n\| \to \|x_*\| \quad \text{and} \quad x'(x_n) \to x'(x_*) = \|x'\| \implies \text{Existence}$$

Uniqueness follows since every such sequence is converging. □

Example 3.3.4 (i) All Hilbert spaces are uniformly convex. Parallelogram identity:

$$\|\tfrac{1}{2}x + y)\|^2 + \underbrace{\|\tfrac{1}{2}(x-y)\|^2}_{\varepsilon^2/4} = \underbrace{\tfrac{1}{2}(\|x\|^2 + \|y\|^2)}_{\leq 1}$$

$$\|\tfrac{1}{2}(x+y)\|^2 < 1 - \tfrac{\varepsilon^2}{4}$$

$$\|\tfrac{1}{2}(x+y)\| < (1 - \tfrac{\varepsilon^2}{4})^{1/2} = 1 - \delta(\varepsilon)$$

(ii) For $p \in]1, \infty[$ all L^p-space are uniformly convex. We consider $L^p(\Omega)$ for $\Omega \subset \mathbb{R}^d$ and $p \geq 2$.

$$\textit{Clarkson's estimate} \quad f, g \in L^p(\Omega) : \|f+g\|_p^p + \|f-g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$$

In $\mathbb{R}^2$ we have

$$|a+b|^p + |a-b|^p \leq 2^{p-1}(|a|^p + |b|^p) \qquad a, b \in \mathbb{R}$$

For $f, g \in L^p(\Omega)$ choose $a = f(x)$ and $b = g(x)$ and integrate the estimate over $x \in \Omega$. The result is Clarkson's estimate. Using Clarkson's estimate and $f, g \in B_1^{L^p(\Omega)}(0)$ we obtain

$$\|f+g\|_p^p \leq 2^{p-1}(1+1) - \|f-g\|_p^p \leq 2^p - \varepsilon^p \implies \|\tfrac{1}{2}(f+g)\|_p \leq (1 - (\tfrac{\varepsilon}{2})^p)^{\frac{1}{p}} = 1 - \delta(\varepsilon)$$

Corollary 3.3.5 (of Thm. Milman and Pattis)
If $p \in]1, \infty[$, then $L^p(\Omega)$ and ℓ_p are reflexive. Moreover, setting $p' = \frac{p}{p-1}$ (the dual exponent to p) we have a norm-preserving isomorphism between $L^p(\Omega)'$ and $L^{p'}(\Omega)$ given by

$$R_p : \begin{cases} L^{p'}(\Omega) \to L^p(\Omega)' \\ g \mapsto \begin{cases} L^p(\Omega) \to \mathbb{R} \\ u \mapsto \int_\Omega gu\, dx \end{cases} \end{cases}$$

Proof: $p \geq 2$: $L^p(\Omega)$ is reflexive (by Clarkson and Milman-Pattis) and hence $L^p(\Omega)'$ is also reflexive. The mapping R as defined above is a bounded linear operator from $L^{p'}(\Omega)$ into $L^p(\Omega)'$. By Hölder's estimate we have $\|\mathbb{R}\|_{\mathcal{L}(L^{p'}(\Omega), L^p(\Omega'))} \leq 1$.

Claim: R is norm-preserving, i.e. $\forall g \in L^{p'}(\Omega) : \|R_p(g)\|_{L^p(\Omega)'} = \|g\|$. By Hölder we know "$\leq$" We have to find a $u \in L^p(\Omega) \setminus \{0\}$ such that Hölder's estimate gives an equality $\int_\Omega gu\, dx = \|u\|_{L^p}\|g\|_{L^{p'}}$.

$$u = g^{\frac{p'}{p'}} = |g|^{\frac{p'}{p}=1} g \qquad \frac{p'}{p} = \frac{p}{(p-1)p} = \frac{1}{p-1}$$

$$\int_\Omega ug\ x = \int |g|^2 |g|^{\frac{p'}{p}-1} = \int |g|^{\frac{p}{p-1}}\, dx = \|g\|^{p'}_{L^{p'}}$$

$$\|u\|_{L^p} = \int_\Omega \left| g^{\frac{p'}{\cancel{p}}} \right|^{\cancel{p}} = \|g\|^{\frac{p'}{p}}_{L^{p'}}$$

We have

$$\int_\Omega ug\, dx = \|g\|^{p'}_{L^p} = \|g\|^{\frac{p'}{p}}_{L^p} = \|g\|^1_{L^{p'}} = \|u\|_{L^p}\|g\|_{L^{p'}}$$

Hence $\|R_p(g)\|_{L^p} \geq \frac{|R_p(g)(u)|}{\|u\|_{L^p}} = \|g\|_{L^{p'}}$ Since the embedding $R_p L^{p'}(\Omega) \subset L^p(\Omega)'$ is norm-preserving, the image is a closed subspace of the reflexive dual space $L^p(\Omega)'$. Hence $R_p L^{p'}(\Omega)$ is reflexive (Thm ii)) and thus $L^{p'}(\Omega)$ is reflexive (Thm i).

We now show that R_p is surjective. Assume $R_p L^{p'}(\Omega) \subsetneq L^p(\Omega)'$. By Hahn-Banach there exists $x'' \in L^p(\Omega)''$ such that $x'' \neq 0$ and $x''(R_p(g)) ==$ for all $g \in L^{p'}(\Omega)$. Since $L^p(\Omega)$ is reflexive, there exists $f \in L^p(\Omega)$ such that

$$x'' = J_{L^p(\Omega)}(f) \quad \text{and} \quad f \neq 0$$

We have

$$0 = x''(R_p(g)) = R_p(g)(f) = \int_\Omega gf\ x$$

Now choose $g = f^{\frac{p}{p'}}$ to obtain a contradiction.

Hence R_p is a norm isomorphism between $L^{p'}(\Omega)$ and $L^p(\Omega)'$. Hence R_p gives a representation of $L^p(\Omega)'$. (Representation theorem of Riesz and Fischer.) Hence, the result is proven. □

Corollary 3.3.6
For $p \in]1, \infty[$ and $k \in \mathbb{N}$ the spaces $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$ are reflexive.

Proof: by Exercise

3.4 The weak and the weak* topology

Again X is Banach space and X' its dual. $(X, \|\cdot\|)$ with the norm topology $\mathcal{T}_{\|\cdot\|}$.

Definition 3.4.1

(i) *The* weak topology $\mathcal{T}_W$ *on X is defined via the system of seminorms given by*

$$\{|x'(\cdot)| \mid x' \in X'\}$$

A sequence $(x_k)_{k\in\mathbb{N}}$ converges weakly *to x_* in X if for all $x' \in X'$ we have $x'(x_k) \to x'(x_*)$.*

(ii) *On the dual space X' we can define the weak topology $\mathcal{T}_W$ as above using $\{|x''(\cdot)| \mid x'' \in X''\}$. Moreover, we define* weak* topology T_{W^*} *to x'_* if $\forall \hat{x} \in X : x'_k(\hat{x}) \to x'_*(\hat{x})$*

"$x_k \to x$" *means norm convergence*

"$x_k \rightharpoonup x$" *means weak convergence*

"$x_k \overset{*}{\rightharpoonup} x$" *means weak* convergence.*

The two seminorms systems are Hausdorff (use Hahn-Banach).

Example 3.4.2 (i) Norm convergence implies convergence in X'. Weak convergence implies weak* convergence

(ii) In a reflexive dual space X' weak convergence and weak* convergence are the same (same topologies)

(iii) $\ell_2 = \{(a_n)_{n\in\mathbb{N}} \mid \sum a_n^2 < \infty\}$. $e_n = (0,\dots,0,1,0,\dots)$. $\|e_n - e_m\| = \sqrt{2}$ for $n \neq m$ That implies bounded sequences without accumulation point in the norm topology. Claim: $e_n \rightharpoonup 0$ By Riesz representation we know that all $x' \in \ell_2'$ have the form $\langle \cdot, b\rangle$. To show: $\langle e_n, b\rangle \to 0$ for all $b \in \ell_2$.

$$\langle e_n, b\rangle = b_n \quad \text{if} \quad b = (b_m)_{m\in\mathbb{N}} \sum_{m=1}^{\infty} b_m^2 < \infty$$

for all $b \in \ell_2$.

(iv) $X = c_0$, $X' = \ell_1$, $X'' = \ell_\infty$ $x_k' = (0,\dots,0,1,0,\dots) \in \ell_1$. $\|x_k' - x_m'\|_{\ell_1} = 2$ if $k \neq m$. Claim: $x_l' \overset{*}{\rightharpoonup} 0'$. To show: $\forall a \in c_0 : x_k'(a) \to 0'(a) = 0$. We have $x_k'(a) = \sum \delta_{km} a_m = a_k \to 0$.

$$x'(a) = \sum a_m b_m \quad \text{if} \quad (b_m) \in \ell_1 \cong c_0'$$

for $a \in c_0$ fixed and $k \to \infty$.

Claim: x_k' does not converge in $\mathcal{T}_W$. We have to find $x'' \in X'' \cong \ell_\infty$ such that $x''(x_k')$ does not converge. We have to find $(L_m)_{m\in\mathbb{N}}$ such that $\sum_{m=1}^{\infty} L_m \delta_{km} = L_k$ does not converge. Take $L_m = (-1)^m$ then $(L_m) \in \ell_\infty$ and no convergence.

(v) In the spaces $L^p(\Omega)$ with $p \in [1,\infty[$ we have $f_n \rightharpoonup g$ if $\forall h \in L^{p'}(\Omega) : \int_\Omega f_n h \, dx \to \int_\Omega g h \, dx$. (use R_p) For a sequence in $L^\infty(\Omega)$ we have $f_n \overset{*}{\rightharpoonup} g$ iff $\forall h \in L^{-1}(\Omega) : \int_\Omega f_n h \, dx \to \int_\Omega g h \, dx$.

Proposition 3.4.3
Closed convex sets in a Banach space are closed, i.e. $x_n \in A$ with A closed and convex and $x_n \rightharpoonup x_*$, then $x_* \in A$.

Example 3.4.4 $X = \ell_2$ and $M = \{a \mid \|a\|_{\ell_2} = 1\}$, then M is closed and bounded, but not weakly closed. $x_n = e_n = (0,\dots,0,1,0,\dots) \in M$. $e_n \rightharpoonup 0 = x_* \notin M$. The norm $\|\cdot\|_{\ell_2}$ is not weakly continuous.

Proof: Assume $x_* \notin A$. Then, we define $B = \{x_*\}$, that is compact and $B \cap A = \emptyset$. Now, the geometric form of the Hahn-Banach principle (separation theorem) provides $\alpha \in \mathbb{R}$ and $x' \in X' : x'(x_*) > \alpha \geq x'(a)$ for all $a \in A$. However $x_n \rightharpoonup x_*$ implies

$$\overbrace{x'(\underbrace{x_n}_{\in A})}^{\leq \alpha} \to \overbrace{x'(x_*)}^{>\alpha} \quad \text{This is a contradiction.}$$

□

Lemma 3.4.5
Let X be Banach space.

(i) $x_n \rightharpoonup x_* \implies \sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$

(ii) $x_n' \overset{*}{\rightharpoonup} x_*' \implies \sup_{n\in\mathbb{N}} \|x_n'\|_{X'} < \infty$

Moreover $\|x_*\| \leq \liminf_{k\to a} \|x_k\|$ *and* $\|x_*'\|_{X'} \leq \liminf_{n\to\infty} \|x_n'\|_{X'}$. *The norms are lower semi-continuous with respect to weak (weak*) convergence.*

Proof:

Step 1: Consider case (ii): $x_n' \overset{*}{\rightharpoonup} x_*'$ means pointwise convergence of the operator family $\{x_n' \mid n \in \mathbb{N}\}$. The uniform boundedness principle of Banach-Steinhaus gives $\sup_{n\in\mathbb{N}} \|x_n'\| < \infty$.

Step 2: Consider (i). We use the embedding into the bidual X''. $x_n'' = J_X(x_n) \in X''$, $x_n \rightharpoonup x_* \iff x_n'' \overset{*}{\rightharpoonup} x_*''$. Hence $\sup \|x_n''\|_{X'} < \infty$ by Step 1. Since J_X is norm-preserving, i.e. $\|x_n\|_X = \|x_n''\|_{X''}$, we have the result.

Step 3: $x_n \rightharpoonup x_* \iff \forall x' \in X' : x'(x_n) \to x'(x_*)$.

$$|x'(x_*)| = \lim_{n\to\infty} \underbrace{|x'(x_n)|}_{\leq \|x'\| \|x_n\|} \leq \liminf_{n\to\infty} \|x'\|_{X'} \|x_n\|_X$$

$$|x'(x_*)| \leq \|x'\|_{X'} \liminf_{n\to\infty} \|x_n\|$$

By Hahn-Banach, choose x with $\|x'\| = 1$ and $x'(x_*) = \|x_*\|$, then the result follows. The other case ($x' \overset{*}{\rightharpoonup} x'_*$) works similarly. □

Theorem 3.4.6
An equivalent and often more useful criterion for weak convergence

$$x_n \rightharpoonup x_* \iff \begin{cases} (a) & \sup_{n\in\mathbb{N}} \|x_n\| < \infty \\ (b) & \exists D \subset X' \textit{ with } \operatorname{span} D \textit{ is dense in } X' \colon \forall y' \in D : y'(x_n) \to y'(x^*) \end{cases}$$

Proof: "$\Rightarrow$" (a) by Lemma and (b) holds with $D = X'$.

"$\Leftarrow$" Follows from the corollary of the Banach-Steinhaus theorem. Boundedness and covergence on a dense set imply pointwise convergence on all of x'. □

Example 3.4.7 (Applications) (i) $X = L^2(]0,\pi[)$, $u_n \rightharpoonup u_*$ in $L^2(]0,\pi[)$

$$\iff \sup \int_0^\pi (u_n)^2\,dx < \infty \quad \text{and} \quad \forall k \in \mathbb{N} : \int_0^\pi u_n(x)\sin(kx)\,dx \xrightarrow{n\to\infty} \int_0^\pi u_*(x)\sin(kx)\,dx$$

Here $D = \{\sin(k\cdot) \mid k \in \mathbb{N}\}$ is a countable set such that the span is dense.

(ii) $X = H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^d$ domain, $v_k \rightharpoonup v_*$ in $H_0^1(\Omega)$

$$\iff \sup_{k\in\mathbb{N}} \int_\Omega v_k^2 + |\nabla v_k|^2\,dx < \infty \quad \text{and} \quad \forall \varphi \in C_c^\infty(\Omega) : \int_\Omega v_k\varphi + \nabla v_k \cdot \nabla\varphi\,dx \xrightarrow{k\to\infty} \int v_*\varphi + \nabla v_* \cdot \nabla\varphi\,dx$$

Hence, it suffices to show $\sup_{k\in\mathbb{N}} \|v_k\|_{H_0^1} < \infty$ and $v_k \to v_*$ in $L^1(\Omega)$.

Theorem 3.4.8 (Banach selection principle, 1932)
If X is a separable Banach space, then every bounded sequence in X' has a weak-convergent subsequence.*

Remark (i) Abstract version of Helly's selection principle ("one can select a converging subsequence")

(ii) Generalisation of Bolzano-Weierstraß

(iii) Existence result: There exists an accumulation point.

(iv) This is a result on sequential weak* compactness.

Theorem 3.4.9 (Weak* compactness Borbaki, 1938, Alaogu 1940)
(More general but less useful version:)

For every Banach space X the closed unit ball in X' (i.e. $B' = \{x' \in X' \mid \|x'\|_{X'} \leq\}$) is weak (covering-)compact. (In general (e.g. if $X = L^\infty(\Omega)$), B' is not sequentially weak*-compact.)*

Proof: (of Banach'S selection principle) Choose a countable set $D = \{z_m \mid m \in \mathbb{N}\}$ that is dense in X (use that X is separable). Let $(x'_n)_{n\in\mathbb{N}}$ be any bounded sequence in X'. Using Cantor's diagonalisation principle we find a subsequence $(x'_{n_l})_{l\in\mathbb{N}}$ such that $\forall m \in \mathbb{N} : x'_{n_l}(z_m) \to \alpha_m$. Now, the corollary of the Banach-Steinhaus principle (using that $(x'_{n_l})_{l\in\mathbb{N}}$ is bounded in X') shows $x'_{n_l} \overset{*}{\rightharpoonup} x_*$. □

We want to get rid of separability and to make direct statements on the space X. For this reflexivity will be helpful. We need the following lemma:

Lemma 3.4.10
If the Banach space X has a separable dual, then X is separable. The opposite is wrong. $\ell_1 \cong c_0'$ are both separable, but $\ell_\infty = \ell_1' = c_0''$ is not separable.

Proof: Let $D = \{x'_k \mid k \in \mathbb{N}\}$ be a dense set in X'. For each $k \in \mathbb{N}$ choose $z_k \in X : |x'_k(z_k)| \geq \frac{1}{2}\|x'_k\|$ and $\|z_k\| = 1$. Let $Z = \overline{\operatorname{span}\{z_k \mid k \in \mathbb{N}\}}$. Then, Z is closed, separable subspace of X. (since $\{qz_k \mid q \in \mathbb{Q}, k \in \mathbb{N}\}$ is dense and countable).

Claim: $Z = X$, then we are finished. Assume $Z \subsetneq X$ to produce a contradiction. By Hahn-Banach (3.1.3) we find $y' \in X'$ such that $y'|_Z = 0$ and $y' \neq 0$. We use that y' can be approximated by $x'_{k_l} \in D$.

$$\begin{aligned}
\|y' - x'_k\|_{X'} &\geq |(y' - x'_k) \underbrace{(z_k)}_{\|\cdot\|=1}| = |\underbrace{y'(z_k)}_{=0} - x'_k(z_k)| \geq \tfrac{1}{2}\|x_k\| \\
&\underset{\text{estimate}}{\overset{\text{triangle}}{\geq}} \tfrac{1}{2}(\|y'\| - \|y' - x'_k\|) \\
\implies \|y'\|_{X'} &\leq 3\|y' - x'_k\|_{X'}
\end{aligned}$$

By density we obtain $\|y'\|_{X'} = 0$ which contradicts $y' \neq 0$. Hence the claim is true. $\square$

Theorem 3.4.11 (Kakutani 1939, Shmulyan 140, Eberlein 1947)
A Banach space is reflexive if and only if any bounded sequence has a weakly convergent subsequence.

Useful direction: reflexivity $\implies$ selection principle. "$\Longleftarrow$" is proven in classical textbooks on Linear Functional Analysis.

Proof: (Only "$\Rightarrow$") We start with a reflexive space X and a bounded sequence $(x_k)_{k\in\mathbb{N}}$. Define

$$Z := \overline{\operatorname{span}\{x_k \mid k \in \mathbb{N}\}}^X$$

then Z is a separable, closed subspace of X since $\{\sum_{k=1}^{\infty} qx_k \mid q_k \in \mathbb{Q}, k \in \mathbb{N}\}$ is countable and dense in Z. $(Z, \|\cdot\|_X)$ is again a reflexive Banach space and that implies $Z'' = J_Z(Z)$ is a separable reflexive Banach space. The previous lemma gives us that $'$ is separable as predual of Z''). Consider $z''_k = J_Z(x_k) \in Z''$ as bounded sequence in Z''. Hence $(z''_k)_{k\in\mathbb{N}}$ has a weak* convergent subsequence $z''_{k_l} \overset{*}{\rightharpoonup} z''_*$. Define $\tilde{z} := J_Z^{-1}(z''_*)$, then $x_{k_l} \rightharpoonup \tilde{z}$ in Z (use $z'(x_k) = J_Z(x_k)(z') = z''_k(z')$).

So $\forall z' \in Z' : z'(x_{k_l}) \to z'(\tilde{z})$. We need $\forall x' \in X' : x'(x_{k_l}) \to x'(\tilde{z})$. This holds since for any $x' \in X'$ the restriction we have $x'|_Z \in Z'$. Hence

$$x'(x_{k_l}) = x'|_Z(x_{z_{k_l}}) \to x'|_Z(\tilde{z}) = x'(\tilde{z})$$

$\square$

Example 3.4.12

(i) X Hilbert space, $L^p(\Omega)$, ℓ_p, $W^{k,p}(\Omega)$ (with $p \in]1, \infty[$) implies selection principle. Every bounded sequence has a weakly converging subsequence.

(ii) Nothing similar for $C^0([0,1])$.

(iii) For $L^\infty(\Omega)$ we can apply Banach's selection principle, since $L^\infty(\Omega) = L^1(\Omega)'$ and $L^1(\Omega)$ is separable. For every bounded sequence in $L^\infty(\Omega)$ there exists a weakly* convergent subsequence.

$$\begin{aligned}
\Omega = \mathbb{R} \quad f_n(x) &= \frac{\cos(\frac{n}{x^2})^2}{1+x^2} + \frac{e^x}{1+e^x}\arctan(n+x) \\
\|f_n\|_{L^\infty} &= \leq \tfrac{1}{1} + 1\tfrac{\pi}{2}
\end{aligned}$$

Claim: $\exists g \in L^\infty(\mathbb{R}) : f_n \overset{*}{\rightharpoonup} g$. This means

$$\forall h \in L^1(\mathbb{R}) : \int_{\mathbb{R}} h(x)f_n(x)\,dx \to \int_{\mathbb{R}} h(x)g(x)\,dx$$

(iv) $X = \ell_1 = \{(a_k)_{k\in\mathbb{N}} \mid \sum_{k=1}^{\infty}|a_k| < \infty\}$, $x^{(n)} = (0, \ldots, 0, 1, 0, \ldots)$. $(x^{(n)})_{n\in\mathbb{N}}$ does not have a weakly convergent subsequence but it is bounded. Take any subsequence $(x^{(n_l)})_{l\in\mathbb{N}}$ and choose $z' \in \ell_1' = \ell_\infty$ as follows:

$$z' = (z_k)_{k\in\mathbb{N}} \qquad z_k = \begin{cases} (-1)^l & \text{if } k = n_l \\ 0 & \text{else} \end{cases}$$

Hence $z'(x^{(n_l)}) = (-1)^l$ does not converge for $l \to \infty$.

However, Banach's selection principle applies, since $X = \ell_1 = c_0'$ and c_0 is separable. In fact, $x^{(n)} \overset{*}{\rightharpoonup} 0$.

Application of these compactness results in the calculus of variations: X Banach space. $I : X \to \mathbb{R}$ nonlinear functional. Aim: find minimizers of I.

Weierstraß' extremal principle

A continuous function I on a compact set M attains its minimum, i.e. $\exists x_0 \in M \forall x \in M : I(x_0) \leq I(x)$

Theorem 3.4.13 (Generalised Weierstraß principle)
X reflexive Banach space, $I : X \to \mathbb{R}$ with

(i) I is coercive, *i.e. $I(x_n) \to \infty$ if $\|x_n\| \to \infty$,*

(ii) I is weakly *lower semi-continuous, i.e. $x_k \rightharpoonup x_* \implies I(x_*) \leq \lim_{k\to\infty} I(x_k)$.*

Then, there exists a minimizer *x_0.*

Proof: We define $\alpha = \inf\{I(x) \mid x \in X\} \in \mathbb{R} \cup \{-\infty\}$. We have to find x_0 such that $I(x_0) \leq \alpha$. Choose $x_k \in X$ such that $I(x_k) \to \alpha$.

$$\begin{aligned} &\implies \sup\{\|x_k\| \mid k \in \mathbb{N}\} < \infty \quad \text{by (i)} \\ &\implies \exists \text{conv. subsequence } (x_{n_l})_{l\in\mathbb{N}} : x_{n_l} \rightharpoonup x_0 \in X \end{aligned}$$

Using weak* convergence (i.e. ii) we obtain $I(x_0) \leq \liminf_{l\to\infty} I(x_{n_l}) = \alpha$. □

Corollary 3.4.14
Let X be a reflexive Banach space, $I : X \to \mathbb{R}$ a norm-continuous, convex, and coercive. Then, I has a minimizer.

Proof: We have to show that norm-continuity and convexity imply weak lower semi-continuity (ii). For $\beta > \alpha = \text{int}\{I(x) \mid x \in X\}$ consider the sublevel

$$S_\beta = \{x \in X \mid I(x) \leq \beta\}$$

I continuous implies S_β is (norm-)closed. I convex implies that S_β is a convex subset of X. Now take any sequence x_k with $x_k \rightharpoonup x_*$. Define $\gamma = \liminf_{k\to\infty} I(x_k)$. There exists a subsequence (x_{k_l}) such that $x_{k_l} \rightharpoonup x_*$ and $I(x_{k_l}) \to \gamma$. For $\varepsilon > 0 \exists l_0 = l_0(\varepsilon) \forall l \geq l_0 : I(x_{k_l}) \leq \gamma + \varepsilon$. Hence $x_{k_l} \in S_{\gamma+\varepsilon}$. Lemma on weak closeness of convex sets implies $x_* \in S_{\gamma+\varepsilon} \implies I(x_*) \leq \gamma + \varepsilon \implies I(x_*) \leq \gamma = \liminf_{k\to\infty} I(x_k)$. □

4 Spectral theory

"Infinite dimensional version of eigenvalues and eigenvectors".

- Characterise the spectrum of a linear operator (= set of all eigenvalues)
- eigenvectors or generalisations
- invertibility of equations $(T - \lambda I)x = g$.

Throughout the whole section, X will be a Banach space and $T \in \mathcal{L}(X) = \mathcal{L}(X, X)$.

4.1 Spectrum of an operator

- $N(T) = \text{kernel}(T) = \{x \in X \mid Tx = 0\}$
- $R(T) = \text{range}(T) = \{Tx \mid x \in X\}$

Definition 4.1.1
Resolvent set $\varrho(T) = \{\lambda \in \mathbb{C} \mid N(T - \lambda I) = \{0\}, \quad R(T - \lambda I) = X\}$

Resolvent equation $(T - \lambda I)x = b$

Resolvent operator $(T - \lambda I)^{-1} : X \to X$ *for* $\lambda \in \varrho(T)$.

Spectrum of T $\sigma(T) = \mathbb{C} \setminus \varrho(T)$

Pointwise spectrum $\sigma_p(T) = \{\lambda \in \mathbb{C} \mid N(T - \lambda I) \neq \{0\}\}$, *i.e. there exists an eigenvector for* λ, $\exists x \neq 0 : Tx = \lambda x$

Continuous spectrum $\sigma_c(T) = \{\lambda \in \mathbb{C} \mid N(T - \lambda I) = \{0\}, R(T - \lambda I) \subsetneq X, \overline{R(T - \lambda I)} = X\}$

Residual spectrum $\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T))$.

Spectral radius $r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\}$

$T - \lambda I \in \mathcal{L}(X)$. $\lambda \in \varrho(T) \iff T - \lambda I$ *bijective. Hence (by Inverse Mapping Principle) the resolvent operator* $(T - \lambda I)^{-1} \in \mathcal{L}(X)$.

Example 4.1.2 (i) $X = \mathbb{C}^n$ or $\mathbb{R}^n$ finite dimensions. Then $T \in \mathbb{K}^{n\times n}$ matrix (see matrix theory).

$$\sigma_p(T) = \text{ set of eigenvalues (at most } n)$$
$$\lambda \notin \sigma_p(T) \implies (T - \lambda I)x = g \quad \text{uniquely solvable} \quad \implies \lambda \in \varrho(T)$$

Only point spectrum $\sigma(T) = \sigma_p(T) \implies \sigma_c(T) = \sigma_r(T) = \emptyset$.

(ii) $X = \ell_2$. $Ta = (a_1, \frac{1}{2}a_2, \frac{1}{3}a_3, \dots, \frac{1}{k}a_k, \dots)$, $T = \operatorname{diag}(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{k}, \dots)$. The eigenpairs $(\frac{1}{k}, e_k)$, $\lambda = \frac{1}{k}$, $Te_k = \lambda e_k$. $\frac{1}{k} \in \sigma_p(T)$ for all $k \in \mathbb{N}$. *Claims:* $0 \in \sigma_c(T)$. Namely:

$$Ta = 0 \implies a = 0 \implies N(T - 0I) = N(T) = \{0\}$$

$R(T - 0I) = R(T) \subsetneq X$ since $Ta = (1, \frac{1}{2}, \frac{1}{3}, \dots) = a_* \in \ell_2$ has no solution $a \in \ell_2$ since $Ta = a_*$ gives $a_k = 1$. However $R(T)$ is dense in X since $D = \{(a_k)_{k\in\mathbb{N}} \mid \exists N \in \mathbb{N} \forall k \geq N, a_k = 0\}$ is dense in ℓ_2 and $D \subset R(T)$. *Claim:* $\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$ with $\sigma_p(T) = \{\frac{1}{k} \mid k \in \mathbb{N}\}$, $\sigma_c(T) = \{0\}$.

$\forall \lambda \notin \sigma(T)$ we show that $(T - \lambda I)a = b$ is uniquely solvable.

$$\forall k \in \mathbb{N} : (\tfrac{1}{k} - \lambda)a_k = b_k \implies a_k = \frac{1}{\frac{1}{k} - \lambda} b_k$$

For $\lambda \notin \sigma(T)$ we have $(\frac{1}{\frac{1}{k}-\lambda}) \in \ell_\infty$. *Resolvent operator* $a = (T - \lambda I)^{-1}b$, $\|(T - \lambda I)^{-1}\|_{\mathcal{L}(\ell_2,\ell_2)} = \sup \left|\frac{1}{\frac{1}{k}-\lambda}\right| < \infty$.

(iii) $X = \mathcal{C}^0([-1, 1])$, $(Tu)(x) = u(-x)$ the reflection with $T^2 = I$.

$$Tu = \lambda u \implies u = T(Tu) = T(\lambda u) = \lambda Tu = \lambda^2 u \implies \sigma_p(T) \subset \{+1, -1\}$$

Claim: $\sigma(T) = \{+1, -1\} = \sigma_p(T)$. u even implies $Tu = u \implies$ infinte dim. eigenspace. u odd (e.g. $e(x) = x^{2m+1}$) implies $Tu = -u$ infinite dim. eigenspace again. Hence $\sigma_p(T) \supset \{+1, -1\}$. Now we calculate the resolvent:

$$\begin{aligned}
\text{Resolvent equation} \quad Tu - \lambda u &= b \in \mathcal{C}^0([-1, 1]) \\
\forall x \quad u(x) - \lambda u(-x) &= b(-x) \\
(u(-x) - \lambda u(x) &= b(x)) \\
u_\lambda &= (T - \lambda I)^{-1} b \\
u_\lambda &= \frac{1}{\lambda^2 - 1}(\lambda b(x) - b(-x)) \in \mathcal{C}^0([-1, 1]) \quad \text{for} \quad \lambda \notin \{-1, 1\} \\
\implies (T - \lambda I)^{-1} &\in \mathcal{L}(X) \quad \text{for} \quad \lambda \notin \{-1, 1\}
\end{aligned}$$

(iv) $X = L^2(]-1, 1[)$, $T \in \mathcal{L}(X)$ with $(Tu)(x) = xu(x)$. Note $\|Tu\|_{L^2} = (\int_{-1}^1 (xu(x))^2\, dx)^{\frac{1}{2}} \leq \|u\|_{L^2}$

$$\text{Eigenpairs:} \quad Tu = \lambda u \implies xu(x) = \lambda u(x) \quad \text{a.e. in }]-1, 1[$$
$$\underbrace{(x - \lambda)}_{\neq 0 \text{ a.e.}} u(x) = 0 \implies u = 0 \quad \text{a.e.} \quad \implies \sigma_p(T) = \emptyset$$

$$\text{Resolvent equation} \quad xu(x) - \lambda u(x) = b(x) \quad b \in L^2(]-1, 1[)$$
$$u(x) = \frac{1}{x - \lambda} b(x) \quad \text{a.e.}$$

Take $b \equiv 1$ and $\lambda = 0 \implies u(x) = \frac{1}{x}$ a.e. which is a contradiction to $u \in L^2(\Omega)$. Assume $\lambda \in \mathbb{C} \setminus [-1, 1]$. Then $x \mapsto \frac{1}{x-\lambda} \in \mathcal{C}^0([-1, 1], \mathbb{C})$

$$\left\|\frac{1}{x - \lambda}\right\|_{L^\infty} = \frac{1}{\operatorname{dist}(\lambda, [-1, 1])}$$

Then $\lambda \in \varrho(T)$ and $\|(T - \lambda I)^{-1}\|_{L^2 \to L^2} \leq \|\frac{1}{x-\lambda}\|_{L^\infty}$ (Hölder estimate). *Claim:* $\sigma(T) = \sigma_c(T) = [-1, 1]$. It remains to show $R(T - \lambda I)$ is dense in $X = L^2(]-1, 1[)$ for all $\lambda \in [-1, 1]$.

$$(T - \lambda I)u(x) = (x - \lambda)u(x)$$

For $\lambda \in [-1,1]$ we define $\chi_n \in L^\infty(]-1,1[)$ such that

$$\chi_n(x) = \begin{cases} 0 & \text{for } |x-\lambda| < \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

For all $b \in L^2(\Omega)$ we have $\|b - \chi_n b\|_{L^2} \to 0$

$$\int \underbrace{|\overbrace{b - \chi_n b}^{\to 0}|^2}_{|\cdot| \le |b|^2 \text{ majorant}} \; dx \to 0 \quad \text{by Lebesgue dominance}$$

$$\text{Moreover} \quad \forall b \in L^2(\Omega) : \chi_n b \in R(T - \lambda I)$$

$$\text{since for} \quad u_n(x) = \underbrace{\frac{\chi_n(x)}{x-\lambda}}_{\in L^\infty(]-1,1[)} \; b(x)$$

$$\left\| \frac{\chi_n}{x-\lambda} \right\|_{L^\infty} \le n$$

we have $(T-\lambda I)u_n = \chi_n b$. Hence $(T-\lambda)u_n$ approximates b in $L^2(\Omega)$

Proposition 4.1.3
Let X be a complex Banach space and $T \in \mathcal{L}(X)$. Then, $\varrho(T)$ is an open subset of $\mathbb{C}$ and the resolvent operator $(T-\lambda I)^{-1}$ depends analytically on $\lambda \in \varrho(T) \subset \mathbb{C}$. Moreover, the spectral radius $r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\} \le \|T\|$. Hence, $\sigma(T)$ is a compact subset of $\mathbb{C}$.

Proof: $\mu \in \varrho(T)$, $|\lambda - \mu|$ small implies that there exists a power series expansion. For $\mu \in \varrho(T)$ we know $(T-\mu I)^{-1} \in \mathcal{L}(X)$.

$$(T-\lambda I)u = b \iff (T-\mu I)u + (\mu - \lambda)u = b$$
$$(T-\lambda I)u = b \iff Iu + (\mu-\lambda)(I-\mu I)^{-1}u = (T-\mu I)^{-1}b$$

Neumann series (Ex. 19)

$$u - Au = \tilde{b} \quad \text{and} \quad \|A\| < 1$$
$$\implies u = \sum_{j=0}^{\infty} A^j \tilde{b} \quad \text{unique solution}$$
$$\left\| \sum_{j=0}^{\infty} A^j \right\| \le \sum \|A\|^j \le \frac{1}{1-\|A\|}$$

For $|\lambda - \mu| \le \frac{1}{\|(T-\mu I)-1\|}$ we have $\|A\| < 1$ and hence $u = \sum_{j=0}^{\infty}((\lambda-\mu)(T-\mu I)^{-1})^j (T-\mu)^{-1} b$ is the unique solution, i.e. $\lambda \in \varrho(T)$. Estimate

$$\|(T-\lambda I)^{-1}\| \le \frac{\|(T-\mu I)^{-1}\|}{1|\lambda-\mu|\|(T-\mu I)^{-1}\|} \implies \quad \text{Analyticity of } \lambda \mapsto (T-\lambda I)^{-1} \text{ and openness of } \varrho(T).$$

Still to be proved: $\sigma(T) \subset \overline{B}^{\mathbb{C}}_{\|T\|}(0)$.

$$(T-\lambda I)u = b \quad \text{where} \quad |\lambda| > \|T\|$$
$$u - \underbrace{\tfrac{1}{\lambda}Tu}_{=:A} = -\tfrac{1}{\lambda}b \quad \text{with} \quad \|A\| = \tfrac{1}{|\lambda|}\|T\| < 1$$

Neumann series implies $(T-\lambda I)^{-1} \in \mathcal{L}(X) \implies \lambda \in \varrho(T)$. □

4.2 Compact operators

Definition 4.2.1
X, Y Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is called compact *, if for every bounded sequence in X its image has a (norm) convergent subsequence in Y. We shortly write $\mathcal{K}(X,Y)$ for the set of compact operators.*

Remark (i) $T_1, T_2 \in \mathcal{K}(X,Y), \alpha_1, \alpha_2 \in \mathbb{K} \implies \alpha_1 T_1 + \alpha_2 T_2 \in \mathcal{K}(X,Y)$. Take $(x_k)_{k\in\mathbb{N}}$ bounded in X and choose a subsequence $T_1 x_{k_l} \xrightarrow{Y} y_1, T_2 x_{k_l} \xrightarrow{Y} y_2$ for $l \to \infty$. Then $(\alpha_1 T_1 + \alpha_2 T_2)x_{k_l} \to \alpha_1 y_1 + \alpha_2 y_2$.

(ii) Composition of "continuous" and "compact" gives "compact". Let $T \in \mathcal{K}(X,Y)$, $R \in \mathcal{L}(X_1, X)$, $S \in \mathcal{L}(Y, Y_2)$. Then $TR \in \mathcal{K}(X_1, Y)$ and $ST \in \mathcal{K}(X, Y_2)$. For TR : Take a bounded sequence $(\tilde{x}_k)_{k\in\mathbb{N}}$ in $X \implies (R\tilde{x}_k)_{k\in\mathbb{N}}$ is bounded in X. By compactness choose subsequence $\tilde{x}_{k_l}$, $(TR)\tilde{x}_{k_l} = T(R\tilde{x}_{k_l}) \to y$. Case ST is similarly.

Special case $X = Y$. $\mathcal{K}(X) \subset \mathcal{L}(X)$. $\mathcal{L}(X)$ is a Banach algebra = Banach space with continuous multiplication.

$$T_1, T_2 \in \mathcal{L}(X) \implies T_1 T_2 \in \mathcal{L}(X) \quad \|T_1 T_2\| \le \|T_1\|\|T_2\| \implies \quad \text{vector space and multiplication group}$$

$\mathcal{K}(X)$ is an ideal in $\mathcal{L}(X)$, hence $\forall K \in \mathcal{K}(X) \forall T \in \mathcal{L}(X) : TK, KT \in \mathcal{K}(X)$.

(iii) $\mathcal{K}(X,Y)$ is a closed subspace of the Banach space $(\mathcal{L}(X,Y), \|\cdot\|_{X\to Y})$. $T_n \in \mathcal{K}(X,Y)$ and $T_n \to T$ in $\mathcal{L}(X,Y)$. To show: $T \in \mathcal{K}(X,Y)$. Take any bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X. To construct: subsequence $(x_{k_l})_{l\in\mathbb{N}} : Tx_{k_l} \to y$. $(T_n x_k)_{k\in\mathbb{N}}$ is countable number of sequences. Using Cantor's diagonal sequence we find the sequence $(x_{k_l})_{l\in\mathbb{N}}$ such for all $n \in \mathbb{N}$: $T_n x_{k_l} \xrightarrow{Y} y_1$ for $l \to \infty$. We estimate

$$\|Tx_{k_{l_1}} - Tx_{k_{l_2}}\|_Y \le \|Tx_{k_{l_1}} - T_n x_{k_{l_1}}\| + \underbrace{\|T_n x_{k_{l_1}} - T_n x_{k_{l_2}}\|}_{=:*} + \|T_n x_{k_{l_2}} - Tx_{k_{l_2}}\|$$

Using $\sup\{\|x_k\| \mid k \in \mathbb{N}\} \le C_X < \infty$ we obtain

$$\|Tx_{k_{l_1}} - Tx_{k_{l_2}}\| \le \|T - T_n\| C_X + \|T_n x_{k_{l_1}} - T_n x_{k_{l_2}}\| + \|T - T_n\| C_X$$

For given $\varepsilon > 0$ find $n_0(\varepsilon)$ such that $\|T - T_n\|_{X\to Y} C_X < \frac{\varepsilon}{3}$ and then choose $l_0 = l_0(\varepsilon, u_0(\varepsilon))$ such that $\forall l_1, l_2 \ge l_0 : * \le \frac{\varepsilon}{3} \implies (Tx_{k_l})_{l\in\mathbb{N}}$ is a Cauchy sequence.

(iv) $T \in \mathcal{L}(X,Y)$ is compact $\iff T' \in \mathcal{L}(Y', X')$ is compact (for proof see textbooks).

(v) If $\dim X < \infty$ or $\dim Y < \infty$ then $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$. For ex. if $\dim X < \infty$ then $I_X : X \to X$ is compact. For $T \in \mathcal{L}(X,Y)$ we have $T = T \circ I_X$. In general, $\mathcal{K}(X,Y) \subsetneq \mathcal{L}(X,Y)$. If $X = Y$ and $\dim X = \infty$, then $\mathcal{K}(X) \subseteq \mathcal{L}(X)$, namely $I_X \notin \mathcal{K}(X)$. This will be proved below. The tool to do so is the so-called "almost orthogonal vector".

Proposition 4.2.2
Let X be a Banach space and $Y \subsetneq X$ is a closed true subspace. Then for all $\theta \in]0,1[$ there exists $x \in X \setminus Y$ with $\|x\| = 1$ and $\operatorname{dist}(x, Y) \ge \theta$. (In Hilbert spaces orthogonal projections yield $\theta = 1$).

Proof: Choose $\tilde{x} \in X \setminus Y$, since Y is a true subspace. Then $\operatorname{dist}(\tilde{x}, Y) = \inf\{\|\tilde{x} - y\| \mid y \in Y\} > 0$ (otherwise $\exists y_k \in Y : \|\tilde{x} - y_k\| \to 0 \implies \tilde{x} = \lim_{k\to\infty} y_k$ and Y closed $\implies \tilde{x} \in Y$). Moreover, for our $\theta \in]0,1[$ there exists $\tilde{y} \in Y : \|\tilde{x} - \tilde{y}\| \le \frac{\theta}{\operatorname{dist}(\tilde{x}, Y)}$. Now define $x = \frac{\tilde{x} - \tilde{y}}{\|\tilde{x} - \tilde{y}\|}$. For all $y \in Y$ we have the estimate

$$\|x - y\| = \left\| \frac{\tilde{x} - \tilde{y}}{\|\tilde{x} - \tilde{y}\|} - y \right\| = \frac{1}{\|\tilde{x} - \tilde{y}\|} \|\tilde{x} - \underbrace{(\tilde{y} + \|\tilde{x} - \tilde{y}\| y)}_{\in Y}\| \ge \frac{\operatorname{dist}(\tilde{x}, Y)}{\|\tilde{x} - \tilde{y}\|} \ge \theta$$

□

Corollary 4.2.3
If X is an infinite dimensional Banach space, then $I_X \notin \mathcal{K}(X)$.

Proof: We construct a bounded sequence $(x_k)_{k\in\mathbb{N}}$ such that $\|x_k - x_n\| \ge \frac{1}{2}$ for $k \ne n$. Then, there does not exists a convergent subsequence. We do this by induction: Choose x_1 arbitrary with $\|x_1\| = 1$, $Y_1 = \operatorname{span}\{x_1\}$. Next choose x_2 with $\|x_2\| = 1$ and $\operatorname{dist}(x_2, Y_1) \ge \frac{1}{2} \implies \|x_1 - x_2\| \ge \frac{1}{2}$. By induction x_{k+1} with $\|x_{k+1}\| = 1$ and $\operatorname{dist}(x_{k+1}, Y_k) \ge \frac{1}{2}$ where $Y_k = \operatorname{span}\{x_1, \ldots, x_k\} \implies \|x_{k+1} - x_n\| \ge \frac{1}{2}$ for $n = 1, \ldots, k$. We have an infinite sequence since for all k we have $X \setminus Y_k \ne \emptyset$. □

Example 4.2.4 (of compact operators) (i) $T \in \mathbb{K}^{m\times n} = \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m) = \mathcal{K}(\mathbb{K}^n, \mathbb{K}^m)$

(ii) *Nuclear operators* $y_k \in Y, x'_k \in X', c_k \in \mathbb{K}$.where $\|y_k\|_Y \le 1$, $\|x'_k\|_{X'} \le 1$ and $\sum_{k=1}^{\infty} |c_k| < \infty$. Now define $T \in \mathcal{L}(X,Y)$ via

$$Tx = \sum_{k=1}^{\infty} c_k x'_k(x) y_k$$

Claim: $T \in \mathcal{K}(X,Y)$. $T_N x = \sum_{k=1}^N c_k x_k'(x) y_k \in \mathcal{K}(X,Y)$, $\operatorname{rank}(T_N) \leq N$ since $R(T_N) = \operatorname{span}\{y_1, \dots, y_N\} = Y_N$ finite dimensional closed subspace of Y. Claim: $T_N \to T$ in $\mathcal{L}(X,Y)$

$$\begin{aligned}
\|T_N - T\|_{X\to Y} &= \sup_{\|x\|\leq 1} \|T_N x - Tx\|_Y \\
&= \sup_{\|x\|\leq 1} \left\| \sum_{k=N+1}^{\infty} c_k x_k'(x) y_k \right\|_Y \\
&\leq \sup_{\|x\|\leq 1} \sum_{N+1}^{\infty} |c_k| |x_k'(x)| \|y_k\|_Y \\
&\leq \sum_{k=N+1}^{\infty} |c_k| \to 0 \quad \text{for} \quad N \to \infty
\end{aligned}$$

(iii) Ex 45 $\Omega \subset \mathbb{R}^n$ domain. For $K \in L^2(\Omega \times \Omega)$ define $T_K : L^2(\Omega) \to L^2(\Omega)$ with $f \mapsto T_K f$ and

$$(T_K f)(x) = \int_\Omega K(x,y) f(y)\, dy \quad \text{integral operator}$$

Integral operators occur as inverse of differential operators. For example $\Omega = B_1(0) \subset \mathbb{R}^3$ and $K(x,y) = \frac{1-|x|^2}{4\pi|x-y|}$, $u = Tf$ is the unique solution of the elliptic equation $-\nabla u = f$ in Ω and $u = 0$ on $\partial\Omega$.

(iv) Compact embeddings. Often one space X is embedded into another space via the natural embedding. $X = W_0^{1,p}(]a,b[) \hookrightarrow Y = \mathcal{C}^{\alpha_p}([a,b])$ (see Exam Ex. 4) with $\alpha_p = 1 - \frac{1}{p}$. $\mathcal{C}^\alpha$ Hölder continuous functions.

$$\|u\|_{\mathcal{C}^\alpha} = \|u\|_{L^\infty} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{u(x)-u(y)}{|x-y|^\alpha} \qquad u \in \mathcal{C}_c^\infty(]a,b[) \subset W_0^{1,p}(]a,b[)$$

$$|u(x) - u(y)| = \int_x^y 1|u'(\xi)|\,dy \leq \underbrace{|\int_x^y 1\, d\xi|^{1-\frac{1}{p}}}_{|x-y|^{1-\frac{1}{p}}} \|u'\|_{L^p}$$

$\|u\|_{\mathcal{C}^\alpha} \leq C\|u\|_{W^{1,p}}$. The embedding $E_1 : W_0^{1,p}(]a,b[) \to \mathcal{C}^\alpha([a,b])$, $u \mapsto u$ is continuous. Now consider $E_2 : \mathcal{C}^\alpha([a,b]) \to \mathcal{C}^0([a,b])$, $u \mapsto u$. $E_2 \in \mathcal{K}(\mathcal{C}^\alpha([a,b]), \mathcal{C}^0([a,b]))$. Arcela-Azcoli theorem: bounded in $\mathcal{C}^\alpha$ implies equicontinuity $\implies$ allows us to extract a uniformly convergent subsequence.

$$W_0^{1,p}(]a,b[) \overset{\text{continuous}}{\hookrightarrow} \mathcal{C}^\alpha \overset{\text{compact}}{\hookrightarrow} \mathcal{C}^0 \overset{\text{continuous}}{\hookrightarrow} L^q(\Omega)$$

Theorem 4.2.5 (Rellich's compactness theorem)
The embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, if Ω is a bounded domain in $\mathbb{R}^d$.

Proof: Ω bounded implies $\exists R : \Omega \subset]-R,R[^d =: \Omega_R$. We embed H_0^1 into $H_0^1(\Omega_R)$ via extension by 0. Note

$$\mathcal{C}_c^\infty(\Omega) \subset \mathcal{C}_c^\infty(\Omega_R)$$

Now take a bounded sequence $(u_k)_{k\in\mathbb{N}}$ in $H_0^1(\Omega)$. Then the extensions $(\tilde{u}_k)_{k\in\mathbb{N}} \subset H_0^1(\Omega_R)$ are still bounded since $\|u_k\|_{H_0^1(\Omega)} = \|\tilde{u}_k\|_{H_0^1(\Omega)}$. Hence, it suffices to proof the result for Ω_R rather than Ω.

We consider a complete ONS, namely $\varphi_n(x) = \prod_{j=1}^d \sin(\frac{\pi n j}{R}(x_j + R))$ with $n \in \mathbb{N}^d$ and $\varphi_n(x) \in H_0^1(\Omega_R)$. $\langle \varphi_n, \varphi_m \rangle_{H^1} = cn\delta_{n,m}$. Our functions $\tilde{u}_k$ can be uniquely written in this cONS

$$\begin{aligned}
&\|\tilde{u}_k\|_{H^1}^2 \overset{\text{Parseval}}{=} \sum |\mu_k^n|^2 \|\varphi_n\|_{H_1}^2 \\
&\|\varphi_n\|_{L^2}^2 = c_0 \qquad \|\nabla\varphi_n\|_{L^2}^2 = (\tfrac{\pi}{R})^2 |n|^2 c_0
\end{aligned}$$

Using $\|\tilde{u}_k\|_{H^1} \leq C$ we find

$$\sum_{n\in\mathbb{N}^d} |\mu_k^n|^2 (1 + |n|^2) \leq \hat{C} \quad \text{for all } k$$

By Cantor's diagonal argument, we find a subsequence such that $\forall n \in \mathbb{N}^d : \mu_{k_l}^n \to \mu_*^n$ as $l \to \infty$.

Claim: $\|\tilde{u}_{k_l} - u_*\|_{L^2} \to 0$ where $u_* = \sum_{n\in\mathbb{N}^d} \mu_*^n \varphi_n$. *Proof:*

$$\|\tilde{u}_{k_l} - u_*\|_{L^2}^2 \overset{\text{Parseval}}{=} \sum_{n\in\mathbb{N}^d} |\mu_{k_l}^n - \mu_*^n|^2 c_0(1+0)$$
$$= \underbrace{\sum_{|n|\le M} (\mu_{k_l}^n - \mu_*^n|^2 c_0}_{\xrightarrow{l\to\infty} 0} + \frac{1}{1+M^2} \underbrace{\sum_{|n|>M} |\mu_k^n - \mu_*^n|^2 c_0(1+|n|^2)}_{\le c_0 \hat{C} < \frac{\varepsilon}{2}}$$

Hence the claim holds and the theorem as well. □

The most important application is solving the elliptic problem $-\operatorname{div}(A(x)\nabla u) = f$ in Ω and $u = 0$ on $\partial\Omega$. Here $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{\text{Sym}})$ and $A(x)\xi\cdot\xi \ge \alpha|\xi|^2$ (ellipticity). The Lax-Milgram Lemma shows that there is a bounded linear operator $\tilde{T} : L^2(\Omega) \to H_0^1(\Omega)$, $f \mapsto u$. $\tilde{T} \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$ solution operator, $E : H_0^1(\Omega) \to L^2(\Omega)$ embedding. We define $T = E\tilde{T} \in \mathcal{L}(L^2(\Omega))$. If Ω is a bounded domain, Rellich's compactness result yields $T \in \mathcal{K}(L^2(\Omega))$.

One is interested in the eigenfunctions of the Laplace operator

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Applying T as solution operator $\iff$ $u = \lambda T u$.

4.3 Spectral and Fredholm theory for compact operators

X Banach space, $T \in \mathcal{K}(X)$, compact operator, resolvent equation $(T - \lambda I)x = f$.

Proposition 4.3.1
$T \in \mathcal{K}(X)$ and $\lambda \ne 0$. Then $N(T-\lambda I) = \{0\}$ implies $(T-\lambda I)^{-1} \in \mathcal{L}(X)$, i.e. $\lambda \in \varrho(T)$. Hence $\lambda \ne 0 \implies$ either $\lambda \in \varrho(T)$ or $\lambda \in \sigma_p(T)$. For $T \in \mathcal{K}(X)$ we have $\sigma(T) \subset \sigma_p(T) \cup \{0\}$.

Proof: To simplify notation, let $L = T - \lambda I$.

Step 1 Claim: $\exists c > 0 \forall x \in X : \|Lx\| \ge c\|x\|$. Proof by contradiction. If no such $c > 0$ exists, there exists $(x_n)_{n\in\mathbb{N}} : \|x_n\| = 1$ and $\|Lx_n\| \to 0$. $T \in \mathcal{K}(X) \implies \exists$ subseq. $(x_{n_k}) \exists z : Tx_{n_k} \xrightarrow{k\to\infty} z$. We have the identity

$$x = \tfrac{1}{\lambda}(Tx - Lx) \quad \text{for } \lambda \ne 0 \text{ we obtain} \quad x_{n_k} = \tfrac{1}{\lambda}(\underbrace{Tx_{n_k}}_{\to z} - \underbrace{Lx_{n_k}}_{\to 0}) \to \tfrac{1}{\lambda}(z - 0) = \tfrac{1}{\lambda} z$$

Hence $Lz = L\lim_{k\to\infty}(\lambda x_{n_k}) = \lambda \lim_{k\to\infty} Lx_{n_k} = \lambda 0 = 0$. Using $N(L) = \{0\}$ we conclude $z = 0$. However

$$\|x\| = 1 \implies \frac{1}{|\lambda|}\|z\| = \lim \|x_{n_k}\| = 1 \qquad \text{Contradiction!}$$

Thus, $\exists c > 0$ as desired.

Step 2 Range$(L) = R(L)$ is closed. (Argument as in the proof of the Lax-Milgram Lemma (2.4.5)). Take a CS (y_n) in X with $y_n \in R(L)$. To show $y_* = \lim y_n \in R(L)$. Step 1 shows $L : X \to R(L)$ is injective. By the inverse mapping principle (2.2.7) $L^{-1} \in \mathcal{L}(R(L), X)$. Define $x_n = L^{-1}y_n$, i.e. $Lx_n = y_n$, then

$$\|x_n - x_m\| \overset{\text{Step 1}}{\le} \tfrac{1}{c}\|Lx_n - Lx_m\| = \tfrac{1}{c}\|y_n - y_m\| \to 0 \implies x_n \to x_* \quad \text{in } X$$

Since $L \in \mathcal{L}(X)$ we have $Lx_* = \lim Lx_n = \lim y_n = y_* \implies y_* \in R(L)$

Step 3 $R(L) = X$ (then $\lambda \in \varrho(T)$). Proof by contradiction. Assume $R(L) =: X_1 \subsetneq X$. Define $X_0 = X$ and $X_{n+1} = LX_n = R(L^n)$. We obtain a decreasing sequence of subspaces $X = X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$. Applying Step 2 to $L_{|X_{k-1}}$ we obtain that all X_{k+1} are closed subspaces of X_k, since

$$L_k = L_{|X_k} = T_{|X_k} - \lambda I_{|X_k} \quad \text{with } T_{|X_k} \in \mathcal{K}(X_k)$$

Choose $x_0 \in X \setminus X_1 = X \setminus R(L)$. Then $x_n = L^n x_0$ satisfies $x_n \in X_n \setminus X_{n+1}$. Proof by contradiction:

$$x_n \in X_{n+1} \implies L^n x_0 \in X_{n+1} \underset{\text{def}}{\overset{\text{by}}{=}} R(L^{n+1}) \implies \exists y \in X : L^n x_0 = L^{n+1} y \implies L^n x_0 - Ly) = 0$$

Using $N(L) = \{0\}$ we find $x_0 = Ly$. This contradicts $x_0 \notin X_1 = R(L)$. Hence, $X_{n+1} \subsetneq X_n$ for all $n \in \mathbb{N}_0$. Using the result on the existence of almost-orthogonal vectors, we find

$$y_n \in X_n \setminus X_{n+1} : \|y_n\| = 1 \quad \text{and} \quad \operatorname{dist}(y_n, X_{n+1}) \geq \tfrac{1}{2}$$

$(y_n)_{n\in\mathbb{N}}$ is a bounded sequence. For $m > n$, we have

$$Ty_n - Ty_m = Ly_n - Ly_m + \lambda(y_n - y_m) = \lambda\big(y_n - \underbrace{[y_m - \tfrac{1}{\lambda} L(y_n - y_m)]}_{\in X_{n+1}}\big)$$

By the choice of y_n, we have

$$\|Ty_n - Ty_m\| \geq |\lambda| \operatorname{dist}(y_n, X_{n+1}) \geq \frac{|\lambda|}{2} > 0 \quad (\text{for } \lambda \neq 0)$$

Hence, there does not exists a convergent subsequence. Since T is compact and $(y_n)_{n\in\mathbb{N}}$ bounded, this is a contradiction.

Hence $R(L) = X$. □

Theorem 4.3.2 (Spectral theorem for compact operators of Riesz-Schauder)
X Banach space, $T \in \mathcal{K}(X)$. Then

(i) if $\dim(X) < \infty$, then $\sigma(T) = \sigma_p(T)$, if $\dim(X) = \infty$, then $\sigma(T) = \sigma_p(T) \cup \{0\}$

(ii) $\sigma_p(T)$ consists of a countable number of $\{\lambda_k \mid k = 1, \dots N\}$ where $N \in \mathbb{N} \cup \{\infty\}$ such that $|\lambda_k| \to 0$ for $k \to \infty$ if $N = \infty$.

It turns out, that the resolvent operators $T - \lambda I \in \mathcal{L}(X)$ for $\lambda \neq 0$ have special properties, which are formulated in terms of the Fredholm theory:

Definition 4.3.3
Let X, Y be Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is called a Fredholm operator, *if*

(i) $R(T)$ is closed in Y

(ii) $\dim N(T) < \infty$

(iii) $\operatorname{codim}(R(T)) < \infty$.

Then, the Fredholm index *is defined via $\operatorname{ind}(T) = \dim N(T) - \operatorname{codim}(R(T))$. Here*

$$\operatorname{codim}_Y(Z) := \inf\{\dim W \mid Y = W + Z\} \in \mathbb{N}_0$$

Remark The Fredholm index is preserved under suitably small perturbations. Moreover, adding arbitrary large compact perturbations, the index does not change.

Example 4.3.4 (i) In linear algebra $T \in \mathcal{L}(X, Y)$ with $\dim X < \infty$ and $\dim Y < \infty$.

$$\begin{aligned} \dim X &= \dim N(T) + \dim(R(T)) \\ \operatorname{codim}(R(T)) &= \dim Y - \dim(R(T)) \\ \operatorname{ind}(T) &= \dim(N(T)) - \operatorname{codim}(R(T)) = \dim X - \cancel{\dim R(T)} - (\dim Y - \cancel{\dim(R(T))}) = \dim X - \dim Y \end{aligned}$$

If $X = Y$ we obtain for $T \in \mathcal{L}(X)$ that $\operatorname{ind}(T) = 0$.

(ii) $X = Y = \ell_2$ with shift operator. $T(a_k)_{k\in\mathbb{N}} = (a_2, a_3, a_4, \dots)$ (Ex 46). $N(T) = \operatorname{span}\{e_1\}$, $R(T) = X$, $\dim N(T) = 1$, $\operatorname{codim} R(T) = 0$, $\operatorname{ind}(T) = +1$. $A \in \mathbb{K}^{n\times n}$, $Ax = f$.

Theorem 4.3.5 (Fredholm alternative)
Let X, Y be Banach spaces. If $T \in \mathcal{L}(X, Y)$ is a Fredholm operator with $\operatorname{ind}(T) = 0$, then

- *either $N(T) = \{0\}$ and $Tx = f$ is uniquely solvable for all $f \in Y$ or*

- $N(T) \supsetneq \{0\}$, *then* $Tx = f$ *is solvable if and only if* $y'(f) = 0$ *for all* $y' \in N(T')$, *where* $\dim N(T') = \dim N(T)$.

  Proof: Using $\operatorname{ind}(T) = 0$, the case $N(T) = \{0\}$ gives $\operatorname{codim}(R(T)) = 0$, i.e. $(R(T)) = Y$. This gives solvability. From $N(T) = \{0\}$ we have uniqueness. If $\dim N(T) \geq 1$, then $\operatorname{codim}(R(T)) \geq 1$ i.e. $R(T) \subsetneq Y$. Hence, $Tx = f$ is not always solvable. For any $y' \in Y'$ we have

  $$y' \in N(T') \iff T'y' = 0 \iff \forall x \in X : (T'y')(x) = 0 \overset{\text{Def of } T'}{\iff} \forall x \in X : y'(Tx) = 0 \overset{\text{Def } R(T)}{\iff} \forall f \in R(T) : y'(f) = 0$$

  The relation $\dim N(T') = \dim N(T)$ follows easily since $\dim N(T) = \operatorname{codim} R(T)$. (Use Hahn-Banach 3.1.3 to extend functionals.) $\square$

Theorem 4.3.6 (Riesz-Schauder)
Let X be a Banach space, $T \in \mathcal{K}(X)$, $\lambda \neq 0$. *Then* $T - \lambda I$ *is a Fredholm operator with* $\operatorname{ind}(T - \lambda I) = 0$. *This is equivalent to*

$$\operatorname{ind}(I - K) = 0$$

We can always reduce to the case $\lambda = 1$*:* $\lambda \neq 0$, $\operatorname{ind}(\lambda L) = \operatorname{ind}(L)$, $K \in \mathcal{K}(X) \implies \frac{1}{\lambda}K \in \mathcal{K}(X)$. $T - \lambda I = -\lambda(I - \frac{1}{\lambda}T)$.

4.4 Selfadjoint operators

Definition 4.4.1
For a Banach space X the adjoint of an operator $T \in \mathcal{L}(X)$ *satisfies* $T' \in \mathcal{L}(X')$. *For a Hilbert space H we can use the Riesz' isomorphism (2.4.2)* $J_R : H' \to H$ *to define a* Hilbert-space adjoint *in* $\mathcal{L}(H)$. *More general, for two Hilbert spaces* H_1 *and* H_2 *and* $T \in \mathcal{L}(H_1, H_2)$, *we define the Hilbert-space adjoint*

$$T^* \in \mathcal{L}(H_2, H_1) \quad \textit{for} \quad T \in \mathcal{L}(H_1, H_2) \quad \textit{via} \quad \forall x \in H_1, \forall y \in H_2 : \quad \langle \underbrace{Tx}_{\in H_2}, \underbrace{y}_{H_2} \rangle_{H_2} = \langle x, T^*y \rangle_{H_1}$$

Using Riesz isomorphism we have

$$T^* = J^R_{H_2} T' (J^R_{H_1})^{-1}$$

Definition 4.4.2
An operator $T \in \mathcal{L}(H)$ *is called* self-adjoint *(symmetric) if* $T^* = T$*; i.e.* $\forall x, y \in H : \langle Tx, y \rangle = \langle x, Ty \rangle$. *Obviously, for self-adjoint T we always have* $\langle Tx, x \rangle \in \mathbb{R}$ *for all* x.

Example 4.4.3 (i) $H = \mathbb{C}^n$, $T \in \mathbb{C}^{n \times n}$, $T^* = T^H = \overline{T^\top}$

(ii) $T : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$, $u \mapsto gu$ with $g \in L^\infty(\Omega, \mathbb{C})$.

$$\begin{aligned} \langle Tu, v \rangle_{L^2} &= \int_\Omega (Tu)\overline{v}\, dx = \int_\Omega gu\overline{v}\, dx \\ &= \int_\Omega u \cdot (g\overline{v})\, dx = \int_\Omega u\overline{(T^*v)}\, dx = \langle u, T^*v \rangle \\ \implies T^*v &= \overline{g}v \end{aligned}$$

Hence $T^* = T$ iff $g(x) \in \mathbb{R}$ a.e.

(iii) Integral operator with kernel $K(x, y) \in L^2(\Omega \times \Omega)$

$$\begin{aligned} (Tu)(x) &= \int_\Omega K(x, y)u(y)\, dy \\ \langle Tu, v \rangle &= \int_\Omega (Tu)(x)\overline{v(x)}\, dx = \int_\Omega \int_\Omega K(x, y)u(y)\overline{v(x)}\, dx \\ &= \int_{y \in \Omega} u(y) \int_\Omega K(x, y)\overline{v(x)}\, dy\, dx = \langle u, T^*v \rangle \\ \implies (T^*v)(z) &= \int_\Omega \overline{K(y(z))}v(x)\, dx \end{aligned}$$

Lemma 4.4.4
H Hilbert space, $T = T^* \in \mathcal{L}(H)$. *Then*

(i) $\sigma(T) \subset [-\|T\|, \|T\|] \subset \mathbb{R} \subset \mathbb{C}$

(ii) Define $\gamma = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}$, then $\|T\| = \gamma$.

Proof: ad (i) : For $\lambda \in \mathbb{C}$ we consider $(T - \lambda I)x = f$. We take the scalar product $\langle \cdot, x\rangle$

$$\langle (T-\lambda I)x, x\rangle = \langle x, f\rangle \qquad \underbrace{\langle Tx, x\rangle}_{\in\mathbb{R}} - \lambda \underbrace{\|x\|^2}_{\in\mathbb{R}} = \langle x, f\rangle$$

Taking the imaginary part yields

$$\begin{aligned} -(\operatorname{Im}\lambda)\|x\|^2 &= \operatorname{Im}\langle f, x\rangle \\ \lambda \notin \mathbb{R} \implies \|x\|^{\cancel{2}} &= \frac{-1}{\operatorname{Im}\lambda}\langle f, x\rangle \underset{\text{estim.}}{\overset{\text{CS}}{\leq}} \frac{1}{|\operatorname{Im}\lambda|}\cancel{\|x\|}\|f\| \\ \implies \|(T-\lambda I)^{-1}\| &\leq \frac{1}{|\operatorname{Im}\lambda|} \quad \text{for} \quad \lambda \notin \mathbb{R} \end{aligned}$$

Using $\sigma(T) \subset \overline{B^{\mathbb{C}}_{\|T\|}(0)}$ we can conclude $\sigma(T) \subset [-\|T\|, \|T\|]$.

ad (ii) It is sufficient to study the quadratic form $u \mapsto \langle Tu, u\rangle$.

$$\gamma \leq \|T\| : \qquad |\langle Tu, u\rangle| \leq \|Tu\| \underbrace{\|u\|}_{=1} \underset{\text{operator norm}}{\overset{\text{def of}}{\leq}} \|T\|\|u\| = \|T\|$$

$$\begin{aligned} \gamma \geq \|T\| : \qquad \|T\| &\overset{\text{Op.}}{=} \sup\{\|Tu\| \mid \|u\| \leq 1\} \\ &= \sup\{\langle Tu, v\rangle \mid \|u\|, \|v\| \leq 1\} \end{aligned}$$

Using $T^* = T$ we have

$$\begin{aligned} \langle Tu, v\rangle &= \tfrac{1}{4}(\langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle) \\ &\leq \tfrac{1}{4}(\gamma\|u+v\|^2 + \gamma\|u-v\|^2) \\ &\underset{\text{identity}}{\overset{\text{parallelogram}}{=}} \tfrac{\gamma}{4}(2\|u\|^2 + 2\|v\|^2) \\ &= \tfrac{\gamma}{4}(2\cdot 1 + 2\cdot 1) = \gamma \end{aligned}$$

From $\langle Tu, v\rangle \leq \gamma$ for all $\|u\|, \|v\| \leq 1$ we obtain $\|T\| \leq \gamma$ as desired. □

For compact operators, we have even better properties, namely the existence of eigenpairs:

Proposition 4.4.5
H Hilbert space, $T = T^* \in \mathcal{K}(H)$ and $\gamma = \sup_{\|u\|=1}|\langle Tu, u\rangle|$. Then, there exists an eigenpair (λ_*, u_*) such that $Tu_* = \lambda u_*$, $\|u_*\| = 1$, $|\lambda_*| = \gamma$.

The "sup" is a "max", attained for the maximal eigenvalue.

Proof: If $\gamma = 0$, then $T = 0$, then for every $u \in H$ we have the eigenpair $(0, u)$. Now assume $\gamma > 0$. Choose $(u_n)_{n\in\mathbb{N}}$ with $\langle Tu_n, u_n\rangle \to \lambda_* \in \{-\gamma, \gamma\}$. Since T is compact,

$$\exists \text{ subseq. } (u_{n_k})_{k\in\mathbb{N}} \exists z \in H : Tu_{n_k} \to z \implies u_{n_k} \rightharpoonup u_*$$

We want to show $Tu_* = z$. We show $Tu_{n_k} \rightharpoonup Tu_*$. In fact

$$\forall x \in H : \langle Tu_{n_k}, x\rangle = \langle u_{n_k}, T^*x\rangle \to \langle u_*, T^*x\rangle = \langle Tu_*, x\rangle$$

Using $Tu_{n_k} \to z$ we conclude $u_* = z$ and hence $Tu_{n_k} \to Tu_* = z$.

$$\lambda_* \xleftarrow{\text{by construction}} \langle Tu_{n_k}, u_{n_k}\rangle \to \langle Tu_*, u_*\rangle$$

Hence, $\lambda_* = \langle Tu_*, u_*\rangle$ attainment. We only know $\|x_*\| \leq \liminf_{k\to\infty}\|u_{n_k}\| = 1$. Let $\rho = \|u_*\|$, then $\rho \in [0, 1]$. We have $\rho > 0$ since $\lambda_* \neq 0$. Take $\tilde{u} = \frac{1}{\rho}$ then $\|\tilde{u}\| = 1$ and

$$|\langle T\tilde{u}, \tilde{u}\rangle| = \frac{1}{\rho^2}|\underbrace{\langle Tu_*, u_*\rangle}_{=\lambda}| = \frac{\gamma}{\rho^2} \leq \gamma \implies \rho^2 \geq 1 \implies \rho = 1$$

In fact, u_* is an eigenvector. W.l.o.g. $\lambda_* = +\gamma > 0$. Then $\forall u \in H : \langle Tu, u\rangle \leq \lambda_* \|u\|^2$. Now insert $u = u_* + \varepsilon v$ (see Calculus of Variations.)

$$
\begin{aligned}
&\langle (T-\lambda_*)(u_* + \varepsilon v, u_* + \varepsilon v\rangle \leq 0 \quad \text{for all} \quad \varepsilon, v \\
&\varepsilon^0 \underbrace{\langle (T-\lambda_*)u_*, u_*\rangle}_{=0} + \varepsilon^1 \langle (T-\lambda_*)u_*, v\rangle + \varepsilon^2 \langle (T-\lambda_*)v, v\rangle \leq 0 \quad \text{for all} \quad \varepsilon \\
&\implies \langle (T-\lambda_*)u_*, v\rangle = 0 \quad \text{for all} \quad v \\
&\implies (T-\lambda_*) = 0 \implies u_* \quad \text{eigenvector}
\end{aligned}
$$

□

Theorem 4.4.6 (Spectral theorem for compact self-adjoint operators)
H Hilbert space, $T = T^* \in \mathcal{K}(X)$. Then, there exists an at most countable ONS $\{e_j \mid j = 1, \dots, N\}$ with $N \in \mathbb{N} \cup \{\infty\}$ and a sequence $(\lambda_j)_{j=1,\dots,N}$ such that

(i) $\lambda_j \in \mathbb{R}$ and $\lambda \to 0$ if $N = \infty$

(ii) $Tu = \sum_{j=1}^{N} \lambda_j \langle u, e_j\rangle e_j$ (spectral representation by eigenvectors)

(iii) If $N(T) = \{0\}$, then the ONS is complete.

Lemma 4.4.7
$T = T^* \in \mathcal{L}(H)$. If $Y \subset H$ with $TY \subset Y$ (invariant subspace), then its orthogonal complement

$$Y^\perp = \{x \in H \mid \forall y \in Y : \langle x, y\rangle = 0\}$$

is also invariant, i.e. $TY^\perp \subset Y^\perp$. Hence for $Y \oplus Y^\perp$, we have $T = T_Y \oplus T_{Y^\perp}$. (Proof is straight forward).

Proof: (of spectral theorem) Inductive construction of eigenvectors:

$$\gamma_0 = \|T\|, \quad H_0 = H$$

The proposition yields an eigenvector (λ_1, e_1) with $|\lambda_1| = \gamma_0$, $\|e_1\| = 1$, $Te_1 = \lambda_1 e_1$.

$$Y_1 = \operatorname{span}\{e_1\} \quad \text{and} \quad H_1 = Y_1^\perp \quad \text{in} \quad H_0$$

Hence $T_1 = T_{|H_1|} \in \mathcal{K}(H_1)$ and $T_1 = T_1^*$. Then

$$\gamma_1 = \sup\{|\langle T_1 u, u\rangle| \,|\, \|u\| = 1, u \in H_1\}$$

As long as $\gamma_k > 0$ we can construct a new eigenpair (λ_{k+1}, e_{k+1}) and proceed.

If $N < \infty$, then the remaining operator T_{N+1} on $\operatorname{span}\{e_1, \dots, e_N\}^\perp$ is trivial and the spectral representation in (ii) holds. If $N = \infty$, we have to prove $\lambda_j \to 0$. Assume $|\lambda_j| \geq c > 0$ to produce a contradiction via compactness. $(e_j)_{j\in\mathbb{N}}$ is a bounded sequence and $Te_j = \lambda_j e_j$ by construction.

$$\|Te_n - Te_m\| = \|\lambda_n e_n - \lambda e_m\| \underset{\text{Pythagoras}}{\overset{n\neq m}{=}} \lambda_n^2 + \lambda_m^2 \geq 2c^2$$

Hence there is no convergent subsequence which is a contradiction to $T \in \mathcal{K}(X)$. This proves (i).

To prove (iii), we define $Y = \overline{\operatorname{span}\{e_j \mid j = 1, \dots, N\}}$. We have to show $Y^\perp = \{0\}$. $TY \subset Y$ by construction which implies (by the lemma) $TY^\perp \subset Y^\perp$. By construction $\gamma(Y^\perp) = 0$, since every $u \in H$ with $|\langle Tu, u\rangle| > 0$ would have been included into Y.

$$\implies T_{|Y^\perp} = 0 \implies Y^\perp \subset N(T) \overset{\text{assumption}}{=} \{0\}$$

□

Main application:

Completeness of the eigenfunctions of the Laplace operator.

$$\begin{aligned} -\Delta u = \lambda u \quad \text{in} \quad \Omega \qquad u \neq 0 \\ u = \quad \text{on} \quad \partial\Omega \end{aligned}$$

(λ, u) is called an eigenpair of the (Dirichlet-)Laplace operator.

We define the solution operator $u = \tilde{T} f$ for the Laplace operator via the weak form

$$\forall v \in H_0^1(\Omega) : \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

By Lax-Milgram Lemma (2.4.5) $\tilde{T} \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$. By Rellich's theorem (4.2.5) the embedding $E : H_0^1(\Omega) \to L^2(\Omega)$, $u \mapsto u$ is compact (for bounded Ω). Hence $T = E\tilde{T} \in \mathcal{K}(L^2(\Omega))$. *Claim:* $T = T^*$ and $N(T) = \{0\}$. Self-adjointness $\langle Tf, g \rangle = \langle f, Tg \rangle$. Let $u = Tf$ and $v = Tg$, then

$$\begin{aligned} \langle Tf, g \rangle = \langle u, g \rangle & \underset{u \text{ test function}}{\overset{\text{Lax-Mil.}}{=}} \int_\Omega \underbrace{\nabla v \cdot \nabla u}_{\text{symmetric}} \, dx \\ & \underset{\text{Lax-Mil.}}{\overset{v \text{ test function}}{=}} \int_\Omega f v \, x = \langle f, v \rangle = \langle f, Tg \rangle \end{aligned}$$

Kernel $N(T)$ $Tf = 0 \implies 0 \, \langle Tf, f \rangle = \int |\nabla u|^2 \, dx \implies u = 0$ by the boundary conditions.

Inverse operators for (elliptic) partial differential operators are typically compact (if Ω bounded). Hence, the spectral theory applies.

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