QUADRIC RANK LOCI ON MODULI OF CURVES AND $K3$ SURFACES

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ABSTRACT. Assuming that $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ is a morphism of vector bundles on a variety X, we compute the class of the locus in X where $\mathrm{Ker}(\phi)$ contains a quadric of prescribed rank. Our formulas have many applications to moduli theory: (i) we find a simple proof of Borcherds' result that the Hodge class on the moduli space of polarized $K3$ surfaces of fixed genus is of Noether-Lefschetz type, (ii) we construct an explicit canonical divisor on the Hurwitz space parametrizing degree k covers of $\mathbf{P}^1$ from curves of genus $2k-1$, (iii) we provide a closed formula for the Petri divisor on $\overline{\mathcal{M}}_g$ of canonical curves which lie on a rank 3 quadric and (iv) construct myriads of effective divisors of small slope on $\overline{\mathcal{M}}_g$.

1. INTRODUCTION

Let X be an algebraic variety and let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles on X having ranks e and f respectively. Assume we are given a morphism of vector bundles

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}.$$

For a positive integer $r \leq e$, we define the subvariety of X consisting of points for which $\mathrm{Ker}(\phi)$ contains a quadric of corank at least r, that is,

$$\overline{\Sigma}_{e,f}^r(\phi) := \Big\{ x \in X : \exists\, 0 \neq q \in \mathrm{Ker}(\phi(x)) \ \ \text{with} \ \ \mathrm{rk}(q) \leq e - r \Big\}.$$

Since the codimension of the variety of symmetric $e \times e$-matrices of corank r is equal to $\binom{r+1}{2}$, it follows that the expected codimension of the locus $\overline{\Sigma}_{e,f}^r(\phi)$ is equal to $\binom{r+1}{2} - \binom{e+1}{2} + f + 1$. A main goal of this paper is to explicitly determine the cohomology class of this locus in terms of the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. This is achieved for every e, f and r in Theorem 4.4, using a localized Atiyah-Bott type formula. Of particular importance in moduli theory is the case when this locus is expected to be a divisor, in which case our general formula has a very simple form:

Theorem 1.1. *We fix integers $0 \leq r \leq e$ and set $f := \binom{e+1}{2} - \binom{r+1}{2}$. Suppose $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ is a morphism of vector bundles over X. The class of the virtual divisor $\overline{\Sigma}_{e,f}^r(\phi)$ is given by the formula*

$$[\overline{\Sigma}_{e,f}^r(\phi)] = A_e^r \Big(c_1(\mathcal{F}) - \frac{2f}{e} c_1(\mathcal{E}) \Big) \in H^2(X, \mathbb{Q}),$$

where

$$A_e^r := \frac{\binom{e}{r} \binom{e+1}{r-1} \cdots \binom{e+r-1}{1}}{\binom{1}{0} \binom{3}{1} \binom{5}{2} \cdots \binom{2r-1}{r-1}}.$$

The quantity A_e^r is the degree of the variety of symmetric $e \times e$-matrices of corank at least r inside the projective space of all symmetric $e \times e$ matrices, see [HT].

Before introducing a second type of degeneracy loci, we give a definition. If V is a vector space, a pencil of quadrics $\ell \subseteq \mathbf{P}(\mathrm{Sym}^2(V))$ is said to be *degenerate* if the intersection of ℓ with the discriminant

divisor $D(V) \subseteq \mathbf{P}(\mathrm{Sym}^2(V))$ is non-reduced. We consider a morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ such that all kernels are expected to be pencils of quadrics and impose the condition that the pencil be degenerate.

Theorem 1.2. *We fix integers e and $f = \binom{e+1}{2} - 2$ and let $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ be a morphism of vector bundles. The class of the virtual divisor $\mathfrak{Dp} := \left\{ x \in X : \mathrm{Ker}(\phi(x)) \text{ is a degenerate pencil} \right\}$ equals*

$$[\mathfrak{Dp}] = (e-1)\Big(ec_1(\mathcal{F}) - (e^2 + e - 4)c_1(\mathcal{E})\Big) \in H^2(X, \mathbb{Q}).$$

Theorems 1.1 and 1.2 are motivated by fundamental questions in moduli theory and in what follows we shall discuss some of these applications, which are treated at length in the paper.

Tautological classes on moduli of polarized $K3$ surfaces. Let $\mathcal{F}_g$ be the moduli space of quasi-polarized $K3$ surfaces $[X, L]$ of genus g, that is, satisfying $L^2 = 2g - 2$. We denote by $\pi : \mathcal{X} \to \mathcal{F}_g$ the universal $K3$ surface and choose a polarization line bundle $\mathcal{L}$ on $\mathcal{X}$. We consider the Hodge class

$$\lambda := c_1\big(\pi_*(\omega_\pi)\big) \in CH^1(\mathcal{F}_g).$$

Note that $CH^1(\mathcal{F}_g) \cong H^2(\mathcal{F}_g, \mathbb{Q})$. Inspired by Mumford's definition of the κ classes on $\mathcal{M}_g$, for integers $a, b \geq 0$, Marian, Oprea and Pandharipande [MOP] introduced the classes $\kappa_{a,b} \in CH^{a+2b-2}(\mathcal{F}_g)$ whose definition we recall in Section 9. In codimension 1, there are two such classes, namely

$$\kappa_{3,0} := \pi_*\Big(c_1(\mathcal{L})^3\Big) \quad \text{and} \quad \kappa_{1,1} := \pi_*\Big(c_1(\mathcal{L}) \cdot c_2(\mathcal{T}_\pi)\Big) \in CH^1(\mathcal{F}_g).$$

Both these classes depend on the choice of $\mathcal{L}$, but the following linear combination

$$\gamma := \kappa_{3,0} - \frac{g-1}{4}\kappa_{1,1} \in CH^1(\mathcal{F}_g)$$

is intrinsic and independent of the polarization line bundle.

For a general element $[X, L] \in \mathcal{F}_g$ one has $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$. Imposing the condition that $\mathrm{Pic}(X)$ be of rank at least 2, one is led to the notion of Noether-Lefschetz (NL) divisor on $\mathcal{F}_g$. For non-negative integers h and d, we denote by $D_{h,d}$ the locus of quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$ such that there exists a primitive embedding of a rank 2 lattice

$$\mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot D \subseteq \mathrm{Pic}(X),$$

where $D \in \mathrm{Pic}(X)$ is a class such that $D \cdot L = d$ and $D^2 = 2h - 2$. From the Hodge Index Theorem $D_{h,d}$ is empty unless $d^2 - 4(g-1)(h-1) > 0$. Whenever non-empty, $D_{h,d}$ is pure of codimension 1.

Maulik and Pandharipande [MP] conjectured that $\mathrm{Pic}(\mathcal{F}_g)$ is spanned by the Noether-Lefschetz divisors $D_{h,d}$. This has been recently proved in [BLMM] using deep automorphic techniques. Note that the rank of $\mathrm{Pic}(\mathcal{F}_g)$ can become arbitrarily large and understanding all the relations between NL divisors remains a daunting task. Borcherds [Bo] using automorphic forms on $O(2, n)$ has shown that the Hodge class λ is supported on NL divisors. A second proof of this fact, via Gromov-Witten theory, is due to Pandharipande and Yin, see [PY] Section 7. Using Theorem 1.1, we find very simple and explicit Noether-Lefschetz representatives of both classes λ and γ. Our methods are within the realm of algebraic geometry and we use no automorphic forms.

We produce relations among tautological classes on $\mathcal{F}_g$ using the projective geometry of embedded $K3$ surfaces of genus g. We study geometric conditions that single out *only* NL special $K3$ surfaces. Let us first consider the divisor in $\mathcal{F}_g$ consisting of $K3$ surfaces which lie on a rank 4 quadric. We fix a $K3$ surface $[X, L] \in \mathcal{F}_g$ with $g \geq 4$ and let $\varphi_L : X \to \mathbf{P}^g$ be the morphism induced by the polarization

L. One computes $h^0(X, L^{\otimes 2}) = 4g - 2$. Assuming that the image $X \subseteq \mathbf{P}^g$ is projectively normal (which holds under very mild genericity assumptions, see again Section 9), we observe that the space $I_{X,L}(2)$ of quadrics containing X has the following dimension:

$$\dim I_{X,L}(2) = \dim \mathrm{Sym}^2 H^0(X, L) - h^0(X, L^{\otimes 2}) = \binom{g-2}{2}.$$

This equals the codimension of the space of symmetric $(g+1) \times (g+1)$ matrices of rank 4. Therefore the condition that $X \subseteq \mathbf{P}^g$ lie on a rank 4 quadric is expected to be divisorial on $\mathcal{F}_g$. This expectation is easily confirmed in Proposition 9.1, and we are led to the divisor:

$$D_g^{\mathrm{rk4}} := \Big\{ [X, L] \in \mathcal{F}_g : \exists\, 0 \neq q \in I_{X,L}(2), \ \ \mathrm{rk}(q) \leq 4 \Big\}.$$

Theorem 1.3. *Set $g \geq 4$. The divisor D_g^{rk4} is an effective combination of NL divisors and its class is*

$$[D_g^{\mathrm{rk4}}] = A_{g+1}^{g-3}\Big((2g-1)\lambda + \frac{2}{g+1}\gamma \Big) \in CH^1(\mathcal{F}_g).$$

In order to get a second relation between λ and γ, we distinguish depending on the parity of g. For odd genus g, we obtain a second relation between λ and γ by considering the locus of $K3$ surfaces $[X, L] \in \mathcal{F}_g$ for which the embedded surface $\varphi_L : X \to \mathbf{P}^g$ has a non-trivial middle linear syzygy. In terms of Koszul cohomology groups, we set

$$\mathfrak{Kos}_g := \Big\{ [X, L] \in \mathcal{F}_g : K_{\frac{g-1}{2}, 1}(X, L) \neq 0 \Big\}.$$

For instance $\mathfrak{Kos}_3$ consists of quartic $K3$ surfaces for which the map $\mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$ is not an isomorphism. Voisin's solution [V1] of the generic Green Conjecture on syzygies of canonical curves ensures that $\mathfrak{Kos}_g$ is a proper locus of NL type. She proved that for a $K3$ surface $[X, L] \in \mathcal{F}_g$ with $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$, the vanishing

$$K_{\frac{g-1}{2}, 1}(X, L) = 0$$

holds, or equivalently, $[X, L] \notin \mathfrak{Kos}_g$. We realize $\mathfrak{Kos}_g$ as the degeneracy locus of a morphism of two vector bundles of the same rank over $\mathcal{F}_g$, whose Chern classes can be expressed in terms of $\kappa_{1,1}, \kappa_{3,0}$ and λ. We then obtain the following formula (see Theorem 9.5)

$$(1) \qquad [\mathfrak{Kos}_g] = \frac{4}{g-1}\binom{g-4}{\frac{g-3}{2}}\Big(\frac{(g-1)(g+7)}{2}\lambda + \gamma \Big) + \alpha \cdot [D_{1,1}] \in CH^1(\mathcal{F}_g),$$

where recall that $D_{1,1}$ is the NL divisor of $K3$ surfaces $[X, L]$ for which the polarization L is not globally generated. Theorems 1.3 and 9.5 then quickly imply (in the case of odd g):

Theorem 1.4. *Both tautological classes λ and γ on $\mathcal{F}_g$ are of Noether-Lefschetz type.*

Theorem 1.4 is proved for even genus $g \geq 8$ in Section 10 using two further geometric relations between tautological classes (in the spirit of Theorem 1.3) involving the geometry of rank 2 Lazarsfeld-Mukai bundle E_L one associates canonically to each NL-general polarized $K3$ surface $[X, L] \in \mathcal{F}_g$. The vector bundle E_L satisfies $\det(E_L) = L$ and $h^0(X, E_L) = \frac{g}{2} + 2$ and has already been put to great use in [La], [Mu], or [V1]. A direct proof of Theorem 1.4 when $g \leq 10$ has already appeared in [GLT].

In Section 11 we discuss an application of Theorem 1.3 to the Geometric Invariant Theory of $K3$ surfaces. The second Hilbert point $[X, L]_2$ of a suitably general polarized $K3$ surface $[X, L]$ is defined

as the quotient $[X, H]_2 := \left[\operatorname{Sym}^2 H^0(X, L) \longrightarrow H^0(X, L^{\otimes 2}) \longrightarrow 0 \right] \in \operatorname{Gr}\left(\operatorname{Sym}^2 H^0(X, H), 4g - 2 \right).$
We establish the following result:

Theorem 1.5. *The second Hilbert point of a polarized K3 surface $[X, L] \in \mathcal{F}_g \setminus D_g^{\mathrm{rk4}}$ is semistable.*

Note that a similar result at the level of canonical curves has been obtained in [FJ].

The Petri class on $\overline{\mathcal{M}}_g$.

A non-hyperelliptic canonical curve $C \subseteq \mathbf{P}^{g-1}$ of genus g is projectively normal and lies on precisely $\binom{g-2}{2}$ quadrics. This number equals the codimension of the locus of symmetric $g \times g$-matrices of rank 3. The condition that C lie on a rank 3 quadric in its canonical embedding is divisorial and leads to the Petri divisor $\mathcal{GP}_g$ of curves $[C] \in \mathcal{M}_g$, having a pencil A such that the Petri map

$$\mu(A) : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$$

is not injective. Using Theorem 1.1, we establish the following result:

Theorem 1.6. *The class of the compactified Petri divisor $\widetilde{\mathcal{GP}}_g$ on $\overline{\mathcal{M}}_g$ is given by the formula*

$$[\widetilde{\mathcal{GP}}_g] = A_g^{g-3}\left(\frac{7g + 6}{g}\lambda - \delta \right) \in CH^1(\overline{\mathcal{M}}_g).$$

Here λ is the Hodge class on $\overline{\mathcal{M}}_g$ and δ denotes the total boundary divisor. The Petri divisor splits into components $D_{g,k}$, where $\lfloor \frac{g+2}{2} \rfloor \leq k \leq g - 1$, depending on the degree of the (base point free) pencil A for which the Petri map $\mu(A)$ is not injective. With a few notable exception when k is extremal, the individual classes $[\overline{D}_{g,k}] \in CH^1(\overline{\mathcal{M}}_g)$ are not known. However, we predict a simple formula for the multiplicities of $\overline{D}_{g,k}$ in the expression of $[\widetilde{\mathcal{GP}}_g]$, see Conjecture 6.3.

Effective divisors on Hurwitz spaces.

We fix an integer $k \geq 4$ and denote by $\mathcal{H}_k$ the Hurwitz space parametrizing degree k covers $[f : C \to \mathbf{P}^1]$ from a smooth curve of genus $2k - 1$. The space $\mathcal{H}_k$ admits a compactification $\overline{\mathcal{H}}_k$ by means of admissible covers, which is defined to be the normalization of the space constructed by Harris and Mumford in [HM]. We refer to [ACV] for details. We denote by $\sigma : \overline{\mathcal{H}}_k \to \overline{\mathcal{M}}_{2k-1}$ the morphism assigning to each admissible cover the stabilization of the source curve. The image $\sigma(\overline{\mathcal{H}}_k)$ is the divisor $\overline{\mathcal{M}}_{2k-1,k}^1$ consisting of k-gonal curves in $\overline{\mathcal{M}}_{2k-1}$, which was studied in great detail by Harris and Mumford [HM] in the course of their proof that $\overline{\mathcal{M}}_g$ is general for large genus. The birational geometry of $\overline{\mathcal{H}}_k$ is largely unknown, see however [ST] for some recent results.

Let us choose a general point $[f : C \to \mathbf{P}^1] \in \mathcal{H}_k$ and denote by $A := f^*(\mathcal{O}_{\mathbf{P}^1}(1)) \in W_k^1(C)$ the pencil inducing the cover. We consider the residual linear system $L := \omega_C \otimes A^\vee \in W_{3k-4}^{k-1}(C)$ and denote by $\varphi_L : C \to \mathbf{P}^{k-1}$ the induced map. Under these genericity assumptions L is very ample, $H^1(C, L^{\otimes 2}) = 0$ and the image curve $\varphi_L(C)$ is projectively normal. In particular,

$$\dim I_{C,L}(2) = \dim \operatorname{Sym}^2 H^0(C, L) - h^0(C, L^{\otimes 2}) = \binom{k - 3}{2},$$

which equals the codimension of the space of symmetric $k \times k$ matrices of rank 4. Imposing the condition that $C \subseteq \mathbf{P}^{k-1}$ be contained in a rank 4 quadric, we obtain a (virtual) divisor

$$\mathfrak{H}_k^{\mathrm{rk4}} := \left\{ [C, A] \in \mathcal{H}_k : \exists\, 0 \neq q \in I_{C,\omega_C \otimes A^\vee}(2),\ \mathrm{rk}(q) \leq 4 \right\}.$$

The condition $[C, A] \in \mathfrak{H}_k^{\mathrm{rk4}}$ amounts to representing the canonical bundle ω_C as a sum

$$\omega_C = A \otimes A_1 \otimes A_2 \tag{2}$$

of *three* pencils, that is, $h^0(C, A_1) \geq 2$ and $h^0(C, A_2) \geq 2$. To show that $\mathfrak{H}_k^{\mathrm{rk4}}$ is indeed a divisor, it suffices to exhibit a point $[C, A] \in \mathcal{H}_k$ such that (2) cannot hold. To that end, we take a general polarized $K3$ surface $[X, L] \in \mathcal{F}_{2k-1}$ carrying an elliptic pencil E with $E \cdot L = k$ (that is, a general element of the NL divisor $D_{1,k} \subseteq \mathcal{F}_{2k-1}$). If $C \in |L|$ is a smooth curve in the polarization class and $A = \mathcal{O}_C(E) \in W_k^1(C)$, we check that one has an isomorphism $I_{C, \omega_C \otimes A^\vee}(2) \cong I_{X, L(-E)}(2)$ between the spaces of quadrics containing C and $X \subseteq \mathbf{P}^{k-1}$ respectively. Showing that this latter space contains no rank 4 quadric becomes a lattice-theoretic problem inside $\mathrm{Pic}(X)$, which we solve.

We summarize our results concerning $\mathfrak{H}_k^{\mathrm{rk4}}$. We denote by $\lambda := \sigma^*(\lambda)$ the Hodge class on $\overline{\mathcal{H}}_k$ and by D_0 the boundary divisor on $\overline{\mathcal{H}}_k$ whose general point corresponds to a 1-nodal singular curve C of genus $2k - 1$ and a locally free sheaf A of degree k with $h^0(C, A) \geq 2$ (see Section 12 for details).

Theorem 1.7. *For each $k \geq 6$, the locus $\mathfrak{H}_k^{\mathrm{rk4}}$ is an effective divisor on $\mathcal{H}_k$. Away from the union of the boundary divisors $\sigma^{-1}(\Delta_i)$ where $i = 1, \ldots, k-1$, one has the relation*

$$K_{\overline{\mathcal{H}}_k} = \frac{k-12}{k-6}\left(7\lambda - [D_0]\right) + \frac{k}{(k-6)A_k^{k-4}}[\overline{\mathfrak{H}}_k^{\mathrm{rk4}}].$$

Theorem 1.7 follows from applying Theorem 1.1 in the context of Hurwitz spaces to compute the class $[\overline{\mathfrak{H}}_k^{\mathrm{rk4}}]$ in terms of certain tautological classes on $\overline{\mathcal{H}}_k$, see Theorem 12.6, then comparing with the formula we find for $K_{\overline{\mathcal{H}}_k}$ in terms of those same classes. Proving that $\mathfrak{H}_k^{\mathrm{rk4}}$ is indeed a genuine divisor on $\overline{\mathcal{H}}_k$ is achieved in Theorem 12.5.

We mention the following consequence to the birational geometry of $\overline{\mathcal{H}}_k$.

Theorem 1.8. *For $k > 12$, there exists an effective $\mathbb{Q}$-divisor class E on $\overline{\mathcal{H}}_k$ supported on the divisor $\sum_{i=1}^{k-1} \sigma^*(\Delta_i)$ of curves of compact type, such that the class $K_{\overline{\mathcal{H}}_k} + E$ is big.*

This result should be compared to the classical result [HM] asserting that $\overline{\mathcal{M}}_{2k-1}$ is of general type for $k \geq 13$, whereas the Kodaira dimension of $\overline{\mathcal{M}}_{23}$ is at least 2, see [F4]. Assuming that the singularities of $\overline{\mathcal{H}}_k$ impose no adjunction conditions (something one certainly expects), Theorem 1.8 should imply that for $k > 12$ the Hurwitz space $\overline{\mathcal{H}}_k$ is a variety of general type.

Effective divisors of small slope on $\overline{\mathcal{M}}_g$.

Theorem 1.1 has multiple applications to the birational geometry of the moduli space of curves. Recall that if $\lambda, \delta_0, \ldots, \delta_{\lfloor \frac{g}{2} \rfloor}$ denote the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, then the *slope* of an effective divisor $D \subseteq \overline{\mathcal{M}}_g$ such that $\Delta_i \not\subseteq \mathrm{supp}(D)$ for all $i = 0, \ldots, \lfloor \frac{g}{2} \rfloor$, is defined as $s(D) := \frac{a}{\min_i b_i} \geq 0$, where $[D] = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i \delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$. The slope of the moduli space, defined as the quantity

$$s(\overline{\mathcal{M}}_g) := \inf\left\{s(D) : D \text{ is an effective divisor of } \overline{\mathcal{M}}_g\right\}$$

is a fundamental invariant encoding for instance the Kodaira dimension of the moduli space. For a long time it was conjectured [HMo] that $s(\overline{\mathcal{M}}_g) \geq 6 + \frac{12}{g+1}$, with equality if and only if $g + 1$ is composite and D is a Brill-Noether divisor on $\overline{\mathcal{M}}_g$ consisting of curves $[C] \in \mathcal{M}_g$ having a linear series $L \in W_d^r(C)$ with Brill-Noether number $\rho(g, r, d) = -1$. This conjecture has been disproved in [F1],

[F2] and [Kh], where for an infinite series of genera g effective divisors of slope less than $6 + \frac{12}{g+1}$ were constructed. At the moment there is no clear conjecture concerning even the asymptotic behavior of $s(\overline{\mathcal{M}}_g)$ as g is large, see also [Pa]. For instance, it is not clear that $\liminf_{g\to\infty} s(\overline{\mathcal{M}}_g) > 0$.

Imposing the condition that a curve C of genus g lie on a quadric of prescribed rank in one of the embeddings $\varphi_L : C \hookrightarrow \mathbf{P}^r$ given by a linear system $L \in W_d^r(C)$ with Brill-Noether number $\rho(g, r, d) := g - (r+1)(g-d+r) = 0$, we obtain an infinite sequence of effective divisors on $\overline{\mathcal{M}}_g$ of very small slope (see condition (20) for the numerical condition g has to satisfy). Theorems 7.1 and 7.2 exemplify two infinite subsequences of such divisors on $\overline{\mathcal{M}}_{(4\ell-1)(9\ell-1)}$ and $\overline{\mathcal{M}}_{4(3\ell+1)(2\ell+1)}$ respectively, where $\ell \geq 1$. We mention the following concrete example on $\overline{\mathcal{M}}_{24}$.

Theorem 1.9. *The following locus defined as*

$$D_{7,3} := \left\{ [C] \in \mathcal{M}_{24} : \exists\, L \in W_{28}^7(C),\ \exists\, 0 \neq q \in I_{C,L}(2),\ \mathrm{rk}(q) \leq 6 \right\}$$

is an effective divisor on $\mathcal{M}_{24}$. The slope of its closure $\overline{D}_{7,3}$ in $\overline{\mathcal{M}}_{24}$ is given by $s(\overline{D}_{7,3}) = \frac{34423}{5320} < 6 + \frac{12}{25}$.

Theorem 7.3 establishes that $D_{7,3}$ is a genuine divisor on $\mathcal{M}_{24}$. We show using *Macaulay* that there exists a smooth curve $C \subseteq \mathbf{P}^7$ of genus 24 and degree 28 which does not lie on a quadric of rank at most 6 in $\mathbf{P}^7$. Using the irreducibility of the space of pairs $[C, L]$, where C is a smooth curve of genus 24 and $L \in W_{28}^7(C)$, we conclude that $D_{7,3} \neq \mathcal{M}_{24}$, hence $D_{7,3}$ is indeed a divisor on $\mathcal{M}_{24}$.

Theorem 1.2 has applications to the slope of $\overline{\mathcal{M}}_{12}$. A general curve $[C] \in \mathcal{M}_{12}$ has a finite number of embeddings $C \subseteq \mathbf{P}^5$ of degree 15. They are all residual to pencils of minimal degree. The curve $C \subseteq \mathbf{P}^5$ lies on a pencil of quadrics and we impose the condition that one of these pencils be degenerate.

Theorem 1.10. *The locus of smooth curves of genus 12 having a degenerate pencil of quadrics*

$$\mathfrak{Dp}_{12} := \left\{ [C] \in \mathcal{M}_{12} : \exists\, L \in W_{15}^5(C)\ \text{with}\ \mathbf{P}\big(I_{C,L}(2)\big)\ \text{degenerate} \right\}$$

is an effective divisor. The slope of its closure $\overline{\mathfrak{Dp}}_{12}$ inside $\overline{\mathcal{M}}_{12}$ equals $s(\overline{\mathfrak{Dp}}_{12}) = \frac{373}{54} < 6 + \frac{12}{13}$.

2. Equivariant fundamental classes, degeneracy loci

2.1. **Equivariant fundamental class.** We consider a connected algebraic group G acting on a smooth variety V, and let Σ be an invariant subvariety. Then Σ represents a fundamental cohomology class—denoted by $[\Sigma]$ or $[\Sigma \subseteq V]$—in the G-equivariant cohomology of V, namely

$$[\Sigma] \in H_G^{2\mathrm{codim}(\Sigma \subseteq V)}(V).$$

Throughout the paper we use cohomology with complex coefficients. There are several equivalent ways to define this fundamental cohomology class, see for example [K], [EG], [FR1], [MS, 8.5], [F] for different flavours and different cohomology theories.

A particularly important case is when V is a vector space and Σ is an invariant cone. Then $[\Sigma]$ is an element of $H_G^*(\text{vector space}) = H_G^*(\text{point}) = H^*(BG)$, that is, $[\Sigma]$ is a G-characteristic class. This

characteristic class has the following well known "degeneracy locus" interpretation. Let $E \to M$ be a bundle with fiber V and structure group G. Since Σ is invariant under the structure group, the notion of *belonging to* Σ makes sense in every fiber. Let $\Sigma(E)$ be the union of Σ's of all the fibers. Let s be a sufficiently generic section. Then the fundamental cohomology class $[s^{-1}(\Sigma(E)) \subseteq M]$ of the "degeneracy locus" $s^{-1}(\Sigma(E))$ in the *ordinary* cohomology $H^*(M)$ is equal to $[\Sigma]$ (as a G-characteristic class) of the bundle $E \to M$.

2.2. **Examples.** We recall two well known formulas for some equivariant cohomology classes. The second one will be used in Sections 4 and 5.

Definition 2.1. *For variables c_i and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r)$ let*

$$s_\lambda(c) = \det(c_{\lambda_i+j-i})_{i,j=1,\ldots,r}$$

be the Schur polynomial. By convention $c_0 = 1$ and $c_{<0} = 0$.

Example 2.2. *The Giambelli-Thom-Porteous formula.* Fix $r \leq n$, $\ell \geq 0$ and let $\Omega^r \subseteq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+\ell})$ be the space of linear maps having an r-dimensional kernel. It is invariant under the group $GL_n(\mathbb{C}) \times GL_{n+\ell}(\mathbb{C})$ acting by $(A, B) \cdot \phi = B \circ \phi \circ A^{-1}$. One has [Po]

$$[\overline{\Omega^r}] = s_\lambda(c),$$

where

$$\lambda = (\underbrace{r+\ell, \ldots, r+\ell}_{r}), \qquad 1 + c_1 t + c_2 t^2 + \ldots = \frac{1 + b_1 t + b_2 t^2 + \ldots + b_{n+\ell} t^{n+\ell}}{1 + a_1 t + a_2 t^2 + \ldots + a_n t^n}.$$

Here a_i (respectively b_i) is the ith universal Chern class of $GL_n(\mathbb{C})$ (respectively $GL_{n+\ell}(\mathbb{C})$).

Example 2.3. *Symmetric 2-forms.* Let $r \leq n$ and let $\Sigma^r = \Sigma_n^r \subseteq \mathrm{Sym}^2(\mathbb{C}^n)$ be the collection of symmetric 2-forms having a kernel of dimension r. It is invariant under the group $GL_n(\mathbb{C})$ acting by $A \cdot M = AMA^T$. One has [JLP, Pr2, HT] that

$$[\overline{\Sigma}_n^r] = 2^r s_{(r,r-1,\ldots,2,1)}(c),$$

where c_i is the ith universal Chern class of $GL_n(\mathbb{C})$.

3. Affine, projective, and restricted projective fundamental classes

In this section we recall the formalism of comparing equivariant fundamental classes in affine and projective spaces.

Consider the representation of the torus $T = (\mathbb{C}^*)^k$ acting by

$$(a_1, \ldots, a_k) \cdot (x_1, \ldots, x_n) = (\prod_{i=1}^k a_i^{s_{1,i}} x_1, \prod_{i=1}^k a_i^{s_{2,i}} x_2, \ldots, \prod_{i=1}^k a_i^{s_{n,i}} x_n).$$

We will assume that the representation "contains the scalars", that is, there exist integers $r_1, \ldots, r_k$ and r such that

$$\sum_{i=1}^k r_i s_{j,i} = r, \qquad \text{for all} \qquad j = 1, \ldots, n.$$

In other words, the action of $(b^{r_1}, \ldots, b^{r_k}) \in T$ ($b \in \mathbb{C}^*$) on $\mathbb{C}^n$ is multiplication by b^r.

Under this assumption we have that the non-zero orbits of the linear representation, and the orbits of the induced action on $\mathbf{P}^{n-1}$ are in bijection. We will compare the (T-equivariant) fundamental class of an invariant subvariety $\Sigma \subseteq \mathbb{C}^n$ with the (T-equivariant) fundamental class of the projectivization $\mathbf{P}(\Sigma) \subseteq \mathbf{P}^{n-1}$. For this we need some notation.

The fundamental class $[\Sigma]$ of Σ is an element of $H_T^*(\mathbb{C}^n) = H^*(BT) = \mathbb{C}[\alpha_1, \dots, \alpha_k]$, where α_i is the equivariant first Chern class of the $\mathbb{C}^*$-action corresponding to the ith factor. Hence we can consider $[\Sigma]$ as a polynomial in the α_i's.

Let $w_j = \sum_{i=1}^k s_{j,i}\alpha_i$, $j = 1, \dots, n$ be the weights of the representation above. Then we have

$$H_T^*(\mathbf{P}^{n-1}) = H^*(BT)[\xi]/\prod_{j=1}^n (\xi - w_j),$$

where ξ is the first Chern class of the tautological line bundle over $\mathbf{P}^{n-1}$.

Proposition 3.1. [FNR1, Thm. 6.1] *Let Σ be a T-invariant subvariety of $\mathbb{C}^n$. For the T-equivariant fundamental class of $\mathbf{P}(\Sigma)$ we have*

$$[\mathbf{P}(\Sigma)] = [\Sigma]|_{\alpha_i \mapsto \alpha_i - \frac{r_i}{r}\xi} \qquad \in H_T^*(\mathbf{P}^{n-1}).$$

Here, and in the future, by $p(\alpha_i)|_{\alpha_i \mapsto \beta_i}$ we mean the substitution of β_i into the variables α_i of the polynomial $p(\alpha_i)$.

We shall need a further twist on this notion. Let F_j be the jth coordinate line of $\mathbb{C}^n$, which is a fixed point of the T-action on $\mathbf{P}^{n-1}$. We have the restriction map $H_T^*(\mathbf{P}^{n-1}) \to H_T^*(F_j) = H^*(BT)$, which we denote by $p \mapsto p|_{F_j}$.

Corollary 3.2. *We have*

$$[\mathbf{P}(\Sigma)]|_{F_j} = [\Sigma]|_{\alpha_i \mapsto \alpha_i - \frac{r_i}{r}w_j} \qquad \in H^*(BT).$$

Proof. The restriction homomorphism $H_T^*(\mathbf{P}^{n-1}) \to H_T^*(F_j)$ is given by substituting w_j for ξ. $\square$

Example 3.3. Let $(\mathbb{C}^*)^3$ act on $\mathbb{C}^2$ by $(a_1, a_2, a_3) \cdot (x_1, x_2) = (a_1^3 a_2^{-1} a_3 \cdot x_1, a_1 a_2^2 a_3^2 \cdot x_2)$. The numbers $r_1 = 2, r_2 = 1, r_3 = 1, r = 6$ prove that this action contains the scalars. Let Σ be the x_1-axis. Then $[\Sigma]$ is the normal Euler class, that is $[\Sigma] = \alpha_1 + 2\alpha_2 + 2\alpha_3$. According to Proposition 3.1 we have

$$[\mathbf{P}(\Sigma)] = \alpha_1 + 2\alpha_2 + 2\alpha_3|_{\alpha_1 \mapsto \alpha_1 - \frac{1}{3}\xi, \alpha_2 \mapsto \alpha_2 - \frac{1}{6}\xi, \alpha_3 \mapsto \alpha_3 - \frac{1}{6}\xi} = \alpha_1 + 2\alpha_2 + 2\alpha_3 - \xi.$$

According to Corollary 3.2 the two fixed point restrictions of this class are

$$[\mathbf{P}(\Sigma)]|_{(1:0)} = \alpha_1 + 2\alpha_2 + 2\alpha_3|_{\alpha_1 \mapsto \alpha_1 - \frac{1}{3}(3\alpha_1 - \alpha_2 + \alpha_3), \alpha_2 \mapsto \alpha_2 - \frac{1}{6}(3\alpha_1 - \alpha_2 + \alpha_3), \alpha_3 \mapsto \alpha_3 - \frac{1}{6}(3\alpha_1 - \alpha_2 + \alpha_3)}$$
$$= -2\alpha_1 + 3\alpha_2 + \alpha_3$$

and

$$[\mathbf{P}(\Sigma)]|_{(0:1)} = \alpha_1 + 2\alpha_2 + 2\alpha_3|_{\alpha_1 \mapsto \alpha_1 - \frac{1}{3}(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_2 \mapsto \alpha_2 - \frac{1}{6}(\alpha_1 + 2\alpha_2 + 2\alpha_3), \alpha_3 \mapsto \alpha_3 - \frac{1}{6}(\alpha_1 + 2\alpha_2 + 2\alpha_3)} = 0.$$

The vanishing of the second one is expected since the x_2-axis is not in Σ, and the first one can be verified by seeing that the action on $\mathbf{P}^1$ in the coordinate $t = x_2/x_1$ is

$$(a_1, a_2, a_3).t = \frac{a_1 a_2^2 a_3^2}{a_1^3 a_2^{-1} a_3} \cdot t = a_1^{-2} a_2^3 a_3 \cdot t.$$

The calculations of this example were deceivingly simple caused by the fact that Σ was smooth.

4. Loci characterised by singular vectors in the kernel

4.1. **The $\Sigma_{e,f}^r$ locus.** For positive integers e, f, let $E := \mathbb{C}^e$ and $F := \mathbb{C}^f$ be the standard representations of $GL_e(\mathbb{C})$ and $GL_f(\mathbb{C})$ respectively. Consider the induced action of $G = GL_e(\mathbb{C}) \times GL_f(\mathbb{C})$ on $\operatorname{Hom}(\operatorname{Sym}^2 E, F)$. Define the locus

$$\Sigma_{e,f}^r = \left\{ \phi \in \operatorname{Hom}(\operatorname{Sym}^2 E, F) : \exists\, q \in \operatorname{Sym}^2 E \text{ with } \dim(\operatorname{Ker} q) = r \text{ and } \phi(q) = 0 \right\}.$$

which is invariant under the G-action. Using the notation of Example 2.3 we have

$$\Sigma_{e,f}^r = \left\{ \phi \in \operatorname{Hom}(\operatorname{Sym}^2 E, F) : \exists\, 0 \neq q \in \Sigma_e^r \cap \operatorname{Ker}(\phi) \right\}.$$

We will assume that $d := \binom{e+1}{2} - f$ is positive, that is, the condition above is not that ϕ *has* a kernel, but rather that this kernel intersects $\Sigma_e^r \subseteq \operatorname{Sym}^2 E$. We shall also assume that this intersection is generically at most 0-dimensional, that is, $d \leq \operatorname{codim}(\Sigma_e^r \subseteq \operatorname{Sym}^2 E) = \binom{r+1}{2}$.

In this section our goal is to find a formula for the G-equivariant fundamental class

$$[\overline{\Sigma_{e,f}^r}] \in H_G^*\left(\operatorname{Hom}(\operatorname{Sym}^2 E, F) \right) = \mathbb{C}[\alpha_1, \dots, \alpha_e, \beta_1, \dots, \beta_f]^{S_e \times S_f}.$$

Here α_i are the Chern roots of $GL_e(\mathbb{C})$ (that is, their elementary symmetric polynomials are the Chern classes), and β_i are the Chern roots of $GL_f(\mathbb{C})$ respectively.

The calculation—which will complete the proof of Theorem 1.1—is done via torus-equivariant localization. To bypass complications caused by a complete resolution of $\overline{\Sigma_{e,f}^r}$ we will use a method of [BSz, FR2] which requires only a partial desingularization exhibited as a vector bundle over a compact space.

4.2. **A partial resolution.** Let $\mathcal{F}$ be the partial flag manifold parametrizing chains of linear subspaces $C \subseteq D \subseteq \operatorname{Sym}^2 E$, where $\dim C = 1$ and $\dim D = d$. Recall that in Example 2.3 we defined the subset $\Sigma^r = \Sigma_e^r \subseteq \operatorname{Sym}^2 E$. Define

$$I = \left\{ \left((C, D), \phi \right) \in \mathcal{F} \times \operatorname{Hom}(\operatorname{Sym}^2 E, F) : C \subseteq \overline{\Sigma^r} \text{ and } \phi|_D = 0 \right\} \text{ and}$$

$$Y = \{ (C, D) \in \mathcal{F} : C \subseteq \overline{\Sigma^r} \}$$

and let $p : I \to Y$ be the map forgetting ϕ. We have the commutative diagram

$$(3) \qquad
\begin{array}{ccc}
I & \xrightarrow{\quad i \quad} \mathcal{F} \times \operatorname{Hom}(\operatorname{Sym}^2 E, F) \xrightarrow{\quad \pi_2 \quad} \operatorname{Hom}(\operatorname{Sym}^2 E, F) \\
\downarrow{\scriptstyle p} \qquad\qquad \downarrow{\scriptstyle \pi_1} \\
Y & \xrightarrow{\quad j \quad} \mathcal{F},
\end{array}$$

with the arc ρ over the top from I to $\operatorname{Hom}(\operatorname{Sym}^2 E, F)$,

with i and j being natural inclusions and π_1, π_2 natural projections. The map $\rho = \pi_2 \circ i$ is birational to $\overline{\Sigma_{e,f}^r}$. We have

$$\dim Y = \binom{e+1}{2} - \binom{r+1}{2} - 1 + (d-1)f, \qquad \dim I = \binom{e+1}{2} - \binom{r+1}{2} - 1 + (d-1)f + f^2.$$

Hence the codimension

$$\mathrm{codim}\left(\overline{\Sigma_{e,f}^r} \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)\right) = \binom{r+1}{2} - \binom{e+1}{2} + f + 1 = \binom{r+1}{2} - d + 1,$$

which is thus the degree of the fundamental class $[\overline{\Sigma_{e,f}^r}]$ we are looking for.

4.3. **Localization and residue formulas.** Let $W = \{\alpha_i + \alpha_j\}_{1 \le i \le j \le e}$ be the set of weights of $\mathrm{Sym}^2 E$. Let $h_r(\alpha_1, \ldots, \alpha_e)$ be the polynomial $2^r s_{(r,r-1,\ldots,1)}(c)$, where $1 + c_1 t + c_2 t^2 + \ldots = \prod_{i=1}^e (1 + \alpha_i t)$ (cf. Example 2.3).

Theorem 4.1. *Using the notations and assumption above we have*

(4)
$$[\overline{\Sigma_{e,f}^r}] = \sum_{\substack{H \subseteq W \\ |H|=d}} \sum_{\gamma \in H} \frac{h_r|_{\alpha_i \mapsto \alpha_i - \gamma/2} \cdot \prod_{j=1}^f \prod_{\delta \in H}(\beta_j - \delta)}{\prod_{\delta \in W - \{\gamma\}}(\delta - \gamma) \cdot \prod_{\delta \in H - \{\gamma\}} \prod_{\epsilon \in W - H}(\epsilon - \delta)}.$$

Proof. To calculate the fundamental class $[\overline{\Sigma_{e,f}^r}]$ it would be optimal to find an equivariant resolution $\tilde{\Sigma} \to \mathrm{Hom}(\mathrm{Sym}^2 E, F)$ of $\overline{\Sigma_{e,f}^r} \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)$, with a well understood Gysin map formula. While the description of such a full resolution is difficult, in diagram (3) we constructed an equivariant partial resolution $\rho : I \to \mathrm{Hom}(\mathrm{Sym}^2 E, F)$ of the locus $\overline{\Sigma_{e,f}^r} \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)$. Although ρ is only a partial resolution (since I is not smooth), it is of special form: I is a *vector bundle* over a (possibly singular) subvariety of a *compact* space $\mathcal{F}$.

In [BSz, Section 3.2] and [FR2, Section 5] it is shown that such a partial resolution reduces the problem of calculating $[\overline{\Sigma_{e,f}^r}]$ to calculating the fundamental class $[Y \subseteq \mathcal{F}]$ near the fixed points of the maximal torus. Namely, [BSz, Proposition 3.2], or equivalently [FR2, Proposition 5.1], applied to diagram (3) gives

(5)
$$[\overline{\Sigma_{e,f}^r}] = \sum_q \frac{[Y \subseteq \mathcal{F}]|_q \cdot [I_q \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)]}{e(T_q \mathcal{F})},$$

where q runs through the finitely many torus fixed points of $\mathcal{F}$ and $I_q = p^{-1}(q)$.

Let us start with the obvious ingredients of this formula. The fixed points of $\mathcal{F}$ are pairs (C, D) where $C \subseteq D$ are coordinate subspaces of $\mathrm{Sym}^2 E$ of dimension 1 and d respectively. The coordinate lines of $\mathrm{Sym}^2 E$ are in bijection with W, and hence the fixed points q are parameterized by choices $H \subset W$ ($|H| = d$) and $\gamma \in H$. Denoting the tautological rank 1 and rank d bundles over $\mathcal{F}$ by $\mathcal{L}$ and $\mathcal{D}$ we have

$$T\mathcal{F} = \mathrm{Hom}(\mathcal{L}, \mathcal{D}/\mathcal{L}) \oplus \mathrm{Hom}(\mathcal{L}, \mathrm{Sym}^2 E/\mathcal{D}) \oplus \mathrm{Hom}(\mathcal{D}/\mathcal{L}, \mathrm{Sym}^2 E/\mathcal{D}).$$

Hence, for a fixed point q corresponding to (H, γ) we have

- $[I_q \subseteq \mathrm{Hom}(\mathrm{Sym}^2 E, F)] = \prod_{j=1}^f \prod_{\delta \in H}(\beta_j - \delta)$,
- $e(T_q \mathcal{F}) = \prod_{\delta \in W - \{\gamma\}}(\delta - \gamma) \cdot \prod_{\delta \in H - \{\gamma\}} \prod_{\epsilon \in W - H}(\epsilon - \delta)$,

both following from the fact that for a G-representation K and invariant subspace $L \subseteq K$ the fundamental class $[L \subset K]$ is the product of the weights of K/L.

It remains to find the non-obvious ingredient of formula (5), the local fundamental class $[Y \subseteq \mathcal{F}]|_q$. However, this problem was essentially solved in Section 3. The space Y is the complete preimage of

$\mathbf{P}(\overline{\Sigma^r})$ under the fibration $z : \mathcal{F} \to \mathbf{P}(\mathrm{Sym}^2 E)$. Hence $[Y \subseteq \mathcal{F}]|_q = [\mathbf{P}(\overline{\Sigma^r})]|_{z(q)}$. We have $[\overline{\Sigma^r}] = h_r(\alpha_1, \ldots, \alpha_e)$ (see Example 2.3), and hence Corollary 3.2 calculates $[\mathbf{P}(\overline{\Sigma^r})]|_{z(q)}$ to be $h_r|_{\alpha_i \mapsto \alpha_i - \gamma/2}$. This completes the proof. $\qquad \square$

Example 4.2. We have

$$[\overline{\Sigma_{2,2}^1}] = \frac{(\beta_1 - 2\alpha_1)(\beta_2 - 2\alpha_1)}{\alpha_2 - \alpha_1} + \frac{(\beta_1 - 2\alpha_2)(\beta_2 - 2\alpha_2)}{\alpha_1 - \alpha_2} = -4(\alpha_1 + \alpha_2) + 2(\beta_1 + \beta_2).$$

More structure of the localization formula (4) will be visible if we rewrite it as a residue formula, with the help of the following lemma, which we prepare by setting some notation.

Let $0 \le k_1 \le k_2 \le \ldots \le k_r$ be integers and let V be a vector bundle of rank k_r on X. Let $p : \mathcal{F}_{k_1, \ldots, k_r}(V) \to X$ be the bundle whose fiber over $x \in X$ is the variety of chains of linear subspaces $V_1^{k_1} \subseteq V_2^{k_2} \subseteq \ldots \subseteq V_r^{k_r} = V_x$, where upper indices indicate dimension and V_x is the fiber of V over x. The Chern roots of the tautological bundle of rank k_i over $\mathcal{F}_{k_1, \ldots, k_r}(V)$ will be denoted by $\sigma_{i,j}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$. The $\sigma_{r,j}$ classes are the pullbacks of the Chern roots of V. In notation we do not indicate the pullback, so $\sigma_{r,j}$ will also denote the Chern roots of V.

Lemma 4.3. *Consider the variables $z_{i,j}$ for $i = 1, \ldots, r-1$, $j = 1, \ldots, k_i$, and let $z_{r,j} = \sigma_{r,j}$. Let $g(z_{i,j})$ be a polynomial symmetric in the sets of variables z_{i*} for all i, and let $D = \sum_{i<j}(k_i - k_{i-1})(k_j - k_{j-1})$ be the dimension of the fiber of p. We have*

$$(6) \qquad p_*(g(\sigma_{i,j})) = (-1)^D \left\{ \frac{g(z_{i,j}) \prod_{i=1}^{r-1} \prod_{1 \le u < v \le k_i} \left(1 - \frac{z_{i,u}}{z_{i,v}}\right)}{\prod_{i=1}^{r-1} \prod_{j=1}^{k_i} z_{i,j}^{k_{i+1}-k_i} \cdot \prod_{i=1}^{r-1} \prod_{u=1}^{k_{r+1}} \prod_{v=1}^{k_r} \left(1 - \frac{z_{i+1,u}}{z_{i,v}}\right)} \right\}_{z_{1*}^0 \cdots z_{k-1,*}^0},$$

where, by $\{P\}_{z_{1*}^0 \cdots z_{k-1,*}^0}$ we mean the constant term in the variables $z_{i,j}$ for $i = 1, \ldots, k-1$ and $j = 1, \ldots, k_i$, of the Laurent expansion of P in the region $|z_{1,j_1}| > |z_{2,j_1}| > \ldots > |z_{r,j_r}|$.

Proof. First we prove the statement for $r = 2$. To that end, we temporarily rename $k_1 = k$, $k_2 = n$, $\sigma_{1,j} = \sigma_j$, $\sigma_{2,j} = \tau_j$, $z_{1,j} = z_j$, and we shall use the abbreviations $\sigma = (\sigma_1, \ldots, \sigma_k)$, $\tau = (\tau_1, \ldots, \tau_n)$, $z = (z_1, \ldots, z_k)$. By [Pr1, Lemma 2.5] we have

$$(7) \qquad p_*(g(\sigma, \tau)) = \sum_I \frac{g(\tau_I, \tau)}{\prod_{j \notin I} \prod_{i \in I} (\tau_j - \tau_i)}$$

where the summation is over k-element subsets $I = \{s_1, \ldots, s_k\}$ of $\{1, \ldots, n\}$ and $\tau_I = (\tau_{s_1}, \ldots, \tau_{s_k})$. Define

$$H = (-1)^{k(n-k)} g(z, \tau) \prod_{1 \le i < j \le k} (z_j - z_i) \cdot \frac{z_1^{k-1} z_2^{k-2} \ldots z_{k-1}}{\prod_{j=1}^n \prod_{i=1}^k (z_i - \tau_j)}$$

and consider the differential form $\omega = H dz_1 \wedge \ldots \wedge dz_k$. Let $R = \mathrm{Res}_{z_k=\infty} \mathrm{Res}_{z_{k-1}=\infty} \ldots \mathrm{Res}_{z_1=\infty}(\omega)$.

First we calculate R by applying the Residue Theorem (the sum of the residues of a meromorphic form on the Riemann sphere is 0) for $z_1, z_2, \ldots, z_k$. We obtain

$$R = (-1)^k \sum_{s_k} \sum_{s_{k-1}} \ldots \sum_{s_1} \mathrm{Res}_{z_k = \tau_{s_k}} \mathrm{Res}_{z_{k-1} = \tau_{s_{k-1}}} \ldots \mathrm{Res}_{z_1 = \tau_{s_1}}(\omega).$$

The terms corresponding to choices with non-distinct s_j's is 0, due to the factor $\prod(z_j - z_i)$ in the numerator of ω. Thus we have

$$R = (-1)^{k(n-k)+k} \sum_I \sum_{w \in S_k} \frac{g(\tau_I, \tau) \prod_{i<j} (\tau_{w(s_j)} - \tau_{w(s_i)}) \tau_{w(s_1)}^{k-1} \tau_{w(s_2)}^{k-2} \cdots \tau_{w(s_{k-1})}}{\prod_{i \neq j} (\tau_{w(s_j)} - \tau_{w(s_i)}) \prod_{j \notin I} \prod_{i=1}^{k} (\tau_{w(s_i)} - \tau_j)},$$

where the summation is over k-element subsets $I = \{s_1, \ldots, s_k\} \subset \{1, \ldots, n\}$. This further equals

$$R = (-1)^k \sum_I \left(\frac{g(\tau_I, \tau)}{\prod_{j \notin I} \prod_{i \in I} (\tau_j - \tau_i)} \underbrace{\sum_{w \in S_k} \frac{\tau_{w(s_1)}^{k-1} \tau_{w(s_2)}^{k-2} \cdots \tau_{w(s_{k-1})}}{\prod_{i>j} (\tau_{w(s_j)} - \tau_{w(s_i)})}}_{(*)} \right).$$

However, the sum marked by $(*)$ is equal to 1—because of the well known product form of a Vandermonde determinant—, and using (7) we obtain that $p_*(g(\sigma, \tau)) = (-1)^k R$. Calculating the residues at infinity as a coefficient of the Laurent expansion we get

$$p_*(g(\sigma, \tau)) = (-1)^k R = \left\{ H \cdot \prod_{i=1}^{k} z_i \right\}_{z_1^0 \ldots z_k^0},$$

where $\{\ \}_{z_1^0 \ldots z_k^0}$ means the constant term of the Laurent-expansion in the $|z_i| > |\tau_j|$ (for all i, j) region. This proves (6) for $r = 2$.

For $r > 2$ the push-forward map p_* can be factored as $p_{1*} \circ p_{2*} \circ \ldots \circ p_{r*}$ for the Grassmanian fibrations

$$p_i : \mathcal{F}_{k_i, k_{i+1}, \ldots, k_r}(V) \to \mathcal{F}_{k_{i+1}, \ldots, k_r}(V),$$

with the notation $\mathcal{F}_\emptyset(V) = X$. The map p_i is a special case of the construction in the theorem for $r = 2$ and the tautological rank k_{i+1} bundle over $\mathcal{F}_{k_{i+1}, \ldots, k_r}(V)$. Hence p_{i*} can be computed with the formula in the theorem (as it is proved for $r = 2$ above). The iterated application of (6) for $r = 2$ gives the general (6), which completes the proof of the theorem. $\square$

Theorem 4.4. *We have*

$$[\overline{\Sigma_{e,f}^r}] = (-1)^{d+1} \left\{ \frac{h_r|_{\alpha_i \mapsto \alpha_i - z/2} \cdot \prod_{1 \leq i < j \leq d} (1 - \frac{u_i}{u_j})}{z^{d-1} \prod_{j=1}^{d} (1 - \frac{u_j}{z})} \cdot \prod_{j=1}^{d} \sum_{i=0}^{\infty} \frac{c_i(F^\vee - \mathrm{Sym}^2 E^\vee)}{u_j^i} \right\}_{z^0 u^0},$$

where $\{P\}_{z^0 u^0}$ means the constant term in P with respect to z and $u_1, \ldots, u_d$.

Proof. The formula (5) for $[\overline{\Sigma_{e,f}^r}]$ is the Atiyah-Bott localization formula for the equivariant push-forward $p_*([Y \subseteq \mathcal{F}]e(\mathrm{Hom}(\mathcal{D}, F)))$, where $\mathcal{D}$ is the tautological rank d bundle over $\mathcal{F}$, and $p : \mathcal{F} \to \mathrm{pt}$. Calculating the equivariant push-forward p_* with the formula in Lemma 4.3, we obtain

$$(8) \qquad (-1)^{d\binom{e+1}{2} - d^2 + d - 1} \left\{ \frac{h_r|_{\alpha_i \mapsto \alpha_i - z/2} \prod_{i=1}^{f} \prod_{j=1}^{d} (\beta_i - u_j) \prod_{1 \leq i < j \leq d} (1 - \frac{u_i}{u_j})}{z^{d-1} (u_1 \cdots u_d)^{\binom{e+1}{2} - d} \prod_{j=1}^{d} (1 - \frac{u_j}{z}) \prod_{j=1}^{d} \prod_{\epsilon \in W} (1 - \frac{\epsilon}{u_j})} \right\}_{z^0 u^0}.$$

Observing that

$$\prod_{j=1}^{d} \frac{\prod_{i=1}^{f}(\beta_i - u_j)}{\prod_{\epsilon \in W}(1 - \epsilon/u_j)} = (-1)^{df} \prod_{j=1}^{d} u_j^f \prod_{j=1}^{d} \frac{\prod_{i=1}^{f}(1 - \beta_i/u_j)}{\prod_{\epsilon \in W}(1 - \epsilon/u_j)}$$

$$= (-1)^{df} \prod_{j=1}^{d} u_j^f \sum_{i=0}^{\infty} \frac{c_i(F^\vee - \mathrm{Sym}^2 E^\vee)}{u_j^i},$$

and that $f = \binom{e+1}{2} - d$, we have that (8) further equals the formula in the theorem. $\qquad\square$

4.4. **The divisorial case.** The residue formula of Theorem 4.4 is more manageable in case the codimension of $\Sigma_{e,f}^r$ is 1—the case relevant for most applications given in this paper. After two technical lemmas we will provide a simple formula for the $[\overline{\Sigma_{e,f}^r}]$ in this case.

Lemma 4.5. *For the z-expansion of the polynomial $h_r|_{\alpha_i \mapsto \alpha_i - z/2}$ we have*

$$(9) \qquad h_r|_{\alpha_i \mapsto \alpha_i - z/2} = (-1)^{\binom{r+1}{2}} \left(A_e^r z^{\binom{r+1}{2}} + B_e^r \cdot \sum_{i=1}^{e} \alpha_i \cdot z^{\binom{r+1}{2}-1} + \mathrm{l.o.t.} \right)$$

where

$$A_e^r = 2^{-\binom{r}{2}} \det\left(\binom{e}{r+1-2i+j} \right)_{i,j=1,\ldots,r} = \frac{\binom{e}{r}\binom{e+1}{r-1}\cdots\binom{e+r-1}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}\cdots\binom{2r-1}{r-1}},$$

$$B_e^r = -\frac{2}{e}\binom{r+1}{2} A_e^r.$$

Proof. The polynomial h_r is a homogeneous degree $\binom{r+1}{2}$ symmetric polynomial in the $\alpha_1,\ldots,\alpha_e$ variables. Hence the expansion (9) must hold for some numbers A_e^r, B_e^r. We will calculate them via the substitution $\alpha_1 = \cdots = \alpha_e$. Let $D = \det\left(\binom{e}{r+1-2i+j} \right)_{i,j=1,\ldots,r}$. From the definition of h_r we see that $h_r(\underbrace{\alpha,\ldots,\alpha}_{e}) = 2^r D \alpha^{\binom{r+1}{2}}$, and hence, for the z-expansion of $h_r(\alpha - \frac{z}{2}, \ldots, \alpha - \frac{z}{2})$ we get

$$2^r D \left(-\frac{1}{2}\right)^{\binom{r+1}{2}} z^{\binom{r+1}{2}} + 2^r D \binom{r+1}{2} \left(-\frac{1}{2}\right)^{\binom{r+1}{2}-1} \frac{1}{e}(e\alpha) z^{\binom{r+1}{2}-1} + \mathrm{l.o.t.},$$

which proves the first expression for A_e^r and the expression for B_e^r. The equivalence of the two displayed expressions for A_e^r is proved in [HT, Proposition 12]. $\qquad\square$

Lemma 4.6. *We have*

$$(10) \qquad \prod_{1 \le i < j \le d} \left(1 - \frac{u_i}{u_j}\right) = 1 - \sum_{i=1}^{d-1} \frac{u_i}{u_{i+1}} + Q,$$

where Q is the sum of u-monomials in which the degree of the denominator is at least two. Also,

$$\left(\sum_{i=1}^{d} u_i\right) \cdot \prod_{1 \le i < j \le d} \left(1 - \frac{u_i}{u_j}\right) = u_d + \mathrm{fractions},$$

where fractions *stands for terms of monomials with at least one u_i in the denominator.*

For example, if $d = 3$ then we have

$$\left(1 - \frac{u_1}{u_2}\right)\left(1 - \frac{u_1}{u_3}\right)\left(1 - \frac{u_2}{u_3}\right) = 1 - \frac{u_1}{u_2} - \frac{u_2}{u_3} + \underbrace{\left(\frac{u_1 u_2}{u_3^2} + \frac{u_1^2}{u_2 u_3} - \frac{u_1^2}{u_3^2}\right)}_{Q},$$

and $(u_1 + u_2 + u_3)\prod_{i<j\le 3}(1 - u_i/u_j) = u_3 +$fractions.

Proof. Arguing by induction on d we have that the left hand side of (10) is

$$\left(1 - \sum_{i=1}^{d-2}\frac{u_i}{u_{i+1}} + Q'\right)\prod_{i=1}^{d-1}\left(1 - \frac{u_i}{u_d}\right) = \left(1 - \sum_{i=1}^{d-2}\frac{u_i}{u_{i+1}} + Q_1\right)\left(1 - \sum_{i=1}^{d-1}\frac{u_i}{u_d} + Q_2\right) =$$

$$1 - \sum_{i=1}^{d-2}\frac{u_i}{u_{i+1}} - \sum_{i=1}^{d-1}\frac{u_i}{u_d} + \sum_{i=1}^{d-2}\frac{u_i}{u_d} + Q = 1 - \sum_{i=1}^{d-1}\frac{u_i}{u_{i+1}} + Q,$$

where Q_1 and Q_2 are sums of terms that multiplied with anything in the other factor will result in monomials with denominator degree at least 2.

The second statement of the lemma follows directly from the first one. □

We now determine the class of $\overline{\Sigma_{e,f}^r}$ when it is a divisor, which leads to a proof of Theorem 1.1.

Theorem 4.7. *Assume that $\overline{\Sigma_{e,f}^r}$ is a divisor, that is,*

$$(11) \qquad \binom{r+1}{2} - d + 1 = \binom{r+1}{2} - \binom{e+1}{2} + f + 1 = 1.$$

Then

$$(12) \qquad [\overline{\Sigma_{e,f}^r}] = A_e^r\left(c_1(F) - \frac{2f}{e}c_1(E)\right).$$

Proof. Under the assumption (11) Theorem 4.4 reads

$$[\overline{\Sigma_{e,f}^r}] = -\left\{\left(A_e^r z^1 + B_e^r \cdot \sum_{i=1}^{e}\alpha_i \cdot z^0 + \text{l.o.t.}\right) \cdot \frac{\prod_{1\le i<j\le d}\left(1 - \frac{u_i}{u_j}\right)}{\prod_{j=1}^{d}\left(1 - \frac{u_j}{z}\right)} \cdot \prod_{j=1}^{d}\sum_{i=0}^{\infty}\frac{c_i(F^\vee - \text{Sym}^2 E^\vee)}{u_j^i}\right\}_{z^0, u^0}.$$

Looking at the z-exponents, this is further equal to

$$-\left\{\left(A_e^r\sum_{j=1}^{d}u_j\prod_{1\le i<j\le d}\left(1 - \frac{u_i}{u_j}\right) + B_e^r\sum_{i=1}^{e}\alpha_i\prod_{1\le i<j\le d}\left(1 - \frac{u_i}{u_j}\right)\right) \cdot \prod_{j=1}^{d}\sum_{i=0}^{\infty}\frac{c_i(F^\vee - \text{Sym}^2 E^\vee)}{u_j^i}\right\}_{u^0}.$$

Looking at u-exponents, and using Lemma 4.6, this is further equal to

$$-\left\{\left(A_e^r\left(u_d + \text{fractions}\right) + B_e^r\sum_{i=1}^{e}\alpha_i(1 + \text{fractions})\right) \cdot \prod_{j=1}^{d}\sum_{i=0}^{\infty}\frac{c_i(F^\vee - \text{Sym}^2 E^\vee)}{u_j^i}\right\}_{u^0},$$

where the term *fractions* stands for terms with at least one u_j variable in the denominator. Hence the formula further equals

$$-A_e^r c_1(F^\vee - \mathrm{Sym}^2 E^\vee) - B_e^r c_1(E).$$

Using that $c_1(F^\vee - \mathrm{Sym}^2 E^\vee) = c_1(F^\vee) - c_1(\mathrm{Sym}^2 E^\vee) = -c_1(F) + (e+1)c_1(E)$, we obtain

$$[\overline{\Sigma_{e,f}^r}] = A_e^r c_1(F) - (A_e^r(e+1) + B_e^r) c_1(E).$$

Using the divisorial condition (11), this expression can be rewritten as (12). $\qquad\square$

Example 4.8. We have

$$[\overline{\Sigma_{2,2}^1}] = -4c_1(E) + 2c_1(F), \quad [\overline{\Sigma_{3,5}^1}] = -10c_1(E) + 3c_1(F), \quad [\overline{\Sigma_{4,9}^1}] = -18c_1(E) + 4c_1(F),$$

$$[\overline{\Sigma_{3,3}^2}] = -8c_1(E) + 4c_1(F), \quad [\overline{\Sigma_{4,7}^2}] = -35c_1(E) + 10c_1(F), \quad [\overline{\Sigma_{5,12}^2}] = -96c_1(E) + 20c_1(F).$$

5. Loci defined by discriminant

Let $e \geq 2$ and use the short hand notation $N = \binom{e+1}{2} - 2$. Let $E := \mathbb{C}^e$ be the standard representations of $GL_e(\mathbb{C})$. Consider the tautological exact sequence of $GL_e(\mathbb{C})$-equivariant bundles $0 \to S \to S^2 E \to Q \to 0$ over the Grassmannian $\mathrm{Gr}(2, \mathrm{Sym}^2 E)$ of 2-planes in $\mathrm{Sym}^2 E$. Recall that we have introduced in Example 2.3 the $GL_e(\mathbb{C})$-invariant subset $\Sigma^1 \subseteq \mathrm{Sym}^2 E$ as the set of *degenerate* symmetric 2-forms. Define

$$\Phi_e := \left\{ W \in \mathrm{Gr}(2, \mathrm{Sym}^2 E) : \mathbf{P}(W) \text{ is tangent to } \mathbf{P}(\Sigma^1) \right\} \subseteq \mathrm{Gr}(2, \mathrm{Sym}^2 E).$$

Notice that we require $\mathbf{P}(W)$ to be tangent to $\mathbf{P}(\Sigma^1)$ (which is a smooth but not closed subvariety of $\mathbf{P}(\mathrm{Sym}^2 E)$), that is we require that the projective line $\mathbf{P}(W)$ intersect the smooth part of $\mathbf{P}(\overline{\Sigma^1})$, and the intersection be tangential. Our goal in this section is to calculate the equivariant fundamental class $[\overline{\Phi}_e] \in H^2(\mathrm{Gr}(2, \mathrm{Sym}^2 E))$.

Denote the $GL_e(\mathbb{C})$-equivariant Chern roots of S by γ_1, γ_2, those of E by $\alpha_1, \ldots, \alpha_e$, and those of Q by $\beta_1, \ldots, \beta_N$. The $GL_e(\mathbb{C})$-equivariant cohomology ring of $\mathrm{Gr}(2, \mathrm{Sym}^2 E)$ can be presented by one of

$$\mathbb{C}[\alpha_1, \ldots, \alpha_e, \gamma_1, \gamma_2]^{S_e \times S_2}/\text{relations} \qquad \text{or} \qquad \mathbb{C}[\alpha_1, \ldots, \alpha_e, \beta_1, \ldots, \beta_N]^{S_e \times S_N}/\text{relations}.$$

Since in each case the relations have degree > 2, the class $[\overline{\Phi}_e]$ is a well-defined linear polynomial $f(\alpha_1, \ldots, \alpha_e, \gamma_1, \gamma_2)$ in the α and γ variables, or a well-defined linear polynomial $g(\alpha_1, \ldots, \alpha_e, \beta_1, \ldots, \beta_N)$ in the α and β variables. The exactness of the $0 \to S \to \mathrm{Sym}^2 E \to Q \to 0$ sequence implies

$$(13) \qquad \sum_{i=1}^{2} \gamma_i + \sum_{i=1}^{N} \beta_i = \sum_{1 \leq i \leq j \leq e} (\alpha_i + \alpha_j),$$

hence $f(\alpha, \gamma)$ and $g(\alpha, \beta)$ determine each other.

The polynomials $f(\alpha, \gamma)$ and $g(\alpha, \beta)$ have "degeneracy locus" interpretations as follows.

- Consider the $GL_2(\mathbb{C}) \times GL_e(\mathbb{C})$ representation $\mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E)$ given by the following action $(A, B) \cdot \phi := \mathrm{Sym}^2 B \circ \phi \circ A^{-1}$, and the locus

$$\Phi_e' := \left\{ \phi \in \mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E) : \mathrm{rk}(\phi) = 2 \text{ and } \mathbf{P}(\mathrm{Im}(\phi)) \text{ is tangent to } \mathbf{P}(\Sigma^1) \right\}.$$

Then

$$[\overline{\Phi'_e}] = f(\alpha, \gamma) \in H^*_{GL_2(\mathbb{C}) \times GL_e(\mathbb{C})}(\mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E))$$
$$= \mathbb{C}[\gamma_1, \gamma_2, \alpha_1, \dots, \alpha_e]^{S_2 \times S_e}.$$

- Consider the $GL_e(\mathbb{C}) \times GL_N(\mathbb{C})$ representation $\mathrm{Hom}(\mathrm{Sym}^2 E, \mathbb{C}^N)$ given by the following action $(A, B) \cdot \phi := B \circ \phi \circ \mathrm{Sym}^2 A^{-1}$, and the locus

$$\Phi''_e := \Big\{ \phi \in \mathrm{Hom}(\mathrm{Sym}^2 E, \mathbb{C}^N) : \dim \mathrm{Ker}(\phi) = 2 \ \text{ and } \ \mathbf{P}(\mathrm{Ker}(\phi)) \text{ is tangent to } \mathbf{P}(\Sigma^1) \Big\}.$$

Then

$$[\overline{\Phi''_e}] = g(\alpha, \beta) \in H^*_{GL_e(\mathbb{C}) \times GL_N(\mathbb{C})}\Big(\mathrm{Hom}(\mathrm{Sym}^2 E, \mathbb{C}^N)\Big)$$
$$= \mathbb{C}[\alpha_1, \dots, \alpha_e, \beta_1, \dots, \beta_N]^{S_e \times S_N}.$$

Theorem 5.1. *We have*

$$f(\alpha, \gamma) = (e - 1)\left(4 \sum_{i=1}^{n} \alpha_i - e \sum_{i=1}^{2} \gamma_i\right).$$

Proof. For $\phi \in \mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^2 E)$ let $\phi((1, 0)) = K$, $\phi((0, 1)) = L$. The equation of the hypersurface $\overline{\Phi'_e}$ in terms of the entries of K and L is the *discriminant* of the polynomial $\det(\lambda K + L) = a_e(K, L)\lambda^e + a_{e-1}(K, L)\lambda^{e-1} + \dots + a_0(K, L)$.

Consider the Sylvester matrix form of the discriminant

$$\frac{1}{a_e} \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{e-1} & a_e & & & \\ & a_0 & a_1 & & \cdots & a_{e-1} & a_e & & \\ & & \ddots & \ddots & & & \ddots & \ddots & \\ & & & a_0 & a_1 & & \cdots & a_{e-1} & a_e \\ a_1 & 2a_2 & \cdots & & ea_e & & & & \\ & a_1 & 2a_2 & \cdots & & ea_e & & & \\ & & \ddots & \ddots & & & \ddots & & \\ & & \ddots & \ddots & & & & \ddots & \\ & & & a_1 & 2a_2 & \cdots & & & ea_e \end{pmatrix}_{((e-1)+e) \times ((e-1)+e)}.$$

One of the terms of its expansion (the one coming from the main diagonal) is a non-zero constant times $(a_0 a_e)^{e-1}$. We have $a_e(K, L) = \det(K)$ and $a_0(K, L) = \det(L)$. Hence one of the monomials appearing in the discriminant is $(\prod_{i=1}^{e} K_{ii})^{e-1}(\prod_{i=1}^{e} L_{ii})^{e-1}$. The weight of this monomial is

$$(14) \qquad (e - 1)\left(\sum_{i=1}^{e}(2\alpha_i - \gamma_1)\right) + (e - 1)\left(\sum_{i=1}^{e}(2\alpha_i - \gamma_2)\right).$$

Since $\overline{\Phi'_e}$ is invariant, all other terms must have the same weight, and this weight is the equivariant fundamental class of $\overline{\Phi'_e}$. Expression (14) simplifies to the formula in the theorem. $\qquad \square$

Remark 5.2. Instead of the Sylvester matrix we could have used specializations of advanced equivariant formulas for more general discriminants, see for instance [FNR2].

Theorem 5.3. *We have*

$$g(\alpha, \beta) = (e - 1)\left(e \sum_{i=1}^{N} \beta_i - (e^2 + e - 4) \sum_{i=1}^{e} \alpha_i\right).$$

Proof. The statement follows from Theorem 5.1 using relation (13). □

This completes the proof of Theorem 1.2.

6. The Petri divisor on the moduli space of curves

An immediate application of the Theorem 1.1 concerns the calculation of the class of the Petri divisor on $\overline{\mathcal{M}}_g$ consisting of genus g curves whose canonical model lies on a rank 3 quadric. We fix some notation. For $1 \leq i \leq \lfloor\frac{g}{2}\rfloor$, let $\Delta_i \subseteq \overline{\mathcal{M}}_g$ be the boundary divisor of $\overline{\mathcal{M}}_g$ whose general point is a union of two smooth curves of genera i and $g - i$ meeting in one point. We denote by Δ_0 the closure of the locus of irreducible stable curves of genus g. As customary, we set $\delta_i = [\Delta_i]_{\mathbb{Q}} \in CH^1(\overline{\mathcal{M}}_g)$ for $i = 0, \ldots, \lfloor\frac{g}{2}\rfloor$ and denote by

$$\delta := \delta_0 + \delta_1 + \cdots + \delta_{\lfloor\frac{g}{2}\rfloor}$$

the class of the total boundary. Often we work with the partial compactification $\widetilde{\mathcal{M}}_g := \mathcal{M}_g \cup \Delta_0$, for which $CH^1(\widetilde{\mathcal{M}}_g) = \mathbb{Q}\langle\lambda, \delta_0\rangle$.

Definition 6.1. *For a projective variety X and a line bundle $L \in \text{Pic}(X)$, for each integer $k \geq 0$ we denote by $I_{X,L}(k) := \text{Ker}\{\text{Sym}^k H^0(X, L) \to H^0(X, L^{\otimes k})\}$ and set $I_{X,L} := \oplus_{k \geq 0} I_{X,L}(k)$.*

We fix a smooth non-hyperelliptic curve C of genus g. From M. Noether's Theorem [ACGH] the multiplication map $\text{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ is surjective. The space $I_C(2) = I_{C,\omega_C}(2)$ of quadrics containing the canonical curve $C \hookrightarrow \mathbf{P}^{g-1}$ has dimension

$$\dim I_C(2) = \binom{g-2}{2}.$$

We conclude that the locus $\mathcal{GP}_g$ of curves whose canonical model lies on a rank 3 quadric is expected to be a divisor. Via the Base Point Free Pencil Trick [ACGH] p. 126, this expectation can be confirmed.

Proposition 6.2. *The locus $\mathcal{GP}_g$ coincides set-theoretically with the divisor of curves $[C] \in \mathcal{M}_g$ having a pencil A such that the Petri map $\mu(A) : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$ is not injective.*

Proof. Let A be a line bundle on C with $h^0(C, A) = 2$. Denote by $F := \text{bs } |A|$ its base locus and set $B := A(-F)$. Thus $H^0(C, B) \cong H^0(C, A)$. Applying the Base Point Free Pencil Trick, we obtain

$$\text{Ker}(\mu(A)) \cong H^0(\omega_C \otimes A^{-2}(F)) \cong H^0(C, \omega_C \otimes B^{-2}(-F)).$$

Thus if $\mu(A)$ is not injective, by possibly enlarging the effective divisor F, we find there exists a base point free pencil B on C and an effective divisor F, such that $\omega_C = B^2(F)$.

Assume the canonical curve $C \subseteq \mathbf{P}^{g-1}$ lies on a rank 3 quadric Q. Denote by $F := C \cdot \text{Sing}(Q)$, where $\text{Sing}(Q) \cong \mathbf{P}^{g-4}$. Then if B is the pull back to C of the unique ruling of Q, we obtain the relation $\omega_C = \mathcal{O}_C(1) \cong B^2(F)$. Setting $A := B(F)$, we obtain that $\mu(A)$ is not injective.

To conclude that $\mathcal{GP}_g$ is a divisor in $\mathcal{M}_g$, we invoke the Gieseker-Petri Theorem which asserts that the Petri map $\mu(A)$ is *injective* for *every* line bundle A on a general curve C of genus g. □

The divisor $\mathcal{GP}_g$ can be extended over $\overline{\mathcal{M}}_g$. Let $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ the universal curve of genus g. We denote by $\mathbb{E} := \pi_*(\omega_\pi)$ the Hodge bundle on $\overline{\mathcal{M}}_g$, having fibres $\mathbb{E}[C] := H^0(C, \omega_C)$. Let $\mathbb{F} := \pi_*(\omega_\pi^{\otimes 2})$. Both sheaves $\mathbb{E}$ and $\mathbb{F}$ are locally free over $\overline{\mathcal{M}}_g$ and denote by

$$\phi : \mathrm{Sym}^2(\mathbb{E}) \to \mathbb{F}$$

the morphism globalizing the multiplication maps $\phi_C : \mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$, as the curve $[C] \in \overline{\mathcal{M}}_g$ varies in moduli. Set

$$\widetilde{\mathcal{GP}}_g := \Big\{ [C] \in \overline{\mathcal{M}}_g : \exists\, 0 \neq q \in \mathrm{Ker}(\phi_C), \ \ \mathrm{rk}(q) \leq 3 \Big\}.$$

Clearly $\widetilde{\mathcal{GP}}_g$ is a divisor on $\overline{\mathcal{M}}_g$ and $\widetilde{\mathcal{GP}}_g \cap \mathcal{M}_g = \mathcal{GP}_g$. For a generic point $[C := C_1 \cup_p C_2] \in \Delta_i$, where C_1 and C_2 are smooth curves of genus i and $g - i$ respectively meeting at one point p, one has $H^0(C, \omega_C) \cong H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})$, that is, every section from $H^0(C, \omega_C)$ vanishes at p. On the other hand,

$$H^0(C, \omega_C^2) \cong \mathrm{Ker}\big\{ H^0(C_1, \omega_{C_1}^2(2p)) \oplus H^0(C_2, \omega_{C_2}^2(2p)) \to \mathbb{C}_p \big\},$$

that is, there exists quadratic differentials on C *not* vanishing at p. It follows that the multiplication map ϕ_C is not surjective, hence for dimension reasons $\mathrm{Ker}(\phi_C)$ contains quadrics of rank 3, whenever $[C] \in \Delta_i$. Thus $\Delta_i \subseteq \widetilde{\mathcal{GP}}_g$, for $i = 1, \ldots, \lfloor \frac{g}{2} \rfloor$. On the other hand, Δ_0 is not contained in $\widetilde{\mathcal{GP}}_g$. In fact, the generic g-nodal rational curve satisfies the Green-Lazarsfeld property $N_{\lfloor \frac{g-3}{2} \rfloor}$, that is, a much stronger property than projective normality, see [V2]. Denoting by $\overline{\mathcal{GP}}_g$ the closure of the Petri divisor $\mathcal{GP}_g$ inside $\overline{\mathcal{M}}_g$, we thus have an equality of effective divisors on $\overline{\mathcal{M}}_g$

$$\widetilde{\mathcal{GP}}_g = \overline{\mathcal{GP}}_g + \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} b_i \Delta_i,$$

where $b_i \geq 1$, for all $i \geq 1$. The class of $\widetilde{\mathcal{GP}}_g$ can now be easily determined.

Proof of Theorem 1.6. We apply Theorem 1.1 in the case of the morphism $\phi : \mathrm{Sym}^2(\mathbb{E}) \to \mathbb{F}$ over $\overline{\mathcal{M}}_g$ given by multiplication. We have $c_1(\mathbb{E}) = \lambda$, whereas by the Grothendieck-Riemann-Roch calculation carried out in [Mum] Theorem 5.10, one has $c_1(\mathbb{F}) = \lambda + \kappa_1 = 13\lambda - \delta$. $\qquad\square$

The Petri divisor decomposes into components depending on the degree of the pencil for which the Petri Theorem fails. For $\lfloor \frac{g+2}{2} \rfloor \leq k \leq g-1$, we denote by $D_{g,k}$ the locus of curves $[C] \in \mathcal{M}_g$ for which there exists a *base point free* pencil $A \in W_k^1(C)$ such that $\mu(A)$ is not injective. It is shown in [F3] that $D_{g,k}$ has at least one divisorial component. In light of Proposition 6.2, we have the decomposition

$$\tag{15} \overline{\mathcal{GP}}_g = \sum_{k=\lfloor \frac{g+2}{2} \rfloor}^{g-1} a_{g,k} \overline{D}_{g,k}.$$

It is an interesting open question to determine the classes $[\overline{D}_{g,k}] \in CH^1(\overline{\mathcal{M}}_g)$ and their multiplicities $a_{g,k}$. For birational geometry application, it is more relevant to compute the slopes $s(\overline{D}_{g,k})$. Few of the individual divisors $D_{g,k}$ are well understood.

By the proof of Proposition 6.2, the divisor $D_{g,g-1}$ consists of curves with an even theta-characteristic $\vartheta \in \mathrm{Pic}^{g-1}(C)$ such that $h^0(C, \vartheta) \geq 2$. The class of its compactification in $\overline{\mathcal{M}}_g$ has been computed in

[T] and we have:

$$[\overline{D}_{g,g-1}] = 2^{g-3}\Big((2^g+1)\lambda - 2^{g-3}\delta_0 - \sum_{i=1}^{\lfloor\frac{g}{2}\rfloor}(2^{g-i}-1)(2^i-1)\delta_i\Big). \tag{16}$$

When k is minimal, for odd $g = 2k-1$, the locus $D_{g,k}$ is the Hurwitz divisor of curves of gonality at most k. Its compactification is the image of the space of admissible covers $\overline{\mathcal{H}}_k$ defined in the Introduction. Harris and Mumford [HM] computed its class, on their way to show that $\overline{\mathcal{M}}_g$ is of general type for large odd genus $g \geq 25$:

$$[\overline{D}_{2k-1,k}] = \frac{1}{(2k-2)(2k-3)}\binom{2k-2}{k-1}\Big(6(k+1)\lambda - k\delta_0 - \sum_{i=1}^{k-1}3i(2k-i-1)\delta_i\Big). \tag{17}$$

For even genus $g = 2k$, the divisor $\overline{D}_{2k,k+1}$ can be viewed as the branch map of the generically finite cover $\overline{\mathcal{H}}_{2k,k+1} \to \overline{\mathcal{M}}_{2k}$ from the space of admissible covers of degree $k+1$. The calculation of its class in [EH] Theorem 2 has been instrumental in proving that $\overline{\mathcal{M}}_g$ is of general type for even genus $g \geq 24$:

$$[\overline{D}_{2k,k+1}] = \frac{2(2k-2)!}{(k-1)!(k+1)!}\Big((6k^2+13k+1)\lambda - k(k+1)\delta_0 - (2k-1)(3k+1)\delta_1 - \cdots\Big). \tag{18}$$

The only case when k is not extremal has been treated in [F3] and it concerns the divisor $D_{2k-1,k+1}$. It is shown in [F3] Corollary 0.6 that its slope equals

$$s(\overline{D}_{2k-1,k+1}) = \frac{6k^2+14k+3}{k(k+1)}. \tag{19}$$

In the range $g \leq 7$, these known cases exhaust all Gieseker-Petri divisors and we can compare Theorem 1.6 with the previously mentioned formulas (16), (17), (18). We denote by $\widetilde{D}_{g,k}$ the closure of $D_{g,k}$ in $\widetilde{\mathcal{M}}_g$. In order to determine the slope of $\overline{D}_{g,k}$, it suffices to compute the class $[\widetilde{D}_{g,k}] \in CH^1(\widetilde{\mathcal{M}}_g)$, for as in the case of $\overline{\mathcal{GP}}_g$, the δ_0-coefficient is smaller in absolute value than the higher boundary coefficients in the expansion of $[\overline{D}_{g,k}]$ in terms of the generators of $CH^1(\overline{\mathcal{M}}_g)$.

For $g = 4$, there is only one component and we obtain $[\widetilde{\mathcal{GP}}_4] = [\widetilde{D}_{4,3}] = 34\lambda - 4\delta_0 \in CH^1(\widetilde{\mathcal{M}}_4)$. For $g = 5$, we obtain $[\widetilde{\mathcal{GP}}_5] = [\widetilde{D}_{5,4}] + 4[\widetilde{D}_{5,3}] = 4(41\lambda - 5\delta_0)$, whereas for $g = 6$, we find

$$[\widetilde{\mathcal{GP}}_6] = [\widetilde{D}_{6,5}] + 4[\widetilde{D}_{6,4}] = 8(112\lambda - 14\delta_0) \in CH^1(\widetilde{\mathcal{M}}_6).$$

Finally, in the case $g = 7$, there are three Petri divisors and we obtain

$$[\widetilde{\mathcal{GP}}_7] = [\widetilde{D}_{7,6}] + 4[\widetilde{D}_{7,5}] + 16[\widetilde{D}_{7,4}] = 96(55\lambda - 7\delta_0) \in CH^1(\widetilde{\mathcal{M}}_7).$$

Based on this formulas for small genus, we make the following conjecture, though we admit that the evidence for it is rather moderate.

Conjecture 6.3. *One has $a_{g,k} = 4^{g-1-k}$ for $\frac{g+2}{2} \leq k \leq g-1$, that is, the following holds:*

$$[\widetilde{\mathcal{GP}}_g] = \sum_{i=1}^{\lceil\frac{g-2}{2}\rceil} 4^{i-1}[\widetilde{D}_{g,g-i}] \in CH^1(\widetilde{\mathcal{M}}_g).$$

7. Effective divisors of small slope on $\overline{\mathcal{M}}_g$

We now present an infinite series of effective divisors on $\overline{\mathcal{M}}_g$ of slope less than $6 + \frac{12}{g+1}$, which recall, is the slope of all the Brill-Noether divisors. We fix integers $r \geq 3$ and $s \geq 1$ and set

$$g := rs + s \quad \text{and} \quad d := rs + r.$$

Observe that $\rho(g, r, d) = g - (r+1)(g-d+r) = 0$, hence by general Brill-Noether Theory a general curve of genus g has a finite number of linear systems of type $\mathfrak{g}_d^r$. Let $\mathcal{M}_g^\sharp$ the open substack of $\mathcal{M}_g$ classifying smooth genus g curves C such that $W_{d-1}^r(C) = \emptyset$, $W_d^{r+1}(C) = \emptyset$ and furthermore $H^1(C, L^{\otimes 2}) = 0$, for every $L \in W_d^r(C)$. Then $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}_g^\sharp, \mathcal{M}_g) \geq 2$. For codimension one calculation, one makes no difference between $\mathcal{M}_g$ and $\mathcal{M}_g^\sharp$. We denote by $\mathfrak{G}_{g,d}^r$ the stack parametrizing pairs $[C, L]$, with $[C] \in \mathcal{M}_g^\sharp$ and $L \in W_d^r(C)$ is a necessarily complete and base point free linear system. Let

$$\sigma : \mathfrak{G}_{g,d}^r \to \mathcal{M}_g^\sharp$$

be the natural projection. It is known from general Brill-Noether Theory that there exists a unique irreducible component of $\mathfrak{G}_{g,d}^r$ which maps dominantly onto $\mathcal{M}_g$.

We pick a general point $[C, L] \in \mathfrak{G}_{g,d}^r$ of the dominating component. It follows from the Maximal Rank Conjecture proved in this case in [F2] or [LOTZ] Theorem 1.4, that the multiplication map

$$\phi_{C,L} : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is surjective. Since $H^1(C, L^{\otimes 2}) = 0$, by Riemann-Roch, the dimension of its kernel $I_{C,L}(2)$ equals

$$\dim I_{C,L}(2) = \binom{r+2}{2} - (2d + 1 - g).$$

We impose the condition that this number equal the codimension of the space

$$\Sigma_{r+1}^{r-a-1} \subseteq \mathrm{Sym}^2 H^0(C, L)$$

of quadrics of rank at most $a + 2$ (that is, corank $r - a - 1$). Since $\mathrm{codim}(\Sigma_{r+1}^{r-a-1}) = \binom{r-a}{2}$, we obtain the following numerical constraint on s and r:

$$(20) \qquad\qquad\qquad s = \frac{a(2r - 1 - a)}{2(r-1)}.$$

For each r and s such that the equation (20) is satisfied, we consider the locus

$$Z_{r,s} := \left\{ [C, L] \in \mathfrak{G}_{g,d}^r : \exists\, 0 \neq q \in I_{C,L}(2), \quad \mathrm{rk}(q) \leq a + 2 \right\}$$

and set $D_{r,s} := \sigma_*(Z_{r,s})$. Then $D_{r,s}$ is expected to be a divisor on $\mathcal{M}_g$, that is, either it is a divisor in which case there exists a smooth curve $[C] \in \mathcal{M}_g$ such that $I_{C,L}(2)$ contains no quadrics of rank at most $a + 2$ for every $L \in W_d^r(C)$, or else $D_{r,s} = \mathcal{M}_g$. We shall determine the slope of the virtual class of its closure in $\overline{\mathcal{M}}_g$.

Before moving further, we discuss some solutions to equation (20). If $a = r - 1$ (that is, when one considers quadrics of maximal rank), then $r = 2s$ and $g = s(2s + 1)$. In this case $D_{2s,s}$ is the locus of curves $[C] \in \mathcal{M}_{s(2s+1)}$ for which there exists a linear series $L \in W_{2s(s+1)}^{2s}(C)$ such that the multiplication map $\phi_{C,L} : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is not an isomorphism. This series of

divisors has been studied in detail in [F2] Theorem 1.5, as well as in [Kh] and shown to contradict the Harris-Morrison Slope Conjecture [HMo].

The first series of genuinely new examples is when for an integer $\ell \geq 1$, we take

(21) $$s = 4\ell - 1, \quad r = 9\ell - 2, \quad a = 2(3\ell - 1), \quad \text{and} \quad g = (4\ell - 1)(9\ell - 1).$$

Specializing to the case $\ell = 1$, we obtain the following effective (virtual) divisor on $\mathcal{M}_{24}$:

$$D_{7,3} := \Big\{ [C] \in \mathcal{M}_{24} : \exists\, L \in W_{28}^7(C), \ \exists\, 0 \neq q \in I_{C,L}(2), \ \mathrm{rk}(q) \leq 6 \Big\}.$$

A second series of examples is when for an integer $\ell \geq 1$, we take the following values

(22) $$s = 3\ell + 1, \ r = 8\ell + 3, \ a = 4\ell + 1, \quad \text{and} \quad g = 4(3\ell + 1)(2\ell + 1).$$

The first example in this series appears produces an effective (virtual) divisor on $\mathcal{M}_{48}$:

$$D_{11,4} := \Big\{ [C] \in \mathcal{M}_{48} : \exists\, L \in W_{55}^{11}(C), \ \exists\, 0 \neq q \in I_{C,L}(2), \ \mathrm{rk}(q) \leq 7 \Big\}.$$

We now describe the (virtual) divisor structure of $D_{r,s}$ and set up some notation that will help compute the class of their closure in $\overline{\mathcal{M}}_g$. We introduce the partial compactification $\widetilde{\mathcal{M}}_g^\sharp$ defined as the union of $\mathcal{M}_g^\sharp$ and the open substack $\Delta_0^\sharp \subseteq \Delta_0$ classifying 1-nodal irreducible genus g curves $C' = C/p \sim q$, where $[C, p, q] \in \mathcal{M}_{g-1,2}$ is a Brill-Noether general 2-pointed curve in the sense of [EH] Theorem 1.1, together with all their degenerations consisting of unions of a smooth genus $g - 1$ curve and a nodal rational curve. Note that $\widetilde{\mathcal{M}}_g$ and $\widetilde{\mathcal{M}}_g^\sharp$ agree outside a set of codimension 2 and we identify the Picard groups of the two stacks. We denote by $\widetilde{\mathfrak{G}}_{g,d}^r$ the parameter space of pairs $[C, L]$, where $[C] \in \mathcal{M}_g^\sharp$ and L is a torsion free sheaf of rank 1 and degree d on C such that $h^0(C, L) \geq r + 1$. We still denote by $\sigma : \widetilde{\mathfrak{G}}_{g,d}^r \to \widetilde{\mathcal{M}}_g^\sharp$ the proper forgetful morphism.

We now consider the universal curve $\pi : \widetilde{\mathcal{M}}_{g,1}^\sharp \to \widetilde{\mathcal{M}}_g^\sharp$ and denote by $\mathcal{L}$ a universal bundle on the fibre product $\widetilde{\mathcal{M}}_{g,1}^\sharp \times_{\widetilde{\mathcal{M}}_g^\sharp} \widetilde{\mathfrak{G}}_{g,d}^r$. If

$$p_1 : \widetilde{\mathcal{M}}_{g,1}^\sharp \times_{\widetilde{\mathcal{M}}_g^\sharp} \widetilde{\mathfrak{G}}_{g,d}^r \to \widetilde{\mathcal{M}}_{g,1}^\sharp \quad \text{and} \quad p_2 : \widetilde{\mathcal{M}}_{g,1}^\sharp \times_{\widetilde{\mathcal{M}}_g^\sharp} \widetilde{\mathfrak{G}}_{g,d}^r \to \widetilde{\mathfrak{G}}_{g,d}^r$$

are the natural projections, then $\mathcal{E} := p_{2*}(\mathcal{L})$ and $\mathcal{F} := p_{2*}(\mathcal{L}^{\otimes 2})$ are locally free sheaves of ranks $r + 1$ and $2d + 1 - g$ respectively. Finally, we denote by

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$$

the sheaf morphism given by multiplication of sections.

Theorem 7.1. *Set $r = 9\ell - 2$ and $s = 4\ell - 1$, therefore $g = (4\ell - 1)(9\ell - 1)$, where $\ell \geq 1$. Then the virtual class of the closure of the divisor $D_{9\ell-2, 4\ell-1}$ inside $\overline{\mathcal{M}}_{(4\ell-1)(9\ell-1)}$ equals*

$$s(\overline{D}_{9\ell-2, 4\ell-1}) = \frac{a}{b},$$

where

$$a := 15116544\ell^8 - 30233088\ell^7 + 26605584\ell^6 - 13594392\ell^5 + 4419720\ell^4 - 899433\ell^3 +$$
$$105656\ell^2 - 6101\ell + 122$$

and

$$b := 2(9\ell - 2)(9\ell - 1)(15552\ell^6 - 25920\ell^5 + 17484\ell^4 - 6102\ell^3 + 1181\ell^2 - 107\ell + 2).$$

In particular, $s(\overline{D}_{9\ell-2,4\ell-1}) < 6 + \frac{12}{g+1}.$

If we look at the difference between the slope of $\overline{D}_{9\ell-2,4\ell-1}$ and that of the Brill-Noether divisors we get a slightly simpler formula:

$$s(\overline{D}_{9\ell-2,4\ell-1}) = 6 + \frac{12}{g+1} -$$

$$-\frac{(13\ell - 2)(36\ell - 13)(27\ell^2 - 19\ell + 2)(36\ell^2 - 13\ell - 1)}{2(9\ell - 2)(9\ell - 1)(15552\ell^6 - 25920\ell^5 + 17484\ell^4 - 6102\ell^3 + 1181\ell^2 - 107\ell + 2)(36\ell^2 - 13\ell + 2)}.$$

We now record the slope of the effective divisors in the second series of examples:

Theorem 7.2. *Set* $r = 8\ell + 3$ *and* $s = 3\ell + 1$, *therefore* $g = 4(3\ell + 1)(2\ell + 1)$. *Then the virtual class of the closure of the divisor* $D_{8\ell+3,3\ell+1}$ *inside* $\overline{\mathcal{M}}_{4(3\ell+1)(2\ell+1)}$ *equals*

$$s(\overline{D}_{8\ell+3,3\ell+1}) = 6 + \frac{12}{g+1} -$$

$$-\frac{(11\ell + 5)(2\ell - 1)(12\ell^2 + 10\ell + 1)(24\ell^2 + 20\ell + 3)}{(3\ell + 2)(8\ell + 3)(2304\ell^6 + 4128\ell^5 + 2992\ell^4 + 1128\ell^3 + 248\ell^2 + 41\ell + 5)(24\ell^2 + 20\ell + 5)}.$$

Proof of Theorems 7.1 and 7.2. We choose integers $r \geq 3$, $s, a \geq 1$ such that (20) holds. Recall that $d = rs + r$ and $g = rs + s$. We shall apply the techniques developed in [F2] and [Kh] in the context of Theorem 4.7. Recall that we have defined the vector bundle morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ over the parameter space $\widetilde{\mathfrak{G}}^r_{g,d}$. Applying Theorem 1.1, if $Z_{r,s}$ is a divisor on $\mathfrak{G}^r_{g,d}$, then the class of its closure $\widetilde{Z}_{r,s}$ inside $\widetilde{\mathfrak{G}}^r_{g,d}$ is given by the formula

$$(23) \qquad\qquad [\widetilde{Z}^r_{g,d}] = \alpha\Big(c_1(\mathcal{F}) - \frac{2(2d + 1 - g)}{r + 1}c_1(\mathcal{E})\Big).$$

We call the right hand side of the formula (23) the virtual class $[\widetilde{Z}^r_{g,d}]^{\mathrm{virt}}$ of the virtual divisor $\widetilde{Z}^r_{g,d}$. Following [F2] we introduce the following tautological divisor classes on $\widetilde{\mathfrak{G}}^r_{g,d}$:

$$\mathfrak{a} := (p_2)_*\Big(c_1^2(\mathcal{L})\Big), \quad \mathfrak{b} := (p_2)_*\Big(c_1(\mathcal{L}) \cdot c_1(\omega_{p_2})\Big) \quad \text{and } \mathfrak{c} := (p_2)_*\Big(c_1^2(\omega_{p_2})\Big) = \sigma^*(\kappa_1),$$

where we recall that $\kappa_1 = 12\lambda - \delta \in CH^1(\overline{\mathcal{M}}_g)$ is Mumford's class, see also [Mum].

Since $R^1(p_2)_*(\mathcal{L}^{\otimes 2}) = 0$, applying Grothendieck-Riemann-Roch to p_2, we compute

$$c_1(\mathcal{F}) = \sigma^*(\lambda) - \mathfrak{b} + 2\mathfrak{a}.$$

The push-forwards of the tautological classes $\mathfrak{a}, \mathfrak{b}$ and $c_1(E)$ under the generically finite proper morphism $\sigma : \widetilde{\mathfrak{G}}^r_{g,d} \to \widetilde{\mathcal{M}}^\sharp_g$ are determined in [F2] Section 2 and [Kh] Theorem 2.11 and we summarize the results: There exists an explicit constant $\beta \in \mathbb{Z}_{>0}$ such that

$$\sigma_*(\mathfrak{a}) = \beta\frac{d}{(g-1)(g-2)}\Big((dg^2 - 2g^2 + 8d - 8g + 4)\lambda - (dg - 2g^2 + 4d - 3g + 2)\delta_0\Big),$$

$$\sigma_*(\mathfrak{b}) = \beta\frac{d}{g-1}\Big(6\lambda - \frac{\delta_0}{2}\Big)$$

and

$$\sigma_*(c_1(\mathcal{E})) = \beta\Big(-\frac{r(r+2)(r^2s^3+2rs^3-r^2s+6rs^2+s^3-2rs+6s^2-8r+3s-8)}{2(r+s+1)(rs+s-2)(rs+s-1)}\lambda+$$

$$+\frac{r(s-1)(s+1)(r+2)(r+1)(rs+s+4)}{12(r+s+1)(rs+s-2)(rs+s-1)}\delta_0\Big).$$

We substitute these formulas in (23) and we obtain a closed formula for $[\widetilde{Z}_{r,s}]$. Substituting the particular values in Theorems 7.1 and 7.2, we obtain the claimed formulas for the slopes.

$\square$

We expect the virtual divisors constructed in Theorems 7.1 and 7.2 to be actual divisors for all ℓ. We can directly confirm this expectation for all bounded ℓ. We illustrate this in the case $\ell = 1$.

Theorem 7.3. *The locus $D_{7,3}$ is a divisor on $\mathcal{M}_{24}$, that is, for a general curve C of genus 24, the image curve $\varphi_L : C \hookrightarrow \mathbf{P}^7$ lies on no quadric of rank at most 6, for any linear system $L \in W^7_{28}(C)$.*

Proof. By residuation, we have a birational isomorphism $\mathfrak{G}^7_{24,28} \cong \mathfrak{G}^2_{24,18}$ of parameter spaces over $\mathcal{M}_{24}$. The latter space is a quotient of the Severi variety of plane curves of genus 24 and degree 18 which is known to be irreducible [H], hence $\mathfrak{G}^7_{24,28}$ is an irreducible, generically finite cover of $\mathcal{M}_{24}$. To show that $D_{7,3}$ is a divisor, that is, $D_{7,3} \neq \mathcal{M}_{24}$, it suffices to produce *one* smooth curve $[C] \in \mathcal{M}_{24}$ and *one* very ample linear system $L \in W^7_{28}(C)$ such that the image curve $\varphi_L : C \hookrightarrow \mathbf{P}^7$ does not lie on any quadric of rank at most 6. The curve we construct lies on a rational surface X in $\mathbf{P}^7$ and has the property that all the quadrics containing C also contain X.

Precisely, we start with 16 general points $p_1, \ldots, p_{16} \in \mathbf{P}^2$. We embed the surface $X := \mathrm{Bl}_{16}(\mathbf{P}^2)$ obtained by blowing-up these points in the space $\mathbf{P}^7$ via the linear system

$$H = 9h - 3E_1 - 2\sum_{i=2}^{14} E_i - E_{15} - E_{16} \in \mathrm{Pic}(X),$$

where h is the hyperplane class and E_i is the exceptional divisor corresponding to the point p_i, for $i = 1, \ldots, 16$. By direct computation we find

$$h^0(X, \mathcal{O}_X(2)) = h^0\Big(X, \mathcal{O}_X\big(18h - 6E_1 - 4\sum_{i=2}^{14} E_i - 2E_{15} - 2E_{16}\big)\Big) = \binom{20}{2} - \binom{7}{2} - 13\binom{5}{2} - 2\binom{3}{2} = 33.$$

By using *Macaulay*, we check that $|H|$ embeds X into $\mathbf{P}^7$ and the map $\mathrm{Sym}^2 H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(2))$ is surjective, hence $\dim I_{X,\mathcal{O}_X(1)}(2) = 3$ and $H^1(\mathbf{P}^7, \mathcal{I}_{X/\mathbf{P}^7}(2)) = 0$. We check furthermore with *Macaulay* that $I_{X,\mathcal{O}_X(1)}(2) \cap \Sigma^2_8 = \emptyset$, that is, $X \subseteq \mathbf{P}^7$ lies on no quadric of rank at most 6.

We construct a curve $C \subseteq X$ as a general element of the linear system

$$C \in \Big|20h - 6E_1 - 5\sum_{i=2}^{13} E_i - 4E_{14} - 3E_{15} - 3E_{16}\Big|.$$

Then $C \cdot H = 28$ and we check by *Macaulay* that such a curve C is smooth. In particular, it follows that $g(C) = 1 + \frac{1}{2}C \cdot (C + K_X) = 24$. Furthermore, one has an exact sequence

$$0 \longrightarrow I_{X,\mathcal{O}_X(1)}(2) \longrightarrow I_{C,\mathcal{O}_C(1)}(2) \longrightarrow H^0(X, \mathcal{O}_X(2H - C)) \longrightarrow 0,$$

obtained by taking cohomology in the exact sequence $0 \to \mathcal{I}_{X/\mathbf{P}^7}(2) \to \mathcal{I}_{C/\mathbf{P}^7}(2) \to \mathcal{O}_X(2H - C) \to 0$, where we use once more that $H^1(\mathbf{P}^7, \mathcal{I}_{X/\mathbf{P}^7}(2)) = 0$. Since $H^0(X, \mathcal{O}_X(2H - C)) = 0$, this induces an isomorphism $I_{X, \mathcal{O}_X(1)}(2) \cong I_{C, \mathcal{O}_C(1)}(2)$. This shows that the smooth curve $C \subseteq \mathbf{P}^7$ lies on no quadric of rank at most 6, which finishes the proof[1].

$\square$

8. The slope of $\overline{\mathcal{M}}_{12}$

We explain in this section how using Theorems 1.2 and 5.3 one can construct an effective divisor on $\overline{\mathcal{M}}_{12}$ having slope less than $6 + \frac{12}{g+1}$.

A general curve $[C] \in \mathcal{M}_{12}$ has finitely many linear systems $L \in W_{15}^5(C)$. As already pointed out, the multiplication map $\phi_{C,L} : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is surjective for each $L \in W_{15}^5(C)$, in particular $\mathbf{P}_L := \mathbf{P}(I_{C,L}(2))$ is a pencil of quadrics in $\mathbf{P}^5$ containing the curve $\varphi_L : C \hookrightarrow \mathbf{P}^5$. By imposing the condition that the pencil $\mathbf{P}_L$ be degenerate, we produce a divisor on $\overline{\mathcal{M}}_{12}$, whose class we ultimately compute.

Proof of Theorem 1.10. We retain the notation of the previous section and recall that $\sigma : \widetilde{\mathfrak{G}}_{12,15}^5 \to \widetilde{\mathcal{M}}_{12}^\sharp$ denotes the proper forgetful morphism from the parameter space of generalized linear series $\mathfrak{g}_{15}^5$ onto (an open subset of) the moduli space of irreducible curves of genus 12. Furthermore, we retain the same notation for the tautological bundles $\mathcal{E}$ and $\mathcal{F}$ over $\widetilde{\mathfrak{G}}_{12,15}^5$, as well as for the vector bundle morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$, globalizing the multiplication maps $\phi_{C,L}$, as $[C, L]$ varies over $\widetilde{\mathfrak{G}}_{12,15}^5$. In particular $\mathbf{P}_L \cong \mathbf{P}(\mathrm{Ker}(\phi_{C,L}))$, for every $[C, L] \in \widetilde{\mathfrak{G}}_{12,15}^5$. Noting that $\mathrm{rk}(\mathcal{E}) = 6$ and $\mathrm{rk}(\mathcal{F}) = 19$, we apply Proposition 5.3. The virtual class of the locus Z of pairs $[C, L] \in \widetilde{\mathfrak{G}}_{12,15}^5$ such that $\mathbf{P}_L$ is a degenerate pencil equals

$$[Z]^{\mathrm{virt}} = 10\Big(6c_1(\mathcal{F}) - 38c_1(\mathcal{E})\Big) \in CH^1(\widetilde{\mathfrak{G}}_{12,15}^5).$$

The pushforward classes $\sigma_*(c_1(\mathcal{E}))$ and $\sigma_*(c_1(\mathcal{F}))$ have been described in the proof of Theorems 7.1 and 7.2. After easy manipulations, we compute the class $[\overline{\mathfrak{Dp}}_{12}]^{\mathrm{virt}} := \sigma_*([Z]^{\mathrm{virt}}) \in CH^1(\widetilde{\mathcal{M}}_{12}^\sharp)$.

It remains to establish that Z is indeed a divisor inside $\widetilde{\mathfrak{G}}_{12,15}^5$. To that end, we observe that one has a birational isomorphism $\mathfrak{G}_{12,15}^5 \cong \mathfrak{G}_{12,7}^1$. The latter being the Hurwitz space of degree 7 covers of $\mathbf{P}^1$, it is well-known to be irreducible, hence $\mathfrak{G}_{12,15}^5$ is irreducible as well. Therefore it suffices to exhibit one projectively normal smooth curve $C \subseteq \mathbf{P}^5$ of genus 12 and degree 15, such that $\mathbf{P}_{\mathcal{O}_C(1)}$ is non-degenerate. This is achieved in a way similar to the proof of Theorem 7.3, by choosing C to lie on a particular rational surface.

We pick 11 general points $p_1, \ldots, p_{11} \in \mathbf{P}^2$. We embed the surface $X := \mathrm{Bl}_{11}(\mathbf{P}^2)$ obtained by blowing-up these points in $\mathbf{P}^5$ via the linear system

$$H = 5h - 2E_1 - 2E_2 - \sum_{i=3}^{11} E_i \in \mathrm{Pic}(X),$$

---

[1] The *Macaulay* file containing all the computations appearing in this proof can be found online at https://www.mathematik.hu-berlin.de/~farkas/computations-gen24.m2.

where h is the hyperplane class and E_i is the exceptional divisor corresponding to the point p_i, for $i = 1, \ldots, 11$. We compute $h^0(X, \mathcal{O}_X(2)) = 19$ and $\dim I_{X, \mathcal{O}_X(1)}(2) = 2$. We check furthermore with *Macaulay* that the pencil $\mathbf{P}_{\mathcal{O}_X(1)}$ is non-degenerate.

We construct a curve $C \subseteq X$ as a general element of the following linear system on X

$$C \in \left| 10h - 4E_1 - 4E_2 - 3E_3 - 3E_4 - 2\sum_{i=5}^{10} E_i - E_{11} \right|.$$

Then C is a smooth curve of genus 12 with $C \cdot H = 15$. Since $H^0(X, \mathcal{O}_X(2H - C)) = 0$, we have an isomorphism $I_{X, \mathcal{O}_X(1)}(2) \cong I_{C, \mathcal{O}_C(1)}(2)$, showing that the pencil $\mathbf{P}_{\mathcal{O}_C(1)}$ is non-degenerate.

$\square$

9. Tautological classes on the moduli space of polarized $K3$ surfaces

For a positive integer g, we denote by $\mathcal{F}_g$ the moduli space of quasi-polarized $K3$ surfaces of genus g classifying pairs $[X, L]$, where X is a smooth $K3$ surface and $L \in \mathrm{Pic}(S)$ is a big and nef line bundle with $L^2 = 2g - 2$. Via the Torelli Theorem for $K3$ surfaces, one can realize $\mathcal{F}_g$ as the quotient Ω_g / Γ_g of a 19-dimensional symmetric domain Ω_g by an arithmetic subgroup Γ_g of $SO(3, 19)$.

We denote by $\pi : \mathcal{X} \to \mathcal{F}_g$ the universal polarized $K3$ surface of genus g and by $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ a universal polarization line bundle. Note that $\mathcal{L}$ is not unique, for it can be twisted by the pull-back of any line bundle coming from $\mathcal{F}_g$. Recall that the Hodge bundle on $\mathcal{F}_g$ is defined by

$$\lambda := \pi_*(\omega_\pi) \in \mathrm{Pic}(\mathcal{F}_g).$$

Following [MOP], for non-negative integers a, b we also consider the κ classes on $\mathcal{F}_g$, by setting

$$\kappa_{a,b} := \pi_*\Big(c_1(\mathcal{L})^a \cdot c_2(\mathcal{T}_\pi)^b\Big) \in CH^{a+2b-2}(\mathcal{F}_g).$$

We shall concentrate on the codimension 1 tautological classes, that is, on $\kappa_{3,0}$ and $\kappa_{1,1}$. Replacing $\mathcal{L}$ by $\widetilde{\mathcal{L}} := \mathcal{L} \otimes \pi^*(\alpha)$, where $\alpha \in \mathrm{Pic}(\mathcal{F}_g)$, the classes $\kappa_{3,0}$ and $\kappa_{1,1}$ change as follows:

$$\widetilde{\kappa}_{3,0} = \kappa_{3,0} + 6(g-1)\alpha \quad \text{and} \quad \widetilde{\kappa}_{1,1} = \kappa_{1,1} + 24\alpha.$$

It follows that the following linear combination of κ classes

$$\gamma := \kappa_{3,0} - \frac{g-1}{4}\kappa_{1,1} \in CH^1(\mathcal{F}_g)$$

is well-defined and independent of the choice of a Poincaré bundle on $\mathcal{X}$.

9.1. $K3$ **surfaces and rank** 4 **quadrics.** Recall that in the Introduction we have introduced the Noether-Lefschetz divisors $D_{h,d}$ consisting of quasi-polarized $K3$ surfaces $[X, L] \in \mathcal{F}_g$ such that there exists a primitive embedding of a rank 2 lattice $\mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot D \subseteq \mathrm{Pic}(X)$, where $D \in \mathrm{Pic}(X)$ is a class with $D \cdot L = d$ and $D^2 = 2h - 2$. In what follows, we fix a quasi-polarized $K3$ surface $[X, L] \in \mathcal{F}_g$ and consider the map

$$\varphi_L : X \to \mathbf{P}^g$$

induced by the polarization. We recall a few classical results on linear systems on $K3$ surfaces. Since L is big and nef, using [SD] Proposition 2.6, we find that L is base point free unless there exists an elliptic curve $E \subseteq X$ with $E \cdot L = 1$. In this case, $L = gE + \Gamma$, where $\Gamma^2 = -2$ and $E \cdot \Gamma = 1$. This case corresponds to the NL divisor $D_{1,1}$. If L is base point free, then L is not very ample if and only if there is a divisor $E \in \mathrm{Pic}(X)$ with $E^2 = -2$ and $E \cdot L = 0$ (which corresponds to the NL divisor

$D_{0,0}$), or there is a divisor $E \in \mathrm{Pic}(X)$ with $E^2 = 0$ and $E \cdot L = 2$, which corresponds to the NL divisor $D_{1,2}$.

When $[X,L] \in D_{0,0}$, the morphism φ_L contracts the smooth rational curve Γ. The NL divisor $D_{1,2}$ consists of *hyperelliptic* $K3$ surfaces, for in this case φ_L maps X with degree 2 onto a surface of degree $g-1$ in $\mathbf{P}^g$. Furthermore, for $[X,L] \in \mathcal{F}_g - \big(D_{0,0} \cup D_{1,1} \cup D_{1,2}\big)$, it is shown in [SD] Theorem 6.1 that the multiplication map
$$\phi_{X,L} : \mathrm{Sym}^2 H^0(X,L) \to H^0(X, L^{\otimes 2})$$
is surjective. By Riemann-Roch, $h^0(X, L^{\otimes 2}) = \chi(X, \mathcal{O}_X) + 2L^2 = 4g - 2$ and we obtain
$$\dim I_{X,L}(2) = \binom{g+2}{2} - (4g-2) = \binom{g-2}{2} = \mathrm{codim}(\Sigma_{g+1}^{g-3}).$$

Recall that we have defined in the Introduction the locus $D_g^{\mathrm{rk}4}$ of quasi-polarized $K3$ surfaces $[X,L] \in \mathcal{F}_g$ such that the image $\varphi_L(X) \subseteq \mathbf{P}^g$ lies on a rank 4 quadric.

Proposition 9.1. *The locus $D_g^{\mathrm{rk}4}$ is a Noether-Lefschetz divisor on $\mathcal{F}_g$. Set-theoretically, it consists of the quasi-polarized $K3$ surfaces $[X,L] \in \mathcal{F}_g$, for which there exists a decomposition $L = D_1 + D_2$ in $\mathrm{Pic}(X)$, with $h^0(X, D_i) \geq 2$, for $i = 1, 2$.*

Proof. Suppose the embedded $K3$ surface $X \hookrightarrow \mathbf{P}^g$ lies on a quadric $Q \subseteq \mathbf{P}^g$ of rank at most 4. Assume $\mathrm{rk}(Q) = 4$, hence $\mathrm{Sing}(Q) \cong \mathbf{P}^{g-4}$. Then Q is isomorphic to the inverse image of $\mathbf{P}^1 \times \mathbf{P}^1$ under the projection $p_{\mathrm{Sing}(Q)} : \mathbf{P}^g \dashrightarrow \mathbf{P}^3$ with center $\mathrm{Sing}(Q)$. Accordingly, Q has two rulings which cut out line bundles D_1 and D_2 on X such that $h^0(X, D_i) \geq 2$ and $L = D_1 + D_2$. The argument is clearly reversible. $\qquad\square$

For $n \geq 1$, we introduce the following tautological bundles
$$\mathcal{U}_n := \pi_*(\mathcal{L}^{\otimes n})$$
on $\mathcal{F}_g$. Note that $R^i\pi_*(\mathcal{L}^{\otimes n}) = 0$ for $i = 1, 2$, hence $\mathcal{U}_n$ is locally free and $\mathrm{rk}(\mathcal{U}_n) = 2 + n^2(g-1)$.

Proposition 9.2. *The following formula holds for every $n \geq 1$:*
$$c_1(\mathcal{U}_n) = \frac{n}{12}\kappa_{1,1} + \frac{n^3}{6}\kappa_{3,0} - \Big(\frac{n^2}{2}(g-1) + 1\Big)\lambda \in CH^1(\mathcal{F}_g).$$

Proof. We apply Grothendieck-Riemann-Roch to the universal $K3$ surface $\pi : \mathcal{X} \to \mathcal{F}_g$ and write:
$$\mathrm{ch}\big(\pi_!\mathcal{L}^{\otimes n}\big) = \pi_*\Big[\Big(1 + nc_1(\mathcal{L}) + \frac{n^2}{2}c_1^2(\mathcal{L}) + \frac{n^3}{6}c_1^3(\mathcal{L}) + \cdots\Big)\cdot$$
$$\Big(1 - \frac{1}{2}c_1(\omega_\pi) + \frac{1}{12}\big(c_1^2(\omega_\pi) + c_2(\Omega_\pi)\big) - \frac{1}{24}c_1(\omega_\pi)c_2(\Omega_\pi) + \cdots\Big)\Big].$$
Note that $\kappa_{2,0} = \pi_*(c_1^2(\mathcal{L})) = 2g - 2 \in CH^0(\mathcal{F}_g)$, hence by looking at degree 2 terms in this formula, we find $\kappa_{0,1} = 24$. We now consider degree 3 terms that get pushed forward under π, and use that $c_1(\Omega_\pi) = \pi^*(\lambda)$, hence $\pi_*\big(c_1(\mathcal{L}) \cdot c_1^2(\omega_\pi)\big) = 0$. Collecting terms, we obtained the desired formula. $\quad\square$

We are now in a position to compute the class of the Noether-Lefschetz divisor $D_g^{\mathrm{rk}4}$.

Proof of Theorem 1.3. On the moduli space $\mathcal{F}_g$ we consider the vector bundle morphism
$$\phi : \mathrm{Sym}^2(\mathcal{U}_1) \to \mathcal{U}_2.$$

The divisor $D_g^{\mathrm{rk}4}$ coincides with the locus where the kernel of ϕ contains a rank 4 quadric. Applying Theorem 4.7, we find the formula

$$[D_g^{\mathrm{rk}4}] = A_{g+1}^{g-3}\Big(c_1(\mathcal{U}_2) - \frac{8g-4}{g+1}c_1(\mathcal{U}_1)\Big).$$

In view of Proposition 9.2, $c_1(\mathcal{U}_1) = \frac{1}{12}\kappa_{1,1} + \frac{1}{6}\kappa_{3,0} - \frac{g+1}{2}\lambda$ and $c_1(\mathcal{U}_2) = \frac{1}{6}\kappa_{1,1} + \frac{4}{3}\kappa_{3,0} - (2g-1)\lambda$. Substituting, we obtain the claimed formula. $\qquad\square$

9.2. Koszul cohomology of polarized $K3$ surfaces of odd genus.

Theorem 1.3 shows that a certain linear combination of the classes λ and γ lies in the span of NL divisors. To conclude that both λ and γ are of NL-type, we find another linear combination of these two classes, that is guaranteed to be supported on NL divisors. To that end, for odd genus, we use Voisin's solution [V1], [V2] to the Generic Green's Conjecture on syzygies of canonical curves.

We fix a quasi-polarized $K3$ surface $[X, L] \in \mathcal{F}_g - D_{1,1}$, so that L is globally generated and we consider the induced morphism $\varphi_L : X \to \mathbf{P}^g$. We introduce the coordinate ring

$$\Gamma_X(L) := \bigoplus_{n \geq 0} H^0(X, L^{\otimes n}),$$

viewed as a graded module over the polynomial algebra $S := \mathrm{Sym}\, H^0(X, L)$. In order to describe the minimal free resolution of $\Gamma_X(L)$, for integers $p, q \geq 0$, we introduce the Koszul cohomology group

$$K_{p,q}(X, L) = \mathrm{Tor}_S^p(\Gamma_X(L), \mathbb{C})_{p+q}$$

of p-th order syzygies of weight q of the pair $[X, L]$. We set $b_{p,q}(X, L) := \dim K_{p,q}(X, L)$. For an introduction to Koszul cohomology in algebraic geometry, we refer to [G] and [AN].

The graded minimal free S-resolution of $\Gamma_X(L)$ has the following shape:

$$0 \longleftarrow \Gamma_X(L) \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_{g-3} \longleftarrow F_{g-2} \longleftarrow 0,$$

where $F_p = \bigoplus_{q>0} S(-p-q) \otimes K_{p,q}(X, L)$, for all $p \leq g-2$.

The resolution is self-dual in the sense that $K_{p,q}(X, L)^\vee \cong K_{g-2-p,3-q}(X, L)$, see [G] Theorem 2.c.6. This shows that the *linear strand* of the Betti diagram of $[X, L]$ corresponding to the case $q = 1$ is dual to the *quadratic strand* corresponding to the case $q = 2$. In [V2], in her course of proving Green's Conjecture for general curves [G], Voisin determined completely the shape of the minimal resolution of a generic quasi-polarized $K3$ surface $[X, H] \in \mathcal{F}_g$ of odd genus $g = 2i + 3$. We summarize in the following table the relevant information contained in the rows of linear and quadratic syzygies of the Betti table.

1	2	...	$i-1$	i	$i+1$	$i+2$	...	$2i$
$b_{1,1}$	$b_{2,1}$	...	$b_{i-1,1}$	$b_{i,1}$	0	0	...	0
0	0	...	0	0	$b_{i+1,2}$	$b_{i+2,2}$	...	$b_{2i,2}$

TABLE 1. The Betti table of a general polarized $K3$ surface of genus $g = 2i + 3$

The crux of Voisin's proof is showing $K_{i+1,1}(X, L) = 0$, which implies $K_{p,1}(X, L) = 0$ for $p > i$. Then by duality, the second row of the resolution has the form displayed above.

Our strategy is to treat this problem variationally and consider the locus of polarized $K3$ surfaces with extra syzygies, that is,

$$\mathfrak{Kos}_{\mathfrak{z}_g} := \Big\{ [X, L] \in \mathcal{F}_g : K_{i+1,1}(X, L) \neq 0 \Big\}.$$

We shall informally refer to $\mathfrak{Kos}_{\mathfrak{z}_g}$ as the *Koszul divisor* on $\mathcal{F}_g$, where $g = 2i + 3$. It is shown in [AN] Corollary 2.17 that the group $K_{i+1,1}(X, L)$ of linear syzygies has the following interpretation

$$K_{i+1,1}(X, L) \cong K_{i,2}\big(I_{X,L}, H^0(X, L)\big),$$

where $I_{X,L} := \oplus_k I_{X,L}(k)$ is the ideal of $X \subseteq \mathbf{P}^g$, cf. Definition 6.1, viewed as a graded Sym $H^0(X, L)$-module. Thus, one has the following identification
(24)

$$K_{i+1,1}(X, L) \cong \mathrm{Ker}\Big\{ \bigwedge^i H^0(X, L) \otimes I_{X,L}(2) \to \bigwedge^{i-1} H^0(X, L) \otimes I_{X,L}(3) \Big\} \cong H^0\Big(\mathbf{P}^g, \Omega_{\mathbf{P}^g}^i(i+2) \otimes \mathcal{I}_{X/\mathbf{P}^g}\Big),$$

where the map in question is given by the Koszul differential. The last identification in (24) is obtained by taking global sections in the exact sequence on $\mathbf{P}^g$

$$0 \longrightarrow \bigwedge^i M_{\mathbf{P}^g} \otimes \mathcal{I}_{X/\mathbf{P}^g}(2) \longrightarrow \bigwedge^i H^0(\mathbf{P}^g, \mathcal{O}_{\mathbf{P}^g}(1)) \otimes \mathcal{I}_{X/\mathbf{P}^g}(2) \longrightarrow \bigwedge^{i-1} M_{\mathbf{P}^g} \otimes \mathcal{I}_{X/\mathbf{P}^g}(3) \longrightarrow 0,$$

where $M_{\mathbf{P}^g} := \Omega_{\mathbf{P}^g}(1)$. More generally, we introduce the *Lazarsfeld bundle* of $[X, L]$ as the kernel of the evaluation map of global sections, that is,

(25) $$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0.$$

Note that $M_L = \Omega_{\mathbf{P}^g|X}(1)$. Via (24), $[X, L] \in \mathfrak{Kos}_{\mathfrak{z}_g}$ if and only the restriction map is not injective:

(26) $$H^0\Big(\mathbf{P}^g, \bigwedge^i M_{\mathbf{P}^g}(2)\Big) \to H^0\Big(X, \bigwedge^i M_L \otimes L^2\Big).$$

The key observation is that the two spaces appearing in (26) have the same dimension, which leads to representing $\mathfrak{Kos}_{\mathfrak{z}_g}$ as the degeneracy locus of a morphism between two vector bundles of the *same* rank over $\mathcal{F}_g$.

We collect a few technical results that will come up in the following calculations:

Lemma 9.3. *Let $[X, L] \in \mathcal{F}_{2i+3}$ be a quasi-polarized $K3$ surface such that L is base point free.*
(1) $H^1\big(X, \bigwedge^j M_L \otimes L^{i+2-j}\big) = 0$, for $j = 0, \ldots, i$.
(2) $h^0\big(X, \bigwedge^i M_L \otimes L^2\big) = h^0\big(\mathbf{P}^{2i+3}, \bigwedge^i M_{\mathbf{P}^{2i+3}}(2)\big) = (i+1)\binom{2i+5}{i+2}$.

Proof. It is proved in [Ca] Corollary 1 that under our assumption, the vector bundle M_L is μ_L-semistable. This implies that $\bigwedge^j M_L \otimes L^{2+i-j}$ is μ_L-semistable for all i and j as well. We take cohomology in the exact sequence

$$0 \longrightarrow \bigwedge^{j+1} M_L \otimes L^{i+1-j} \longrightarrow \bigwedge^{j+1} H^0(X, L) \otimes L^{i+1-j} \longrightarrow \bigwedge^j M_L \otimes L^{i+2-j} \longrightarrow 0.$$

Since $H^1(X, L^{i+1-j}) = 0$ and $H^2(X, L^{i+1-j}) = 0$ for $j \leq i$, we obtain the isomorphism

$$H^1\big(X, \bigwedge^j M_L \otimes L^{i+2-j}\big) \cong H^2\big(X, \bigwedge^{j+1} M_L \otimes L^{i+1-j}\big).$$

Since $\mathrm{rk}(M_L) = g$ and $c_1(M_L) = -L$, by standard Chern class calculation, we find

$$\mu_L\Big(\bigwedge^{j+1} M_L \otimes L^{i+1-j}\Big) = \frac{i+2}{2i+3}(2i - 2j + 1) > 0,$$

which establishes $H^2\big(X, \bigwedge^{j+1} M_L \otimes L^{i+1-j}\big) = 0$ by the stability of the vector bundle in question.

The fact that $h^0\big(\mathbf{P}^g, \bigwedge^i M_{\mathbf{P}^{2i+3}}(2)\big) = h^0\big(\mathbf{P}^g, \Omega_{\mathbf{P}^{2i+3}}(i+2)\big) = (i+1)\binom{2i+5}{i+2}$ follows directly from Bott's formula on the cohomology of spaces of twisted holomorphic forms on projective spaces, see e.g. [OSS] page 4. To compute the last quantity appearing, noting that $c_2(M_L) = 2g - 2$, after a Riemann-Roch calculation on X, we obtain

$$h^0\big(X, \bigwedge^i M_L \otimes L^2\big) = \chi\big(X, \bigwedge^i M_L \otimes L^2\big) = (i+1)\binom{2i+5}{i+2},$$

where we have used the standard formulas $c_1\big(\bigwedge^i M_L\big) = \binom{2i+2}{i-1}c_1(M_L)$ and

$$c_2\big(\bigwedge^i M_L\big) = \frac{1}{2}\binom{2i+2}{i-1}\Big(\binom{2i+2}{i-1} - 1\Big)c_1^2(M_L) + \binom{2i+1}{i-1}c_2(M_L).$$

$\qquad\square$

Taking exterior powers in the short exact sequence (25) and using the first part of Lemma 9.3, for $j = 0, \ldots, i$, we obtain the exact sequences, valid for $[X, L] \in \mathcal{F}_g - D_{1,1}$:

$$0 \longrightarrow H^0\big(X, \bigwedge^j M_L \otimes L^{i+2-j}\big) \longrightarrow \bigwedge^j H^0(X, L) \otimes H^0(X, L^{i+2-j}) \longrightarrow H^0\big(X, \bigwedge^{j-1} M_L \otimes L^{i+3-j}\big) \longrightarrow 0.$$

Globalizing these exact sequences over the moduli space, for $j = 0, \ldots, i$, we define inductively the vector bundles $\mathcal{G}_{j,i+2-j}$ over $\mathcal{F}_g$ via the exact sequences

$$(27) \qquad 0 \longrightarrow \mathcal{G}_{j,i+2-j} \longrightarrow \bigwedge^j \mathcal{U}_1 \otimes \mathcal{U}_{i+2-j} \longrightarrow \mathcal{G}_{j-1,i+3-j} \longrightarrow 0,$$

starting from $\mathcal{G}_{0,i+2} := \mathcal{U}_{i+2}$.

Similarly, taking exterior powers in the Euler sequence on $\mathbf{P}^g$, we find the exact sequences

$$0 \longrightarrow H^0\big(\bigwedge^j M_{\mathbf{P}^g}(i+2-j)\big) \longrightarrow \bigwedge^j H^0(\mathcal{O}_{\mathbf{P}^g}(1)) \otimes H^0(\mathcal{O}_{\mathbf{P}^g}(i+2-j)) \longrightarrow H^0\big(\bigwedge^{j-1} M_{\mathbf{P}^g}(i+3-j)\big) \longrightarrow 0,$$

which can also be globalizes to exacts sequences over $\mathcal{F}_g$. We define inductively the vector bundles $\mathcal{H}_{j,i+2-j}$ for $j = 0, \ldots, i$, starting from $\mathcal{H}_{0,i+2} := \mathrm{Sym}^{i+2}(\mathcal{U}_1)$ and then via the exact sequences

$$(28) \qquad 0 \longrightarrow \mathcal{H}_{j,i+2-j} \longrightarrow \bigwedge^j \mathcal{U}_1 \otimes \mathrm{Sym}^{i+2-j}(\mathcal{U}_1) \longrightarrow \mathcal{H}_{j-1,i+3-j} \longrightarrow 0.$$

In particular, there exist restriction morphisms $\mathcal{H}_{j,i+2-j} \to \mathcal{G}_{j,i+2-j}$ for all $j = 0, \ldots, i$. Setting $j = i$, we observe that the second part of Lemma 9.3 yields $\mathrm{rk}(\mathcal{H}_{i,2}) = \mathrm{rk}(\mathcal{G}_{i,2})$, and the degeneracy locus of the morphism

$$\phi : \mathcal{H}_{i,2} \to \mathcal{G}_{i,2}$$

is precisely the locus $\mathfrak{Kos}_{3g}$ of quasi-polarized $K3$ surfaces having extra syzygies.

Proposition 9.4. *The locus $\mathfrak{Kos}_{3g}$ is an effective divisor on $\mathcal{F}_g$ of NL type.*

Proof. Let $[X, L] \in \mathcal{F}_g$ be a quasi-polarized $K3$ surface with $\text{Pic}(X) = \mathbb{Z} \cdot L$ and choose a general curve $C \in |L|$. Using the Koszul duality $K_{i,2}(X, L) \cong K_{i+1,1}(X, L)^\vee$, in order to conclude, it suffices to show that $K_{i,2}(X, L) = 0$. Using the main result of [V2], we have that $K_{i,2}(X, L) \cong K_{i,2}(C, \omega_C) = 0$, for the genus g curve $C \in |L|$ is known to be Brill-Noether general, in particular it has maximal Clifford index $\text{Cliff}(C) = i + 1$. $\square$

In what follows, we shall repeatedly use that if E is a vector bundle of rank r on a stack X, then

$$(29) \qquad c_1\Big(\bigwedge^n E\Big) = \binom{r-1}{n-1} c_1(E) \quad \text{and} \quad c_1\big(\text{Sym}^n(E)\big) = \binom{r+n-1}{r} c_1(E).$$

Theorem 9.5. *Set $g = 2i + 3$. The class of the Koszul divisor of $K3$ surfaces with extra syzygies is given by*

$$[\mathfrak{Kos}_g] = \frac{2}{i+2} \binom{2i-1}{i} \Big(2(i+1)(i+5)\lambda + \gamma \Big) + \alpha \cdot [D_{1,1}] \in CH^1(\mathcal{F}_g),$$

for some coefficient $\alpha \in \mathbb{Z}$.

Proof. As explained, off the divisor $D_{1,1}$, the locus $\mathfrak{Kos}_g$ is the degeneracy locus of the morphism $\phi : \mathcal{H}_{i,2} \to \mathcal{G}_{i,2}$, therefore $[\mathfrak{Kos}_g] = c_1(\mathcal{G}_{i,2}) - c_1(\mathcal{H}_{i,2}) + \alpha \cdot [D_{1,1}]$, for a certain integral coefficient α. Using repeatedly the exact sequences (27) and the formulas for the ranks of the vector bundles $\mathcal{U}_{2+j}$, we find

$$c_1(\mathcal{G}_{i,2}) = \sum_{j=0}^i (-1)^j c_1\Big(\bigwedge^{i-j} \mathcal{U}_1 \otimes \mathcal{U}_{2+j}\Big) =$$

$$\sum_{j=0}^i (-1)^j \Big(\big(2 + (j+2)^2(g-1) \big) \binom{g}{i-j-1} c_1(\mathcal{U}_1) + \binom{g+1}{i-j} c_1(\mathcal{U}_{2+j}) \Big).$$

Similarly, in order to compute the first Chern class of $\mathcal{H}_{i,2}$, we use the exact sequences (28):

$$c_1(\mathcal{H}_{i,2}) = \sum_{j=0}^i (-1)^j c_1\Big(\bigwedge^{i-j} \mathcal{U}_1 \otimes \text{Sym}^{j+2}\mathcal{U}_1\Big) =$$

$$\sum_{j=0}^i (-1)^j \Big(\binom{g+j+2}{g} \binom{g}{i-j-1} + \binom{g+1}{i-j} \binom{g+j+2}{g+1} \Big) c_1(\mathcal{U}_1) = \frac{i+1}{2} \binom{2i+5}{i+2} c_1(\mathcal{U}_1).$$

Substituting in these formulas the Chern classes computed in Proposition 9.2, after some manipulations we obtain the claimed formula for $[\mathfrak{Kos}_g]$. $\square$

10. LAZARSFELD-MUKAI BUNDLES ON $K3$ SURFACES OF EVEN GENUS AND TAUTOLOGICAL CLASSES

For even genus, in order to obtain a Noether-Lefschetz relation between the classes λ and γ which is different than the one in Theorem 1.3, we use the geometry of the rank 2 Lazarsfeld-Mukai vector bundle one associates to a sufficiently general polarized $K3$ surface. We denote by $D_{\text{NL}} \subseteq \mathcal{F}_g$ the Noether-Lefschetz divisor consisting of $K3$ surfaces $[X, L]$ of genus g, such that $L = \mathcal{O}_X(D_1 + D_2)$, with both D_1 and D_2 being non-trivial effective divisors on X. We set $\mathcal{F}_g^\sharp := \mathcal{F}_g - D_{\text{NL}}$ and slightly abusing notation, we denote by $\pi : \mathcal{X}^\sharp \to \mathcal{F}_g^\sharp$ the corresponding restriction of the universal $K3$ surface. Throughout this subsection we fix an even genus $g = 2i$, with $i \geq 4$. Our aim is to show that the

restriction of both classes λ and γ to $\mathcal{F}_g^\sharp$ is trivial. The geometric source of such a relation lies in the geometry of Lazarsfeld-Mukai vector bundles that have proved to be instrumental in Lazarsfeld's proof [La] of the Petri Theorem.

Definition 10.1. *For a polarized $K3$ surface $[X, L] \in \mathcal{F}_g^\sharp$, we denote by E_L the unique stable rank 2 vector bundle on X, satisfying $\det(E_L) = L$, $c_2(E_L) = i + 1$ and $h^0(X, E_L) = i + 2$.*

The vector bundle $E := E_L$, which we refer to as the *Lazarsfeld-Mukai vector bundle* of $[X, L]$ has been first considered in [Mu] and [La]. In order to construct it, one chooses a smooth curve $C \in |L|$ and a pencil of minimal degree $A \in W_{i+1}^1(C)$. By Lazarsfeld [La], it is known that C verifies the Brill-Noether Theorem, in particular $\mathrm{gon}(C) = i + 1$. We define the dual Lazarsfeld-Mukai bundle via the following exact sequence on X

$$(30) \qquad 0 \longrightarrow E_L^\vee \longrightarrow H^0(C, A) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} \iota_* A \longrightarrow 0,$$

where $\iota : C \hookrightarrow X$ denotes the inclusion map. Dualizing the previous sequence, we obtain the short exact sequence

$$0 \longrightarrow H^0(C, A)^\vee \otimes \mathcal{O}_X \longrightarrow E_L \longrightarrow \omega_C \otimes A^\vee \longrightarrow 0.$$

We summarize the properties of this vector bundle and refer to [La] for proofs:

Proposition 10.2. *Let $[X, L] \in \mathcal{F}_g^\sharp$ and $E = E_L$ be the corresponding rank 2 Lazarsfeld-Mukai bundle.*
  (1) *E is globally generated and $H^1(X, E) = H^2(X, E) = 0$.*
  (2) *$h^0(X, E) = h^0(C, \omega_C \otimes A^\vee) + h^0(C, A) = i + 2$.*
  (3) *E is μ_L-stable, in particular $h^0(X, E \otimes E^\vee) = 1$ as well as rigid, that is, $H^1(X, E \otimes E^\vee) = 0$.*
  (4) *The vector bundle E is independent of the choice of C and of that of the pencil $A \in W_{i+1}^1(C)$.*

In particular, Proposition 10.2 implies that E is the only μ_L-semistable sheaf on X having Mukai vector $v = v(E) = (2, L, i)$. We denote by $\det : \bigwedge^2 H^0(X, E) \to H^0(X, L)$ the determinant map.

Let $\mathcal{E}$ be the universal rank 2 Lazarsfeld-Mukai vector bundle over $\mathcal{X}^\sharp$, that is, $\mathcal{E}_{|X} = E_L$, for every $[X, L] \in \mathcal{F}_g^\sharp$. In this case $\mathcal{L} := \det(\mathcal{E})$ can be taken to be the polarization line bundle, and apart from the classes $\kappa_{1,1} = \pi_*\big(c_1(\mathcal{E}) \cdot c_2(\mathcal{T}_\pi)\big)$ and $\kappa_{3,0} = \pi_*\big(c_1(\mathcal{E})^3\big)$, we also have a third tautological class

$$\vartheta := \pi_*\big(c_1(\mathcal{E}) \cdot c_2(\mathcal{E})\big).$$

To show that both classes λ and $\gamma = \kappa_{1,1} - \frac{g-1}{4}\kappa_{3,0}$ are of NL type, we need *two* further sources of geometric relations in terms of Lazarsfeld-Mukai bundles. These provide two relations involving $\lambda, \kappa_{3,0}, \kappa_{1,1}$ and ϑ, hence by eliminating ϑ, one relation between λ and γ, which turns out to be different than the one given by Theorem 1.3.

10.1. The Chow form of the Grassmannian and Lazarsfeld-Mukai bundles.
One such source of relations is provided by the recent work [AFPRW], where among other things a Schubert-theoretic description of the Cayley-Chow form of the Grassmannian $\mathrm{Gr}(2, n)$ of lines is provided.

Let V be an n-dimensional complex vector space and choose a linear subspace $K \subseteq \bigwedge^2 V$. Then $K^\perp \subseteq \bigwedge^2 V^\vee$. It is shown in [AFPRW] Theorem 3.1 that the condition $\mathbf{P}(K^\perp) \cap \mathrm{Gr}(V, 2) = \emptyset$, the intersection being taken inside $\mathbf{P}(\bigwedge^2 V^\vee)$, is equivalent to the exactness of the complex

$$K \otimes \mathrm{Sym}^{n-3}(V) \xrightarrow{\delta_2} V \otimes \mathrm{Sym}^{n-2}(V) \xrightarrow{\delta_1} \mathrm{Sym}^{n-1}(V),$$

where $\delta_i : \bigwedge^i V \otimes \mathrm{Sym}(V) \to \bigwedge^{i-1} V \otimes \mathrm{Sym}(V)$ denotes the Koszul differential, for $i = 1, 2$.

We apply this result for polarized $K3$ surfaces, when we take $V := H^0(X, E)^\vee$ and

$$K^\perp := \mathrm{Ker}\Big\{ \det : \bigwedge^2 H^0(X, E) \to H^0(X, L) \Big\}$$

is the kernel of the determinant map. Note that $\det$ does not vanish on any element of rank 2, see [V2] page 380, for the existence of an element $0 \neq s_1 \wedge s_2 \in \bigwedge^2 H^0(X, E)$ such that $\det(s_1 \wedge s_2) = 0$, would imply a splitting of L as a sum of two pencils. By dualizing, we conclude that the complex

$$(31) \qquad \mathrm{Sym}^{i+1} H^0(X, E) \longrightarrow H^0(X, E) \otimes \mathrm{Sym}^i H^0(X, E) \xrightarrow{\beta} H^0(X, L) \otimes \mathrm{Sym}^{i-1} H^0(X, E)$$

is exact for every point $[X, L] \in \mathcal{F}_g^\sharp$. The map β is obtained by composing the (dual) Koszul differential

$$H^0(E) \otimes \mathrm{Sym}^i H^0(X, E) \to \bigwedge^2 H^0(X, E) \otimes \mathrm{Sym}^{i-1} H^0(X, E)$$

with the map $\det \otimes \mathrm{id}_{\mathrm{Sym}^{i-1} H^0(X, E)} : \bigwedge^2 H^0(X, E) \otimes \mathrm{Sym}^{i-1} H^0(X, E) \to H^0(X, L) \otimes \mathrm{Sym}^{i-1} H^0(X, E)$.

We globalize this geometric fact. For $n \geq 1$, we introduce the vector bundle $\mathcal{V}_n := \pi_*\big(\mathrm{Sym}^n \mathcal{E}\big)$ on $\mathcal{F}_g^\sharp$, where we observe that $R^i \pi_*\big(\mathrm{Sym}^n \mathcal{E}\big) = 0$, for $i = 1, 2$. We shall make use of the following formulas:

Proposition 10.3. *The following formulas hold in $CH^1(\mathcal{F}_{2i}^\sharp)$:*

$$c_1(\mathcal{V}_1) = \frac{1}{12} \kappa_{1,1} + \frac{1}{6} \kappa_{3,0} - \frac{i+2}{2} \lambda - \frac{1}{2} \vartheta \quad and \quad c_1(\mathcal{V}_2) = \frac{1}{4} \kappa_{1,1} + \frac{3}{2} \kappa_{3,0} - \frac{6i-3}{2} \lambda - 4\vartheta.$$

Proof. We only discuss the calculation of $c_1(\mathcal{V}_2)$. For any $[X, L] \in \mathcal{F}_g^\sharp$, observe that

$$h^0\big(X, \mathrm{Sym}^2(E)\big) = \chi\big(X, \mathrm{Sym}^2(E)\big) = \frac{c_1^2\big(\mathrm{Sym}^2(E)\big)}{2} - c_2(\mathrm{Sym}^2(E)) + 3\chi(X, \mathcal{O}_X) = 6i - 3,$$

where we use the formulas $c_1\big(\mathrm{Sym}^2(E)\big) = 3c_1(E)$ and $c_2\big(\mathrm{Sym}^2(E)\big) = 2c_1^2(E) + 4c_2(E)$. Applying Grothendieck-Riemann-Roch to the universal family $\pi : \mathcal{X}^\sharp \to \mathcal{F}_g^\sharp$, we find:

$$c_1(\mathcal{V}_2) = c_1\big(\pi_!(\mathrm{Sym}^2 \mathcal{E})\big) = \pi_*\Big[\Big(3 + 3c_1(\mathcal{E}) + \frac{5c_1^2(\mathcal{E}) - 8c_2(\mathcal{E})}{2} + \frac{9c_1^3(\mathcal{E}) - 24c_1(\mathcal{E})c_2(\mathcal{E})}{6}\Big) \cdot$$
$$\Big(1 - \frac{1}{2} c_1(\omega_\pi) + \frac{1}{12}\big(c_1^2(\omega_\pi) + c_2(\Omega_\pi)\big) - \frac{1}{24} c_1(\omega_\pi) c_2(\Omega_\pi) + \cdots\Big)\Big]_2.$$

Expanding the product and using again that $\pi_*(c_2(\Omega_\pi)) = 24$, we obtain the claimed formula. $\qquad \square$

In order to treat the complex (31) variationally, we consider the following vector bundles over $\mathcal{F}_{2i}^\sharp$

$$\mathcal{A} := \frac{\mathcal{V}_1 \otimes \mathrm{Sym}^i(\mathcal{V}_1)}{\mathrm{Sym}^{i+1}(\mathcal{V}_1)} \quad and \quad \mathcal{B} := \mathcal{U}_1 \otimes \mathrm{Sym}^{i-1}(\mathcal{V}_1).$$

Note that $\mathrm{rk}(\mathcal{A}) = (i+2)\binom{2i+1}{i} - \binom{2i+2}{i+1} = (2i+1)\binom{2i}{i-1} = \mathrm{rk}(\mathcal{B})$ and there is a sheaf morphism

$$\beta : \mathcal{A} \to \mathcal{B},$$

which over a point $[X, L] \in \mathcal{F}_g^\sharp$ is precisely the map

$$\beta_{X,L} : \frac{H^0(X, E) \otimes \mathrm{Sym}^i H^0(X, E)}{\mathrm{Sym}^{i+1} H^0(X, E)} \to H^0(X, L) \otimes \mathrm{Sym}^{i-1} H^0(X, E)$$

induced by (31). As explained, the morphism β is everywhere non-degenerate over $\mathcal{F}_g^\sharp$.

Theorem 10.4. *One has the following formula*

$$\vartheta = \frac{i}{8i+4}\kappa_{1,1} + \frac{i}{4i+2}\kappa_{3,0} - \frac{i+2}{2}\lambda.$$

Proof. The morphism $\beta : \mathcal{A} \to \mathcal{B}$ being everywhere non-degenerate, we find that $c_1(\mathcal{A}) = c_1(\mathcal{B})$. Applying systematically the formulas (29), we write:

$$c_1(\mathcal{A}) = \left(\binom{2i+1}{i} + (i+2)\binom{2i+1}{i+2} - \binom{2i+2}{i+2}\right)c_1(\mathcal{V}_1), \ c_1(\mathcal{B}) = \binom{2i}{i-1}c_1(\mathcal{U}_1) + (2i+1)\binom{2i}{i+2}c_1(\mathcal{V}_1),$$

hence, after manipulations

$$0 = c_1(\mathcal{B} - \mathcal{A}) = \binom{2i}{i-1}\left(c_1(\mathcal{U}_1) - \frac{4i+2}{i+2}c_1(\mathcal{V}_1)\right).$$

We then replace $c_1(\mathcal{U}_1)$ and $c_1(\mathcal{V}_1)$ with their respective expressions provided by Propositions 9.2 and 10.3, clear denominators (our Chow groups are with $\mathbb{Q}$-coefficients), then conclude. $\square$

10.2. **Lazarsfeld-Mukai bundles and rank 6 quadrics.** A second source of relations between the classes λ, $\kappa_{1,1}$, $\kappa_{3,0}$ and ϑ is obtained by studying the kernel the multiplication map

$$\mu_E : \mathrm{Sym}^2 H^0(X, E) \to H^0\big(X, \mathrm{Sym}^2(E)\big)$$

associated to the Lazarsfeld-Mukai bundle $E = E_L$ corresponding to an element $[X, L] \in \mathcal{F}_{2i}^\sharp$. We assume throughout that $i \geq 4$.

Lemma 10.5. *One has $H^i\big(X, \mathrm{Sym}^2(E)\big) = 0$, for $i = 1, 2$.*

Proof. We choose a general curve $C \in |L|$ and a minimal pencil $A \in W_{i+1}^1(C)$. Tensoring the exact sequence (30) by $E^\vee$ and taking global sections implies $H^2\big(X, E \otimes E\big) \cong H^0(X, E^\vee \otimes E^\vee) = 0$. Similarly, we can prove that $H^1(X, E \otimes E) = 0$, which implies $H^1\big(X, \mathrm{Sym}^2(E)\big) = 0$. We tensor again (30) by $E^\vee$ and take cohomology. The vanishing of $H^1(X, E^\vee \otimes E^\vee) \cong H^1(X, E \otimes E)^\vee$ is implied by $H^0(C, A \otimes E_{|C}^\vee) = 0$, which follows because $E_{|C}$ is stable on C and $\mu\big(A \otimes E_{|C}^\vee\big) = 4 - 2i < 0$. $\square$

Using Lemma 10.5, we compute $h^0\big(X, \mathrm{Sym}^2(E)\big) = 6i - 3$, then observe that

$$\dim \mathrm{Sym}^2 H^0(X, E) - h^0\big(X, \mathrm{Sym}^2(E)\big) = \binom{i+3}{2} - (6i-3) = \binom{i-3}{2},$$

that is, the locus

$$D_{2i}^{\mathrm{rk6}} := \left\{[X, L] \in \mathcal{F}_{2i}^\sharp : \exists\, 0 \neq q \in \mathrm{Ker}(\mu_E),\ \mathrm{rk}(q) \leq 6\right\},$$

is expected to be a divisor on $\mathcal{F}_{2i}^\sharp$. We confirm this expectation in a very precise form.

Theorem 10.6. *For a polarized K3 surface $[X, L] \in \mathcal{F}_{2i}^\sharp$, the kernel of the map μ_E contains no non-zero elements of rank at most 6, that is, $D_{2i}^{\mathrm{rk6}} = \emptyset$.*

Proof. We start with a $K3$ surface $[X, L] \in \mathcal{F}_{2i}^{\sharp}$ and assume we have an element $0 \neq q \in \mathrm{Ker}(\mu_E)$, where $\mathrm{rk}(q) = n \leq 6$. We write $q = s_1^2 + \cdots + s_n^2$, where $s_i \in H^0(X, E)$. Denoting by $\mathbf{P}(E) \to X$ the projective bundle associated to E, we have the canonical identifications

$$H^0(X, E) \cong H^0\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(1)\big) \quad \text{and} \quad H^0(X, \mathrm{Sym}^2(E)) \cong H^0\big(\mathbf{P}(E), \mathcal{O}_{\mathbf{P}(E)}(2)\big).$$

Let $V := \langle s_1, \ldots, s_n \rangle \subseteq H^0(X, E)$. Since $H^0(X, E(-L)) = 0$, the cokernel of the evaluation map $V \otimes \mathcal{O}_X \to E$ is supported along finitely many points and we denote by

$$\varphi_V : X \dashrightarrow \mathrm{Gr}(V, 2) \subseteq \mathbf{P}\big(\overset{2}{\bigwedge} V^{\vee}\big)$$

the induced rational map. Note that $\varphi_V^*(\mathcal{O}(1)) = L$. We further denote by $Q \subseteq \mathbf{P}(V^{\vee})$ the quadric given by the equation $q = 0$. The condition $q \in \mathrm{Ker}(\mu_E)$ can be interpreted geometrically as saying that the image of $\mathbf{P}(E)$ under the composition

$$\mathbf{P}(E) \xrightarrow{|\mathcal{O}(1)|} \mathbf{P}\big(H^0(E)^{\vee}\big) \dashrightarrow \mathbf{P}(V^{\vee})$$

lies on the quadric Q. This in turn, amounts to saying that $\varphi_V(X)$ is contained in orthogonal Grassmannian $\mathrm{Gr}_Q \subseteq \mathrm{Gr}(V, 2)$ of lines in $\mathbf{P}(V^{\vee})$ contained in Q. The essential observation is that for $n \leq 6$, the pull-back $\mathcal{O}_{\mathrm{Gr}_Q}(1)$ splits *non-trivially* as the sum of two effective line bundles, which in turn, induces a decomposition of the polarization class L on X, contradicting the assumption that $[X, L]$ is NL-general.

We discuss in detail the case $n = 6$, the situation for $n \leq 5$ being similar. Thus $Q \subseteq \mathbf{P}(V^{\vee}) = \mathbf{P}^5$ is a rank 6 quadric and we may assume that $Q = \mathrm{Gr}(2, U) \subseteq \mathbf{P}\big(\bigwedge^2 U\big) \cong \mathbf{P}(V^{\vee})$ is the Grassmannian of lines in $\mathbf{P}^3 \cong \mathbf{P}(U)$, where U is a 4-dimensional complex vector space such that $\bigwedge^2 U \cong V^{\vee}$. Then every line inside Gr_Q is of the form $L_{\ell, H} := \{\Pi \in \mathrm{Gr}(2, U) : \ell \subseteq \Pi \subseteq H\}$, where $\ell \subseteq U$ is 1-dimensional and $H \subseteq U$ is a 3-dimensional subspace. Accordingly, one has an isomorphism between Gr_Q and the incidence correspondence $\Sigma \subseteq \mathbf{P}(U) \times \mathbf{P}(U^{\vee})$, assigning to the pair $(\ell, H) \in \Sigma$ with $\ell \subseteq H$ the line $L_{\ell, H}$ defined above. Denoting by

$$\mathbf{P}(U) \xleftarrow{\pi_1} \mathrm{Gr}_Q \cong \Sigma \subseteq \mathbf{P}(U) \times \mathbf{P}(U^{\vee}) \xrightarrow{\pi_2} \mathbf{P}(U^{\vee})$$

the two projections, we have $\mathcal{O}_{\mathrm{Gr}_Q}(1) \cong \pi_1^*\big(\mathcal{O}_{\mathbf{P}(U)}(1)\big) \otimes \pi_2^*\big(\mathcal{O}_{\mathbf{P}(U^{\vee})}(1)\big)$. Let $q_1 := \pi_1 \circ \varphi_V : X \dashrightarrow \mathbf{P}(U)$ and $q_2 := \pi_2 \circ \varphi_V : X \dashrightarrow \mathbf{P}(U^{\vee})$. It is now enough to observe that $h^0(X, q_i^*(\mathcal{O}(1)) \geq 2$, for $i = 1, 2$. Indeed, else the image of one of the maps q_i, say q_1, is a point, hence there exists $\ell_0 \in \mathbf{P}(U)$, such that $\mathrm{Im}(\varphi_V) \subseteq \pi_1^{-1}(\ell_0)$. But $\pi_1^{-1}(\ell_0) \cong \mathbf{P}^2 \subseteq \mathbf{P}\big(\bigwedge^2 U\big)$, that is, $\varphi_V(X) \subseteq \mathrm{Gr}(2, 3)$, which is impossible, for V is 6-dimensional. We conclude that $L = q_1^*\big(\mathcal{O}_{\mathbf{P}(U)}(1)\big) \otimes q_2^*\big(\mathcal{O}_{\mathbf{P}(U^{\vee})}(1)\big)$ is NL special.

We briefly mention the cases $n \leq 5$. For $n = 4$, we have $Q \subseteq \mathbf{P}^3$ and the variety of lines Gr_Q consists of two copies of $\mathbf{P}^1$. For $n = 5$, when $Q \subseteq \mathbf{P}^4$ is a rank 5 quadric, the variety of lines Gr_Q is identified with $\mathbf{P}^3$ in its second Veronese embedding $\mathbf{P}^3 \hookrightarrow \mathbf{P}^9 \cong \mathbf{P}\big(\bigwedge^2 V^{\vee}\big)$. The assumption that there exist $0 \neq q \in \mathrm{Ker}(\mu_E)$ with $\mathrm{rk}(q) \in \{4, 5\}$ implies that L is non-primitive, a contradiction. $\square$

Theorem 10.7. *The relation* $\frac{1}{2i+1}\gamma + (i+2)\lambda = 0$ *holds in* $CH^1(\mathcal{F}_{2i}^{\sharp})$.

Proof. We first use the fact that the divisor D_{2i}^{rk6} is of Noether-Lefschetz type, that is, by applying Theorem 1.1 coupled with proposition 10.3, we obtain the relation

$$0 = [D_{2i}^{\mathrm{rk6}}] = c_1(\mathcal{V}_2) - \frac{2(6i-3)}{i+2}c_1(\mathcal{V}_1) = \frac{3}{2}(2i-1)\lambda + \frac{2i-11}{i+2}\vartheta - \frac{i-8}{2i+4}\kappa_{3,0} - \frac{3i-4}{4i+8}\kappa_{1,1}.$$

Substituting ϑ obtained in this way in the formula provided by Theorem 10.4, we obtained the desired relation between λ and γ. $\qquad\square$

Proof of Theorem 1.4 for even g. Theorem 1.3 provides the relation $(4i-1)\lambda + \frac{2}{2i+1}\gamma = 0 \in CH^1(\mathcal{F}_{2i}^\sharp)$. Coupled with Theorem 10.7, we conclude that both classes λ and γ vanish on $\mathcal{F}_{2i}^\sharp$, hence they are of Noether-Lefschetz type on $\mathcal{F}_{2i}$. $\qquad\square$

11. Semistability of the second Hilbert point of a polarized $K3$ surface

A simple and somewhat surprising application of the techniques developed in Subsection 9.1 concerns the GIT semistability of polarized $K3$ surfaces. We fix once and for all a vector space $V \cong \mathbb{C}^{g+1}$.

Definition 11.1. *For a polarized $K3$ surface $[X, L] \in \mathcal{F}_g$ such that $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$, we define its second Hilbert point to be the quotient*

$$[X, H]_2 := \left[\mathrm{Sym}^2 H^0(X, L) \longrightarrow H^0(X, L^{\otimes 2}) \longrightarrow 0 \right] \in \mathrm{Gr}\left(\mathrm{Sym}^2 H^0(X, H), 4g - 2 \right).$$

The group $SL(V)$ acts linearly on the Grassmannian $\mathrm{Gr}\left(\mathrm{Sym}^2 V, 4g - 2 \right) \subseteq \mathbf{P}\left(\bigwedge^{4g-2} \mathrm{Sym}^2 V^\vee \right)$, where the last inclusion is given by the Plücker embedding. Let $\overline{\mathrm{Hilb}}_g$ be the closure inside the quotient

$$\mathbf{P}\left(\overset{4g-2}{\bigwedge} \mathrm{Sym}^2 V^\vee \right) /\!/ SL(V)$$

of the locus of semistable second Hilbert points $[X, H]_2$ of quasi-polarized $K3$ surfaces of genus g. Then the GIT quotient

$$\overline{\mathcal{F}}_g := \overline{\mathrm{Hilb}}_g^{\mathrm{ss}} /\!/ SL(V)$$

is a projective birational model of the moduli space $\mathcal{F}_g$, provided the locus $\overline{\mathrm{Hilb}}_g^{\mathrm{ss}}$ of semistable 2nd Hilbert points is not empty.

Theorem 11.2. *Let $[X, L] \in \mathcal{F}_g$ be a polarized $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot L$. Then the second Hilbert point $[X, H]_2$ is semistable. In particular, $\overline{\mathcal{F}}_g$, defined as above, exists.*

Proof. By definition of semistability, since the Grassmannian $\mathrm{Gr}(\mathrm{Sym}^2 V, 4g-2)$ has Picard numer 1, it suffices to construct an $SL(V)$-invariant effective divisor $\mathcal{D}$ of $\mathrm{Gr}(\mathrm{Sym}^2 V, 4g-2)$ such that $[X, L]_2 \notin \mathcal{D}$. Theorem 1.3 provides such a divisor. We take $\mathcal{D}$ to be the locus of $(4g-2)$-dimensional quotients $\phi : \mathrm{Sym}^2 V \twoheadrightarrow Q$ such that $\mathrm{Ker}(\phi)$ contains a quadric of rank at most 4. The parameter count from Subsection 9.1 shows that $\mathcal{D}$ is indeed a divisor. If $[X, H] \notin D_g^{\mathrm{rk}4}$, then $I_{X,L}(2)$ contains no quadrics of rank at most 4, in particular $[X, L]_2 \notin \mathcal{D}$, hence its second Hilbert point is semistable. $\qquad\square$

Remark 11.3. By the analogy with the much studied case of the moduli space of curves [HH], we expect that, apart from smooth $K3$ surfaces, $\overline{\mathcal{F}}_g$ also parametrizes degenerate $K3$ surfaces with various singularities. It is also likely that the 2nd Hilbert point of NL special smooth $K3$ surfaces is not semistable, that is, the natural map $\mathcal{F}_g \dashrightarrow \overline{\mathcal{F}}_g$ might not be regular along NL special loci.

12. The geometry of the Hurwitz spaces of admissible covers

In what follows, for a Deligne-Mumford stack M, we shall denote by $\mathcal{M}$ its coarse moduli space. If $X \subseteq \mathcal{M}$ is an irreducible subvariety, we denote by $[X] \in CH^*_{\mathbb{Q}}(\mathcal{M})$ its class in the stack sense, that is, we divide by the order of the automorphism group of a general element in X.

We denote by $\mathcal{H}^{\mathfrak{o}}_k$ the Hurwitz space of degree k covers $f : C \to \mathbf{P}^1$ with simple ramification from a smooth curve C of genus $2k - 1$, together with an *ordering* $(p_1, \ldots, p_{6k-4})$ of its branch points. Let $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ denote the compactification of $\mathcal{H}^{\mathfrak{o}}_k$ by admissible covers. By [ACV], the stack $\overline{H}^{\mathfrak{o}}_k$ (whose coarse moduli space is precisely $\overline{\mathcal{H}}^{\mathfrak{o}}_k$) is isomorphic to the stack of *twisted stable* maps into the classifying stack $\mathcal{B}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$, that is,

$$\overline{H}^{\mathfrak{o}}_k := \overline{M}_{0,6k-4}\Big(\mathcal{B}\mathfrak{S}_k\Big).$$

Points of $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ are admissible covers $[f : C \to R, p_1, \ldots, p_{6k-4}]$, where C and R are nodal curves of genus $2k - 1$ and 0 respectively, and $p_1, \ldots, p_{6k-4} \in R_{\mathrm{reg}}$ are the branch points of f. Let $\mathfrak{b} : \overline{\mathcal{H}}^{\mathfrak{o}}_k \to \overline{\mathcal{M}}_{0,6k-4}$ be the *branch* morphism. The symmetric group $\mathfrak{S}_{6k-4}$ acts on $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ by permuting the branch points of each cover. Denoting by

$$\overline{\mathcal{H}}_k := \overline{\mathcal{H}}^{\mathfrak{o}}_k / \mathfrak{S}_{6k-4}$$

the quotient parametrizing admissible covers without an ordering of the branch points, the projection $q : \overline{\mathcal{H}}^{\mathfrak{o}}_k \to \overline{\mathcal{H}}_k$ is a principal $\mathfrak{S}_{6k-4}$-bundle. We denote by $\sigma : \overline{\mathcal{H}}_k \to \overline{\mathcal{M}}_{2k-1}$ the map assigning to an admissible cover the stable model of its source curve. We shall use throughout the isomorphism $CH^1_{\mathbb{Q}}(\overline{\mathcal{H}}_k) \cong H^2(\overline{\mathcal{H}}_k, \mathbb{Q})$, see [DE] Theorem 5.1 and Proposition 2.2.

For $i = 0, \ldots, 3k - 2$, let B_i be the boundary divisor of $\overline{\mathcal{M}}_{0,6k-4}$ whose general point is the union of two smooth rational curves meeting at one point, such that precisely i of the marked points lie on one component. The boundary divisors of $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ are parametrized by the following combinatorial data:

(1) A partition $I \sqcup J = \{1, \ldots, 6k - 4\}$, such that $|I| \geq 2$, $|J| \geq 2$.
(2) Transpositions $\{w_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ in $\mathfrak{S}_k$, with $\prod_{i \in I} w_i = u$, $\prod_{j \in J} w_j = u^{-1}$, for some $u \in \mathfrak{S}_k$.

To this data, we associate the locus of admissible covers with labeled branch points

$$\big[f : C \to R, \ p_1, \ldots, p_{6k-4}\big] \in \overline{\mathcal{H}}^{\mathfrak{o}}_k,$$

where $[R = R_1 \cup_p R_2, p_1, \ldots, p_{6k-4}] \in B_{|I|} \subseteq \overline{\mathcal{M}}_{0,6k-4}$ is a pointed union of two smooth rational curves R_1 and R_2 meeting at the point p. The marked points lying on R_1 are precisely those labeled by the set I. Let $\mu := (\mu_1, \ldots, \mu_\ell) \vdash k$ be the partition induced by $u \in \mathfrak{S}_k$ and denote by $E_{i:\mu}$ the boundary divisor on $\overline{\mathcal{H}}^{\mathfrak{o}}_k$ classifying twisted stable maps with underlying admissible cover as above, with $f^{-1}(p)$ having partition type μ, and precisely i of the points $p_1, \ldots, p_{6k-4}$ lying on R_1. We denote by $D_{i:\mu}$ the reduced boundary divisor of $\overline{\mathcal{H}}_k$ pulling back to $E_{i:\mu}$ under the map q.

For $i = 2, \ldots, 3k - 2$, we have the following relation, see [HM] p. 62, as well as [GK1] Lemma 3.1:

$$(32) \qquad \qquad \mathfrak{b}^*(B_i) = \sum_{\mu \vdash k} \mathrm{lcm}(\mu) E_{i:\mu}.$$

The class of the Hodge class $\lambda := (\sigma \circ q)^*(\lambda)$ on $\overline{\mathcal{H}}_k^{\mathfrak{o}}$ has been determined in [KKZ] and [GK1]:

$$(33) \qquad \lambda = \sum_{i=2}^{3k-2} \sum_{\mu \vdash k} \mathrm{lcm}(\mu) \Big(\frac{i(6k-4-i)}{8(6k-5)} - \frac{1}{12} \Big(k - \sum_{j=1}^{\ell(\mu)} \frac{1}{\mu_j} \Big) \Big) [E_{i:\mu}] \in CH^1(\overline{\mathcal{H}}_k^{\mathfrak{o}}).$$

The sum is taken over partitions μ that correspond to permutations that can be written as products of i transpositions. Furthermore, $\ell(\mu)$ denotes the length of the partition μ and $\mathrm{lcm}(\mu)$ is the lowest common multiple of the parts of μ.

We now discuss in detail the divisors of $\overline{\mathcal{H}}_k^{\mathfrak{o}}$ lying over the boundary divisor B_2. We pick a cover

$$[f : C = C_1 \cup C_2 \to R = R_1 \cup_p R_2, \ p_1, \dots, p_{6k-4}] \in \mathfrak{b}^*(B_2),$$

where $C_i = f^{-1}(R_i)$. We assume $I = \{1, \dots, 6k-6\}$, thus $p_1, \dots, p_{6k-6} \in R_1$ and $p_{6k-5}, p_{6k-4} \in R_2$.

We denote by $E_{2:(1^k)}$ the closure in $\overline{\mathcal{H}}_k^{\mathfrak{o}}$ of the set of admissible covers for which the transpositions w_{6k-5} and w_{6k-4} corresponding to the points p_{6k-5} and p_{6k-4} are equal. Let E_0 be the component of $E_{2:(1^k)}$ corresponding to the case when C_1 is connected, which happens precisely when $\langle w_1, \dots, w_{6k-6} \rangle = \mathfrak{S}_k$. Clearly E_0 is irreducible. When w_{6k-5} and w_{6k-4} are distinct but not disjoint then $\mu = (3, 1^{k-3}) \vdash k$ and one is led to the boundary divisor $E_{2:(3,1^{k-3})}$. We denote by E_3 the (irreducible) subdivisor of $E_{2:(3,1^{k-3})}$ corresponding to the case $\langle w_1, \dots, w_{6k-6} \rangle = \mathfrak{S}_k$. Finally, the case when w_{6k-5} and w_{6k-4} are disjoint corresponds to the boundary divisor $E_{2:(2,2,1^{k-4})}$ and we denote by E_2 the irreducible component of $E_{2:(2,2,1^{k-4})}$ parametrizing covers for which $\langle w_1, \dots, w_{6k-6} \rangle = \mathfrak{S}_k$.

We discuss the behavior of the divisors E_0, E_2 and E_3 under the map q and we have

$$q^*(D_0) = 2E_0, \ q^*(D_2) = E_2 \ \text{ and } q^*(D_3) = 2E_3.$$

Indeed the general point of both E_0 and E_3 has no automorphism that fixes all branch points, but admits an automorphism of order two that fixes C_1 and permutes the branch points p_{6k-4} and p_{6k-5}. The general admissible cover in E_2 has an automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (each of the two components of C_2 mapping $2:1$ onto R_2 has an automorphism of order 2). In the stack $\overline{H}_k^{\mathfrak{o}}$ we have two points lying over this admissible cover and each of them has an automorphism group of order 2. In particular the map $\overline{H}_k^{\mathfrak{o}} \to \overline{\mathcal{H}}_k^{\mathfrak{o}}$ from the stack to its coarse moduli space is simply ramified along E_2.

The Hurwitz formula applied to the finite ramified cover $\mathfrak{b} : \overline{H}_k^{\mathfrak{o}} \to \overline{\mathcal{M}}_{0,6k-4}$, coupled with the expression $K_{\overline{\mathcal{M}}_{0,6k-4}} = \sum_{i=2}^{3k-2} \big(\frac{i(6k-4-i)}{6k-5} - 2 \big) [B_i]$ for the canonical class of $K_{\overline{\mathcal{M}}_{0,6k-4}}$, yields the following formula (on the stack!):

$$(34) \qquad K_{\overline{H}_k^{\mathfrak{o}}} = \mathfrak{b}^* K_{\overline{\mathcal{M}}_{0,6k-4}} + \mathrm{Ram}(\mathfrak{b}),$$

where $\mathrm{Ram}(\mathfrak{b}) = \sum_{i,\mu} (\mathrm{lcm}(\mu) - 1) E_{i:\mu}$.

12.1. A partial compactification of the Hurwitz space.

It turns out to be convenient to work with a partial compactification of $\mathcal{H}_k$. We denote by $\widetilde{\mathcal{H}}_k$ the (quasi-projective) parameter space of pairs $[C, A]$, where C is an irreducible nodal curve of genus $2k-1$ and A is a base point free locally free sheaf of degree k on C with $h^0(C, A) = 2$. There exists a rational map $\overline{\mathcal{H}}_k \dashrightarrow \widetilde{\mathcal{H}}_k$, well-defined on the general point of each of the divisors D_0, D_2 and D_3 respectively. Under this map, to the general point $[f : C_1 \cup C_2 \to R_1 \cup_p R_2]$ of D_3 (respectively D_2) we assign the pair $[C_1, A_1 := f^* \mathcal{O}_{R_1}(1)] \in \widetilde{\mathcal{H}}_k$. Note that C_1 is a smooth curve of genus $2k-1$ and A_1 is a pencil with a triple point (respectively with two ramification points in the fibre over p). The two partial compactifications $\mathcal{H}_k \cup D_0 \cup D_2 \cup D_3$

and $\widetilde{\mathcal{H}}_k$ differ outside a set of codimension at least 2 and for divisor class calculations they will be identified. Using this, formula (33) has the following translation at the level of $\widetilde{\mathcal{H}}_k$:

$$(35) \qquad \lambda = \frac{3(k-1)}{4(6k-5)}[D_0] - \frac{1}{4(6k-5)}[D_2] + \frac{3k-7}{12(6k-5)}[D_3] \in CH^1(\widetilde{\mathcal{H}}_k).$$

We now record the formula for the canonical class of $\widetilde{\mathcal{H}}_k$:

Proposition 12.1. *One has* $K_{\widetilde{\mathcal{H}}_k} = 8\lambda + \frac{1}{6}[D_3] - \frac{3}{2}[D_0]$.

Proof. We combine the equation (34) with the Hurwitz formula applied to $q : \overline{H}_k^{\mathfrak{o}} \dashrightarrow \widetilde{\mathcal{H}}_k$ and write:

$$q^*(K_{\widetilde{\mathcal{H}}_k}) = K_{\overline{H}_k^{\mathfrak{o}}} - [E_0] - [E_2] - [E_3] = q^*\Big(-\frac{2}{6k-5}[D_2] - \frac{6k-3}{2(6k-5)}[D_0] + \frac{6k-11}{2(6k-5)}[D_3]\Big).$$

The divisors E_0 and E_3 lie in the ramification locus of q, so they are subtracted from $K_{\overline{H}_k^{\mathfrak{o}}}$. Furthermore, the morphism $\overline{H}_k^{\mathfrak{o}} \to \widetilde{\mathcal{H}}_k^{\mathfrak{o}}$ is simply ramified along E_2, so this divisor has to be subtracted as well. We now use (35) to express $[D_2]$ in terms of λ, $[D_0]$ and $[D_3]$ and reach the claimed formula. $\qquad\square$

Let $f : \mathcal{C}_k \to \widetilde{\mathcal{H}}_k$ be the universal curve and we choose a universal degree k line bundle $\mathcal{L}$ on $\mathcal{C}_k$, that is, satisfying $\mathcal{L}_{|f^{-1}[C,A]} = A$, for every $[C,A] \in \widetilde{\mathcal{H}}_k$. Just like in Section 7, we define the following codimension one tautological classes:

$$\mathfrak{a} := f_*\big(c_1^2(\mathcal{L})\big) \text{ and } \mathfrak{b} := f_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_f)\big) \in CH^1(\widetilde{\mathcal{H}}_k).$$

Note that $\mathcal{V} := f_*\mathcal{L}$ is a vector bundle of rank two on $\widetilde{\mathcal{H}}_k$. Although $\mathcal{L}$ is not unique, the class

$$(36) \qquad \gamma := \mathfrak{b} - \frac{2k-2}{k}\mathfrak{a} \in CH^1(\widetilde{\mathcal{H}}_k)$$

is well-defined and independent of the choice of $\mathcal{L}$.

Proposition 12.2. *We have that* $\mathfrak{a} = kc_1(\mathcal{V}) \in CH^1(\widetilde{\mathcal{H}}_k)$.

Proof. Recall that $\widetilde{\mathcal{H}}_k$ has been defined to consist of pairs $[C,A]$ such that A is a base point free pencil of degree k. In particular, the image under f of the codimension 2 locus in $\mathcal{C}_k$ where the morphism of vector bundles $f^*(\mathcal{V}) \to \mathcal{L}$ is not surjective is empty, hence by Porteous' formula

$$0 = f_*\Big(c_2(f^*\mathcal{V}) - c_1(f^*\mathcal{V}) \cdot c_1(\mathcal{L}) + c_1^2(\mathcal{L})\Big) = -kc_1(\mathcal{V}) + \mathfrak{a}.$$

$$\square$$

We now introduce the following locally free sheaves on $\widetilde{\mathcal{H}}_k$:

$$\mathcal{E} := f_*(\omega_f \otimes \mathcal{L}^{\vee}) \text{ and } \mathcal{F} := f^*(\omega_f^2 \otimes \mathcal{L}^{-2})$$

Proposition 12.3. *The following formulas hold*

$$c_1(\mathcal{E}) = \lambda - \frac{1}{2}\mathfrak{b} + \frac{k-2}{2k}\mathfrak{a} \text{ and } c_1(\mathcal{F}) = 13\lambda + 2\mathfrak{a} - 3\mathfrak{b} - [D_0].$$

Proof. We apply Grothendieck-Riemann-Roch twice to f. Since $R^1 f_*(\omega_f^2 \otimes \mathcal{L}^{-2}) = 0$, we write:

$$c_1(\mathcal{F}) = f_*\Big[\Big(1 + 2c_1(\omega_f) - 2c_1(\mathcal{L}) + 2(c_1(\omega_f) - c_1(\mathcal{L})^2\Big) \cdot \Big(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\Omega_f^1) + c_2(\Omega_f^1)}{12}\Big)\Big]_2.$$

Now use Proposition 12.2 as well as $f_*\big(c_1^2(\Omega_f^1) + c_2(\Omega_f^1)\big) = 12\lambda$, see [HM] p. 49, in order to conclude. To compute the first Chern class of $\mathcal{E}$, note that $c_1\big(R^1 f_*(\omega_f \otimes \mathcal{L}^\vee)\big) = -c_1(\mathcal{V})$, hence applying again Grothendieck-Riemann-Roch together with Proposition 12.2, we write:

$$c_1(\mathcal{E}) = -kc_1(\mathcal{V}) + f_*\Big[\Big(1 + c_1(\omega_f) - c_1(\mathcal{L}) + \frac{(c_1(\omega_f) - c_1(\mathcal{L})^2}{2}\Big) \cdot \Big(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\Omega_f^1) + c_2(\Omega_f^1)}{12}\Big)\Big]_2,$$

which quickly leads to the claimed formula.

$\square$

We summarize the relation between the class γ and the classes $[D_0], [D_2]$ and $[D_3]$ as follows:

Proposition 12.4. *One has the formula* $[D_3] = 6\gamma + 24\lambda - 3[D_0]$.

Proof. We form the fibre product of the universal curve $f : \mathcal{C}_k \to \widetilde{\mathcal{H}}_k$ together with its projections:

$$\mathcal{C}_k \xleftarrow{\ \pi_1\ } \mathcal{C}_k \times_{\widetilde{\mathcal{H}}_k} \mathcal{C}_k \xrightarrow{\ \pi_2\ } \mathcal{C}_k.$$

For $\ell \geq 1$, we introduce the sheaf of principal parts $\mathcal{P}_f^\ell(\mathcal{L}) := (\pi_2)_*\Big(\pi_1^*(\mathcal{L}) \otimes \mathcal{I}_{(\ell+1)\Delta}\Big)$. Observe that $\mathcal{P}_f^\ell(\mathcal{L})$ is not locally free along the codimension 2 locus in $\mathcal{C}_k$ where f is not smooth. The jet bundle $J_f^\ell(\mathcal{L}) := \big(\mathcal{P}_f^\ell(\mathcal{L})\big)^{\vee\vee}$, viewed as a *locally free* replacement of $\mathcal{P}_f^\ell(\mathcal{L})$, sits in a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_f^{\otimes \ell} \otimes \mathcal{L} & \longrightarrow & \mathcal{P}_f^\ell(\mathcal{L}) & \longrightarrow & \mathcal{P}_f^{\ell-1}(\mathcal{L}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega_f^{\otimes \ell} \otimes \mathcal{L} & \longrightarrow & J_f^\ell(\mathcal{L}) & \longrightarrow & J_f^{\ell-1}(\mathcal{L}) & \longrightarrow & 0
\end{array}
$$

We introduce the vector bundle morphism $\nu_2 : f^*(\mathcal{V}) \to J_f^2(\mathcal{L})$, which for points $[C, A, p] \in \mathcal{C}_k$ such that $p \in C_{\mathrm{reg}}$ is just the evaluation morphism $H^0(C, A) \to H^0(A \otimes \mathcal{O}_{3p})$. We consider the codimension 2 locus $Z \subseteq \mathcal{C}_k$ where $\nu_2 : f^*(\mathcal{V}) \to J_f^2(\mathcal{L})$ is not injective. Over the locus of smooth curves, D_3 is the set-theoretic image of Z. A simple local analysis shows that the morphism ν_2 is simply degenerate for each point $[C, A, p]$, where $p \in C_{\mathrm{sing}}$, that is, the divisor D_0 also appears (with multiplicity 1) in the degeneracy locus of ν_2. Assuming this fact for a moment, via the Porteous formula we obtain:

$$[D_3] = f_* c_2\left(\frac{J_f^2(\mathcal{L})}{f^*(\mathcal{V})}\right) - [D_0] \in CH^1(\widetilde{\mathcal{H}}_k).$$

One computes $c_1(J_f^2(\mathcal{L})) = 3c_1(\mathcal{L}) + 3c_1(\omega_f)$ and $c_2(J_f^2(\mathcal{L})) = 3c_1^2(\mathcal{L}) + 6c_1(\mathcal{L}) \cdot c_1(\omega_f) + 2c_1^2(\omega_f)$, therefore

$$f_* c_2\left(\frac{J_f^2(\mathcal{L})}{f^*(\mathcal{V})}\right) = 3\mathfrak{a} + 6\mathfrak{b} - 3(5k - 4)c_1(\mathcal{V}) + 2\kappa_1 = 6\gamma + 2\kappa_1.$$

Recalling that $\kappa_1 = 12\lambda - [D_0] \in CH^1(\widetilde{\mathcal{H}}_k)$, the claimed formula follows by substitution.

We are left with concluding that D_0 appears with multiplicity 1 in the degeneracy locus Z. Let $F : \mathcal{X} \to B$ be a family of curves of genus $2k - 1$ over a 1-dimensional base B, with $\mathcal{X}$ smooth, such that there is a point $b_0 \in B$ with $X_b := F^{-1}(b)$ smooth for $b \in B \setminus \{b_0\}$, whereas $X_{b_0} := F^{-1}(b_0)$ has a unique node $u \in \mathcal{X}$. Assume we are given $\mathcal{A} \in \mathrm{Pic}(\mathcal{X})$ such that $A_b := \mathcal{A}_{|X_b} \in W_k^1(X_b)$, for each $b \in B$. We pick a parameter $t \in \mathcal{O}_{B,b_0}$ and $x, y \in \mathcal{O}_{\mathcal{X},u}$, such that $xy = t$ represents the local

equation of $\mathcal{X}$ around u. Then ω_F is locally generated by the meromorphic differential τ given by $\frac{dx}{x}$ outside the divisor $x = 0$ and by $-\frac{dy}{y}$ outside the divisor $y = 0$. We choose a $\mathbb{C}[[t]]$-basis (s_1, s_2) of $H^0(\mathcal{X}, A)$, where $s_1(u) \neq 0$, whereas s_2 vanishes with order 1 at the node u of X_{b_0}, along both its branches. Passing to germs at u, we may assume that $s_{2,u} = (x + y)s_{1,u}$. Denoting by $\partial : \mathcal{O}_{\mathcal{X}} \to \omega_F$ the canonical derivation, we note that $\partial(x) = x\tau$ and $\partial(y) = -y\tau$. Then Z is locally given by the 2×2 minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ x + y & x - y & x + y \end{pmatrix},$$

which proves our claim, that D_0 appears with multiplicity 1.

$\square$

12.2. The divisor $\mathfrak{H}_k^{\mathrm{rk4}}$. We fix a cover $[f : C \to \mathbf{P}^1] \in \mathcal{H}_k$ and set $A := f^*(\mathcal{O}_{\mathbf{P}^1}(1))$. First we observe that by choosing $[f]$ outside a subset of codimension 2 in $\mathcal{H}_k$, we may assume that $\omega_C \otimes A^\vee$ is very ample. Otherwise by Riemann-Roch there exist points $p, q \in C$ such that $A(p + q) \in W_{k+1}^2(C)$. The Brill-Noether number of this linear series equals $\rho(2k - 1, 2, k + 1) = -1 - k < -3$ and it follows from [Ed] that the locus of curves $[C] \in \mathcal{M}_{2k-1}$ possessing such a linear system has codimension at least 3 in moduli, which establishes the claim.

Theorem 12.5. *Fix a general point $[C, A] \in \mathcal{H}_k$. Then the embedded curve $\varphi_{\omega_C \otimes A^\vee} : C \hookrightarrow \mathbf{P}^{k-1}$ lies on no quadrics of rank 4 or less. It follows that $\mathfrak{H}_k^{\mathrm{rk4}}$ is a divisor on $\mathcal{H}_k$.*

Proof. We choose a polarized $K3$ surface X such that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot E$, where $L^2 = 4k - 4$, the curve E is elliptic with $E^2 = 0$ and $L \cdot E = k$. First we observe that X contains no (-2)-curves, hence an effective line bundle $\alpha \in \mathrm{Pic}(X)$ must necessarily be nef and satisfy $\alpha^2 \geq 0$.

Since $(L - 2E)^2 = -4$, we compute $\chi(X, L(-2E)) = 0$. Furthermore, as we have just pointed out $H^0(X, L(-2E)) = 0$, whereas obviously $H^2(X, L(-2E)) = 0$, which implies that $H^1(X, L(-2E)) = 0$, as well. We choose a general curve $C \in |L|$ and set $A := \mathcal{O}_C(E) \in W_k^1(C)$. By taking cohomology in the exact sequence

$$0 \longrightarrow L(-2E) \longrightarrow L^{\otimes 2}(-2E) \longrightarrow \omega_C^{\otimes 2}(-2A) \longrightarrow 0,$$

we obtain an isomorphism $H^0(X, L^{\otimes 2}(-2E)) \cong H^0(C, \omega_C^{\otimes 2}(-2A))$. Since clearly, the isomorphism $H^0(X, L(-E)) \cong H^0(C, \omega_C(-A))$ also holds, we obtain

$$I_{X,L}(2) \cong I_{C,\omega_C(-A)}(2),$$

so it suffices to show that the embedded surface $\varphi_{L(-E)} : X \hookrightarrow \mathbf{P}^{k-1}$ lies on no quadric of rank 4. This amounts to showing that one cannot have a decomposition $L - E = D_1 + D_2$, where D_1 and D_2 are divisor classes on X with $h^0(X, D_i) \geq 2$, for $i = 1, 2$. Assume we have such a decomposition and write $D_i = x_i C + y_i E$, where $x_1 + x_2 = 1$ and $y_1 + y_2 = -1$. Since E is nef, we obtain that both x_1 and x_2 have to be non-negative and we assume $x_1 = 0$ and $x_2 = 1$. Then $D_1 \equiv y_1 E$, therefore $y_1 \geq 1$, yielding $y_2 \leq -2$, which implies that $h^0(X, D_2) \leq h^0(X, L(-2E)) = 0$, which leads to a contradiction. $\square$

The divisor $\mathfrak{H}_k^{\mathrm{rk4}}$ decomposes into components, depending on the degrees of the pencils A_1 and A_2 for which the decomposition (2) holds. For instance, when $\deg(A_1) = \deg(A) = k$, we obtain the component denoted in [FK] by $\mathfrak{BN}$ and which consists of pairs $[C, A] \in \mathcal{H}_k$, such that C carries a second pencil of degree k. It is shown in [FK] that $\mathfrak{BN}$ has a syzygy-theoretic incarnation that makes reference only to the canonical bundle, being equal to the Eagon-Northcott divisor on $\mathcal{H}_k$ of curves

for which $b_{k-1,1}(C, \omega_C) \geq k$. It is an interesting question whether the remaining components of $\mathfrak{H}_k^{\mathrm{rk4}}$ have a similar intrinsic realization.

We now compute the class of the closure of $\mathfrak{H}_k^{\mathrm{rk4}}$ inside $\widetilde{\mathcal{H}}_k$:

Theorem 12.6. *The following formula holds:* $[\overline{\mathfrak{H}}_k^{\mathrm{rk4}}] = A_k^{k-4} \left(\frac{5k+12}{k} \lambda + \frac{k-6}{k} \gamma - [D_0] \right)$.

Proof. We are in a position to apply Theorem 1.1 and then $[\overline{\mathfrak{H}}_k^{\mathrm{rk4}}] = A_k^{k-4} \left(c_1(\mathcal{F}) - \frac{4}{k}(2k-3)c_1(\mathcal{E}) \right)$ and we substitute these Chern classes with the formulas provided in Proposition 12.3. $\square$

The proof of Theorem 1.7 from the Introduction now follows. We substitute the formula for the class γ obtained from Theorem 12.6 in the expression provided by Proposition 12.4, then compare it to the formula for $K_{\widetilde{\mathcal{H}}_k}$.

Proof of Theorem 1.8. It is enough to observe that for $k \geq 12$, the class $7\lambda - \delta_0$ is big on $\overline{\mathcal{M}}_{2k-1}$ and there exists an effective divisor of this slope that does not contain $\mathrm{Im}(\sigma) = \overline{\mathcal{M}}_{2k-1,k}^1$ as a component. This follows from results in [F3] Corollary 0.6, where it is proved that the divisor $\overline{D}_{2k-1,k+1}$ on $\overline{\mathcal{M}}_{2k-1}$ already considered in (19), has support distinct from that of $\overline{\mathcal{M}}_{2k-1,k}^1$ and slope $\frac{6k^2+14k+3}{k(k+1)} < 7$. $\square$

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