MINIMAL RESOLUTIONS, CHOW FORMS OF $K3$ SURFACES AND ULRICH BUNDLES

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The Minimal Resolution Conjecture (MRC) for points on a projective variety $X \subset \mathbf{P}^r$ predicts that the minimal graded free resolution of a general set $\Gamma \subset X$ of points is as simple as the geometry of X allows. Originally, the most studied case has been that when $X = \mathbf{P}^r$, see [EPSW]. The general form of the MRC for subvarieties $X \subset \mathbf{P}^r$ was formulated in [Mus] and [FMP]. The Betti diagram of a large enough set $\Gamma \subset X$ consisting of γ general points is obtained from the Betti diagram of X, by adding two rows, indexed by $u-1$ and u, where u is an integer depending on γ. All differences $b_{i+1,u-1}(\Gamma) - b_{i,u}(\Gamma)$ are known and depend on the Hilbert polynomial P_X and i, u and γ, see [FMP]. The *Minimal Resolution Conjecture* for γ general points on X predicts that

$$b_{i+1,u-1}(\Gamma) \cdot b_{i,u}(\Gamma) = 0,$$

for each $i \geq 0$, in which case, the Betti numbers of Γ are explicitly given in terms of P_X and γ. The *Ideal Generation Conjecture* (IGC) predicts the same vanishing but only for $i = 1$, that is, $b_{2,u-1}(\Gamma) \cdot b_{1,u}(\Gamma) = 0$; equivalently, the number of generators of the ideal I_Γ / I_X is minimal.

In [FMP], the Minimal Resolution Conjecture for points on curves is reformulated in geometric terms. For a globally generated linear series $\ell = (L, V) \in G^r_d(C)$, we consider the kernel vector bundle M_V defined via the evaluation sequence

$$0 \longrightarrow M_V \longrightarrow V \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0.$$

Then MRC holds for $C \overset{|V|}{\hookrightarrow} \mathbf{P}^r$ if and only if M_V satisfies the *Raynaud property* (R)

$$H^0\Big(C, \bigwedge^i M_V \otimes \xi\Big) = 0, \qquad (1)$$

for each $i = 0, \ldots, r$ and a general line bundle ξ on C with $\deg(\xi) = g - 1 + \lfloor \frac{id}{r} \rfloor$, see [FMP] Corollary 1.8. When $\mu := \frac{d}{r} \in \mathbb{Z}$ (in which case we refer to $C \subset \mathbf{P}^r$ as being a curve of integer slope), property (R) is satisfied if and only if for $i = 0, \ldots, r$, the cycle

$$\Theta_{\bigwedge^i M_V} := \Big\{\xi \in \mathrm{Pic}^{g-1+i\mu}(C) : h^0\Big(C, \bigwedge^i M_V \otimes \xi\Big) \neq 0\Big\}$$

is a divisor in $\mathrm{Pic}^{g-1+i\mu}(C)$. Equivalently, $\bigwedge^i M_V$ has a theta divisor for all $i \geq 0$.

Our first result is a proof of MRC for curves $C \subset \mathbf{P}^r$ of integer slope $\mu := \frac{d}{r} \in \mathbb{Z}_{\geq 1}$.

Theorem 0.1. *The Minimal Resolution Conjecture holds for a general embedding $C \hookrightarrow \mathbf{P}^r$ of degree μr of any curve C with general moduli, for any integers $\mu, r \geq 1$.*

The hypothesis on the generality of C implies that its genus g satisfies the inequality $g \leq (r+1)(\mu-1)$ imposed by Brill-Noether theory. We have similarly complete results for curves $C \subset \mathbf{P}^r$ of degree $d \equiv \pm 1 \bmod r$, see Theorem 1.6.

In the case of curves $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^{d-g}$ embedded by a complete linear system of degree $d \geq 2g+5$, counterexamples to MRC for points on C were found in [FMP][1]; observe that in these cases $\mu = \frac{d}{d-g} < 2$. On the other hand, MRC holds for *all* smooth canonical curves $C \subset \mathbf{P}^{g-1}$, see [FMP], as well as for general line bundles of degree $2g$, see [B1]. In both these cases, one has $\mu = 2$. This confusing state of affairs is reminiscent of the situation for the projective space $\mathbf{P}^r$, where it is known [HS] that MRC holds for $r \geq 4$ and γ very large with respect to r, but fails for each $r \geq 6, r \neq 9$ for many values of γ, see [EPSW]. Our next result show that for curves, independently of the genus, the *Clifford line* line $d = 2r$ in the (d,r)-plane governs whether MRC holds for a general curve $C \subset \mathbf{P}^r$ of genus g and degree d.

Theorem 0.2. *Let C be a curve of genus g with general moduli and integers $d, r \geq 1$ such that $d \geq 2r$. The Minimal Resolution Conjecture holds for a general embedding $C \hookrightarrow \mathbf{P}^r$ of degree d, whenever the following condition is satisfied:*

$$d + r\lfloor \frac{d}{r} \rfloor \geq 2g + 2r - 2. \tag{2}$$

Note that no assumption is made regarding the completeness of the linear series (L, V) inducing the map $\varphi_V : C \hookrightarrow \mathbf{P}^r$. Inequality (2) in Theorem 0.2 is satisfied when $d \geq g + \frac{3r}{2} - 2$. It is also satisfied in the range $d \geq 2g-1$, when all line bundles in question are non-special. The condition $d \geq 2r$ is certainly necessary, for as already pointed out, in the other cases counterexamples to MRC were produced using complete linear series, see [FMP] Theorem 2.2.

We now turn our attention to the IGC for a set Γ of γ general points lying on an embedded curve $\varphi_V : C \hookrightarrow \mathbf{P}^r$. Assume $\gamma \geq d \cdot \mathrm{reg}(C) - g + 1$ and set $u := 1 + \lfloor \frac{\gamma+g-1}{d} \rfloor$; thus u is the integer uniquely determined by the condition $P_C(u-1) \leq \gamma < P_C(u)$, see also Section 1 for details. The resolution of the zero-dimensional scheme $\Gamma \subset \mathbf{P}^r$ has the following form, see also [Mus] Proposition 1.6,

$$\cdots \to S(-u)^{\oplus(du+1-g-\gamma)} \oplus S(-u-1)^{\oplus b_{1,u}(\Gamma)} \to S \to S(\Gamma) \to 0,$$

where $b_{2,u-1}(\Gamma) - b_{1,u}(\Gamma) = r(du - \gamma + 1 - g) - d$. The Ideal Generation Conjecture for C and Γ amounts to the multiplication map

$$V \otimes H^0\big(C, \mathcal{I}_{\Gamma/C}(u)\big) \to H^0\big(C, \mathcal{I}_{\Gamma/C}(u+1)\big)$$

having maximal rank, or equivalently, the number of generators of the ideal I_Γ/I_C being minimal, precisely $b_{1,u}(\Gamma) = \max\big\{d - r(du - \gamma + 1 - g), 0\big\}$. The following result gives a complete solution to IGC for general curves.

[1]Note that it is precisely in this range the Minimal Resolution Conjecture fails to hold for C itself as well; if $L \in \mathrm{Pic}^d(C)$ is a line bundle of degree $d \geq 2g+5$, then the resolution of $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^{d-g}$ is not natural in the sense of [CEFS], that is, there exists i such that $b_{i+1,1}(C) \cdot b_{i,2}(C) \neq 0$.

Theorem 0.3. *Fix integers $g, r, d \geq 0$. Then the Ideal Generation Conjecture holds for a general embedding $C \hookrightarrow \mathbf{P}^r$ of degree d of any genus g curve C having general moduli.*

It should be pointed out that Theorems 0.1, 0.2 and 0.3 are optimal in the sense that they establish MRC or IGC for a *general* curve $[C] \in \mathcal{M}_g$ and a *general* linear series $\ell \in G^r_d(C)$. Having fixed g, r and d, one cannot expect a more precise statement. It suffices indeed to consider the situation in genus 0. To a non-degenerate rational curve $R \subset \mathbf{P}^r$ of degree d, one associates the splitting type $a_1 \leq \ldots \leq a_r$ of the vector bundle $T_{\mathbf{P}^r|R}(-1) = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_r)$. The splitting type of a general R as above is *balanced*, that is, with $0 \leq a_r - a_1 \leq 1$, and then $a_1 = \lfloor \frac{d}{r} \rfloor$ and $a_r = \lceil \frac{d}{r} \rceil$; the locus of curves with non-balanced splitting type is a divisor in the (irreducible) Hilbert scheme of rational curves $R \subset \mathbf{P}^r$ of degree d. On the other hand, it is easy to see cf. [Mus] Corollary 3.8, that R verifies MRC if and only if its splitting type is balanced. Such examples can be constructed on every curve of positive genus, by considering linear series with exceptional secant behaviour; systematically MRC fails along certain proper subvarieties of the corresponding Hilbert schemes, but holds generically.

The second topic we investigate in this paper concerns Chow forms and Ulrich bundles. We fix a k-dimensional variety $X \subset \mathbf{P}^r$ of degree d. Following [ESW], a vector bundle E on X is said to be an *Ulrich bundle* if E admits a *completely linear* $\mathcal{O}_{\mathbf{P}^r}$-resolution

$$0 \to \mathcal{O}_{\mathbf{P}^r}(-r+k)^{\oplus a_{r-k}} \to \cdots \to \mathcal{O}_{\mathbf{P}^r}(-1)^{\oplus a_1} \to \mathcal{O}_{\mathbf{P}^r}^{\oplus a_0} \to E \to 0,$$

where $a_0 = d \cdot \mathrm{rk}(E)$ and $a_i = \binom{r-k}{i} a_0$ for $i \geq 1$. In terms intrinsic to X, this amounts to requiring E to be an ACM bundle, that is, $H^i(X, E(t)) = 0$ for all t and $i = 1, \ldots, k-1$, and the module $\mathbf{\Gamma}_*(E) := \oplus_{t \in \mathbb{Z}} H^0(X, E(t))$ to have the maximum number of generators, which equals $d \cdot \mathrm{rk}(E)$, all appearing in degree 0. It is conjectured in [ESW] that every k-dimensional projective variety $X \subset \mathbf{P}^r$ carries an Ulrich bundle. As explained in [ES], the existence of an Ulrich bundle on X implies that the *cone of cohomology tables*

$$C(X, \mathcal{O}_X(1)) := \mathbb{Q}_{\geq 0} \Big\langle \Big(h^i(X, F(m)) \Big)_{0 \leq i \leq k, m \in \mathbb{Z}} : F \text{ sheaf on } X \Big\rangle \subset \mathrm{Mat}_{k+1, \infty}(\mathbb{Q})$$

is the same as that for the projective space $\mathbf{P}^k$. This conjecture has been confirmed so far only in few cases. A hypersurface carries an Ulrich bundle of exponential rank, see [BHU]. Curves also carry Ulrich line bundles [ESW]; a vector bundle E on a smooth curve $C \subset \mathbf{P}^r$ having slope $\mu(E) = d + g - 1$ is an Ulrich bundle, if and only

$$H^0(C, E(-1)) = 0 \Leftrightarrow \mathcal{O}_C(-1) \notin \Theta_E.$$

When $X \subset \mathbf{P}^r$ is a hypersurface, the existence of Ulrich bundles is related to classical problems in algebraic geometry, see [B2]. If $\mathrm{rk}(E) = 1$, then one has a determinantal presentation of $X : \{\det(M) = 0\}$, where $M = (\ell_{ij})_{1 \leq i,j \leq d}$ is a matrix of linear forms; a bundle E with $\mathrm{rk}(E) = 2$ corresponds to a Pfaffian equation of $X : \{\mathrm{pf}(M) = 0\}$, where M is a $(2d) \times (2d)$ skew-symmetric linear matrix.

Del Pezzo surfaces $X_d \subset \mathbf{P}^d$ of degree d have Ulrich bundles of any rank $r \geq 2$, see [MP], [CHGS]. A remarkable connection between Ulrich bundles and the Ideal Resolution Conjecture as studied in this paper is established in [CKM1]. Precisely, there exists an Ulrich bundle E on X with $\det(E) = \mathcal{O}_X(C)$, if and only if the curve $C \subset X$

has degree $d \cdot \mathrm{rk}(E)$ and IRC holds for C. Finally, we mention that using the techniques of [AF], Coskun, Kulkarni and Mustopa [CKM2] have shown that every smooth quartic surface $X \subset \mathbf{P}^3$ carries a rank 2 Ulrich bundle, thus generalizing work of Beauville [B2].

In this paper we describe the moduli space of Ulrich bundle on a polarized $K3$ surface. First, we show that $K3$ surfaces satisfying a mild generality condition carry Ulrich bundles of rank 2, satisfying the skew-symmetry requirement of [ESW].

Theorem 0.4. *Let $S \subset \boldsymbol{P}^{s+1}$ be a polarized $K3$ surface. If the Clifford index of cubic sections of S is computed by $\mathcal{O}_S(1)$, then S carries a $(2s+10)$-dimensional family of stable rank 2 Ulrich bundles E with $\det(E) = \mathcal{O}_S(3)$.*

A smooth cubic section $C \in |\mathcal{O}_S(3)|$ is a curve of genus $9s+1$ and degree $6s$, hence the inequality $\mathrm{Cliff}(C) \leq \mathrm{Cliff}(\mathcal{O}_C(1)) = 4s-2$ holds. If $\mathrm{Cliff}(C) < 4s-2$, then from [GL] it follows that there exists an effective class $D \in \mathrm{Pic}(S)$ such that

$$h^0(S, \mathcal{O}_S(D)) \geq 2, \ \ D \cdot H \leq 3s \ \text{ and } \ 3D \cdot H - D^2 \leq 4s-1. \tag{3}$$

The existence of rank 2 Ulrich bundles is established for all $K3$ surfaces in the complement of the Noether-Lefschetz locus singled out by (3). In particular, Theorem 0.4 holds for every $K3$ surface of Picard number one. Using [L2] page 185, condition (3) is never satisfied for smooth quartics (that is, $s = 2$), for in this case we have complete intersection curves, whose Clifford index is computed by multisecants. The restriction to rank 2 is natural, for a very general polarized $K3$ surface carries *no* Ulrich bundles of odd rank, see Section 2.

The case $s = 2$ of Theorem 0.4 was proved in [CKM2]. Our proof of Theorem 0.4 partly grew out of an attempt to generalize that result. The bundles E are *special Ulrich bundles* in the sense of [ESW] Proposition 6.2; when $\det(E) = \mathcal{O}_S(3)$ the Ulrich condition is equivalent to E being 0-regular. The candidate bundles are *Lazarsfeld-Mukai bundles* $E := E_{C,A}$, where $C \in |\mathcal{O}_S(3)|$ is a suitable cubic section of S and $A \in W^1_{5s+4}(C)$ is a complete base point free pencil. Since C is far from being Brill-Noether general, showing that a general cubic section $C \subset S$ carries a pencil $\mathfrak{g}^1_{5s+4}$ inducing a simple Ulrich bundle, becomes a rather tricky variational problem, which we solve in a way reminiscent of our proof [AF] of Green's conjecture for curves on arbitrary $K3$ surfaces. The role of the required equality $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{O}_C(1))$ is that it ensures the existence of a complete base point free pencil $\mathfrak{g}^1_{5s+4}$.

By taking direct sums of Ulrich bundles, Theorem 0.4 implies the existence of Ulrich bundles of any even rank on S. We show that for very general $K3$ surfaces these direct sums can be deformed to stable Ulrich bundles:

Theorem 0.5. *Let $S \subset \boldsymbol{P}^{s+1}$ be a polarized $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot [H]$. For every integer $a \geq 1$, there exists an $(8a^2 + 2a^2 s + 2)$-dimensional family of* stable *Ulrich bundles of rank $2a$. Furthermore, there exists a component of the corresponding moduli space $M_S\big(2a, \mathcal{O}_S(3a), 9a^2 s - 4a(s-1)\big)$ of the moduli space of vector bundles on S, whose general point corresponds to a stable Ulrich bundle of rank $2a$.*

The case $s = 2$ of this result has been recently established by Coskun [C] and our method of proof is very similar to his. Since $K3$ surfaces with Picard number one carry no Ulrich bundles of odd rank, this answers completely the question which numbers appear as ranks of stable Ulrich bundles on S. In the language of [MP], Theorem 2.7 establishes that every $K3$ surface of Picard number one has *wild representation type*.

Our results on Ulrich bundles imply via [ESW] that the Chow form of a polarized $K3$ surface admits a pfaffian Bézout form in Plücker coordinates. We fix as before a polarized $K3$ surface $S \subset \mathbf{P}^{s+1}$ and let $\mathbf{G} := G(s-1, H^0(S, \mathcal{O}_S(1))^\vee)$ be the Grassmannian of codimension 3 planes in $\mathbf{P}^{s+1}$ and $\mathcal{U}$ the rank 3 tautological bundle on $\mathbf{G}$ sitting in the exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow H^0(S, \mathcal{O}_S(1)) \otimes \mathcal{O}_{\mathbf{G}} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Let $\mathbf{\Lambda} := \bigwedge^{\bullet} H^0(S, \mathcal{O}_S(1))^\vee$ be the exterior algebra and $\mathbf{\Lambda}^\vee := \bigwedge^{\bullet} H^0(S, \mathcal{O}_S(1))$ its dual. Using the identification $\bigwedge^{s-1} H^0\big(S, \mathcal{O}_S(1)\big)^\vee \cong \bigwedge^3 H^0\big(S, \mathcal{O}_S(1)\big)$, we view elements from $\bigwedge^3 H^0\big(S, \mathcal{O}_S(1)\big)^\vee$ as Plücker coordinates on $\mathbf{G}$. We recall that the *Cayley-Chow* form of S is the degree $2s$ hypersurface

$$\mathcal{Z}(S) := \{L \in \mathbf{G} : L \cap S \neq \emptyset\}.$$

Putting together Theorem 0.4 and [ESW] Corollary 3.4, we obtain the following result:

Theorem 0.6. *Let $S \subset \mathbf{P}^{s+1}$ be a polarized $K3$ surface such that the Clifford index of a cubic section is computed by $\mathcal{O}_S(1)$ and E a rank two Ulrich bundle on S. Then there exists a skew-symmetric morphism of vector bundles of rank $4s$ on $\mathbf{G}$*

$$\mathbf{U}_3(\varphi) : H^0(S, E)^\vee \otimes \bigwedge^3 \mathcal{U} \longrightarrow H^0(S, E) \otimes \mathcal{O}_{\mathbf{G}}$$

whose pfaffian cuts out precisely the Cayley-Chow form of S.

Via the identification $H^2(S, E(-3)) \cong H^0(S, E)^\vee$, the morphism $\mathbf{U_3}(\varphi)$ is given by applying the functor $\mathbf{U}_3$ from [ESW] to the map

$$\varphi : H^2(S, E(-3)) \otimes \bigwedge^3 H^0(S, \mathcal{O}_S(1)) \to H^0(S, E).$$

Precisely, there exists an exact sequence of $\mathbf{\Lambda}$-modules

$$\mathbf{\Lambda}^\vee(3) \otimes H^2(S, E(-3)) \xrightarrow{\varphi} \mathbf{\Lambda}^\vee \otimes H^0(S, E) \xrightarrow{\psi} \mathbf{\Lambda}^\vee(-1) \otimes H^0(S, E(1)), \tag{4}$$

where $\psi = (\psi^j)_{j \geq 0}$ is the map given by the Koszul differentials

$$\psi^j : \bigwedge^j H^0(S, \mathcal{O}_S(1)) \otimes H^0(S, E) \longrightarrow \bigwedge^{j-1} H^0(S, \mathcal{O}_S(1)) \otimes H^0(S, E(1)).$$

As a map of $\mathbf{\Lambda}$-modules, ψ is given by the same tensor as the multiplication map

$$H^0(S, \mathcal{O}_S(1)) \otimes H^0(S, E) \to H^0(S, E(1)).$$

The complex (4) is part of the *Tate resolution* of the module $\mathbf{\Gamma}_*(E)$ over the exterior algebra $\mathbf{\Lambda}$, as constructed in [EFS]. The vector bundle morphism in Theorem 0.6 is obtained by applying the functor of Eisenbud and Schreyer [ESW] to the Tate resolution. This reduces the complex to a single morphism of vector bundles on $\mathbf{G}$, which degenerates exactly along $\mathcal{Z}$. For an explicit description of φ, we refer to Section 2.

Apart from serving as Ulrich bundles on $K3$ surfaces, the Lazarsfeld-Mukai (LM) bundles are also studied from a different angle in the last section of the paper. For a $K3$

surface S, a smooth curve $C \subset S$ and a globally generated linear series $A \in W^r_d(C)$ with $h^0(C,A) = r+1$, the *Lazarsfeld-Mukai* vector bundle $E_{C,A}$ is defined via the following elementary modification on S

(5) $$0 \longrightarrow E^{\vee}_{C,A} \longrightarrow H^0(C,A) \otimes \mathcal{O}_S \longrightarrow A \longrightarrow 0.$$

The bundles $E_{C,A}$ have been intensely studied in the context of Brill-Noether theory [L1], moduli of sheaves [Mu], and more recently, in connection with Mercat's conjecture [FO]. Recall that the Clifford index of a semistable vector bundle $E \in \mathcal{U}_C(n,d)$ on a curve C of genus g is defined as $\gamma(E) := \mu(E) - \frac{2}{n}h^0(C,E) + 2$. Then the *higher Clifford indices* of the curve C are defined as the quantities

$$\mathrm{Cliff}_n(C) := \min\Big\{\gamma(E) : E \in \mathcal{U}_C(n,d),\ \ d \le n(g-1),\ \ h^0(C,E) \ge 2n\Big\}.$$

Mercat [Me1] predicted that for any smooth curve C of genus g, the following equality

$$(M_n):\ \ \mathrm{Cliff}_n(C) = \mathrm{Cliff}(C).$$

should hold. For background on this problem, see [Me1], [LN], [GMN] and [FO].

The restricted LM bundle $E|_C := E_{C,A} \otimes \mathcal{O}_C$ sits in the following exact sequence

(6) $$0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^{\vee} \longrightarrow 0,$$

where $Q_A = M_A^{\vee}$ is the dual of the kernel bundle appearing in the formulation of MRC. One then shows [V], [FO] that the sequence (6) is exact on global section, that is,

$$h^0(C,E|_C) = h^0(C,K_C \otimes A^{\vee}) + h^0(C,Q_A) = g-d+2r+1.$$

By choosing d minimal such that $W^r_d(C) \neq \emptyset$, precisely $d = r + \lfloor \frac{r(g+1)}{r+1} \rfloor$, it becomes clear that for sufficiently high g, one has $\gamma(E|_C) < \mathrm{Cliff}(C)$, that is, $E|_C$, when semistable, is a counterexample to Mercat's conjecture (M_{r+1}). We prove the following result, extending to rank 4 a picture studied in smaller ranks in [Mu], [V], respectively [FO].

Theorem 0.7. *Let S be a $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot L$ where $L^2 = 2g-2$ and write $g = 4i-4+\rho$ and $d = 3i+\rho$, with $\rho \ge 0$ and $i \ge 6$. Then for a general curve $C \in |L|$ and a globally generated linear series $A \in W^3_d(C)$ with $h^0(C,A) = 4$, the restriction to C of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable.*

Note that in Theorem 0.7, dim $W^3_d(C) = \rho$. We record the following consequence.

Corollary 0.8. *For $C \subset S$ with $g \ge 20$ and $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, we set $d := \lfloor \frac{4g+14}{3} \rfloor$ and $A \in W^3_d(C)$ with $h^0(C,A) = 4$. Then $E|_C$ is a stable rank 4 bundle with $\gamma(E|_C) < \lfloor \frac{g-1}{2} \rfloor$. It follows that the statement (M_4) fails for C.*

The curves C appearing in Corollary 0.8 satisfy $\mathrm{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$, see [L1]. In Section 4 of the paper, we also show that under mild restrictions, on a very general $K3$ surface, the extension (6) is non-trivial and the restricted LM bundle $E|_C$ is simple (see Theorem 3.2). We expect that the bundle $E|_C$ remains stable also for higher ranks $r+1 = h^0(C,A)$, at least when $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$. However, our method of proof based on the Bogomolov inequality, seem not to extend easily for $r \ge 4$.

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1. MINIMAL RESOLUTIONS OF SETS OF POINTS ON CURVES AND THETA DIVISORS

The aim of this section is to prove Theorems 0.1 and 0.2 and we begin with some preliminaries. Let us fix a subscheme $Z \subset \mathbf{P}^r$. The *graded Betti numbers* of Z, counting the i-th order syzygies of degree j in the minimal free resolution of the coordinate ring $S(Z)$ over the polynomial ring $S := \mathbb{C}[x_0, \ldots, x_r]$ are denoted as usual by

$$b_{i,j}(Z) := \dim_{\mathbb{C}} \mathrm{Tor}^i_{i+j}\big(S(Z), \mathbb{C}\big).$$

The graded Betti diagram of Z is obtained by placing $b_{i,j}(Z)$ in the j-th row and i-th column. The number of non-trivial rows in the Betti diagram of Z equals the Castelnuovo-Mumford regularity reg(Z), that is, $b_{i,j}(Z) = 0$, for $j \geq \mathrm{reg}(Z) + 1$.

Let $C \subset \mathbf{P}^r$ be a smooth curve of genus g embedded by a linear series $\ell := (L, V) \in G^r_d(C)$. Via the Euler sequence, the kernel bundle $M_V := \mathrm{Ker}\{V \otimes \mathcal{O}_C \to L\}$ of the evaluation map can be interpreted as the restriction $M_V = \Omega^1_{\mathbf{P}^r|C}(1)$. We fix a set $\Gamma \subset C$ of γ general points, where $\gamma \geq d \cdot \mathrm{reg}(C) + 1 - g$, then set

$$u := 1 + \lfloor \frac{\gamma + g - 1}{d} \rfloor \geq 1 + \mathrm{reg}(C).$$

It is proved in [FMP] Theorem 1.2 that the Betti diagram of Γ is obtained from that of C by adding two rows, indexed by $u - 1$ and u respectively. Precisely, one has that

$$b_{i,j}(\Gamma) = b_{i,j}(C), \ \text{ for } i \geq 0, \ j \leq u - 2, \quad \text{and}$$
$$b_{i,j}(\Gamma) = 0, \ \text{ for } \ i \geq 0 \text{ and } j \geq u + 1.$$

The Betti numbers of Γ in rows $u - 1$ and u have the following interpretation:

$$b_{i+1,u-1}(\Gamma) = h^0\Big(C, \bigwedge^i M_V \otimes L^{\otimes u}(-\Gamma)\Big) \ \text{ and } \ b_{i,u}(\Gamma) = h^1\Big(C, \bigwedge^i M_V \otimes L^{\otimes u}(-\Gamma)\Big).$$

The difference of the two Betti numbers on each diagonal can be computed via Riemann-Roch, being equal to the Euler characteristic of a vector bundle on C:

$$b_{i+1,u-1}(\Gamma) - b_{i,u}(\Gamma) = \chi\Big(C, \bigwedge^i M_V \otimes L^{\otimes u}(-\Gamma)\Big) = \binom{r}{i}\Big(-\frac{id}{r} + du - \gamma + 1 - g\Big).$$

The *Minimal Resolution Conjecture* (MRC) for C predicts that $b_{i+1,u-1}(\Gamma) \cdot b_{i,u}(\Gamma) = 0$ for all i, that is, the number of syzygies of Γ is as small as the parameters g, d, r, u and γ allow. The *Ideal Generation Conjecture* (IGC) predicts the same vanishing, but only for $i = 1$. The MRC (respectively IGC) for C break up into generic vanishing statements for exterior powers of kernel bundles.

Proposition 1.1. *(a) The Minimal Resolution Conjecture holds for a smooth curve $C \subset \mathbf{P}^r$, if and only if $H^0\big(C, \bigwedge^i M_V \otimes \xi\big) = 0$ for all $i = 1, \ldots, r-1$ and a general line bundle $\xi \in \mathrm{Pic}(C)$ with $\deg(\xi) = g - 1 + \lfloor \frac{id}{r} \rfloor$.*
(b) The Ideal Resolution Conjecture holds for $C \subset \mathbf{P}^r$, if and only if the previous generic vanishing statement holds for $i = 1, r - 1$.

As already observed in [FMP], the vanishing statements in Proposition 1.1 are closely related to work of Raynaud [R].

Definition 1.2. Let C be a smooth curve of genus g and E a vector bundle on C with slope $\mu(E)=\mu$. Then E is said to satisfy condition (R), if $H^0\big(C,\bigwedge^i E\otimes\xi\big)=0$, for all $i=1,\ldots,r-1$ and for a general line bundle $\xi\in \mathrm{Pic}^{g-1-\lceil i\mu\rceil}(C)$.

When $\mu\in\mathbb{Z}$, condition (R) implies the semistability of the vector bundle E and it is in general a much stronger property. Raynaud [R] has given the first examples of stable vector bundles on curves of genus at least 4 that do not satisfy condition (R). Popa [P] showed that if $\deg(L)\geq 2g+2$, then the kernel bundle M_L fails to verify condition (R). When $\mu(E)=\mu\in\mathbb{Z}$, the bundle E verifies condition (R) if and only if $\bigwedge^i E$ admits a theta divisor $\Theta_{\bigwedge^i E}\subset \mathrm{Pic}^{g-i\mu-1}(C)$ for all i.

Let us fix integers $g,r,d\geq 1$, such that the Brill-Noether number

$$\rho(g,r,d):=g-(r+1)(g-d+r)$$

is non-negative. The Hilbert scheme $\mathrm{Hilb}_{g,r,d}$ of curves $C\subset \mathbf{P}^r$ of genus g and degree d has a unique component $\mathcal{H}_{g,r,d}$ with general point corresponding to a smooth curve and which maps dominantly onto $\mathcal{M}_g$ under the forgetful map $\sigma:\mathcal{H}_{g,r,d}\dashrightarrow\mathcal{M}_g$. In order to prove MRC for a general embedding of a curve of genus g with general moduli, it suffices, for given r and d, to construct a smooth curve $\big[C\xrightarrow{|V|}\mathbf{P}^r\big]$ such that (i) C lies in the component $\mathcal{H}_{g,r,d}$ and (ii) the bundle M_V verifies the conditions (R). Condition (i) is implied by the injectivity of the Petri map $\mu_0(V):V\otimes H^0(C,K_C(-1))\to H^0(C,K_C)$, which is automatically satisfied in the non-special range $d\geq 2g-1$.

We now prove Theorem 0.1 for curves of integral slope $\mu=\frac{d}{r}\in\mathbb{Z}$. For an integer $\mu\geq 1$, the inequality $\rho(g,r,\mu r)\geq 0$ is equivalent to $g\leq (r+1)(\mu-1)$. If $C\subset\mathbf{P}^r$ is a nodal curve, when there is no danger of confusion, we write $M_C:=\Omega^1_{\mathbf{P}^r|C}(1)=M_V$, where $V\subset H^0(C,\mathcal{O}_C(1))$ is the space of sections inducing the embedding of C.

Proof of Theorem 0.1. When $\mu=1$, then $C\subset\mathbf{P}^r$ is necessarily a rational normal curve and $M_C=\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus r}$. The conclusion of the theorem is immediate.

Suppose $\mu\geq 2$ and we specialize to a μ-gonal curve of genus g; this has the effect of splitting the corresponding kernel bundle into a direct sum of line bundles of the same slope. Let $[C]\in\mathcal{M}^1_{g,\mu}$ be a general member of the μ-gonal locus in $\mathcal{M}_g$. Then the *scrollar invariants* of a suitably general pencil $\mathfrak{g}^1_\mu$ on C are as balanced as possible. Precisely, C possesses a base point free pencil $(A,W)\in G^1_\mu(C)$, such that $H^0(C,A^{\otimes j})=j+1$ if and only if $g\geq j(\mu-1)$; else, that is, when $g\leq j(\mu-1)$, we have that $H^1(C,A^{\otimes j})=0$. In particular, the assumption $\rho(g,r,\mu r)\geq 0$ implies that $H^1(C,A^{\otimes(r+1)})=0$, see [CM] Proposition 2.1.1. We consider the following triple

$$\big[C,\ L:=A^{\otimes r},\ V:=\mathrm{Sym}^r(W)\big]\in\mathrm{Hilb}_{g,r,\mu r},$$

where we identify $\mathrm{Sym}^r(W)$ with its image under the injection $\mathrm{Sym}^r(W)\hookrightarrow H^0(C,A^{\otimes r})$. This point corresponds to a complete linear series, that is, $V=H^0(C,A^{\otimes r})$, if and only $g\in[r(\mu-1),(r+1)(\mu-1)]$, or equivalently, when $g-d+r\geq 0$. Geometrically, the constructed curve is given by the map $\nu_r\circ\varphi:C\to\mathbf{P}^r$, where $\varphi:C\to\mathbf{P}^1$ is the degree μ

map corresponding to the pencil $|W|$ and $\nu_r : \mathbf{P}^1 \to \mathbf{P}^r$ is the rth Veronese map, whose image is a rational normal curve $R \subset \mathbf{P}^r$.

The kernel bundle $M_R = \Omega^1_{\mathbf{P}^r|R}(1)$ splits into a sum of line bundles of the same degree, precisely, $M_R = \mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus r}$. Moreover, $M_V = \varphi^*(M_R) = (A^\vee)^{\oplus r}$, hence

$$\bigwedge^i M_V = \left(A^{\otimes(-i)}\right)^{\oplus \binom{r}{i}},$$

for $i = 1, \ldots, r-1$. Since a direct sum of line bundles of the same degree has a (reducible) theta divisor, we are left with proving that $[C, L, V]$ belongs to the main component $\mathcal{H}_{g,r,\mu r}$ of the Hilbert scheme. It suffices to show that the Petri map

$$\mu_0(V) : \mathrm{Sym}^r(W) \otimes H^0(C, K_C \otimes A^{\otimes(-r)}) \to H^0(C, K_C)$$

is injective. This is automatic when $g \leq \mu r - r$, because then $H^1(C, A^{\otimes r}) = 0$.

We consider the case $r(\mu-1) \leq g \leq (r+1)(\mu-1)$, when A is complete, $h^0(C, A^{\otimes r}) = r+1$ and the map $\nu_r \circ \varphi$ corresponds to a complete linear series.

We prove by induction that for each $1 \leq j \leq r$, the multiplication map

$$\chi_j : \mathrm{Sym}^j H^0(C, A) \otimes H^0(C, K_C \otimes A^{\otimes(-r)}) \to H^0(C, K_C)$$

is injective. Note that $\chi_r = \mu_0(V)$ is simply the Petri map, which will conclude the proof. Suppose χ_{j-1} is known to be injective and assume that $\mathrm{Ker}(\chi_r) \neq 0$. After choosing a basis (s_1, s_2) for the 2-dimensional space $H^0(C, A)$, we find sections $u_1, \ldots, u_{j+1} \in H^0(K_C \otimes A^{\otimes(-r)})$ such that

$$s_1^j \cdot u_1 + (s_1^{j-1} s_2) \cdot u_2 + \cdots + (s_1 s_2^{j-1}) \cdot u_j = s_2^j \cdot u_{j+1}. \tag{7}$$

Then $u_{j+1} \neq 0$, for else, $\sum_{k=1}^j (s_1^{j-k} s_2^{k-1}) \otimes u_k \in \mathrm{Sym}^{j-1} H^0(A) \otimes H^0(K_C \otimes (A^\vee)^{\otimes(-r)})$ is a non-zero element in the kernel of χ_{j-1}, a contradiction. Applying the Base Point Free Pencil Trick to equality (7), we obtain a non-zero section $x_1 \in H^0\big(K_C \otimes (A^\vee)^{\otimes(r-j+2)}\big)$ such that the following equalities hold in $H^0\big(C, K_C \otimes (A^\vee)^{\otimes(r-j+1)}\big)$:

$$s_1 \cdot x_1 = s_2^{j-1} \cdot u_{j+1} \ \text{ and } \ s_1^{j-1} \cdot u_1 + \cdots + s_2^{j-1} \cdot u_j = -s_2 \cdot x_1.$$

Applying again the Base Point Free Pencil Trick to the first of these equalities, we find a section $0 \neq x_2 \in H^0\big(C, K_C \otimes (A^\vee)^{\otimes(r-j+3)}\big)$, such that

$$x_1 = -s_2 \cdot x_2 \ \text{ and } \ s_2^{j-2} \cdot u_{j+1} = s_1 \cdot x_2.$$

Repeating the same argument $(j-1)$ times, we obtain a non-zero section $x_{j-1} \in H^0\big(C, K_C \otimes (A^\vee)^{\otimes(-r)}\big)$, such that $s_2 \cdot u_{j+1} = s_1 \cdot x_{j-1}$. So, we can write

$$s_1 \otimes x_{j-1} - s_2 \otimes u_{j+1} \in \mathrm{Ker}(\chi_1) \cong H^0(C, K_C \otimes (A^\vee)^{\otimes(-r-1)}) = 0.$$

Therefore $u_{j+1} = 0$. This is a contradiction, hence $\nu_r \circ \varphi : C \to \mathbf{P}^r$ lies in $\mathcal{H}_{g,r,\mu r}$. □

The following result must be well-known and it follows easily from Atiyah's classification of vector bundles on elliptic curves.

Proposition 1.3. *Let E be an elliptic curve, $B \in \mathrm{Pic}^b(E)$ a line bundle of degree $b \geq 2$ and an integer $1 \leq r \leq b-1$. Then the kernel bundle M_V corresponding to a general $(r+1)$-dimensional subspace $V \subset H^0(E, B)$ is semistable.*

Proof. We fix a semistable vector bundle F on E of rank r with $\det(F) = B$. Note that $\mu(F) = \frac{b}{r} > 1$. For every point $p \in E$, one has $\mu(F(-p)) = \mu(F) - 1 > 0$, therefore $H^1(E, F(-p)) = 0$. In particular, F is globally generated. By Riemann-Roch, $h^0(E, F) = b \geq r + 1$. A globally generated vector bundle F on a curve is generated by a general set of $(\mathrm{rk}(F) + 1)$-global sections. We choose a generating subspace $W \subset H^0(C, F)$ with $\dim(W) = r + 1$ and write the exact sequence

$$0 \longrightarrow B^\vee \longrightarrow W \otimes \mathcal{O}_E \longrightarrow F \longrightarrow 0.$$

By dualizing, we take $V := W^\vee \subset H^0(E, B)$ and then $M_V = F^\vee$ is semistable. □

Next we use a specialization to the bielliptic locus in $\mathcal{M}_g$ that will be of use in the proof of Theorem 0.2 for curves not of integral slope.

Proposition 1.4. *Let $f : C \to E$ be a bielliptic curve of genus g and $(B, V) \in G^r_b(E)$ a general linear series, where $r + 1 \leq b$. Then the kernel bundle corresponding to the pair $\ell := (f^*(B), f^*(V)) \in G^r_{2b}(C)$ verifies condition (R). Moreover, for $b \geq g - 2$, the Petri map corresponding to ℓ is injective, hence $\ell \in \mathcal{H}_{g,r,2b}$.*

Proof. From Proposition 1.3 it follows that we can choose the pair (B, V) such that M_V is semistable. The cover $f : C \to E$ is characterized by a line bundle $\delta \in \mathrm{Pic}^{g-1}(E)$ with

$$f_*(\mathcal{O}_C) = \mathcal{O}_E \oplus \delta^\vee \text{ and } \delta^{\otimes 2} = \mathcal{O}_E(\mathfrak{b}),$$

where $\mathfrak{b} \in E_{2g-2}$ is the branch divisor of f. By pulling-back to C the exact sequence

$$0 \longrightarrow M_{V,B} \longrightarrow V \otimes \mathcal{O}_E \longrightarrow B \longrightarrow 0,$$

we find that $M_{f^*(V),f^*(B)} = f^*(M_{V,B})$. Since $K_C = f^*(\delta)$, via the push-pull formula we obtain $H^0(C, K_C \otimes f^*(B^\vee)) = f^*H^0(E, \delta \otimes B^\vee)$; the Petri map corresponding to ℓ is essentially the multiplication map $V \otimes H^0(E, \delta \otimes B^\vee) \to H^0(E, \delta)$. This is injective when $h^0(E, \delta \otimes B^\vee) \leq 1$, that is, $b \geq g - 2$ (Note that $f^*(B)$ is non-special for $b \geq g - 1$).

It remains to check that $M_{f^*(V)}$ verifies property (R). Pick an integer $1 \leq i \leq r - 1$ and a general line bundle $\xi \in \mathrm{Pic}^{g-1+\lfloor \frac{id}{r} \rfloor}(C)$. From the formula $\det(f_*\xi) = \mathrm{Nm}_f(\xi) \otimes \delta^\vee$, coupled with Lemma 2.5 from [CEFS], it follows that $f_*\xi$ is a general *semistable* vector bundle on E of rank 2 and degree $\lfloor \frac{id}{r} \rfloor$. Then because of the semistability of the exterior powers of M_V we obtain that

$$H^0\Big(C, \bigwedge^i M_{f^*(V)} \otimes \xi\Big) = H^0\Big(E, \bigwedge^i M_V \otimes f_*\xi\Big) = 0,$$

for $\bigwedge^i M_V \otimes f_*\xi$ is a general semistable vector bundle of slope $\frac{1}{2}\lfloor \frac{2ib}{r} \rfloor - \frac{ib}{r} \leq 0$. □

1.1. **Smoothing techniques.** The proof of Theorems 0.2 and 0.3 is by induction on the degree and genus. The inductive step uses the smoothing techniques of Hartshorne-Hirschowitz and Sernesi [Se] and we recall a few basic things. We fix a nodal curve $X \subset \mathbf{P}^r$ with $p_a(X) = g$ and $\deg(X) = d$, then denote by T^1_X the *Lichtenbaum-Schlessinger* sheaf defined via the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_{\mathbf{P}^r|X} \longrightarrow N_X \longrightarrow T^1_X \longrightarrow 0.$$

Setting $N'_X := \mathrm{Ker}\{N_{X/\mathbf{P}^r} \to T^1_X\}$, the vanishing $H^1(X, N'_X) = 0$ is a sufficient condition for $X \subset \mathbf{P}^r$ to be flatly smoothable and for $\mathrm{Hilb}_{g,r,d}$ to be smooth and of expected dimension $(r+1)d - (r-3)(g-1)$ at the point $[X]$, cf. [Se] Proposition 1.6.

Suppose $X := C \cup_\Delta D$ is the union of two smooth curves $C, D \subset \mathbf{P}^r$, meeting transversally at a set of points $\Delta := \{p_1, \ldots, p_\delta\}$. From [Se] Lemma 5.1, one writes the following exact sequence on X

$$0 \longrightarrow N_{D/\mathbf{P}^r}\Big(-\sum_{i=1}^{\delta} p_i\Big) \longrightarrow N'_X \longrightarrow N_{C/\mathbf{P}^r} \longrightarrow 0. \tag{8}$$

If both $H^1(C, N_{C/\mathbf{P}^r}) = 0$ and $H^1(D, N_{D/\mathbf{P}^r}(-p_1 - \cdots - p_\delta)) = 0$, then $H^1(X, N'_X) = 0$ and X is flatly smoothable in $\mathbf{P}^r$. The next result is essentially contained in [Se]:

Lemma 1.5. *Suppose $C \subset \mathbf{P}^r$ is a non-special smooth curve of genus g and $p_1, \ldots, p_\delta \in C$ distinct points in general linear position, with $\delta \leq r+1$. If $R \subset \mathbf{P}^r$ is a rational normal curve passing through $p_1, \ldots, p_\delta$, then $X := C \cup R$ is a flatly smoothable non-special nodal curve in $\mathbf{P}^r$ satisfying $H^1(X, N'_X) = 0$.*

Proof. Under the isomorphism $\nu_r : \mathbf{P}^1 \xrightarrow{\cong} R \subset \mathbf{P}^r$ (hence $\nu_r^*(\mathcal{O}_R(1)) = \mathcal{O}_{\mathbf{P}^1}(r)$), it is well-known that $N_{R/\mathbf{P}^r} = \mathcal{O}_{\mathbf{P}^1}(r+2)^{\oplus(r-1)}$. The condition $H^1\big(R, N_R(-p_1 - \cdots - p_\delta)\big) = 0$ is satisfied precisely when $\delta \leq r+3$. Since C is non-special, $H^1(C, N_{C/\mathbf{P}^r}) = 0$ and from (8) it follows that X is smoothable in $\mathbf{P}^r$. From the exact sequence

$$\cdots \longrightarrow H^1\Big(R, \mathcal{O}_R(1)(-\sum_{i=1}^{\delta} p_i)\Big) \longrightarrow H^1(X, \mathcal{O}_X(1)) \longrightarrow H^1(C, \mathcal{O}_C(1)) \longrightarrow \cdots,$$

we obtain that X is non-special precisely when $\delta \leq r+1$. □

We turn our attention to the Ideal Generation Conjecture for a linear series $(L, V) \in G^r_d(C)$. Via Proposition 1.1, this is equivalent to the generic vanishing statements

$$H^0(C, M_V \otimes \xi) = 0, \quad \text{for a general} \quad \xi \in \mathrm{Pic}^{g-1+\lfloor \frac{d}{r} \rfloor}(C), \quad \text{and} \tag{9}$$

$$H^0\Big(C, \bigwedge^{r-1} M_V \otimes \xi\Big) = 0, \ \text{for a general} \ \ \xi \in \mathrm{Pic}^{g-1+d-\lceil \frac{d}{r} \rceil}(C). \tag{10}$$

We shall prove this for a nodal curve in $\mathbf{P}^r$ obtained by attaching to a curve of integer slope at most $r-1$ general secant lines.

Proof of Theorem 0.3. We fix positive integers g, r and d such that $\rho := \rho(g, r, d) \geq 0$ and set $d_1 := d - r\lfloor \frac{d}{r} \rfloor < r$ and $g_1 := \max\{g - d_1, 0\}$. By direct computation, we find $\rho(g_1, r, \lfloor \frac{d}{r} \rfloor r) \geq \min\{\rho - d_1, 0\}$. This last quantity is non-negative whenever $\rho \geq r$. In this case, by using Theorem 0.1, we can construct a smooth curve $C_1 \subset \mathbf{P}^r$ of genus g_1 and degree $r\lfloor \frac{d}{r} \rfloor$ with general moduli and with the bundle M_{C_1} verifying condition (R). When on the other hand $0 \leq \rho \leq r-1$, then $s := g - d + r \geq 0$ and one writes

$$g = rs + s + \rho \quad \text{and} \quad d = rs + r + \rho.$$

Observe that $\rho(rs+s, r, rs+r) = 0$ and use again Theorem 0.1 to choose a curve $C_1 \subset \mathbf{P}^r$ of genus $rs+s$ and degree $rs+r$ enjoying the exact same properties as above.

To summarize the two cases, one can find integers $a \geq 1$ and $0 \leq d_1 \leq r-1$ such that

$$g = g_1 + d_1 \quad \text{and} \quad d = ar + d_1,$$

for which there exists a smooth curve with general moduli $C_1 \subset \mathbf{P}^r$ with $\deg(C_1) = ar$ and $g(C_1) = g_1$, such that M_{C_1} verifies condition (R). To C_1 we attach d_1 general 2-secant lines $\ell_1, \ldots, \ell_{d_1} \subset \mathbf{P}^r$. The resulting nodal curve

$$X := C_1 \cup \ell_1 \cup \cdots \cup \ell_{d_1}$$

has $\deg(X) = d$ and $p_a(X) = g$, and is flatly smoothable in $\mathbf{P}^r$ to a curve with general moduli. It remains to check conditions (9) and (10) and we explain only the first part, omitting the details for the second. We pick a line bundle $\eta \in \mathrm{Pic}^{g_1-1+a}(C_1)$ such that $H^0(C_1, M_{C_1} \otimes \eta) = 0$; the existence of such η is implied by the property (R). We create a line bundle ξ on the curve X such that ξ_{ℓ_j} is of degree -1 for each $j = 1, \ldots, d_1$, whereas

$$\xi_{C_1} = \eta \otimes \mathcal{O}_{C_1}\Big(\sum_{j=1}^{d_1} \ell_j \cdot C_1\Big). \tag{11}$$

We claim that $H^0(X, M_X \otimes \xi) = 0$. This indeed follows by tensoring and taking cohomology in the Mayer-Vietoris sequence on X, while using (11), together with the fact that since $M_{\ell_j} = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}^{\oplus(r-1)}$, one has that $H^0(\ell_j, M_{\ell_j} \otimes \xi_{\ell_j}) = 0$.

Finally, note that $g - 1 + \lfloor \frac{d}{r} \rfloor = g - 1 + a = \deg(\xi)$, which shows that ξ has precisely the correct degree to establish IGC. □

A variation of this idea gives a proof of MRC for general curves of degrees that are congruent to ± 1 modulo r.

Theorem 1.6. *Let C be a general curve of genus g and fix positive integers r, μ and $d := \mu r \pm 1$. Then the Minimal Resolution Conjecture holds for a general embedding $C \hookrightarrow \mathbf{P}^r$ of degree d.*

Proof. We treat only the case $d = \mu r + 1$, the other case being similar. From Brill-Noether theory, we obtain that $g \leq (r+1)(\mu-1)+1$. Applying Theorem 0.1, there exists a smooth curve with general moduli $C_1 \subset \mathbf{P}^r$ of genus $g-1$ and degree $d - 1 = \mu r$, such that the kernel bundle M_{C_1} enjoys property (R).

Let ℓ be a general 2-secant line to C_1 and set $X := C_1 \cup \ell \subset \mathbf{P}^r$. It is easy to verify that $H^0(X, \mathcal{O}_X(1)) \cong H^0(C_1, \mathcal{O}_{C_1}(1))$ and $H^0(X, \omega_X(-1)) \cong H^0(C_1, K_{C_1}(-1))$, so the Petri map $\mu_0(X)$ can be assumed to be injective and X deforms in $\mathbf{P}^r$ to a curve of genus g with general moduli. By assumption, C_1 possesses for each $1 \leq i \leq r-1$ a line bundle $\eta \in \mathrm{Pic}^{g-2+i\mu}(C_1)$ such that $H^0\big(C_1, \bigwedge^i M_{C_1} \otimes \eta\big) = 0$. Observing that for all $i \leq r-1$

$$g - 1 + \lfloor \frac{id}{r} \rfloor = g - 1 + i\mu$$

(and this is the point where the assumption $d \equiv 1 \bmod r$ is essential!), we can construct a line bundle $\xi \in \mathrm{Pic}^{g-1+i\mu}(X)$, such that $\xi_\ell = \mathcal{O}_{\mathbf{P}^1}(-1)$ and $\xi_{C_1} = \eta(C_1 \cdot \ell)$. Now one checks directly that $H^0\big(X, \bigwedge^i M_X \otimes \xi\big) = 0$, thus finishing the proof. □

After this preparations, we are finally ready to prove Theorem 0.2.

Proof of Theorem 0.2. We fix $d, r \geq 1$ such that $d \geq 2r$. Using Theorems 0.1 and 1.6, we need to consider only the case when $\frac{d}{r} \not\equiv 0, \pm 1 \bmod r$ and inequality (2) holds. We set $a := \lfloor \frac{d}{r} \rfloor - 2 \geq 0$ and write $d = ar + d_1$, where $2r + 2 \leq d_1 \leq 3r - 2$. We set $g_1 := \max\{g - ar, 0\}$. Inequality (2) implies that $d_1 \geq 2g_1 - 2$. If d_1 is even, applying Proposition 1.4, there exists a smooth non-special curve $C_1 \subset \mathbf{P}^r$ of genus g_1 and degree d_1, such that $\Omega^1_{\mathbf{P}^r|C_1}$ verifies condition (R). If, on the other hand, d_1 is odd, then there is a curve of degree $d_1 - 1$ and genus g_1 with the same property. We treat only the case when d_1 is even and indicate at the end the modifications in the proof needed in the remaining case.

Setting, as usual, $M_{C_1} := \Omega^1_{\mathbf{P}^r|C_1}(1)$, condition (R) amounts to the following vanishing

$$(12)\quad H^0\Big(C_1, \bigwedge^i M_{C_1} \otimes \eta\Big) = 0, \text{ for } i = 1, \ldots, r-1 \text{ and a general } \eta \in \mathrm{Pic}^{g_1 - 1 + \lfloor \frac{id_1}{r} \rfloor}(C_1).$$

To C_1 we attach a rational normal curves as follows. We fix subsets $\Delta_1, \ldots, \Delta_a \subset C_1$ consisting of general points such that $|\Delta_j| \leq r + 1$ for $j = 1, \ldots, a$ and $g = g_1 + \sum_{j=1}^a |\Delta_j| - a$. For each $1 \leq j \leq a$, we choose a general rational curve $R_j \subset \mathbf{P}^r$ intersecting C_1 transversally along the set Δ_j, then set

$$X := C_1 \cup R_1 \cup \ldots \cup R_a \subset \mathbf{P}^r.$$

Clearly $p_a(X) = g_1 + \sum_{j=1}^a |\Delta_j| - a = g$ and $\deg(X) = d$. Applying Lemma 1.5, we conclude that X is non-special and flatly smoothable in $\mathbf{P}^r$.

Let us fix an index $1 \leq i \leq r - 1$. Via the surjection

$$\mathrm{Pic}^{g-1+\lfloor \frac{id}{r} \rfloor}(X) \longrightarrow \mathrm{Pic}^{g-1+a+\lfloor \frac{id_1}{r} \rfloor}(C_1) \times \prod_{j=1}^a \mathrm{Pic}^{i-1}(R_j) \longrightarrow 0,$$

we consider a line bundle ξ on X of degree $g - 1 + \lfloor \frac{id}{r} \rfloor$, such that $\deg(\xi_{R_j}) = i - 1$, for all j. We claim that ξ_{C_1} can be chosen so that $H^0\big(X, \bigwedge^i M_X \otimes \xi\big) = 0$.

Indeed, we first observe that $\bigwedge^i M_{R_j}$ is a sum of line bundles of degree $-i$, hence $H^0\big(R_j, \bigwedge^i M_{R_j} \otimes \xi_{R_j}\big) = 0$ for degree reasons. Considering the inclusion

$$H^0\Big(X, \bigwedge^i M_X \otimes \xi\Big) \hookrightarrow H^0\Big(C_1, \bigwedge^i M_{C_1} \otimes \xi_{C_1}\Big) \oplus \Big(\bigoplus_{j=1}^a H^0\Big(R_j, \bigwedge^i M_{R_j} \otimes \xi_{R_j}\Big)\Big)$$

induced by the Mayer-Vietoris sequence on X, from the previous observation it follows that a non-zero section in $H^0\big(X, \bigwedge^i M_X \otimes \xi\big)$ corresponds to a non-zero section in $H^0\big(C_1, \bigwedge^i M_{C_1} \otimes \xi_{C_1}(-\sum_{j=1}^a \Delta_j)\big)$. Observing that $\deg(\xi_{C_1}) - \sum_{j=1}^a |\Delta_j| = g_1 - 1 + \lfloor \frac{id_1}{r} \rfloor$, we choose ξ_{C_1} so that the vanishing (12) holds for $\eta = \xi_{C_1}\big(-\sum_{j=1}^a \Delta_j\big)$. We conclude that the kernel bundle of a general smoothing of $X \subset \mathbf{P}^r$ verifies condition (R). □

Remark 1.7. In the previous proof, if d_1 is odd, then we start with a smooth curve of degree $d_1 - 1$ and genus g_1, to which we attach as before $a - 1$ rational normal curves and one *linearly normal* elliptic curve $E \subset \mathbf{P}^r$. Since the restricted cotangent bundle $\Omega^1_{\mathbf{P}^r|E}$ is stable, the rest of the proof follows along similar lines.

We close this section by explaining how our methods solve in many cases *Butler's Conjecture*, as stated in [BBPN] Conjecture 9.5. We have the following result:

Theorem 1.8. *Let C be a general curve of genus $g \geq 1$ and integers $r, d \geq 1$ such that*

$$d + r\lfloor\frac{d}{r}\rfloor \geq 2g + 2r - 2.$$

Then the bundle M_V corresponding to a general linear series $\ell := (L, V) \in G^r_d(C)$ is semistable.

This result has already been known in many important cases. The semistability of M_V when $d \leq g + r$, in particular if $V = H^0(C, L)$ is complete, follows from a filtration argument of Lazarsfeld [L1], [EL]; for this reason, M_V is sometimes referred to as a *Lazarsfeld bundle*. The case $d \leq 2r$ of Theorem 1.8 is due to Mercat [Me2]; further cases of the conjecture, on which Theorem 1.8 improves, were established in [BH].

Proof of Theorem 1.8. The property of the restricted cotangent bundle $\Omega^1_{\mathbf{P}^r|X_t}$ being semistable is obviously open in any flat family of nodal curves $\{X_t \subset \mathbf{P}^r\}_{t \in T}$, hence it suffices to construct one example of a nodal curve $X \subset \mathbf{P}^r$ with $p_a(X) = g$ and $\deg(X) = d$, such that M_X is semistable (with respect to subsheaves of constant rank) and X flatly deforms in $\mathbf{P}^r$ to a curve with general moduli. The curve constructed in the proof of Theorem 0.2 plays exactly this role. First, the Lazarsfeld bundle of the curve C constructed in Proposition 1.4 is semistable, for $M_C = f^*(M_V)$, where $f : C \to E$ was a bielliptic cover and $(B, V) \in G^r_b(E)$; since M_V is semistable, so is $f^*(M_V)$. Then one deforms C to a smooth curve $C_1 \subset \mathbf{P}^r$, to which one attaches rational normal curves $R_1, \ldots, R_a \subset \mathbf{P}^r$, whose respective Lazarsfeld bundles are clearly semistable. Thus the Lazarsfeld bundle of the resulting curve $X = C_1 \cup R_1 \cup \ldots \cup R_a$ is also semistable; since in the course of establishing Theorem 0.2 it was proved that X deforms to a curve with general moduli, this finishes the proof. □

2. ULRICH BUNDLES ON $K3$ SURFACES

Let $X \subset \mathbf{P}^r$ be a smooth arithmetically Cohen-Macaulay projective variety of degree d. A vector bundle E on X is said to be an *Ulrich sheaf* if

$$H^i(X, E(-i)) = 0 \text{ for } i > 0 \text{ and } H^i(X, E(-i-1)) = 0 \text{ for } i < \dim(X). \qquad (13)$$

This definition is equivalent to the one mentioned in the Introduction, see [ESW] Proposition 2.1. An Ulrich sheaf E enjoys a number of properties we list, see [CHGS]:

(i) The restriction E_H to a general hyperplane section H of X is again an Ulrich bundle.

(ii) $h^0(X, E) = d \cdot \text{rk}(E)$ and $\deg(E|_C) = \text{rk}(E)(d + g - 1)$, where g is the genus of a general curvilinear section $C = X \cap \mathbf{P}^{r-\dim(X)+1}$ of X. Furthermore, $\mathcal{O}_C(-1) \notin \Theta_{E_C}$, hence the restriction E_C admits a theta divisor.

(iii) Ulrich bundles are semistable with respect to the polarization $\mathcal{O}_X(1)$.

Combining properties (i) and (iii), one obtains rational maps between moduli spaces of semistable bundles on X and on the hyperplane section H respectively. From now on let $X = S$ be a smooth surface, in which case the condition (13) amounts to the vanishing of the following cohomology groups

$$H^0(S, E(-1)),\ H^1(S, E(-1)),\ H^1(S, E(-2)),\ H^2(S, E(-2)).$$

This implies the further vanishing $H^0(S, E(-2)) = 0$ and $H^2(S, E(-1)) = 0$ (the bundle E being 0-regular, is 1-regular as well), hence $\chi(S, E(-1)) = \chi(S, E(-2)) = 0$, see also

[ESW] Corollary 2.2. Applying Riemann-Roch to both $E(-1)$ and $E(-2)$ and taking the difference of the Euler characteristics, we obtain the relation

$$H \cdot \left(c_1(E) - \frac{\mathrm{rk}(E)}{2}(K_S + 3H)\right) = 0. \tag{14}$$

This calculation motivates the following:

Definition 2.1. A *special Ulrich bundle* on a surface S is a 0-regular rank 2 vector bundle E with determinant $\det(E) = K_S(3)$.

It is proved in [ESW] Corollary 2.3 that such bundles are indeed Ulrich. If E is a special rank 2 Ulrich bundle on a $K3$ surface $S \subset \mathbf{P}^{s+1}$, from Riemann-Roch $c_2(E) = \frac{5}{2}H^2 + 4$. Moreover, E being 0-regular it is globally generated. A parameter count performed in [ESW] Remark 6.4 suggests that $K3$ surfaces could possess rank 2 Ulrich bundles. Our Theorem 0.4 confirms this expectation and we show that the hypothesis of [ESW] Proposition 6.2 is verified for Lazarsfeld-Mukai vector bundles on many polarized $K3$ surfaces. An immediate consequence of the relation (14) is the following fact:

Corollary 2.2. *A $K3$ surface with Picard number 1 carries no Ulrich bundles of odd rank.*

In even rank, for each $a \geq 1$ one looks for Ulrich bundles E on S with

$$\mathrm{rk}(E) = 2a, \ \ \det(E) = \mathcal{O}_S(3a) \text{ and } c_2(E) = 9a^2 s - 4a(s-1). \tag{15}$$

If $\mathrm{Pic}(S) = \mathbb{Z} \cdot [H]$ with $H^2 = 2s$, every Ulrich bundle E on S satisfies (15). Natural candidates for E are the LM bundles $E_{C,A}$, where $C \in |\mathcal{O}_S(3a)|$ is a smooth curve and $A \in W^{2a-1}_{9a^2 s - 4a(s-1)}(C)$ is a complete and base point free linear series. The curve C has $K_C = \mathcal{O}_C(3a)$ and $\mathrm{Cliff}(C) \leq \mathrm{Cliff}(\mathcal{O}_C(1)) = 6as - 2s - 2$, with equality for instance when $\mathrm{Pic}(S) = \mathbb{Z} \cdot [H]$. Note that it is by no means certain that such an A exists, and if so, that it leads to an Ulrich bundle. Theorem 0.4 establishes these facts in the most important case, $a = 1$. Then we construct stable Ulrich bundles of higher even rank, by deforming bundles sitting in extensions of two Ulrich bundles of smaller rank.

Remark 2.3. The hypothesis in Theorem 0.4 that the Clifford index of a cubic section $C \in |\mathcal{O}_S(3)|$ be computed by $\mathcal{O}_S(1)$ is not restrictive. For instance, it is satisfied if $S \subset \mathbf{P}^3$ is a quartic surface, when C is a $(3,4)$ complete intersection in $\mathbf{P}^3$. From [L2] page 185 we obtain that $\mathrm{gon}(C) \geq 8$. Since C has Clifford dimension 1, cf. [CP], it follows that $\mathrm{Cliff}(C) \geq 6 = \mathrm{Cliff}(\mathcal{O}_C(1))$. The only place in the proof where this condition is used is to ensure that C carries a base point free pencil of degree $\frac{5H^2}{4} + 4$.

Lemma 2.4. *Let (S, H) be a polarized $K3$ surface of genus g and $C \in |H|$ a general curve in its linear system having gonality k. Then C carries a complete, base point free pencil $\mathfrak{g}^1_{g-k+3}$.*

Proof. The case $\rho(g,1,k) > 0$ follows immediately, for in this situation $g = 2k-3$ and hence $g-k+3 = k$. We may assume that $\rho(g,1,k) \leq 0$. When the Clifford dimension of a general curve in $|H|$ equals 1, from [AF] Theorem 3.12 and Remark 3.13, one obtains that for a general $C \in |H|$, every component of $W^1_{g-k+2}(C)$ has dimension $g-2k+2$. Via excess linear series it then follows that each component of $W^1_{g-k+3}(C)$ has dimension $\dim(W^1_{g-k+2}(C)) + 2 = g - 2k + 4 \big(= \rho(g,1,g-k+3)\big)$.

Since $\dim\big(C + W^1_{g-k+2}(C)\big) = g - 2k + 3$, we conclude that the general element in every component of $W^1_{g-k+3}(C)$ is base point free and complete. The case when the general curve in $|H|$ has Clifford dimension at least 2, will not be needed in this paper, but it can be deduced along the lines of [AF] Section 5. $\Box$

The following result is needed in the proof of Theorem 0.4.

Lemma 2.5. *Let (S, H) be a polarized $K3$ surface with $H^2 = 2s \geq 4$ and $D \in |2H|$ a smooth quadric section. Then the following estimate holds*

$$\dim \big\{\Gamma \in D_{5s+4} : h^0(D, \mathcal{O}_D(\Gamma - H)) \geq 1\big\} \leq 2s + 7.$$

Proof. By direct calculation, $\varphi_H : D \hookrightarrow \mathbf{P}^{s+1}$ is a smooth half-canonical curve with $\deg(D) = D \cdot H = 4s$ and $g(D) = 4s + 1$. We consider the incidence variety

$$\mathcal{V} := \Big\{(\Gamma, \zeta) \in D_{5s+4} \times D_{s+4} : \Gamma \in |\mathcal{O}_D(H + \zeta)|\Big\},$$

together with the projections $\pi_1 : \mathcal{V} \to D_{5s+4}$ and $\pi_2 : \mathcal{V} \to D_{s+4}$. Note that $\pi_1(\mathcal{V})$ is precisely the variety whose dimension we have to compute. To estimate $\dim(\mathcal{V})$ we look at the fibres of π_2. By Riemann-Roch, $h^0(D, \mathcal{O}_D(H + \zeta)) = h^0(D, \mathcal{O}_D(H - \zeta)) + s + 4$, for every $\zeta \in D_{s+4}$. In particular, for a general divisor $\zeta \in D_{s+4}$, we obtain that $\pi_2^{-1}(\zeta) = \mathbf{P}H^0(\mathcal{O}_D(H + \zeta)) \cong \mathbf{P}^{s+3}$, and hence $\mathcal{V}$ has a unique irreducible component of dimension $2s + 7$ that dominates D_{s+4}.

For $i \geq 1$, the locally closed variety $\Sigma_i := \{\zeta \in D_{s+4} : h^0(D, \mathcal{O}_D(H - \zeta)) = i\}$ has dimension at most $\dim |\mathcal{O}_D(H)| - i + 1 = s + 2 - i$. If $\zeta \in \Sigma_i$ then $\pi_2^{-1}(\zeta) \cong \mathbf{P}^{s+i+3}$, hence $\dim \pi_2^{-1}(\Sigma_i) \leq \dim \Sigma_i + s + i + 3 \leq 2s + 5$. To sum up, all components of $\mathcal{V}$ are of dimension $\leq 2s + 7$, implying the same conclusion for $\dim(\pi_1(\mathcal{V}))$. $\Box$

We now proceed to show that polarized $K3$ surfaces satisfying a mild Brill-Noether genericity condition carry stable rank 2 Ulrich bundles.

Proof of Theorem 0.4. We start with a $K3$ surface $S \subset \mathbf{P}^{s+1}$ and let $H \in |\mathcal{O}_S(1)|$ be a hyperplane section with $H^2 = 2s$. We fix a smooth curve $C \in |\mathcal{O}_S(3)|$ and compute its genus $g(C) = 9s + 1$. Invoking [CP], note that C has Clifford dimension 1 and clearly $\mathrm{Cliff}(\mathcal{O}_C(1)) = 4s - 2$. Our hypothesis implies $\mathrm{gon}(C) = 4s$, hence by Lemma 2.4, C possesses a complete base point free pencil $A \in W^1_{5s+4}(C)$. The candidate Ulrich bundle is the Lazarsfeld-Mukai bundle $E := E_{C,A}$. More precisely, the general point (C, A) of any dominating component $\mathcal{W}$ of the relative space $\mathcal{W}^1_{5s+4}(|\mathcal{O}_S(3)|)$ over the linear system $|\mathcal{O}_S(3)|$ corresponds to a complete and base point free pencil $\mathfrak{g}^1_{5s+4}$.

Since the Ulrich condition (13) is open, we need to ensure that the *non-Ulrich locus* does not coincide with the whole $\mathcal{W}$.

Step 1. For a general point $(C, A) \in \mathcal{W}$, we verify the partial Ulrich condition

$$H^0(S, E_{C,A}(-1)) = 0. \tag{16}$$

We shall find an explicit parametrization of the failure locus of (16) and count parameters. Consider the following Grassmann bundle over the moduli space of LM bundles

$$\mathcal{G} := \Big\{(E_{C,A}, \Lambda) : (C, A) \in \mathcal{W},\ \Lambda \in G\big(2, H^0(S, E_{C,A})\big)\Big\}.$$

Recall from [L1] or [AF] the following dimension estimate

$$\dim(\mathcal{W}) \geq \dim|\mathcal{O}_S(3)| + \rho(9s+1, 1, 5s+4) = 10s+6.$$

Since the projection $\mathcal{G} \to \mathcal{W}$ is dominant with fibre $\mathbf{P}H^0(S, E_{C,A} \otimes E_{C,A}^{\vee})$ over a general point $(C, A) \in \mathcal{W}$, the estimate $\dim(\mathcal{G}) \geq 10s+6$ holds as well. Since $h^0(S, E_{C,A}) = h^0(C, A) + h^1(C, A) = 4s$, the dimension of the space of LM bundles $E_{C,A}$ corresponding to pairs $(C, A) \in \mathcal{W}$ has dimension at least $\dim(\mathcal{G}) - \dim G(2, H^0(E_{C,A})) \geq 2s+10$. Observe that the local dimension at $E_{C,A}$ of the moduli space $\mathrm{Spl}(2, \mathcal{O}_S(3), 5s+4)$ of simple vector bundles of rank 2 on S with first Chern class $\mathcal{O}_S(3)$ and second Chern class $5s+4$ is also equal to $c_1^2(E) - 2\mathrm{rk}(E)\chi(E) + 2\mathrm{rk}(E)^2 + 2 = 2s+10$, that is, a general point of $\mathrm{Spl}(2, \mathcal{O}_S(3), 5s+4)$ corresponds to a bundle $E_{C,A}$.

Next, we consider the projective bundle

$$\mathcal{P} := \big\{(E_{C,A}, \ell) : (C, A) \in \mathcal{W},\ \ell \in \mathbf{P}H^0(S, E_{C,A})\big\},$$

with $\dim(\mathcal{P}) \geq 6s+9$. Any LM bundle $E_{C,A}$ is given by an extension

$$0 \longrightarrow \mathcal{O}_S \stackrel{\ell}{\longrightarrow} E_{C,A} \longrightarrow \mathcal{I}_{\Gamma/S}(3) \longrightarrow 0,$$

where $\Gamma \in S^{[5s+4]}$ is a 0-dimensional subscheme which satisfies the *Cayley-Bacharach* (CB) condition with respect to $|\mathcal{O}_S(3)|$. This condition is necessary in order to obtain locally free extensions, cf. [L2] page 177. Note that

$$\dim \mathrm{Ext}^1(\mathcal{I}_{\Gamma/S}(3), \mathcal{O}_S) = 1;$$

indeed, from the exact sequence defining Γ and from $h^0(S, E_{C,A}^{\vee}) = h^1(S, E_{C,A}) = 0$, we obtain an isomorphism $H^0(S, \mathcal{O}_S) = \mathrm{Ext}^1(\mathcal{I}_{\Gamma/S}(3), \mathcal{O}_S)$. In particular, Γ determines uniquely the LM bundle $E_{C,A}$ and the map $\varphi : \mathcal{P} \to S^{[5s+4]}$ given by $\varphi([E_{C,A}, \ell]) := \Gamma$ is generically injective onto its image.

Since $H^0(S, E_{C,A}(-1)) \cong H^0(S, \mathcal{I}_{\Gamma/S}(2))$, we shall show that cycles $\Gamma \in \mathrm{Im}(\varphi)$ with $H^0(S, \mathcal{I}_{\Gamma/S}(2)) \neq 0$ depend on at most $6s+8 \leq \dim(\mathcal{P}) - 1$ parameters. To this end, we consider the incidence variety, see also [CKM2] Proposition 3.18 for the case $s = 2$,

$$\mathcal{Z} := \Big\{(D, \Gamma) : D \in |\mathcal{O}_S(2)|,\ \Gamma \subset D \text{ satisfies CB with respect to } |\mathcal{O}_S(3)|\Big\}.$$

We fix a smooth section $D \in |\mathcal{O}_S(2)|$ and an effective divisor $\Gamma \in D_{5s+4}$. For a point $p \in \mathrm{supp}(\Gamma)$, we write $\Gamma = \Gamma_p + p$, where $\Gamma_p \in D_{5s+3}$. Via the exact sequence

$$0 \longrightarrow H^0(S, \mathcal{O}_S(1)) \stackrel{+D}{\longrightarrow} H^0(S, \mathcal{I}_{\Gamma/S}(3)) \longrightarrow H^0(D, \mathcal{O}_D(3H-\Gamma)) \longrightarrow 0,$$

we rephrase the Cayley-Bacharach condition for Γ as requiring that the isomorphism $H^0(D, \mathcal{O}_D(3H-\Gamma)) \cong H^0(D, \mathcal{O}_D(3H-\Gamma_p))$ hold, or equivalently by Riemann-Roch,

$$h^0(D, \mathcal{O}_D(\Gamma_p - H)) = h^0(D, \mathcal{O}_D(\Gamma - H)) - 1, \quad \text{for each } p \in \mathrm{supp}(\Gamma).$$

In particular, $h^0(D, \mathcal{O}_D(\Gamma - H)) \geq 1$. Via Lemma 2.5, we conclude that the dimension of each fibre of the map $\mathcal{Z} \to |\mathcal{O}_S(2)|$ does not exceed $2s+7$; thus $\dim(\mathcal{Z}) \leq \dim|\mathcal{O}_S(2)| + 2s+7 = 6s+8$, which establishes condition (16) for a general $(C, A) \in \mathcal{W}$.

Step 2. The partial Ulrich condition (16) implies the full Ulrich condition (13).

By Serre duality, using the isomorphism $E^\vee \cong E(-3)$, the condition (13) reduces to $H^0(S,E(-1)) = H^1(S,E(-1)) = 0$. Twisting the sequence (5) by $\mathcal{O}_S(2)$ and taking cohomology, we obtain the exact sequence

$$0 \to H^0(S,E(-1)) \to H^0(C,A)\otimes H^0(S,\mathcal{O}_S(2)) \to H^0(C,A(2)) \to H^1(S,E(-1)) \to 0,$$

and an isomorphism

$$H^1(C,A(2)) = H^0(S,E(-2))^\vee = 0,$$

from (16); hence, for a general pair $(C,A)\in\mathcal{W}$, the bundle $A(2)$ is non-special, which implies $h^0(C,A(2)) = 8s+4$. Since $h^0(S,\mathcal{O}_S(2)) = 4s+2$, we obtain $H^1(S,E(-1)) = 0$.

We have proved that the failure locus of the Ulrich condition for $E_{C,A}$ is a genuine effective divisor in $\mathcal{W}$ (or rather in the open subset of $\mathcal{W}$ given by the vanishing of $H^0(S,E_{C,A}(-2))$).

Step 3. We prove that $E = E_{C,A}$ is stable. Simplicity already follows from [AF] Remark 3.13 and suppose E is not stable. Following [CHGS] Theorem 2.9, any such E is presented as an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0,$$

where M, N are Ulrich line bundles. Since $\chi(S,M(-1))$, $\chi(S,M(-2))$, $\chi(S,N(-1))$ and $\chi(S,N(-2))$ vanish, we obtain the following numerical conditions:

$$M^2 = N^2 = 4s-4 \ \text{ and } M\cdot H = N\cdot H = 3s.$$

Furthermore, $M\otimes N = \mathcal{O}_S(3)$ and $M \ncong N$, because the bundle E is simple. In particular, $h^0(S,M\otimes N^\vee) = h^0(S,N\otimes M^\vee) = 0$ which implies that $h^1(S,M\otimes N^\vee) = 2s+6$. Hence $\dim \mathbf{P}(\mathrm{Ext}^1(N,M)) = 2s+5$. Since the space of special Ulrich bundles has dimension $2s+10$ and $\mathrm{Pic}(S)$ is discrete, we conclude that a general E is stable.

2.1. **Existence of Ulrich stable bundles of even rank.** We now restrict ourselves to a K3 surface $S \subset \mathbf{P}^{s+1}$ with Picard number 1. By Corollary 2.2, S does not admit Ulrich bundles of odd rank. Recall that if E is an Ulrich bundle on S of even rank $2a$, then $\det(E) = \mathcal{O}_S(3a)$ and $c_2(E) = 9a^2s - 4a(s-1)$. We record the following consequence of Riemann-Roch, see also [CHGS] Proposition 2.12.

Lemma 2.6. *Let E and F be Ulrich bundles on S of ranks $2a$ and $2b$ respectively. Then*

$$\chi(S,E^\vee\otimes F) = -2abs - 8ab.$$

We now show that such an S carries stable Ulrich bundles of every even rank.

Theorem 2.7. *For any K3 surface $S\subset \mathbf{P}^{s+1}$ with $\mathrm{Pic}(S) = \mathbb{Z}\cdot[H]$ and any integer $a \geq 1$, there exists an $(8a^2+2a^2s+2)$-dimensional family of stable Ulrich bundles on S of rank $2a$.*

Proof. Our proof follow closely the lines of [CHGS] Theorem 5.7 and especially those of [C] Theorem 3.1. We proceed by induction on a. The case $a=1$ is part of Theorem 0.4. Suppose there exist stable Ulrich bundles of any even rank smaller that $2a$. We choose stable Ulrich bundles F_1 and F_2 of ranks 2 and $2a-2$ respectively, with $F_1 \ncong F_2$ in the case $a=2$. From Lemma 2.6, we find that $\dim \mathrm{Ext}^1(F_1,F_2) > 0$, that is, a general extension $E \in \mathbf{P}\mathrm{Ext}^1(F_2,F_1)$ is a *simple* Ulrich bundle of rank $2a$, cf. [CHGS] Lemma 4.2. Let $\mathrm{Def}(E)$ be the universal deformation space of E; since the deformation functor for simple bundles is pro-representable, Def(E) can be constructed in the étale topology and its dimension equals

$$\dim \mathrm{Def}(E) = h^1(S, E^\vee \otimes E) = 2 - \chi(S, E^\vee \otimes E) = 2 + 2a^2 s + 8a^2. \tag{17}$$

By a parameter count, we show that the general element of Def(E) corresponds to a stable bundle. Since the Jordan-Hölder filtration of a strictly semistable Ulrich bundle is a direct sum of stable Ulrich bundles (of the same slope), we are led to count the number of moduli of vector bundles appearing in this way. We fix stable, pairwise non-isomorphic Ulrich bundles $E_1, \ldots, E_n$ where $\mathrm{rk}(E_i) = 2a_i$ for $i = 1, \ldots, n$. By (15), the Chern classes of each E_i are uniquely determined by their ranks. We consider the family $\mathcal{U}$ of Ulrich bundles E of rank $2a$ such that

$$\mathrm{gr}(E) = E_1^{\oplus k_1} \oplus \cdots \oplus E_n^{\oplus k_n}.$$

Thus we have $a = k_1 a_1 + \cdots + k_n a_n$ and let $m := k_1 + \cdots + k_n$ be the total number of bundles employed.

Let F be an Ulrich bundle of rank $2b$ constructed from successive extensions of some of the E_i's, such that for a given $j \in \{1, \ldots, n\}$, precisely k copies of the bundle E_j are used. Since $\mathrm{Hom}(E_i, E_j) = 0$ for $i \neq j$, we find that $h^0(S, E_j^\vee \otimes F) \leq k$ and similarly, by Serre duality, $h^2(S, E_j^\vee \otimes F) \leq k$. Thus,

$$\dim \mathrm{Ext}^1(E_j, F) \leq k - \chi(S, E_j^\vee \otimes F) = k + (2s + 8)a_j b. \tag{18}$$

Using the bounds (17) and (18), we conclude that the space $\mathcal{U}$ of Ulrich bundles obtained by a succession of extensions involving k_i times the bundle E_i for $i = 1, \ldots, n$ has dimension at most

$$\sum_{i=1}^{n}\bigl(2 + 2a_i^2(s+4)\bigr) + (2s+8)\Bigl(\sum_{i<j} k_i k_j a_i a_j + \sum_{i=1}^{n} \binom{k_i}{2} a_i^2\Bigr) + \sum_{i=1}^{n} \binom{k_i}{2} - (m-1),$$

where the first sum is the moduli count for the universal deformation spaces Def(E_i), whereas the other terms account for the dimension of the isomorphism classes of the $m - 1$ successive extensions. It is not difficult to check that this quantity is smaller that

$$2 + 2(s+4)\left(\sum_{i=1}^{n} k_i^2 a_i^2 + 2\sum_{i<j} k_i k_j a_i a_j\right) = 2 + 2a^2(s+4)$$

which is the dimension of the moduli space of simple Ulrich bundles of rank $2a$. Thus, there exist simple Ulrich bundles of rank $2a$ which are stable. □

2.2. **The Chow form of a $K3$ surface.** Let $S \subset \mathbf{P}^{s+1}$ be a $K3$ surface for which the hypothesis of Theorem 0.4 applies. We choose a special rank two Ulrich bundle E on S, hence $\det(E) = \mathcal{O}_S(3)$ and $h^0(S, E) = 4s$. We have defined in the introduction the exterior algebras $\mathbf{\Lambda} := \bigwedge^\bullet H^0(S, \mathcal{O}_S(1))^\vee$ and $\mathbf{\Lambda}^\vee := \bigwedge^\bullet H^0(S, \mathcal{O}_S(1))$, with gradings

$$\mathbf{\Lambda}_{-i} := \bigwedge^i H^0(S, \mathcal{O}_S(1))^\vee \text{ and } \mathbf{\Lambda}_i^\vee := \bigwedge^i H^0(S, \mathcal{O}_S(1))$$

respectively. To E, viewed as a sheaf on $\mathbf{P}^{s+1}$, one associates a minimal bi-infinite exact sequence of free graded $\mathbf{\Lambda}$-modules called the *Tate resolution*:

$$T^{\bullet}(E) : \cdots \to T^{-2}(E) \to T^{-1}(E) \to T^{0}(E) \to T^{1}(E) \to T^{2}(E) \to \cdots$$

It is shown in [EFS] that one has isomorphisms of graded $\mathbf{\Lambda}$-modules

$$T^{p}(E) = \bigoplus_{i=0}^{2} \mathbf{\Lambda}^{\vee} \otimes H^{i}(S, E(p-i)).$$

For an Ulrich bundle, the Tate resolution is particularly simple, for instance

$$T^{-1}(E) = \mathbf{\Lambda}^{\vee}(3) \otimes H^{2}(S, E(-3)) \ \text{ and } \ T^{0}(E) = \mathbf{\Lambda}^{\vee} \otimes H^{0}(S, E).$$

To pass from the Tate resolution of E to the Chow form $\mathcal{Z}(S)$ one applies the functor $\mathbf{U}_3$ of [ESW] from the category of free graded $\mathbf{\Lambda}$-modules to that of vector bundles over the Grassmannian $\mathbf{G} := G\big(s-1, H^0(S, \mathcal{O}_S(1))^{\vee}\big)$ of projective codimension 3 planes in $\mathbf{P}^{s+1}$. This functor replaces the module $\mathbf{\Lambda}^{\vee}(p)$ by the p-th exterior power of the rank 3 tautological bundle $\mathcal{U}$ on $\mathbf{G}$. The resulting complex $\mathbf{U}_3 T^{\bullet}(E)$ consists of a single morphism of vector bundles over $\mathbf{G}$

$$\mathbf{U}_3(\varphi) : H^2(S, E(-3)) \otimes \bigwedge^{3} \mathcal{U} \to H^0(S, E) \otimes \mathcal{O}_{\mathbf{G}}.$$

The determinant of φ gives the equations in Plücker coordinates of the Chow form of S, see [ESW] Corollary 3.4. In what follows we describe explicitly the linear map

$$\varphi : \bigwedge^{3} H^0(S, \mathcal{O}_S(1)) \otimes H^2(S, E(-3)) \to H^0(S, E). \tag{19}$$

Proof of Theorem 0.6. We fix sections $x^1, x^2, x^3 \in H^0(S, \mathcal{O}_S(1))$ and $u \in H^2(S, E(-3))$. Let $\mathcal{V} = \{V_\alpha\}_\alpha$ be a covering of S and $\{u_{\alpha\beta\gamma}\} \in \check{Z}^2(\mathcal{V}, E(-3))$ a Čech cocycle representative of u. Since $H^2(S, E(-2)) = 0$, we obtain that $\{x^i \cdot u_{\alpha\beta\gamma}\} \in \check{Z}^2(\mathcal{V}, E(-2)) = \check{B}^2(\mathcal{V}, E(-2))$. In particular, there exist 1-cocycles $\{t^i_{\alpha\beta}\} \in \check{C}^1(\mathcal{V}, E(-2))$ such that the following hold for $i = 1, 2, 3$:

$$x^i \cdot u_{\alpha\beta\gamma} = t^i_{\alpha\beta} - t^i_{\beta\gamma} + t^i_{\gamma\alpha} \in \Gamma(U_{\alpha\beta\gamma}, E(-2)).$$

Since $H^1(S, E(-1)) = 0$, we find $\{x^i \cdot t^j_{\alpha\beta} - x^j \cdot t^i_{\alpha\beta}\} \in \check{Z}^1(\mathcal{V}, E(-1)) = \check{B}^1(\mathcal{V}, E(-1))$, therefore for $i \neq j$ there exist cocycles $\{v^{ij}_\alpha\} \in \check{C}^0(\mathcal{V}, E(-1))$, such that

$$x^i \cdot t^j_{\alpha\beta} - x^j \cdot t^i_{\alpha\beta} = v^{ij}_\alpha - v^{ij}_\beta.$$

Therefore we obtain a section $w \in H^0(S, E)$ such that

$$w = x^1 v^{23}_\alpha - x^2 v^{13}_\alpha + x^3 v^{12}_\alpha = x^1 v^{23}_\beta - x^2 v^{13}_\beta + x^3 v^{12}_\beta \in \Gamma(V_{\alpha\beta}, E).$$

We set $\varphi(x_1 \otimes x_2 \otimes x_3 \otimes s) := w$. Clearly this construction vanishes on symmetric tensors, hence it gives rise to the map (19), which is the degree zero part of the $\mathbf{\Lambda}$-module map

$$\Big\{\varphi^i : \bigwedge^{i+3} H^0(S, \mathcal{O}_S(1)) \otimes H^2(S, E(-3)) \longrightarrow \bigwedge^{i} H^0(S, \mathcal{O}_S(1)) \otimes H^0(S, E)\Big\}_i$$

appearing in the Tate resolution of E. It is straightforward to check that the image of $\{\varphi^i\}_i$ is precisely the kernel of the Koszul differential $T^0(E) \to T^1(E)$, which finishes the proof. □

Remark 2.8. One may also view the above constructed $\mathbf{\Lambda}$-linear differential as a linear map $\varphi : H^0(S,E)^\vee \otimes H^0(S,E)^\vee \to \bigwedge^3 H^0(S,\mathcal{O}_S(1))^\vee$, given by a $4s \times 4s$ matrix of linear forms in the Plücker coordinates of $\mathbf{G}$. This matrix is antisymmetric and its pfaffian gives the equation for $\mathcal{Z}(S)$.

3. Restricted Lazarsfeld-Mukai bundles

We fix a $K3$ surface S, a curve $C \subset S$ of genus g and a globally generated linear series $A \in W^r_d(C)$, with $h^0(C,A) = r+1$. Using the sequence (5) we form the vector bundle $F = F_{C,A}$; by dualizing, we obtain an exact sequence for $E = E_{C,A} := F^\vee_{C,A}$:

$$0 \longrightarrow H^0(C,A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow K_C \otimes A^\vee \longrightarrow 0. \tag{20}$$

It is well-known [Mu], [L1] that $c_1(E) = [C]$ and $c_2(E) = d$; moreover $h^0(S,F) = 0$ and $h^1(S,E) = h^1(S,F) = 0$. Finally, one also has that $\chi(S, E \otimes F) = 2 - 2\rho(g,r,d)$; in particular, if E is a simple bundle, then $\rho(g,r,d) \geq 0$. Assuming furthermore that $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, it is also well-known that both E and F are C-stable bundles on S.

We begin by showing that in rank 2, irrespective of the structure of $\mathrm{Pic}(S)$, a splitting of the restriction $E|_C$ can only be induced by an elliptic pencil on the surface.

Theorem 3.1. *Let $C \subset S$ be as above and a base point free pencil $A \in W^1_d(C)$ of degree $2 < d < g-1$ with $K_C \otimes A^\vee$ globally generated. The following conditions are equivalent:*

(i) *$E|_C \cong A \oplus (K_C \otimes A^\vee)$;*
(ii) *There exists an elliptic pencil $N \in \mathrm{Pic}(S)$ such that $N|_C = A$.*

Proof. (ii)⇒(i). Let N be an elliptic pencil with $N|_C = A$ and write the exact sequence

$$0 \longrightarrow N^\vee \longrightarrow F \longrightarrow N(-C) \longrightarrow 0,$$

whose restriction to C provides a splitting of the dual of the sequence (6) characterizing $E|_C$. Observe that since $d < g-1$, there is no morphism from $A^\vee$ to $K_C^\vee \otimes A$.

(i)⇒(ii). Conversely, suppose that $E|_C = A \oplus (K_C \otimes A^\vee)$. Applying $\mathrm{Hom}(K_C \otimes A^\vee, -)$ to the sequence (5), we obtain an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(K_C \otimes A^\vee, F) \longrightarrow \mathrm{Ext}^1(K_C \otimes A^\vee, H^0(C,A) \otimes \mathcal{O}_S) \longrightarrow \mathrm{Ext}^1(K_C \otimes A^\vee, A).$$

Since the extension class $[E] \in \mathrm{Ext}^1(K_C \otimes A^\vee, H^0(C,A) \otimes \mathcal{O}_S)$ maps to the trivial extension in $\mathrm{Ext}^1(K_C \otimes A^\vee, A)$, it follows that there exists a rank 2 bundle G on S which fits into a commutative diagram

$$(21)\qquad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & H^0(A)\otimes \mathcal{O}_S & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & K_C\otimes A^\vee & = & K_C\otimes A^\vee & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Using that $H^0(S,F)=H^1(S,F)=0$, we obtain $H^0(S,G)\cong H^0(C,K_C\otimes A^\vee)$. Since $h^0(S,E)=h^0(C,A)+h^1(C,A)=h^0(C,A)+h^0(S,G)$, and $h^1(S,E)=0$, it follows that $H^1(S,G)=0$. From the second row of (21), we find that $H^0(S,G(-C))=0$.

Furthermore, we compute $c_1(G)=0$ and $c_2(G)=2d-2g+2$. So $c_2(G)<0=c_1^2(G)$, that is, G violates Bogomolov's inequality, and then it sits in an extension

$$(22)\qquad 0\longrightarrow M\longrightarrow G\longrightarrow M^\vee\otimes \mathcal{I}_{\Gamma/S}\longrightarrow 0,$$

where Γ is a zero-dimensional subscheme of S, and $M\in \mathrm{Pic}(S)$ is such that $M^2>0$ and $M\cdot H>0$ for any ample line bundle H on S. In particular, $H^0(S,M^\vee)=0$, and hence $H^0(S,M)\cong H^0(S,G)\cong H^0(C,K_C\otimes A^\vee)\neq 0$. On the other hand $H^0(S,F)=0$, which implies that the composed map $M\to G\to K_C\otimes A^\vee$ is non-zero; in fact, we claim that it is surjective, that is, $M|_C=K_C\otimes A^\vee$.

Suppose that $M|_C=K_C\otimes A^\vee(-D')$, with $D'\neq 0$ an effective divisor on C. Since $h^0(S,G(-C))=0$, we have $h^0(S,M(-C))=0$, which implies $h^0(S,M)\le h^0(C,M|_C)$. Since we assumed $K_C\otimes A^\vee$ to be globally generated, $h^0(S,M)\le h^0(C,K_C\otimes A^\vee(-D'))<h^0(C,K_C\otimes A^\vee)=h^0(S,M)$, a contradiction.

Setting $N:=M^\vee(C)$, we have shown that $N|_C=A$ and there is an exact sequence

$$0\longrightarrow M^\vee\longrightarrow N\longrightarrow A\longrightarrow 0.$$

Since $h^0(S,M^\vee)=h^1(S,M^\vee)=0$, it follows that $H^0(S,N)=H^0(C,A)$. To see that N is globally generated, we observe that N is a quotient of E.

□

3.1. **Lazarsfeld-Mukai bundles of higher rank.** We study when the restriction $E|_C$ is a simple vector bundle. Our main tool is a variant of the Bogomolov instability theorem.

Theorem 3.2. *Let S be a $K3$ surface and $C\subset S$ a smooth curve of genus $g\ge 4$ such that $\mathrm{Pic}(S)=\mathbb{Z}\cdot C$. We fix integers r and d such that $\rho(g,r,d)\ge 0$, $g\ge 2r+4$ and $d\le \frac{3r(g-1)}{2r+2}$. Then for any linear series $A\in W^r_d(C)$ such that $h^0(C,A)=r+1$ and $K_C\otimes A^\vee$ is globally generated, the restricted LM bundle $E|_C$ is simple.*

Note that in the special case $\rho(g,r,d)=0$, the constraints from the previous statement give rise to the bound $g>2r+5$.

Proof. Step 1. We first establish that the natural extension (6), that is,

$$0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0$$

is non-trivial. Assuming that (6) is trivial. Then there is an injective morphism from $K_C \otimes A^\vee$ to $E|_C$ and hence a surjective map $F(C) \to A$. Then $G := \text{Ker}\{F(C) \to A\}$ is a vector bundle of rank $r+1$ with Chern classes $c_1(G) = (r-1)[C]$ and

$$c_2(G) = c_2(F(C)) - c_1(F(C)) \cdot C + \deg(A) = 2d + r(r-3)(g-1).$$

We compute the discriminant of G

$$\Delta(G) = 2\text{rk}(G)c_2(G) - (\text{rk}(G)-1)c_1^2(G) = 4d(r+1) - 8r(g-1) < 0,$$

hence G is unstable. Applying [HL] Theorem 7.3.4, there exists a subsheaf $M \subset G$ with

$$\xi^2_{M,G} \geq -\frac{\Delta(G)}{r(r+1)^2},$$

where $\xi_{M,G} = c_1(M)/\text{rk}(M) - c_1(G)/\text{rk}(G)$. Setting $c_1(M) = k \cdot [C]$ and $s := \text{rk}(M)$, the previous inequality becomes

$$\left(\frac{k}{s} - \frac{r-1}{r+1}\right)^2 (2g-2) \geq \frac{8r(g-1) - 4d(r+1)}{r(r+1)^2}.$$

Note that M destabilizes G, which coupled with the stability of $F(C)$ yields

$$\frac{r-1}{r+1} \leq \frac{k}{s} < \frac{r}{r+1},$$

implying after manipulations $2d(r+1) > 3(g-1)r$, thus contradicting the hypothesis.

Step 2. Assuming that $E|_C$ is non-simple, we deduce that the extension (6) splits. We consider the exact sequence

$$H^0(S, E \otimes F) \longrightarrow H^0(C, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).$$

and it suffices to show that $H^1(S, E \otimes F(-C)) = 0$. Assuming this not to be the case, twisting (5) by $E(-C)$ induces the exact sequence

$$H^0(C, A \otimes E|_C \otimes K_C^\vee) \longrightarrow H^1(S, E \otimes F(-C)) \longrightarrow H^0(C, A) \otimes H^1(S, E(-C)).$$

Since $H^1(S, E(-C)) = 0$, we obtain that $H^0(C, A \otimes E|_C \otimes K_C^\vee) \neq 0$. Furthermore, Q_A is a stable bundle and since $\mu(Q_A \otimes A \otimes K_C^\vee) < 0$, we find $H^0(C, Q_A \otimes A \otimes K_C^\vee) = 0$, hence we also have the sequence induced from (6) after twisting with $A \otimes K_C^\vee$

$$0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^1(C, K_C^\vee \otimes A \otimes Q_A).$$

We conclude that the coboundary map $H^0(C, \mathcal{O}_C) \to H^1(C, Q_A \otimes A \otimes K_C^\vee)$ is trivial, that is, $E|_C \cong Q_A \oplus (K_C \otimes A^\vee)$, which completes the proof. □

4. Stability of restricted Lazarsfeld-Mukai bundles

4.1. **The rank 2 case.** If $C \subset S$ is an ample curve, then with one exception ($g = 10$ and C a smooth plane sextic), Cliff(C) is computed by a pencil, see [CP] Proposition 3.3. We show that in rank 2 the semistability of the LM bundle is preserved under restriction.

Theorem 4.1. *Let S be a $K3$ surface, $C \subset S$ an ample curve of genus $g \geq 4$ and $A \in W^1_d(C)$ a pencil computing* $\mathrm{Cliff}(C)$. *If $E_{C,A}$ is C-semistable on S, then $E|_C$ is also semistable on C.*

Proof. We write $A = \mathcal{O}_C(D)$, where D is an effective divisor on C. Suppose $E|_C$ is unstable and consider an exact sequence

$$0 \longrightarrow L_1 \longrightarrow E|_C \longrightarrow K_C \otimes L_1^\vee \longrightarrow 0,$$

with $\deg(L_1) \geq g$. Since $L_1 \nsubseteq A$, the composed map $L_1 \to E|_C \to K_C \otimes A^\vee$ must be non-zero, that is, $L_1 = K_C(-D - D_1)$, where D_1 is an effective divisor on C. Set $d_1 := \deg(D_1)$. Consider the elementary modification

$$0 \longrightarrow V \longrightarrow E \longrightarrow A(D_1) \longrightarrow 0 \tag{23}$$

induced by the composition $E \to E|_C \to A(D_1)$. Then $c_1(V) = 0$ and $c_2(V) = 2d + d_1 - 2g + 2 < 0$, hence V is unstable with respect to any polarization and fits in a sequence

$$0 \longrightarrow M \longrightarrow V \longrightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \longrightarrow 0, \tag{24}$$

where $\Gamma \subset S$ is a 0-dimensional subscheme and M is a divisor class that intersects positively any ample class on S and with $M^2 > 0$. From (23) and (24) we find that $H^0(S, M) \cong H^0(S, V)$ and $H^0(S, M(-C)) = 0$. Dualizing (23), we obtain the sequence

$$0 \longrightarrow F \longrightarrow V^\vee \longrightarrow K_C(-D - D_1) \longrightarrow 0,$$

from which, using that $V \cong V^\vee$, we obtain $H^0(S, V) = H^0(C, K_C(-D - D_1))$.

We claim that $\mathrm{Cliff}(A(D_1)) = \mathrm{Cliff}(C)$. Recall that $h^0(S, E) = h^0(C, A) + h^1(C, A)$, and, from the sequence (23) we write $h^0(S, E) \leq h^0(C, A(D_1)) + h^1(C, A(D_1))$. By assumption, the pencil A computes $\mathrm{Cliff}(C)$, which implies

$$\mathrm{Cliff}(C) = g + 1 - h^0(A) - h^1(A) \geq g + 1 - h^0(A(D_1)) - h^1(A(D_1)) = \mathrm{Cliff}(A(D_1)).$$

It follows that $\mathrm{Cliff}(A(D_1)) = \mathrm{Cliff}(C)$, in particular $K_C(-D-D_1)$ is globally generated.

Clearly, $M \nsubseteq F$, hence the composition $\varphi : M \to V \to K_C(-D - D_1)$ is non-zero and one writes $\mathrm{Im}(\varphi) = K_C(-D - D_1 - D_2)$, where D_2 is an effective divisor on C. If $D_2 \neq 0$, then one has the sequence of inequalities

$$h^0(S, M) \leq h^0(C, K_C(-D - D_1 - D_2)) < h^0(C, K_C(-D - D_1)) = h^0(S, M),$$

a contradiction. Therefore $M|_C = K_C(-D - D_1)$, Viewing M as a subsheaf of E, we find $\mu(M) = M \cdot C = \deg(L_1) > \mu(E)$, thus bringing the proof to an end. □

Remark 4.2. The same proof shows that if $E_{C,A}$ is C-stable, then the restriction $E|_C$ is stable too. Observe that in this case, $E_{C,A}$ being simple, necessarily $d = \lfloor \frac{g+3}{2} \rfloor$, see [L1]. Conversely, if $C' \subset S$ is an ample curve of genus g and gonality $\lfloor \frac{g+3}{2} \rfloor$, then it was shown in [LC] that the LM bundle $E_{C,A}$ corresponding to a general curve $C \in |\mathcal{O}_S(C')|$ and a pencil $A \in W^1_{\lfloor \frac{g+3}{2} \rfloor}(C)$ is C-semistable (even stable when g is odd).

4.2. **Stability of rank 4 Lazarsfeld-Mukai bundles.** We show that restrictions of LM bundles of rank 4 on very general $K3$ surfaces of genus $g \geq 20$ are stable. Similar results were established in [V] and [FO] for rank 2 and 3 respectively. We fix integers $i \geq 6$ and $\rho \geq 0$ and write

$$g := 4i - 4 + \rho \quad \text{and} \quad d := 3i + \rho,$$

so that $\rho(g, 3, d) = \rho$. Let S be a $K3$ surface and $C \subset S$ a curve of genus g such that $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$, and pick a globally generated linear series $A \in W^3_d(C)$ with $h^0(C, A) = 4$.

Proof of Theorem 0.7. Our previous results show that $E|_C$ is simple, hence indecomposable. Suppose $E|_C$ is not stable and fix a maximal destabilizing sequence

$$0 \longrightarrow M \longrightarrow E|_C \longrightarrow N \longrightarrow 0.$$

Put $d_N := \deg(N)$ and $d_M := \deg(M) = 2g - 2 - d_N$. Since M is destabilizing,

$$\frac{d_M}{\mathrm{rk}(M)} \geq \frac{g-1}{2}, \quad \frac{d_N}{\mathrm{rk}(N)} \leq \frac{g-1}{2}. \tag{25}$$

The bundle N, being a quotient of E, is globally generated. Since $H^0(C, E|_C^\vee) = 0$, clearly $N \neq \mathcal{O}_C$, therefore $h^0(C, N) \geq 2$. From the inequalities (25) it follows that $\mathrm{rk}(N) > 1$, because C has maximal gonality.

Step 1. We prove that M is a line bundle. Assume that, on the contrary, $\mathrm{rk}(M) = \mathrm{rk}(N) = 2$ and consider the elementary modification $G := \mathrm{Ker}\{E \to N\}$. Its Chern classes are given as follows:

$$c_1(G) = -[C], \quad c_2(G) = d + d_N - 2(g-1),$$

and its discriminant equals $\Delta(G) = -64i + 110 + 8d_N - 14\rho < 0$, because of (25). In particular, there exists a saturated subsheaf $F \subset G$ which verifies the inequalities

$$\mu(G) \leq \mu(F) < \mu(E), \quad \text{and} \tag{26}$$

$$\xi^2_{F,G} \geq -\frac{\Delta(G)}{48}. \tag{27}$$

Write $c_1(F) = \alpha \cdot [C]$ and $\mathrm{rk}(F) = \beta \leq 3$. The above inequality (27) becomes

$$\left(\frac{\alpha}{\beta} + \frac{1}{4}\right)^2 (2g-2) \geq -\frac{\Delta(G)}{48}.$$

We apply (26) for $\mu(F) = \alpha(2g-2)/\beta$ and obtain

$$-\frac{1}{4} \leq \frac{\alpha}{\beta} < \frac{1}{4},$$

hence $\alpha = 0$, and the inequality (27) reads in this case $d_N \geq 5i - 10 + \rho$. Recalling that $d_N \leq g - 1 = 4i - 5 + \rho$, we obtain a contradiction whenever $i \geq 6$.

Step 2. We construct an elementary modification, in order to reach a contradiction.

From (25), we have $d_M \geq \frac{g-1}{2}$. The composite map $M \to E|_C \to K_C \otimes A^\vee$ is not zero, for else $M \hookrightarrow Q_A$ and since $\mu(Q_A \otimes M^\vee) < 0$, one contradicts the semistability of Q_A. We set $A_1 := K_C \otimes A^\vee \otimes M^\vee$ and obtain a surjection $F(C)|_C \to A \otimes A_1$ inducing, as before, an elementary modification $V := \mathrm{Ker}\{F(C) \to A \otimes A_1\}$.

By direct computation we show that $\Delta(V) < 0$. Indeed, we compute

$$c_1(V) = 2 \cdot [C], \quad c_2(V) = d + 2g - 2 - d_M, \quad \text{hence}$$

$$\Delta(V) = 8c_2(V) - 3c_1^2(V) = 8(d - d_M - g + 1) = 8(5 - d_M - i) < 0.$$

We obtain a destabilizing sheaf $P \subset V$, with $\mathrm{rk}(P) = b \leq 3$ and $c_1(P) := a \cdot [C]$, such that the following inequalities are both satisfied

$$\left(\frac{a}{b} - \frac{1}{2}\right)^2 (2g-2) \geq -\frac{\Delta(V)}{48} \quad \text{and} \quad \mu(V) \leq \mu(P) < \mu(F(C)). \tag{28}$$

The second inequality gives $\frac{1}{2} \leq \frac{a}{b} < \frac{3}{4}$, which leaves with two possibilities: either $a = 1$ and $b = 2$, when via (28) one finds that $\Delta(V) \geq 0$, a contradiction, or else $a = 2$ and $b = 3$, when inequalities (28) and (25) clash. □

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