THE UNIVERSAL ABELIAN VARIETY OVER $\mathcal{A}_5$

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ABSTRACT. We establish a structure result for the universal abelian variety over $\mathcal{A}_5$. This implies that the boundary divisor of $\overline{\mathcal{A}}_6$ is unirational and leads to a lower bound on the slope of the cone of effective divisors on $\overline{\mathcal{A}}_6$.

The general principally polarized abelian variety $[A, \Theta] \in \mathcal{A}_g$ of dimension $g \leq 5$ can be realized as a Prym variety. Abelian varieties of small dimension can be studied in this way via the rich and concrete theory of curves, in particular, one can establish that $\mathcal{A}_g$ is unirational in this range. In the case $g = 5$, the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$ is finite of degree 27, see [DS]; three different proofs [Don], [MM], [Ve1] of the unirationality of $\mathcal{R}_6$ are known. The moduli space $\mathcal{A}_g$ is of general type for $g \geq 7$, see [Mu], [T]. Determining the Kodaira dimension of $\mathcal{A}_6$ is a notorious open problem.

The aim of this paper is to give a simple unirational parametrization of the universal abelian variety over $\mathcal{A}_5$ and hence of the boundary divisor of a compactification of $\mathcal{A}_6$. We denote by $\phi : \mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ the universal abelian variety of dimension $g-1$. The moduli space $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension g and their rank 1 degenerations is a partial compactification of $\mathcal{A}_g$ obtained by blowing-up $\mathcal{A}_{g-1}$ in the Satake compactification, cf. [Mu]. Its boundary $\partial\widetilde{\mathcal{A}}_g$ is isomorphic to the universal Kummer variety in dimension $g-1$ and there exist a surjective double covering $j : \mathcal{X}_{g-1} \to \partial\widetilde{\mathcal{A}}_g$. We establish a simple structure result for the boundary $\partial\widetilde{\mathcal{A}}_6$:

Theorem 0.1. *The universal abelian variety $\mathcal{X}_5$ is unirational.*

This immediately implies that the boundary divisor $\partial\widetilde{\mathcal{A}}_6$ is unirational as well. What we prove is actually stronger than Theorem 0.1. Over the moduli space $\mathcal{R}_g$ of smooth Prym curves of genus g, we consider the universal Prym variety $\varphi : \mathcal{Y}_g \to \mathcal{R}_g$ obtained by pulling-back $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ via the Prym map $P : \mathcal{R}_g \to \mathcal{A}_{g-1}$. Let $\overline{\mathcal{R}}_g$ be the moduli space of stable Prym curves of genus g together with the universal Prym curve $\tilde{\pi} : \widetilde{\mathcal{C}} \to \overline{\mathcal{R}}_g$ of genus $2g-1$. If $\widetilde{\mathcal{C}}^{g-1} := \widetilde{\mathcal{C}} \times_{\overline{\mathcal{R}}_g} \ldots \times_{\overline{\mathcal{R}}_g} \widetilde{\mathcal{C}}$ is the $(g-1)$-fold product, one has a universal *Abel-Prym* rational map $\mathfrak{ap} : \widetilde{\mathcal{C}}^{g-1} \dashrightarrow \mathcal{Y}_g$, whose restriction on each individual Prym variety is the usual Abel-Prym map. The rational map $\mathfrak{ap}$ is dominant and generically finite (see Section 4 for details). We prove the following result:

Theorem 0.2. *The five-fold product $\widetilde{\mathcal{C}}^5$ of the universal Prym curve over $\overline{\mathcal{R}}_6$ is unirational.*

The key idea in the proof of Theorem 0.2 is to view smooth Prym curves of genus 6 as discriminants of conic bundles, via their representation as symmetric determinants of quadratic forms in three variables. We fix four general points $u_1, \ldots, u_4 \in \mathbf{P}^2$, set $w_i := (u_i, u_i) \in \mathbf{P}^2 \times \mathbf{P}^2$, then consider the linear system

$$\mathbf{P}^{15} := \left|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\right| \subset \left|\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)\right|$$

of hypersurfaces $Q\subset \mathbf{P}^2\times\mathbf{P}^2$ of bidegree $(2,2)$ which are nodal at $w_1,\ldots,w_4$. Fixing the points $w_1,\ldots,w_4\in\mathbf{P}^2\times\mathbf{P}^2$ neutralizes the action of $\mathrm{Aut}(\mathbf{P}^2)\times\mathrm{Aut}(\mathbf{P}^2)$. For a general threefold $Q\in\mathbf{P}^{15}$, the first projection $p:Q\to\mathbf{P}^2$ induces a conic bundle structure with a sextic discriminant curve $\Gamma\subset\mathbf{P}^2$ such that $p(\mathrm{Sing}(Q))=\mathrm{Sing}(\Gamma)$. The discriminant curve Γ is nodal precisely at the points $u_1,\ldots,u_4$. Furthermore, Γ is equipped with an unramified double cover $p_\Gamma:\widetilde{\Gamma}\to\Gamma$, parametrizing the lines which are components of the singular fibres of $p:Q\to\mathbf{P}^2$. By normalizing, p_Γ lifts to an unramified double cover $f:\widetilde{C}\to C$ between the normalization $\widetilde{C}$ of $\widetilde{\Gamma}$ and the normalization C of Γ respectively. Note that there exists an exact sequence of generalized Prym varieties

$$0\longrightarrow(\mathbf{C}^*)^4\longrightarrow P(\widetilde{\Gamma}/\Gamma)\longrightarrow P(\widetilde{C}/C)\longrightarrow 0.$$

One can show without much effort that the assignment $\mathbf{P}^{15}\ni Q\mapsto[\widetilde{C}\stackrel{f}{\to}C]\in\mathcal{R}_6$ is dominant. This offers an alternative, much simpler, proof of the unirationality of $\mathcal{R}_6$. However, much more can be obtained with this construction.

Let $\mathbf{G}:=\mathbf{P}^2\times(\mathbf{P}^2)^\vee=\bigl\{(o,\ell):o\in\mathbf{P}^2,\ell\in\{o\}\times(\mathbf{P}^2)^\vee\bigr\}$ be the Hilbert scheme of lines in the fibres of the first projection $p:\mathbf{P}^2\times\mathbf{P}^2\to\mathbf{P}^2$. Since containing a given line in a fibre of p imposes three conditions on the linear system $\mathbf{P}^{15}$ of threefolds $Q\subset\mathbf{P}^2\times\mathbf{P}^2$ as above, it follows that imposing the condition $\{o_i\}\times\ell_i\subset Q$ for *five* general lines, singles out a *unique* conic bundle $Q\in\mathbf{P}^{15}$. This induces an étale double cover $f:\widetilde{C}\to C$, as above, over a smooth curve of genus 6. Moreover, f comes equipped with five marked points $\ell_1,\ldots,\ell_5\in\widetilde{C}$. To summarize, we have a well-defined rational map

$$\zeta:\mathbf{G}^5\dashrightarrow\widetilde{\mathcal{C}}^5,\quad \zeta\Bigl((o_1,\ell_1),\ldots,(o_5,\ell_5)\Bigr):=\Bigl(f:\widetilde{C}\to C,\ell_1,\ldots,\ell_5\Bigr),$$

between two 20-dimensional varieties.

Theorem 0.3. *The morphism $\zeta:\mathbf{G}^5\dashrightarrow\widetilde{\mathcal{C}}^5$ is dominant, so that $\widetilde{\mathcal{C}}^5$ is unirational.*

More precisely, we show that $\mathbf{G}^5$ is birationally isomorphic to the fibre product $\mathbf{P}^{15}\times_{\overline{\mathcal{R}}_6}\widetilde{\mathcal{C}}^5$. In order to set Theorem 0.3 on the right footing and in view of enumerative calculations, we introduce a $\mathbf{P}^2$-bundle $\pi:\mathbf{P}(\mathcal{M})\to S$ over the quintic del Pezzo surface S obtained by blowing-up $\mathbf{P}^2$ at the points $u_1,\ldots,u_4$. The rank 3 vector bundle $\mathcal{M}$ on S is obtained by making an elementary transformation along the four exceptional divisors $E_1,\ldots,E_4$ over $u_1,\ldots,u_4$. The nodal threefolds $Q\subset\mathbf{P}^2\times\mathbf{P}^2$ considered above can be thought of as sections of a tautological linear system on $\mathbf{P}(\mathcal{M})$, and via the identification

$$\Bigl|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\Bigr|=\Bigl|\mathcal{O}_{\mathbf{P}(\mathcal{M})}(2)\Bigr|,$$

we can view 4-nodal conic bundles in $\mathbf{P}^2\times\mathbf{P}^2$ as *smooth* conic bundles over S. In this way the numerical characters of a pencil of such conic bundles can be computed (see Sections 2 and 3 for details).

Theorem 0.3 is then used to give a lower bound for the slope of the effective cone of $\overline{\mathcal{A}}_6$ (though we stop short of determining the Kodaira dimension of $\overline{\mathcal{A}}_6$). Recall that if E is an effective divisor on the perfect cone compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ with no component supported on the boundary $D_g:=\overline{\mathcal{A}}_g-\mathcal{A}_g$ and $[E]=a\lambda_1-b[D_g]$, where $\lambda_1\in CH^1(\widetilde{\mathcal{A}}_g)$ is the Hodge class, then the slope of E is defined as $s(E):=\frac{a}{b}\geq 0$. The slope $s(\overline{\mathcal{A}}_g)$ of the effective cone of divisors of $\overline{\mathcal{A}}_g$ is the infimum of the slopes of all effective divisors on $\overline{\mathcal{A}}_g$. This important invariant governs to a large extent the birational geometry of $\mathcal{A}_g$; for instance, since $K_{\overline{\mathcal{A}}_g} = (g+1)\lambda_1 - [\partial\overline{\mathcal{A}}_g]$, the variety $\overline{\mathcal{A}}_g$ is of general type if $s(\overline{\mathcal{A}}_g) < g+1$, and uniruled when $s(\overline{\mathcal{A}}_g) > g+1$.

It is known that $s(\overline{\mathcal{A}}_4) = 8$ and that the Jacobian locus $\overline{\mathcal{M}}_4 \subset \overline{\mathcal{A}}_4$ achieves the minimal slope [SM]; one of the results of [FGSMV] is the calculation $s(\overline{\mathcal{A}}_5) = \frac{54}{7}$. Furthermore, the only irreducible effective divisor on $\overline{\mathcal{A}}_5$ of minimal slope is the closure of the Andreotti-Mayer divisor N_0' consisting of 5-dimensional ppav $[A, \Theta]$ for which the theta divisor Θ is singular at a point which is not 2-torsion. Concerning $\overline{\mathcal{A}}_6$, we establish the following estimate:

Theorem 0.4. *The following lower bound holds:* $s(\overline{\mathcal{A}}_6) \geq \frac{53}{10}$.

Note that this is the first concrete lower bound on the slope of $\overline{\mathcal{A}}_6$. The idea of proof of Theorem 0.4 is very simple. Since $\widetilde{\mathcal{C}}^5$ is unirational, we choose a suitable sweeping rational curve $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$, which we then push forward to $\overline{\mathcal{A}}_6$ as follows:

$$\begin{tikzcd} \mathbf{P}^1 \arrow[r, "i"] \arrow[rrrr, bend left, "h"] & \widetilde{\mathcal{C}}^5 \arrow[r, "\mathfrak{ap}"] & \widetilde{\mathcal{Y}}_6 \arrow[r] & \widetilde{\mathcal{X}}_5 \arrow[r, "j"] & \partial\overline{\mathcal{A}}_6 \end{tikzcd}$$

Here $\widetilde{\mathcal{Y}}_6$ and $\widetilde{\mathcal{X}}_5$ are partial compactifications of $\mathcal{Y}_6$ and $\mathcal{X}_5$ respectively, which are described in Section 4. The curve $h(\mathbf{P}^1)$ sweeps the boundary divisor of $\overline{\mathcal{A}}_6$ and intersects non-negatively any effective divisor on $\overline{\mathcal{A}}_6$ not contained in $\partial\overline{\mathcal{A}}_6$. In particular,

$$s(\overline{\mathcal{A}}_6) \geq \frac{h(\mathbf{P}^1)\cdot[\partial\overline{\mathcal{A}}_6]}{h(\mathbf{P}^1)\cdot\lambda_1}.$$

To define $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$, we fix general points $(o_1, \ell_1), \dots, (o_4, \ell_4) \in \mathbf{G}$ and a further general point $o \in \mathbf{P}^2$. Then we consider the image under ζ of the pencil of lines in $\mathbf{P}^2$ through o, that is, the sweeping curve i is defined as

$$\mathbf{P}(T_o(\mathbf{P}^2)) \ni \ell \mapsto \zeta\Big((o_1,\ell_1),\dots,(o_4,\ell_4),(o,\ell)\Big) \in \widetilde{\mathcal{C}}^5.$$

The calculation of the numerical characters of $h(R) \subset \overline{\mathcal{A}}_6$ is a consequence of the geometry of the map ζ and is completed in Section 4.

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1. Determinantal nodal sextics and a parametrization of $\mathcal{X}_5$

In this section we prove Theorem 0.3. We begin by recalling basic facts about determinantal representation of nodal plane sextics, see [B2], [Dol], [DIM]. Let $\Gamma \subset \mathbf{P}^2$ be an integral 4-nodal sextic and $\nu : C \to \Gamma$ the normalization map, thus C is a smooth curve of genus 6. One has an exact sequence at the level of 2-torsion groups

$$0 \longrightarrow \mathbb{Z}_2^{\oplus 4} \longrightarrow \mathrm{Pic}^0(\Gamma)[2] \xrightarrow{\nu^*} \mathrm{Pic}^0(C)[2] \longrightarrow 0.$$

In particular, *unsplit* étale double covers $f : \Gamma' \to \Gamma$ are induced by 2-torsion points $\eta \in \mathrm{Pic}^0(\Gamma)[2]$, such that $\eta_C := \nu^*(\eta) \neq \mathcal{O}_C$.

Definition 1.1. We denote by $\mathcal{P}_6$ the quasi-projective moduli space of pairs (Γ, η) as above, where $\Gamma \subset \mathbf{P}^2$ is an integral 4-nodal sextic and $\eta \in \mathrm{Pic}^0(\Gamma)[2]$ is a torsion point inducing a non-split double cover $\Gamma' \to \Gamma$, or equivalently, $\eta_C \neq \mathcal{O}_C$.

Starting with a general element $[C, \eta_C] \in \mathcal{R}_6$, since $|W^2_6(C)| = 5$, there are five sextic nodal plane models $\nu : C \to \Gamma$. For each of them, there are 2^4 further ways of choosing $\eta \in (\nu^*)^{-1}(\eta_C)$. Thus there is a degree $80 = 5 \cdot 2^4$ covering $\rho : \mathcal{P}_6 \to \mathcal{R}_6$.

Suppose now that $(\Gamma, \eta) \in \mathcal{P}_6$ is a general point[1]. In particular $h^0(\Gamma, \eta(1)) = 0$, or equivalently, $h^0(\Gamma, \eta(2)) = 3$. Indeed, the condition $h^0(\Gamma, \eta(1)) \geq 1$ implies that $\Gamma \subset \mathbf{P}^2$ possesses a totally tangent conic, that is, there exists a reduced conic $B \subset \mathbf{P}^2$ such that $\nu^*(B) = 2b$, with b being an effective divisor of C. This condition is satisfied only if $\rho(\Gamma, \eta)$ lies in the ramification divisor of the Prym map $P : \mathcal{R}_6 \to \mathcal{A}_5$, see [FGSMV]. Thus we may assume that $h^0(\Gamma, \eta(2)) = 3$, for a general point $(\Gamma, \eta) \in \mathcal{P}_6$.

Following [B2] Theorem B, it is known that such a sheaf η admits a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-4)^{\oplus 3} \xrightarrow{A} \mathcal{O}_{\mathbf{P}^2}(-2)^{\oplus 3} \longrightarrow \eta \longrightarrow 0, \tag{1}$$

where the map A is given by a symmetric matrix $\Big(a_{ij}(x_1, x_2, x_3)\Big)_{i,j=1}^3$ of quadratic forms. More precisely, we can view the resolution (1) as a twist of the exact sequence

$$0 \longrightarrow H^0(\Gamma, \eta(2))^\vee \otimes \mathcal{O}_{\mathbf{P}^2}(-2) \xrightarrow{A} H^0(\Gamma, \eta(2)) \otimes \mathcal{O}_{\mathbf{P}^2} \xrightarrow{\mathrm{ev}} \eta(2) \longrightarrow 0, \tag{2}$$

where ev is the evaluation map. Since η is invertible, for each point $x \in \Gamma$ one has

$$1 = \dim_{\mathbb{C}} \eta(x) = 3 - \mathrm{rk}\ A(x),$$

where, as usual, $\eta(x) := \eta_x \otimes_{\mathcal{O}_{\Gamma,x}} \mathbb{C}(x)$ is the fibre of the sheaf η at the point x. Thus $\mathrm{rk}\ A(x) = 2$, for each $x \in \Gamma$.

To the matrix $A \in M_3\big(H^0(\mathcal{O}_{\mathbf{P}^2}(2))\big)$ we can associate the following $(2,2)$ threefold in $\mathbf{P}^2_{[x_1:x_2:x_3]} \times \mathbf{P}^2_{[y_1:y_2:y_3]} = \mathbf{P}^2 \times \mathbf{P}^2$

$$Q : \sum_{i,j=1}^3 a_{ij}(x_1, x_2, x_3) y_i y_j = 0,$$

which is a conic bundle with respect to the two projections. Alternatively, if

$$A : H^0(\Gamma, \eta(2))^\vee \otimes H^0(\Gamma, \eta(2))^\vee \to H^0(\Gamma, \mathcal{O}_\Gamma(2))$$

is the symmetric map appearing in (2), then A induces the $(2,2)$ hypersurface

$$Q \subset \mathbf{P}\Big(H^0(\Gamma, \mathcal{O}_\Gamma(1))^\vee\Big) \times \mathbf{P}\Big(H^0(\Gamma, \eta(2))^\vee\Big) = \mathbf{P}^2 \times \mathbf{P}^2.$$

We denote by $p : Q \to \mathbf{P}^2$ the first projection and then $\Gamma \subset \mathbf{P}^2$ is precisely the discriminant curve of Q given by determinantal equation $\Gamma := \big\{\det A(x_1, x_2, x_3) = 0\big\}$. Let Γ' denote the Fano scheme of lines $F_1\big(p^{-1}(\Gamma)/\Gamma\big)$ over the discriminant curve Γ and $f : \Gamma' \to \Gamma$ be the forgetful map $f(x, \ell) := x$, where ℓ is an irreducible component of $p^{-1}(x)$. Since $\mathrm{rk}\ A(x) = 2$ for all $x \in \Gamma$, it follows that f is an étale double cover.

Proposition 1.2. *For $(\Gamma, \eta) \in \mathcal{P}_6$, the restriction map $p_{|\mathrm{Sing}(Q)} : \mathrm{Sing}(Q) \to \mathrm{Sing}(\Gamma)$ is bijective.*

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[1] We shall soon establish that $\mathcal{P}_6$ is irreducible, but here we just require that $\rho(\Gamma, \eta)$ be a general point of the irreducible variety $\mathcal{R}_6$.

Proof. Let $x \in \Gamma$ and $R := \mathcal{O}_{\mathbf{P}^2,x}$ be the local ring of $\mathbf{P}^2$ and $\mathfrak{m}$ its maximal ideal. After a linear change of coordinates, we may assume that the matrix A mod $\mathfrak{m} =: A(x)$ equals

$$A(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Suppose $(x, y = [y_1, y_2, y_3]) \in \mathrm{Sing}(Q)$. Then $A(x) \cdot^t y = 0$, hence $y_1 = y_2 = 0$. Imposing that the partials of the defining equation of Q with respects to x_1, x_2, x_3 vanish, we obtain that $a_{33} \in \mathfrak{m}^2$. Since $\det(a_{ij}) \equiv a_{33}$ mod $\mathfrak{m}^2$, this implies that Γ is singular at x. Conversely, for $x \in \mathrm{Sing}(\Gamma)$, we obtain that $\mathrm{Sing}(Q) \cap p^{-1}(x) = \{(x, y)\}$, where $y \in \mathbf{P}^2$ is uniquely determined by the condition $A(x) \cdot^t y = 0$ (use once more that rk $A(x) = 2$). □

Proposition 1.3. *$f_*(\mathcal{O}_{\Gamma'}) = \mathcal{O}_\Gamma \oplus \eta$, that is, the double cover f is induced by η.*

Proof. Essentially identical to [B1] Lemme 6.14. □

Summarizing the discussion so far, to give a general point $(\Gamma, \eta) \in \mathcal{P}_6$ is equivalent to specify a 4-nodal conic bundle as above. Let $\mathbf{T} \subset \left|\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)\right|$ be the subvariety consisting of 4-nodal hypersurfaces of bidegree $(2, 2)$. This is an irreducible 31-dimensional variety endowed with an action of $\mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2)$.

Theorem 1.4. *A general Prym curve $(\Gamma, \eta) \in \mathcal{P}_6$ is the discriminant of a 4-nodal conic bundle $p : Q \to \boldsymbol{P}^2$, where $Q \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$ is a 4-nodal threefold of bidegree $(2, 2)$. More precisely, we have a birational isomorphism $\boldsymbol{T}//\mathrm{Aut}(\boldsymbol{P}^2 \times \boldsymbol{P}^2) \overset{\cong}{\dashrightarrow} \mathcal{P}_6$.*

Remark 1.5. A similar isomorphism between the moduli space of Prym curves over *smooth* plane sextics and the quotient $\left|\mathcal{O}_{\mathbf{P}^2 \times \mathbf{P}^2}(2,2)\right|//\mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2)$ has already been established and used in [B1] and [Ve2].

Remark 1.6. Theorem 1.4 yields another (shorter) proof of the unirationality of $\mathcal{R}_6$.

The automorphism group of $\mathbf{P}^2 \times \mathbf{P}^2$ sits in an exact sequence

$$0 \longrightarrow PGL(3) \times PGL(3) \longrightarrow \mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

In particular, we can fix four general points $u_1, \ldots, u_4 \in \mathbf{P}^2$, as well as diagonal points $w_i := (u_i, u_i) \in \mathbf{P}^2 \times \mathbf{P}^2$, and consider the linear system $\mathbf{P}^{15} := \left|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\right|$ of $(2, 2)$ threefolds with assigned nodes at these points. Theorem 1.4 implies the existence of a dominant discriminant map $\mathfrak{d} : \mathbf{P}^{15} \dashrightarrow \mathcal{P}_6$ assigning $\mathfrak{d}(Q) := (\Gamma' \xrightarrow{f} \Gamma)$.

Proof of Theorem 0.3. Using the notation introduced in this section and in the Introduction, setting $\mu := \varphi \circ \mathfrak{ap} : \widetilde{\mathcal{C}}_5 \dashrightarrow \overline{\mathcal{R}}_6$, one has the following commutative diagram:

$$\begin{tikzcd}
\mathbf{G}^5 \arrow[r] \arrow[d] & \mathcal{P}_6 \times_{\mathcal{R}_6} \widetilde{\mathcal{C}}^5 \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}^5 \arrow[d, "\mu"'] \\
\mathbf{P}^{15} \arrow[r, "\mathfrak{d}"] & \mathcal{P}_6 \arrow[r, "\rho"] & \overline{\mathcal{R}}_6
\end{tikzcd}$$

The dominance of the composite map $\zeta : \mathbf{G}^5 \dashrightarrow \widetilde{\mathcal{C}}^5$ follows once we observe, that the above diagram is birationally a fibre product, that is, $\mathbf{G}^5 \overset{\cong}{\dashrightarrow} \mathbf{P}^{15} \times_{\overline{\mathcal{R}}_6} \widetilde{\mathcal{C}}^5$. □

2. Conic bundles over a del Pezzo surface

With view to further applications, we analyze the linear system of conic bundles of type $(2,2)$ in $\mathbf{P}^2\times\mathbf{P}^2$ which are singular at four fixed general points and birationally, we reconstruct such a linear system as the complete linear system of smooth conic bundles in a certain $\mathbf{P}^2$-bundle over a smooth quintic del Pezzo surface.

We fix four general points $u_1,\ldots,u_4\in\mathbf{P}^2$ and set $w_i:=(u_i,u_i)\in\mathbf{P}^2\times\mathbf{P}^2$. Let S be the Del Pezzo surface defined by the blow-up $\sigma: S\to\mathbf{P}^2$ of $u_1,\ldots,u_4$. For $i=1,\ldots,4$, we denote by $E_i:=\sigma^{-1}(u_i)$ the exceptional line over u_i. Set $E:=E_1+\cdots+E_4$ and denote by $L\in|\sigma^*\mathcal{O}_{\mathbf{P}^2}(1)|$ the pull-back of a line in $\mathbf{P}^2$ under σ. An important role is played by the rank 3 vector bundle $\mathcal{M}$ on S defined by the following sequence

$$0\longrightarrow\mathcal{M}\xrightarrow{j}H^0(S,L)\otimes\mathcal{O}_S(L)\xrightarrow{r}\bigoplus_{i=1}^4\mathcal{O}_{E_i}(2L)\longrightarrow 0. \tag{3}$$

Here $r_i: H^0(S,\mathcal{O}_S(L))\otimes\mathcal{O}_S(L)\to\mathcal{O}_{E_i}(2L)$ is the evaluation map and $r:=\oplus_{i=1}^4 r_i$. Since $\mathcal{O}_{E_i}(L)$ is trivial, it follows that $h^0(r)$ is surjective. Passing to cohomology, we write the exact sequence

$$0\longrightarrow H^0(S,\mathcal{M})\xrightarrow{h^0(j)}H^0(S,O_S(L))\otimes H^0(S,\mathcal{O}_S(L))\xrightarrow{h^0(r)}\bigoplus_{i=1}^4 H^0(\mathcal{O}_{E_i}(2L))\longrightarrow 0.$$

In particular, we obtain that $h^0(S,\mathcal{M})=5$. By direct calculation, we also find that

$$c_1(\mathcal{M})=\mathcal{O}_S(-K_S)\quad\text{and}\quad c_2(\mathcal{M})=3. \tag{4}$$

Under the decomposition $H^0(\mathcal{O}_S(L))\otimes H^0(\mathcal{O}_S(L))=\wedge^2H^0(\mathcal{O}_S(L))\oplus H^0(\mathcal{O}_S(2L))$ into symmetric and anti-symmetric tensors, the space $j(H^0(S,\mathcal{M}))$ decomposes as

$$H^0(S,\mathcal{M})=H^0(S,\mathcal{M})^-\oplus H^0(S,\mathcal{M})^+=\bigwedge^2 H^0(S,\mathcal{O}_S(L))\oplus H^0(S,\mathcal{O}_S(2L-E)).$$

Lemma 2.1. *The vector bundle $\mathcal{M}$ is globally generated.*

Proof. Clearly, we only need to address the global generation of $\mathcal{M}$ along $\bigcup_{i=1}^4 E_i$ and to that end, we consider the restriction of the sequence (3) to E_i,

$$\mathcal{M}_{|E_i}\xrightarrow{j_{|E_i}}H^0(S,\mathcal{O}_S(L))\otimes\mathcal{O}_{E_i}\xrightarrow{r_{|E_i}}\mathcal{O}_{E_i}\longrightarrow 0.$$

The sheaf $H^0(\mathcal{O}_S(L-E_i))\otimes\mathcal{O}_{E_i}=\mathcal{O}_{\mathbf{P}^1}^{\oplus 2}$ is the kernel of $r_{|E_i}$. Since $\det(\mathcal{M}_{|E_i})=\mathcal{O}_{E_i}(1)$, it follows that $\mathcal{M}_{|E_i}$ fits into an exact sequence of bundles on $\mathbf{P}^1$:

$$0\longrightarrow\mathcal{O}_{\mathbf{P}^1}(1)\longrightarrow\mathcal{M}_{|E_i}\xrightarrow{j|E_i}\mathcal{O}_{\mathbf{P}^1}^{\oplus 2}\longrightarrow 0.$$

This sequence is split, so that $\mathcal{M}_{|E_i}=\mathcal{O}_{\mathbf{P}^1}(1)\oplus\mathcal{O}_{\mathbf{P}^1}^{\oplus 2}$, which is globally generated. The same holds for $\mathcal{M}$ if the map $H^0(\mathcal{M})\to H^0(\mathcal{M}_{|E_i})$ is surjective; this is implied by the vanishing $H^1(S,\mathcal{M}(-E_i))=0$. We twist by $\mathcal{O}_S(-E_i)$ the sequence (3), and write

$$0\longrightarrow\mathcal{M}(-E_i)\longrightarrow H^0(S,\mathcal{O}_S(L))\otimes\mathcal{O}_S(L-E_i)\xrightarrow{r}\bigoplus_{i=1}^4\mathcal{O}_{E_i}(2L-E_i)\longrightarrow 0.$$

Since $h^0(r)$ is surjective and $h^1(S,\mathcal{O}_S(L-E_i))=0$, it follows $h^1(S,\mathcal{M}(-E_i))=0$. $\square$

From now on we set $\mathbf{P} := \mathbf{P}(\mathcal{M})$ and consider the $\mathbf{P}^2$-bundle $\pi : \mathbf{P} \to S$. The linear system $|\mathcal{O}_{\mathbf{P}}(1)|$ is base point free, for $\mathcal{M}$ is globally generated. We reserve the notation

$$h := \phi_{\mathcal{O}_{\mathbf{P}}(1)} : \mathbf{P} \to \mathbf{P}^4 := \mathbf{P}H^0(S, \mathcal{M})^\vee.$$

for the induced morphism. A Chern classes count implies that $\deg(h) = 2$. The map j from the sequence (3) induces a birational morphism

$$\epsilon : S \times \mathbf{P}^2 \dashrightarrow \mathbf{P}.$$

We describe a factorization of ϵ. Since j is an isomorphism along $U := S - \bigcup_{i=1}^4 E_i$, it follows that $\epsilon : U \times \mathbf{P}^2 \to \pi^{-1}(U)$ is biregular. The behaviour of ϵ along $E_i \times \mathbf{P}^2$ can be understood in terms of the restriction of the sequence (3) to E_i. Following the proof of Lemma 2.1, one has the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(1) \longrightarrow \mathcal{M}_{|E_i} \xrightarrow{j_{|E_i}} H^0(S, \mathcal{O}_S(L)) \otimes \mathcal{O}_{E_i} \xrightarrow{r_{|E_i}} \mathcal{O}_{E_i} \longrightarrow 0,$$

where $\mathrm{Im}(j_{|E_i}) = H^0(\mathcal{O}_S(L - E_i)) \otimes \mathcal{O}_{E_i}$. Now $j_{|E_i}$ induces a rational map

$$\epsilon_{|E_i \times \mathbf{P}^2} : E_i \times \mathbf{P}^2 \dashrightarrow \mathbf{P}(\mathcal{M}_{|E_i}) \subset \mathbf{P}.$$

For a point $x \in E_i$, the restriction of ϵ to $\mathbf{P}^2 \times \{x\}$ is the projection $\{x\} \times \mathbf{P}^2 \to \mathbf{P}^1$ of center (x, u_i). This implies that:

Lemma 2.2. *The birational map ϵ contracts $E_i \times \mathbf{P}^2$ to a surface which is a copy of $\mathbf{P}^1 \times \mathbf{P}^1$. Furthermore, the indeterminacy scheme of ϵ is equal to $\bigcup_{i=1}^4 E_i \times \{u_i\}$.*

Let $D_i := E_i \times \{u_i\} \subset S \times \mathbf{P}^2$ and $D := D_1 + \cdots + D_4$. We consider the blow-up

$$\alpha : \widetilde{S \times \mathbf{P}^2} \to S \times \mathbf{P}^2$$

of $S \times \mathbf{P}^2$ along D, and the birational map

$$\epsilon_2 := \epsilon \circ \alpha : \widetilde{S \times \mathbf{P}^2} \to \mathbf{P}.$$

The restriction of ϵ_2 to the strict transform $\widetilde{E_i \times \mathbf{P}^2}$ of $E_i \times \mathbf{P}^2$ is a regular morphism, for $\epsilon_{|E_i \times \mathbf{P}^2}$ is defined by the linear system $\left|\mathcal{I}_{E_i \times \{u_i\}/S \times \mathbf{P}^2}(1,1)\right|$. This implies that ϵ_2 itself is a regular morphism:

Proposition 2.3. *The following commutative diagram solves the indeterminacy of ϵ:*

$$\begin{tikzcd} & \widetilde{S \times \boldsymbol{P}^2} \arrow[dl, "\alpha"'] \arrow[dr, "\epsilon_2"] & \\ S \times \boldsymbol{P}^2 \arrow[rr, dotted, "\epsilon"] & & \boldsymbol{P} \end{tikzcd}$$

In the sequel, it will be useful to consider the exact commutative diagram

$$\begin{tikzcd} H^0(S, \mathcal{M}) \arrow[r] \arrow[d] & H^0(\mathcal{O}_S(L)) \otimes H^0(\mathcal{O}_S(L)) \arrow[r] \arrow[d] & \bigoplus_{i=1}^4 H^0(\mathcal{O}_{E_i}(2L)) \arrow[d] \\ H^0\big(\mathcal{I}_{\{w_1 \dots w_4\}}(1,1)\big) \arrow[r] & H^0(\mathcal{O}_{\mathbf{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbf{P}^2}(1)) \arrow[r] & \bigoplus_{i=1}^4 H^0(\mathcal{O}_{w_i}(2)) \end{tikzcd}$$

where the vertical arrows are isomorphisms induced by $\sigma : S \to \mathbf{P}^2$. Starting from the left arrow, one can construct the commutative diagram

$$\begin{tikzcd} H^0(S,\mathcal{M}) \otimes \mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "j"] \\ H^0\big(\mathcal{I}_{\{w_1\ldots w_4\}}(1,1)\big) \otimes \mathcal{O}_S \arrow[r] & H^0(S,L)\otimes \mathcal{O}_S(L) \end{tikzcd}$$

Passing to evaluation maps, we obtain the morphism $h : \mathbf{P} \to \mathbf{P}^4$ and the rational map $h_D : S \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4$ defined by the space $\big(\sigma \times \mathrm{id}_{\mathbf{P}^2}\big)^* H^0\Big(\mathbf{P}^2 \times \mathbf{P}^2, \mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\Big)$.

The discussion above is summarized in the following commutative diagram:

$$\begin{tikzcd} & \widetilde{S \times \mathbf{P}^2} \arrow[dl, "\alpha"'] \arrow[dr, "\epsilon_2"] & \\ S \times \mathbf{P}^2 \arrow[rr, dotted, "\epsilon"] \arrow[dr, dotted, "h_D"'] & & \mathbf{P} \arrow[dl, "h"] \\ & \mathbf{P}^4 & \end{tikzcd}$$

We derive a few consequences. Let $\pi_1 : S \times \mathbf{P}^2 \to S$ and $\pi_2 : S \times \mathbf{P}^2 \to \mathbf{P}^2$ be the two projections, then define the following effective divisors of $\widetilde{S \times \mathbf{P}^2}$:

$$\widetilde{H} \in \left|(\pi_1 \circ \alpha)^*(\mathcal{O}_S(-K_S))\right| \,,\ \widetilde{H}_1 \in \left|(\pi_1 \circ \alpha)^*(\mathcal{O}_S(L))\right| \,,\ \widetilde{H}_2 \in \left|(\pi_2 \circ \alpha)^*(\mathcal{O}_{\mathbf{P}^2}(1))\right|,$$

as well as

$$\widetilde{N}_i := \alpha^{-1}(D_i) \ \text{ and } \ \widetilde{N} = \sum_{i=1}^4 \widetilde{N}_i.$$

Applying push-forward under ϵ_2, we obtain the following divisors on $\mathbf{P}$:

$$H := \epsilon_{2*}(\widetilde{H}),\ \ H_i := \epsilon_{2*}(\widetilde{H}_i),\ \ N_i := \epsilon_{2*}(\widetilde{N}_i),\ \text{and} \ \ N := \sum_{i=1}^4 N_i.$$

Proposition 2.4. $|\mathcal{O}_{\boldsymbol{P}}(1)| = |H_1 + H_2 - N|$.

Proof. Using for instance [Ma] Theorem 1.4, we have $\epsilon_2^*(\mathcal{O}_{\mathbf{P}}(1)) = \mathcal{O}_{\widetilde{S\times\mathbf{P}^2}}(\widetilde{H}_1 + \widetilde{H}_2 - \widetilde{N})$. By pushing forward, we obtain the desired result. $\square$

We have already remarked that $h : \mathbf{P} \to \mathbf{P}^4$ is a morphism of degree 2. The inverse image $E \subset \mathbf{P}$ under h of a general line in $\mathbf{P}^4$ is a smooth elliptic curve. The restriction h_E has 4 branch points and the branch locus of h is a quartic hypersurface $B \subset \mathbf{P}^4$.

Proposition 2.5. *For each $d \geq 0$, one has $h^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(d)) = \binom{d+4}{4} + \binom{2d}{4}$.*

Proof. We pass to the Stein factorization $h := s \circ f$, where $f : \overline{\mathbf{P}} \to \mathbf{P}^4$ is a double cover and $s : \mathbf{P} \to \overline{\mathbf{P}}$ is birational. In particular, $h^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d)) = h^0(f^*\mathcal{O}_{\mathbf{P}^4}(d))$. The involution $\iota : \overline{\mathbf{P}} \to \overline{\mathbf{P}}$ induced by f acts on $H^0(f^*\mathcal{O}_{\mathbf{P}^4}(d))$ and the eigenspaces are $f^*H^0(\mathcal{O}_{\mathbf{P}^4}(d))$ and $b \cdot f^*H^0(\mathcal{O}_{\mathbf{P}^4}(2d-4))$ respectively, where $b \in H^0(f^*\mathcal{O}_{\mathbf{P}^4}(2))$ and $\mathrm{div}(b) = f^{-1}(B)$. $\square$

We can now relate the 15-dimensional linear system $|\mathcal{O}_{\mathbf{P}}(2)|$ of *smooth* conic bundles in $\mathbf{P}$ to the linear system of 4-*nodal* conic bundles of type $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. Let $\tilde{I}$ be the moving part of the total transform $\big((\sigma \times \mathrm{id}_{\mathbf{P}^2}) \circ \alpha\big)^* \Big|\mathcal{I}^2_{\{w_1,\dots,w_4\}}(2,2)\Big|$. Over $\mathbf{P}$, we consider the linear system $I' := (\epsilon_2)_* \tilde{I}$, and conclude that:

Proposition 2.6. *One has the equality $I' = |\mathcal{O}_{\mathbf{P}}(2)|$ of linear systems on $\mathbf{P}$.*

Proof. Consider a general threefold $Y \in \big|\mathcal{I}_{\{w_1,\dots,w_4\}}(1,1)\big|$. Its strict transform $\widetilde{Y}$ under the morphism $(\sigma \times \mathrm{id}_{\mathbf{P}^2}) \circ \alpha$ is smooth and has class $\widetilde{H}_1 + \widetilde{H}_2 - \widetilde{N}$. Therefore we obtain $(\epsilon_2)_*(\widetilde{Y}) \in |H_1 + H_2 - N| = |\mathcal{O}_{\mathbf{P}}(1)|$, and then $I' = |\mathcal{O}_{\mathbf{P}}(2)|$. □

To conclude this discussion, the identification

$$\big|\mathcal{O}_{\mathbf{P}}(2)\big| = \Big|\mathcal{I}^2_{\{w_1,\dots w_2\}}(2,2)\Big| := \mathbf{P}^{15},$$

induced by the birational map ϵ, will be used throughout the rest of the paper.

Remark 2.7. One can describe $h : \mathbf{P} \to \mathbf{P}^4$ in geometric terms. Consider the rational map $h' := h_D \circ \big(\sigma \times \mathrm{id}_{\mathbf{P}^2}\big)^{-1} : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4$, where h_D appears in a previous diagram. Then h' is defined by the linear system $\big|\mathcal{I}_{\{w_1,\dots,w_4\}}(1,1)\big|$. If $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ is the Segre embedding and $\Lambda \subset \mathbf{P}^8$ the linear span of $w_1, \dots, w_4$, then h' is the restriction to $\mathbf{P}^2 \times \mathbf{P}^2$ of the linear projection having center Λ.

One can also recover the tautological bundle $\mathcal{M}$ as follows. Consider the family of planes $\Big\{\mathbf{P}_x := h'_*\big(\{x\} \times \mathbf{P}^2\big)\Big\}_{x \in \mathbf{P}^2}$. Its closure in the Grassmannian $\mathbf{G}(2,4)$ of planes of $\mathbf{P}^4$ is equal to the image of S under the classifying map of $\mathcal{M}$. We omit further details.

Proposition 2.8. *The following relations hold in $CH^4(\widetilde{S \times \mathbf{P}^2})$:*

$$\widetilde{N}^4 = -4,\ \widetilde{N}^3 \cdot \widetilde{H} = 4,\ \widetilde{N}^3 \cdot \widetilde{H}_1 = \widetilde{N}^3 \cdot \widetilde{H}_2 = 0,\ \widetilde{N}^2 \cdot \widetilde{H}^2 = \widetilde{N}^2 \cdot \widetilde{H}_1^2 = \widetilde{N}^2 \cdot \widetilde{H}_2^2 = 0.$$

Proof. These are standard calculations on blow-ups. We fix $i \in \{1, \dots, 4\}$ and note that $\widetilde{N}_i = \mathbf{P}\big(\mathcal{O}_{D_i}^{\oplus 2} \oplus \mathcal{O}_{D_i}(1)\big)$. We denote by $\xi_i := c_1(\mathcal{O}_{\widetilde{N}_i}(1)) \in CH^1(\widetilde{N}_i)$ the class of the tautological bundle on the exceptional divisor, by $\alpha_i := \alpha_{|\widetilde{N}_i} : \widetilde{N}_i \to D_i$ the restriction of α, and by $j_i : \widetilde{N}_i \hookrightarrow \widetilde{S \times \mathbf{P}^2}$ the inclusion map. Then for $k = 1, \dots, 4$, the formula $\widetilde{N}_i^k = (-1)^{k-1}(j_i)_*(\xi_i^{k-1})$ holds in $CH^k(\widetilde{S \times \mathbf{P}^2})$. In particular,

$$\widetilde{N}_i^4 = -(j_i)_*(\xi_i^3) = -c_1(\mathcal{O}_{D_i}^{\oplus 2} \oplus \mathcal{O}_{D_i}(1)) = -1,$$

which implies that $\widetilde{N}^4 = \widetilde{N}_1^4 + \cdots + \widetilde{N}_4^4 = -4$. Furthermore, based on dimension reasons, $\widetilde{N}_i^2 \cdot \alpha^*(\gamma) = -(j_i)_*\big(\xi_i \cdot \alpha_i^*(\gamma_{|D_i})\big) = 0$, for each class $\gamma \in CH^2(S \times \mathbf{P}^2)$. Finally, for a class $\gamma \in CH^1(S \times \mathbf{P}^2)$, we have that $\widetilde{N_i}^3 \cdot \alpha^*(\gamma) = (j_i)_*\big(\xi_i^2 \cdot \alpha_i^*(\gamma_{|D_i})\big) = (\alpha_i)_*(\xi_i^2) \cdot \gamma_{|D_i} = \gamma \cdot D_i$, where the last intersection product is computed on $S \times \mathbf{P}^2$. This determines all top intersection numbers involving $\widetilde{N}^3$, which finishes the proof.

□

Remark 2.9. Since ϵ_2 contracts the divisors $\widetilde{E_i \times \mathbf{P}^2}$, clearly $H = 3H_1 - N$. An immediate consequence of Proposition 2.8 is that the degree of the morphism $h : \mathbf{P} \to \mathbf{P}^4$ equals $\deg(h) = (H_1 + H_2 - N)^4 = 6H_1^2 \cdot H_2^2 + N^4 = 2$.

2.1. **Pencils of conic bundles in the projective bundle P.** In this section we determine the numerical characters of a pencil of 4-nodal conic bundle of type $(2,2)$. Let

$$P \subset \left|\mathcal{O}_{\mathbf{P}}(2)\right| = \left|\mathcal{O}_{\mathbf{P}}(2H_1 + 2H_2 - 2N)\right|$$

be a Lefschetz pencil in **P**. We may assume that its base locus $B \subset \mathbf{P}$ is a smooth surface. We are primarily interested in the number of singular conic bundles and those having a double line respectively. We first describe B.

Lemma 2.10. *For the base surface $B \subset \mathbf{P}$ of a pencil of conic bundles, the following hold:*
 (i) $K_B = \mathcal{O}_B(H_1 + H_2 - N) \in \mathrm{Pic}(B)$.
 (ii) $K_B^2 = 8$ *and* $c_2(B) = 64$.

Proof. The surface B is a complete intersection in **P**, hence by adjunction

$$K_B = K_{\mathbf{P}|B} \otimes \mathcal{O}_B(4H_1 + 4H_2 - 4N).$$

Furthermore, $K_{\widetilde{S\times \mathbf{P}^2}} = \alpha^*\bigl(\mathcal{O}_S(-H) \boxtimes \mathcal{O}_{\mathbf{P}^2}(-3)\bigr) \otimes \mathcal{O}_{\widetilde{S\times \mathbf{P}^2}}(2\widetilde{N})$, and by push-pull

$$K_{\mathbf{P}} = (\epsilon_2)_*\bigl(K_{\widetilde{S\times \mathbf{P}^2}}\bigr) = \mathcal{O}_{\mathbf{P}}(-H - 3H_2 + 2N) = \mathcal{O}_{\mathbf{P}}(-3H_1 - 3H_2 + 3N),$$

for $H = 3H_1 - N$. We find that $K_B = \mathcal{O}_B(H_1 + H_2 - N)$. From Lemma 2.8, we compute

$$K_B^2 = 4(H_1 + H_2 - N)^2 \cdot (H_1 + H_2 - N)^2 = 24H_1^2 \cdot H_2^2 + 4N^4 = 8.$$

Finally, from the Euler formula applied for B, we obtain $12\chi(B, \mathcal{O}_B) = K_B^2 + c_2(B)$. Since $\chi(B, \mathcal{O}_B) = 6$, this yields $c_2(B) = 64$. □

For a variety Z we denote as usual by $e(Z)$ its topological Euler characteristic.

Lemma 2.11. *For a general conic bundle $Q \in \left|\mathcal{O}_{\mathbf{P}}(2)\right|$, we have that $e(Q) = 4$, whereas for conic bundle Q_0 with a single ordinary quadratic singularity, $e(Q_0) = 5$.*

Proof. We fix a conic bundle $\pi_1 : Q \to S$ with smooth discriminant curve $C \in |-2K_S|$. We then write the relation $e(Q - \pi_1^*(C)) = 2e(S - C)$. Since $e(\pi_1^*(C)) = 3e(C)$, we find that $e(Q) = 2e(S) + e(C) = 2\cdot 7 - 10 = 4$.

Similarly, if $\pi_1 : Q_0 \to \mathbf{P}^2$ is a conic bundle such that the discriminant curve $C_0 \subset S$ has a unique node, then $e(Q_0) = 2e(S) + e(C_0) = 14 - 9 = 5$. □

In the next statement we use the notation from [FL] for divisors classes on $\overline{\mathcal{R}}_g$, see also the beginning of Section 3 for further details.

Theorem 2.12. *In a Lefschetz pencil of conic bundles $P \subset |\mathcal{O}_{\mathbf{P}}(2)|$ there are precisely* 77 *singular conic bundles and* 32 *conic bundles with a double line.*

Proof. Retaining the notation from above, $B \subset \mathbf{P}$ is the base surface of the pencil. The number δ of nodal conic bundles in P is given by the formula:

$$\delta = e(\mathbf{P}) + e(B) - 2e(Q) = 3e(S) + 64 - 2\cdot 4 = 77,$$

where the relation $e(\mathbf{P}) = 3e(S)$ follows because $\pi : \mathbf{P} \to S$ is a $\mathbf{P}^2$-bundle.

The number of conic bundles in the pencil P having a double line equals the number of discriminant curves in the family induced by P in $\overline{\mathcal{R}}_6$, that lie in the ramification divisor Δ_0^{ram} of the projection map $\pi : \overline{\mathcal{R}}_6 \to \overline{\mathcal{M}}_6$. We choose general conic bundles $Q_1, Q_2 \in P$, and let $A = \bigl(a_{ij}(x_1, x_2, x_3)\bigr)_{i,j=1}^3$ and $B = \bigl(b_{ij}(x_1, x_2, x_3)\bigr)_{i,j=1}^3$ be the symmetric matrices of quadratic forms giving rise to Prym curves $(\Gamma_1, \eta_1) := \mathfrak{d}(Q_1)$ and

$(\Gamma_2, \eta_2) := \mathfrak{d}(Q_2) \in \mathcal{P}_6$ respectively. Note that both curves Γ_1 and Γ_2 are nodal precisely at the points $u_1, \ldots, u_4$. Let us consider the surface

$$Y := \Big\{ \Big([x_1 : x_2 : x_3], [t_1 : t_2] \Big) \in \mathbf{P}^2 \times \mathbf{P}^1 : \det\big((t_1 a_{ij} + t_2 b_{ij})(x_1, x_2, x_3) \big) = 0 \Big\},$$

together with the projection $\gamma : Y \to \mathbf{P}^1$. If $h_1, h_2 \in CH^1(\mathbf{P}^2 \times \mathbf{P}^1)$ are the pull-backs of the hyperplane classes under the two projections, then $Y \equiv 6h_1 + 3h_2$. Therefore $\omega_Y = \mathcal{O}_Y(3h_1 + h_2)$ and $h^0(Y, \mathcal{O}_Y) = 20$. Observe that the surface Y is singular along the curves $L_j := \{u_j\} \times \mathbf{P}^1$ for $j = 1, \ldots, 4$, and let $\nu_Y : \mathcal{Y} \to Y$ be the normalization. From the exact sequence

$$0 \longrightarrow H^0(\mathcal{Y}, \omega_{\mathcal{Y}}) \longrightarrow H^0(Y, \omega_Y) \longrightarrow \bigoplus_{j=1}^{4} H^0(L_j, \omega_{Y|L_j}) \longrightarrow 0,$$

taking also into account that $\omega_{Y|L_j} = \mathcal{O}_{L_j}(1)$, we compute that $h^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = 12$, and hence $\chi(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = 13$. The morphism $\tilde{\gamma} := \gamma \circ \nu_Y : \mathcal{Y} \to \mathbf{P}^1$ is a family of Prym curves of genus 6 and it induces a moduli map $m(\tilde{\gamma}) : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$.

The points $u_1, \ldots, u_4 \in \mathbf{P}^2$ being general, the curve $\mathfrak{e} := m(\tilde{\gamma})(\mathbf{P}^1) \subset \overline{\mathcal{R}}_6$ is disjoint from the pull-back $\pi^*(\overline{\mathcal{GP}}_6) \subset \overline{\mathcal{R}}_6$ of the Gieseker-Petri divisor consisting of curves of genus 6 lying on a singular quintic del Pezzo surface, see [FGSMV] for details on the geometry of $\pi^{-1}(\overline{\mathcal{GP}}_6)$. Since $\pi^*\big([\overline{\mathcal{GP}}_6]\big)_{|\widetilde{\mathcal{R}}_6} = 94\lambda - 12(\delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}}) \in CH^1(\widetilde{\mathcal{R}}_6)$, and $\mathfrak{e} \cdot \delta_0^{\mathrm{ram}} = 77$ (this being the already computed number of nodal conic bundles in P), whereas $\mathfrak{e} \cdot \delta_0^{'} = 0$, we obtain the following relation

$$47\mathfrak{e} \cdot \lambda - 6\mathfrak{e} \cdot \delta_0^{'} - 12\mathfrak{e} \cdot \delta_0^{\mathrm{ram}} = 0.$$

Finally, we observe that $\mathfrak{e} \cdot \lambda = \chi(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) + g - 1 = 18$, which leads to $\mathfrak{e} \cdot \delta_0^{\mathrm{ram}} = 32$. $\square$

3. A SWEEPING RATIONAL CURVE IN THE BOUNDARY OF $\overline{\mathcal{A}}_6$

In this section we construct an explicit sweeping rational curve in $\widetilde{\mathcal{C}}^5$, whose numerical properties we shall use in order to bound the slope of $\overline{\mathcal{A}}_6$. Before doing that, we quickly review basic facts concerning the moduli space $\overline{\mathcal{R}}_g$ of stable Prym curves of genus g, while referring to [FL] for details.

Geometric points of $\overline{\mathcal{R}}_g$ correspond to triples (X, η, β), where X is a quasi-stable curve of arithmetic genus g, η is a line bundle on X of degree 0, such that $\eta_E = \mathcal{O}_E(1)$ for each smooth rational component $E \subset X$ with $|E \cap \overline{(X - E)}| = 2$, and $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component of X is an isomorphism. Denoting by $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ the forgetful map, one has the following formula [FL] Example 1.4

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}} \in CH^1(\overline{\mathcal{R}}_g), \tag{5}$$

where $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$, and $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$ are boundary divisor classes on $\overline{\mathcal{R}}_g$ whose meaning we recall. Let us fix a general point $[C] \in \Delta_0$ corresponding to a smooth 2-pointed curve (N, x, y) of genus $g - 1$ with normalization map $\nu : N \to C$, where $\nu(x) = \nu(y)$. A general point of $\Delta_0^{'}$ (respectively of $\Delta_0^{''}$) corresponds to a stable Prym curve $[C, \eta]$, where $\eta \in \mathrm{Pic}^0(C)[2]$ and $\nu^*(\eta) \in \mathrm{Pic}^0(N)$ is non-trivial (respectively, $\nu^*(\eta) = \mathcal{O}_N$). A general point of Δ_0^{ram} is of the form (X, η), where $X := N \cup_{\{x,y\}} \mathbf{P}^1$

is a quasi-stable curve of arithmetic genus g, whereas $\eta \in \mathrm{Pic}^0(X)$ is a line bundle characterized by $\eta_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_N^{\otimes 2} = \mathcal{O}_N(-x-y)$. Throughout this paper, we only work on the partial compactification $\widetilde{\mathcal{R}}_g := \pi^{-1}(\mathcal{M}_g \cup \Delta_0)$ of $\mathcal{R}_g$ and ignore the boundary divisors of $\overline{\mathcal{R}}_g$ corresponding to curves of compact type. We denote by δ_0', δ_0'' and δ_0^{ram} the restrictions of the corresponding boundary classes to $\widetilde{\mathcal{R}}_g$. Note that $CH^1(\widetilde{\mathcal{R}}_g) = \mathbf{Q}\langle \lambda, \delta_0', \delta_0'', \delta_0^{\mathrm{ram}}\rangle$.

Recall that we use the identification $\mathbf{P}^{15} := \bigl|\mathcal{I}^2_{\{w_1,\ldots,w_4\}}(2,2)\bigr| = |\mathcal{O}_{\mathbf{P}}(2)|$ for the linear system of $(2,2)$ threefolds in $\mathbf{P}^2 \times \mathbf{P}^2$ which are nodal at $w_1, \ldots, w_4$. Recall also that $\mathbf{P} \to S$ is the $\mathbf{P}^2$-bundle constructed in section 2.

We start constructing a sweeping curve $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$, by fixing general elements $(o_1, \ell_1), \ldots, (o_4, \ell_4) \in \mathbf{P}^2 \times (\mathbf{P}^2)^\vee$ and a general point $o \in \mathbf{P}^2$. We introduce the net

$$T := \Bigl\{Q \in \mathbf{P}^{15} : (o,o) \in Q \ \text{ and } \{o_j\} \times \ell_i \subset Q \text{ for } j = 1, \ldots, 4\Bigr\},$$

consisting of conic bundles containing the lines $\{o_1\} \times \ell_1, \ldots, \{o_4\} \times \ell_4$ and passing through the point $(o,o) \in \mathbf{P}^2 \times \mathbf{P}^2$. Because of the genericity of our choices, the restriction

$$\mathrm{res}_{|\{o\}\times \mathbf{P}^2} : T \to \bigl|\mathcal{O}_{\{o\}\times\mathbf{P}^2}(2)\bigr|$$

is an injective map and we can view T as a general net of conics in $\mathbf{P}^2$ passing through the fixed point $o \in \mathbf{P}^2$. The discriminant curve of the net is a nodal cubic curve $\Delta_T \subset T$; its singularity corresponds to the only conic of type $\ell_0 + m_0 \in T$, consisting of a pair of lines ℓ_0 and m_0 passing through o.

To ease notation, we identify $\{o\} \times \mathbf{P}^2$ with $\mathbf{P}^2$ in everything that follows. Denoting by $\mathbf{P}^1 := \mathbf{P}(T_o(\mathbf{P}^2))$ the pencil of lines through o, it is clear that the map

$$\tau : \mathbf{P}^1 \to \Delta_T, \quad \tau(\ell) := Q_\ell \in T, \text{ such that } Q_\ell \supset \{o\} \times \ell,$$

is the normalization map of Δ_T. In particular, we have $\tau(\ell_0) = \tau(m_0) = \ell_0 + m_0$, where, abusing notation, we identify Q_ℓ with its singular conic $\{o\} \times (\ell + m) = Q_\ell \cdot \bigl(\{o\} \times \mathbf{P}^2\bigr)$. For $\ell \in \mathbf{P}^1$, the double cover $f_\ell : \widetilde{\Gamma}_\ell \to \Gamma_\ell$ over the discriminant curve Γ_ℓ of Q_ℓ is an element of $\mathcal{P}_6$ (see Definition 1.1). Clearly $\widetilde{\Gamma}_\ell$ carries the marked points $\ell_1, \ell_2, \ell_3, \ell_4$ and ℓ. This procedure induces a moduli map into the universal symmetric product

$$i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5, \quad i(\ell) := \bigl[\rho(\widetilde{\Gamma}_\ell/\Gamma_\ell),\ \ell_1,\ \ell_2,\ \ell_3,\ \ell_4,\ \ell\bigr].$$

We explicitly construct the family of discriminant curves Γ_ℓ of the conic bundles Q_ℓ, where $\tau(\ell) \in \Delta_T$. Setting coordinates $x := [x_1 : x_2 : x_3], y := [y_1 : y_2 : y_3]$ in $\mathbf{P}^2$, let

$$Z := \Bigl\{(x,y,t) \in \mathbf{P}^2 \times \mathbf{P}^2 \times \Delta_T : y \in \mathrm{Sing}\bigl(\pi_1^{-1}(x) \cap Q_t\bigr)\Bigr\} \subset \mathbf{P}^2 \times \mathbf{P}^2 \times T.$$

Concretely, if Q_1, Q_2, Q_3 is a basis of T, then the surface Z is given by the equations

$$\frac{\partial}{\partial y_i}\Bigl(t_1 Q_1(x,y) + t_2 Q_2(x,y) + t_3 Q_3(x,y)\Bigr) = 0, \quad \text{for } i = 1,2,3,$$

where $[t_1 : t_2 : t_3] \in T$. It follows immediately that Z is a complete intersection of three divisors of multidegree $(2,1,1)$, defined by the partial derivatives, and the divisor $\mathbf{P}^2 \times \mathbf{P}^2 \times \Delta_T$ of multidegree $(0,0,3)$.

Lemma 3.1. *The first projection $\gamma_1 : Z \to \mathbf{P}^2$ is a map of degree* 9.

Proof. Denoting by $h_1, h_2, h_3 \in \mathrm{Pic}(\mathbf{P}^2 \times \mathbf{P}^2 \times T)$ the pull-backs of the hyperplane bundles from the three factors, we find that $\deg(\gamma_1) = (2h_1 + h_2 + h_3)^3 \cdot (3h_3) \cdot h_1^2 = 9$. $\square$

The third projection $\gamma_3 : Z \to \Delta_T$ is a birational model of the family $\{C_\ell\}_{\ell \in \mathbf{P}^1}$ of underlying genus 6 curves, induced by the map $i : \mathbf{P}^1 \to \widetilde{\mathcal{C}}^5$. However, the surface Z is not normal. It has singularities along the curves $\{w_j\} \times \Delta_T$ for $j = 1, \dots, 4$, as well as along the fibre $\gamma_3^{-1}(\ell_0 + m_0)$ over the point $\tau(\ell_0) = \tau(m_0) = \ell_0 + m_0 \in T$. To construct a smooth model of Z, we pass instead to its natural birational model in the 5-fold $\mathbf{P} \times \mathbf{P}^1$.

Abusing notation, we still denote by $Q_\ell \subset \mathbf{P}$ the strict transform of the conic bundle Q_ℓ in $\mathbf{P}^2 \times \mathbf{P}^2$; its discriminant curve C_ℓ is viewed an an element of $|-K_S|$. We denote by $\pi_\ell : Q_\ell \to S$ the restriction of $\pi : \mathbf{P} \to S$, then consider the surface

$$\mathcal{Z} := \Big\{(z, \ell) \in \mathbf{P} \times \mathbf{P}^1 : z \in \mathrm{Sing}\ \pi_\ell^{-1}(C_\ell)\Big\}.$$

Clearly $\mathcal{Z}$ is endowed with the projection $q_{\mathbf{P}^1} : \mathcal{Z} \to \mathbf{P}^1$. We have the following commutative diagram, where $u := \big(\sigma \times \mathrm{id}_{\mathbf{P}^2}\big) \circ \epsilon^{-1} : \mathbf{P} \to \mathbf{P}^2 \times \mathbf{P}^2$ and the horizontal arrows are the natural inclusions or projections and $\nu_Z : \mathcal{Z} \to Z$ is the normalization map:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d, "\nu_Z"'] \arrow[rr, bend left, "q_{\mathbf{P}^1}"] & \mathbf{P} \times \mathbf{P}^1 \arrow[r] \arrow[d, "u \times \tau"] & \mathbf{P}^1 \arrow[d, "\tau"'] \\
Z \arrow[r] \arrow[rr, bend right, "\gamma_3"'] & \mathbf{P}^2 \times \mathbf{P}^2 \times \Delta_T \arrow[r] & \Delta_T
\end{tikzcd}$$

Since $u \times \tau$ is birational, it follows that $\deg(\mathcal{Z}/S) = \deg(Z/\mathbf{P}^2) = 9$. The fibration $q_{\mathbf{P}^1} : \mathcal{Z} \to \mathbf{P}^1$ admits sections

$$\tau_j : \mathbf{P}^1 \to \mathcal{Z} \ \text{ for } j = 1, \dots, 5,$$

which we now define. For $1 \le j \le 4$ and each $\ell \in \mathbf{P}^1$, the fibre $Q_\ell \cdot \big(\{o_j\} \times \mathbf{P}^2\big)$ contains the line ℓ_j. Hence ℓ_j defines a point in the covering curve of $f_\ell : \widetilde{C}_\ell \to C_\ell$. By definition $\tau_j(\ell)$ is this point. Tautologically, $\tau_5(\ell)$ is the point corresponding to the line ℓ.

Finally, we consider the universal family $\mathcal{Q} \subset \mathbf{P} \times T$ defined by T. The pull-back of the projection $\mathcal{Q} \to T$ by the morphism $\mathrm{id}_{\mathbf{P}} \times \tau$ induces a flat family of conic bundles $\mathcal{Q}' \subset \mathbf{P} \times \mathbf{P}^1$ and a projection $q' : \mathcal{Q}' \to \mathbf{P}^1$. Clearly, $\mathcal{Z} \subset \mathcal{Q}'$ and $q_{\mathbf{P}^1} = q'_{|\mathcal{Z}}$.

Definition 3.2. A conic bundle $Q \in |\mathcal{O}_{\mathbf{P}}(2)|$ is said to be *ordinary* if both Q and its discriminant cover curve C are nodal. A subvariety in $|\mathcal{O}_{\mathbf{P}}(2)|$ is said to be a *Lefschetz family*, if each of its members is an ordinary conic bundles.

Postponing the proof, we assume that the fibration $q' : \mathcal{Q} \to \mathbf{P}^1$ constructed above is a Lefschetz family of conic bundles, and we determine the properties of the Prym moduli map $m : \mathbf{P}^1 \to \overline{\mathcal{R}}_6$, where $m(\ell) := [f_\ell : \widetilde{C}_\ell \to C_\ell] = \rho\big(\widetilde{\Gamma}_\ell/\Gamma_\ell\big)$.

Proposition 3.3. *The numerical features of $m : \boldsymbol{P}^1 \to \overline{\mathcal{R}}_6$ are as follows:*

$$m(\boldsymbol{P}^1) \cdot \lambda = 9 \cdot 6,\ m(\boldsymbol{P}^1) \cdot \delta_0^{'} = 3 \cdot 77,\ m(\boldsymbol{P}^1) \cdot \delta_0^{\mathrm{ram}} = 3 \cdot 32,\ m(\boldsymbol{P}^1) \cdot \delta_0^{''} = 0.$$

Proof. We consider the composite map $\rho\circ\mathfrak{d}_{|T}: T \dashrightarrow \overline{\mathcal{R}}_6$, assigning to a conic bundle from the net $T\subset \mathbf{P}^{15}$ the double covering of its (normalized) discriminant curve. This map is well-defined outside the codimension two locus in T corresponding to conic bundles with non-nodal discriminant. Furthermore, $m=\rho\circ\mathfrak{d}\circ\tau:\mathbf{P}^1\to\overline{\mathcal{R}}_6$, where we recall that $\tau(\mathbf{P}^1)=\Delta_T\subset T$ is a nodal cubic curve. It follows that the intersection number of $m(\mathbf{P}^1)\subset\overline{\mathcal{R}}_6$ with any divisor class on $\overline{\mathcal{R}}_6$ is three times the intersection number of the corresponding class in $CH^1(\overline{\mathcal{R}}_6)$ with the curve of discriminants induced by a *pencil* of conic bundles in $|\mathcal{O}_{\mathbf{P}}(2)|$. The latter numbers have been determined in Theorem 2.12. $\square$

The composition of the map $i:\mathbf{P}^1\to\widetilde{\mathcal{C}}^5$ with the projection $\widetilde{\mathcal{C}}^5\to\overline{\mathcal{R}}_6$ is the map $m:\mathbf{P}^1\to\overline{\mathcal{R}}_6$ discussed in Proposition 3.3. We discuss the numerical properties of i:

Proposition 3.4. *The moduli map $i:\boldsymbol{P}^1\to\widetilde{\mathcal{C}}^5$ induced by the pointed family of Prym curves*
$$\big(q_{\boldsymbol{P}^1}:\mathcal{Z}\to\boldsymbol{P}^1,\tau_1,\ldots,\tau_5:\boldsymbol{P}^1\to\mathcal{Z}\big)$$
sweeps the five-fold product $\widetilde{\mathcal{C}}^5$. Furthermore $i(\boldsymbol{P}^1)\cdot\psi_{x_j}=9$, for $j=1,\ldots,5$.

Proof. For $1\le j\le 4$, the image of the section $\tilde{\tau}_j:=\nu_Z\circ\tau_j:\mathbf{P}^1\to Z$ is the curve
$$L_j:=\big\{(o_j,\ y_j(\ell),\ \nu(\ell))\in\mathbf{P}^2\times\mathbf{P}^2\times T:\ell\in\mathbf{P}^1\big\},$$
where $y_j(\ell)=\ell_j\cap m_j(\ell)$, with $m_j(\ell)$ being the line in $\mathbf{P}^2$ defined by the equality of cycles $Q_\ell\cdot\big(\{o_j\}\times\mathbf{P}^2\big)=\{o_j\}\times(\ell_j+m_j(\ell))$. Here, recall that $\ell\in\mathbf{P}^1=\mathbf{P}(T_o(\mathbf{P}^2)$ is a point corresponding to a line in $\mathbf{P}^2$ passing through o. In particular, noting that by the adjunction formula $\omega_Z=\mathcal{O}_Z(3h_1+3h_3)$, we compute that $L_j\cdot h_1=0$ and $L_j\cdot h_3=3$, hence $L_j\cdot\omega_Z=L_j\cdot(3h_1+3h_3)=9$.

By definition $i(\mathbf{P}^1)\cdot\psi_{x_j}=\tau_j^*\big(c_1(\omega_{q_{\mathbf{P}^1}})\big)$. To evaluate the dualizing class, we note that $\omega_{q_{\mathbf{P}^1}}=\omega_{\mathcal{Z}}\otimes q_{\mathbf{P}^1}^*(T_{\mathbf{P}^1})$, therefore $\deg\tau_j^*\big(c_1(\omega_{q_{\mathbf{P}^1}})\big)=\deg\tau_j^*(\omega_{\mathcal{Z}})+2$. Furthermore,
$$\nu_Z^*(\omega_Z)=\omega_{\mathcal{Z}}\otimes\mathcal{O}_{\mathcal{Z}}\big(q_{\mathbf{P}^1}^{-1}(\ell_0)+q_{\mathbf{P}^1}^{-1}(m_0)+D\big),$$
where $D\subset\mathcal{Z}$ is a curve disjoint from $\nu_Z^{-1}(L_j)$. We compute that
$$\deg\tau_j^*(\omega_{\mathcal{Z}})=\deg\tilde{\tau}_j^*(\omega_Z)-\deg\tau_j^*q_{\mathbf{P}^1}^*(\ell_0)-\deg\tau_j^*q_{\mathbf{P}^1}^*(m_0)=\omega_Z\cdot L_j-2,$$
and finally, $i(\mathbf{P}_1)\cdot\psi_{x_j}=\omega_Z\cdot L_j=9$. The calculation of $i(\mathbf{P}^1)\cdot\psi_{x_5}$ is largely similar and we skip it. $\square$

Proof of the claim. We show that $q':\mathcal{Q}\to\mathbf{P}^1$ is a Lefschetz family, that is, it consists entirely of ordinary conic bundles. For $1\le j\le 4$, let $\ell_j'\subset\mathbf{P}$ be the inverse image of the line $\{o_j\}\times\ell_j$ under the map $u:\mathbf{P}\dashrightarrow\mathbf{P}^2\times\mathbf{P}^2$ and set $W:=\big|\mathcal{I}_{\{\ell_1',\ldots,\ell_4'\}}(2)\big|\subset|\mathcal{O}_{\mathbf{P}}(2)|$. The net $T:=T_o$ of conic bundles passing through the point $(o,o)\in\mathbf{P}^2\times\mathbf{P}^2$ is a plane in W. Let Δ_{no} denote the locus of non-ordinary conic bundles $Q\in W$. We aim to show that $\Delta_{\mathrm{no}}\cap\Delta_{T_o}=\emptyset$, for a general point $o\in\mathbf{P}^2$.

We consider the incidence correspondence
$$\Sigma:=\Big\{\big(Q,(o,\ell)\big)\in\Delta_{\mathrm{no}}\times\mathbf{G}:\{o\}\times\ell\subset u(Q),\ \ o\in\ell\Big\}$$
together with the projection map $p_1:\Sigma\to\Delta_{\mathrm{no}}$. Over a conic bundle $Q\in\Delta_{\mathrm{no}}$ for which the image $u(Q)\subset\mathbf{P}^2\times\mathbf{P}^2$ is transversal to a general fibre $\{o\}\times\mathbf{P}^2$, the fibre $p_1^{-1}(Q)$ is

finite. To account for the conic bundles not enjoying this property, we define $\Delta_{\rm hr}$ to be the union of the irreducible components of $\Delta_{\rm no}$ consisting of conic bundles $Q \in W$ such that the branch locus of $Q \to S$ is equal to S.

To conclude that $\Delta_{\rm no} \cap \Delta_{T_o} = \emptyset$ for a general $o \in \mathbf{P}^2$, it suffices to show that (1) $\Delta_{\rm no}$ has codimension at least 2 in W, and (2) $\Delta_{\rm hr}$ has codimension at least 3 in W. The next two lemmas are devoted to the proof of these assertions. $\square$

Lemma 3.5. $\Delta_{\rm no}$ *has codimension at least* 2 *in* W.

Proof. We have established that $h : \mathbf{P} \to \mathbf{P}^4$ is a morphism of degree two. We claim that the 4 lines $l_i := h(\ell_i') \subset \mathbf{P}^4$ are general, in the sense that $V := \left|\mathcal{I}_{\{l_1,\ldots,l_4\}}(2)\right|$ is a net of quadrics. Granting this and denoting by $L_{ij} \in H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(1))$ the linear form vanishing along $l_i \cup l_j$, the space V is generated by the quadrics $L_{12} \cdot L_{34}, L_{13} \cdot L_{24}$ and $L_{14} \cdot L_{23}$ respectively. The base locus bs $|V|$ of the net is a degenerate canonical curve of genus 5, which is a union of 8 lines, namely $l_1, \ldots, l_4$ and $b_1, \ldots, b_4$, where if $\{1,2,3,4\} = \{i,j,k,l\}$, then the line $b_l \subset \mathbf{P}^4$ is the common transversal to the lines l_i, l_j and l_k. Then by direct calculation, the pull-back P of a general pencil in V is a Lefschetz family of conic budles in $|\mathcal{O}_{\mathbf{P}}(2)|$. Since $P \cap \Delta_{\rm no} = \emptyset$, it follows that $\operatorname{codim}(\Delta_{\rm no}, W) \geq 2$. It remains to show that the lines $l_1, \ldots, l_4$ are general. To that end, we observe that the construction can be reversed. Four general lines $m_1, \ldots, m_4 \in \mathbf{G}(1,4) \subset \mathbf{P}^9$ define a codimension 4 linear section S' of $\mathbf{G}(1,4)$ which is isomorphic to S. The projectivized universal bundle $\mathbf{P}' \to S'$ is a copy of $\mathbf{P}$ and the projection $h' : \mathbf{P}' \to \mathbf{P}^4$ is the tautological map. This completes the proof. $\square$

The second lemma follows from a direct analysis in $\mathbf{P}^2 \times \mathbf{P}^2$.

Lemma 3.6. $\Delta_{\rm hr}$ *has codimension at least* 3 *in* W.

Proof. If Q is a general element of an irreducible component of $\Delta_{\rm hr}$, then the discriminant locus of the projection $p : Q \to S$ equals S, and necessarily $Q = D + D'$, where $p(D) = p(D') = S$. By a dimension count, it follows that W contains only *finitely many* elements $Q \in \Delta_{\rm hr}$, such that $D, D' \in |\mathcal{O}_{\mathbf{P}}(1)|$, and assume that we are not in this case.

Recall that $h' : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow \mathbf{P}^4$ is the map defined by $\left|\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\right|$. The case when both $u(D), u(D') \in |\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)|$ having been excluded, we may assume that one of the components of $u(Q)$, say $u(D) \subset \mathbf{P}^2 \times \mathbf{P}^2$, has type $(0,1)$. In particular, $u(D) = \mathbf{P}^2 \times n$, where $n \subset \mathbf{P}^2$ is a line. Observe that $u(D)$ has degree three in the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ and the base scheme of $\left|\mathcal{I}_{\{w_1,\ldots,w_4\}}(1,1)\right|$ consists of the simple points $w_1, \ldots, w_4$. Since $h'(D)$ lies on a quadric, it follows $u(D) \cap \{w_1, \ldots, w_4\} \neq \emptyset$, therefore we have $u_i \in n$ for some i, say $i = 4$. Since the lines $\{o_i\} \times \ell_i$ are general, they do not lie on $u(D)$, for $\ell_i \neq n$. Hence $u(D') \subset \mathbf{P}^2 \times \mathbf{P}^2$ is a $(2,1)$ hypersurface which contains $\{o_1\} \times \ell_1, \ldots \{o_4\} \times \ell_4$, is singular at w_1, w_2, w_3 and such that $w_4 \in u(D')$. This contradicts the generality of the lines $\{o_i\} \times \ell_i$. $\square$

4. THE SLOPE OF $\overline{\mathcal{A}}_6$

For $g \geq 2$, let $\overline{\mathcal{A}}_g$ be the first Voronoi compactification of $\mathcal{A}_g$ — this is the toroidal compactification of $\mathcal{A}_g$ constructed using the perfect fan decomposition, see [SB]. The rational Picard group of $\overline{\mathcal{A}}_g$ has rank 2 and it is generated by the first Chern class λ_1 of the Hodge bundle and the class of the *irreducible* boundary divisor $D = D_g := \overline{\mathcal{A}}_g - \mathcal{A}_g$. Following Mumford [Mu], we consider the moduli space $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties of dimension g together with their rank 1 degenerations. Precisely, if $\xi : \overline{\mathcal{A}}_g \to \mathcal{A}_g^s = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$ is the projection from the toroidal to the Satake compactification of $\mathcal{A}_g$, then

$$\widetilde{\mathcal{A}}_g := \overline{\mathcal{A}}_g - \xi^{-1}\Bigl(\bigcup_{j=2}^{g} \mathcal{A}_{g-j}\Bigr) := \mathcal{A}_g \sqcup \widetilde{D}_g,$$

where $\widetilde{D}_g$ is an open dense subvariety of D_g isomorphic to the universal Kummer variety $\mathrm{Kum}(\mathcal{X}_{g-1}) := \mathcal{X}_{g-1}/\pm$. Furthermore, if $\phi : \widetilde{\mathcal{X}}_{g-1} \to \widetilde{\mathcal{A}}_{g-1}$ is the extended universal abelian variety, there exists a degree two morphism $j : \widetilde{\mathcal{X}}_{g-1} \to \overline{\mathcal{A}}_g$, extending the Kummer map $\widetilde{\mathcal{X}}_{g-1} \stackrel{2:1}{\to} \widetilde{D}_g$. The geometry of the boundary divisor $\partial \widetilde{\mathcal{X}}_{g-1} = \phi^{-1}(\mathrm{Kum}(\mathcal{X}_{g-2}))$ is discussed in [vdG] and [EGH]. In particular, $\mathrm{codim}(D_g - j(\partial \widetilde{\mathcal{X}}_{g-1}), D_g) = 2$. As usual, let $\mathbb{E}_g$ be the Hodge bundle on $\overline{\mathcal{A}}_g$.

Denoting by $\varphi : \widetilde{\mathcal{Y}}_g \to \widetilde{\mathcal{R}}_g$ the universal Prym variety restricted to the partial compactification $\widetilde{\mathcal{R}}_g$ of $\overline{\mathcal{R}}_g$ introduced in Section 3, we have the following commutative diagram summarizing the situation, where the lower horizontal arrow is the Prym map:

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}_g \arrow[r, "\chi"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{X}}_{g-1} \arrow[r, "j"] \arrow[d, "\phi"'] & \overline{\mathcal{A}}_g \\
\widetilde{\mathcal{R}}_g \arrow[r, "P"] & \widetilde{\mathcal{A}}_{g-1} &
\end{tikzcd}$$

Furthermore, let us denote by $\theta \in CH^1(\widetilde{\mathcal{X}}_{g-1})$ the class of the universal theta divisor trivialized along the zero section and by $\theta_{\mathrm{pr}} := \chi^*(\theta) \in CH^1(\widetilde{\mathcal{Y}}_g)$ the Prym theta divisor. The following formulas have been pointed out to us by Sam Grushevsky:

Proposition 4.1. *The following relations at the level of divisor classes hold:*

(i) $j^*([D]) = -2\theta + \phi^*([D_{g-1}]) \in CH^1(\widetilde{\mathcal{X}}_{g-1})$.
(ii) $(j \circ \chi)^*(\lambda_1) = \varphi^*\bigl(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}\bigr) \in CH^1(\widetilde{\mathcal{Y}}_g)$.
(iii) $(j \circ \chi)^*([D]) = -2\theta_{\mathrm{pr}} + \varphi^*(\delta_0^{'}) \in CH^1(\widetilde{\mathcal{Y}}_g)$.

Proof. At the level of the restriction $j : \mathcal{X}_{g-1} \to \overline{\mathcal{A}}_g$, the formula

$$j^*(D) \equiv -2\theta \in CH^1(\mathcal{X}_{g-1})$$

is proven in [Mu] Proposition 1.8. To extend this calculation to $\widetilde{\mathcal{X}}_{g-1}$, it suffices to observe that the boundary divisor $\partial \widetilde{\mathcal{X}}_{g-1} = \phi^*(\widetilde{D}_{g-1})$ is mapped under j to the locus in $\overline{\mathcal{A}}_g$ parametrizing *rank* 2 degenerations and it will appear with multiplicity one in $j^*(D)$.

To establish relation (ii), we observe that $j^*(\lambda_1) = \phi^*(\lambda_1)$, where we use the same symbol to denote the Hodge class on $\mathcal{A}_g$ and that on $\mathcal{A}_{g-1}$. Indeed, there exists an exact sequence of vector bundles on $\widetilde{\mathcal{X}}_g$, see also [vdG] p.74:

$$0 \longrightarrow \phi^*(\mathbb{E}_{g-1}) \longrightarrow j^*(\mathbb{E}_g) \longrightarrow \mathcal{O}_{\widetilde{\mathcal{X}}_{g-1}} \longrightarrow 0.$$

It follows that $\chi^* j^*(\lambda_1) = \varphi^* P^*(\lambda_1) = \varphi^*(\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}})$, where $P^*(\lambda_1) = \lambda - \frac{1}{4}\delta_0^{\mathrm{ram}}$, see [FL], [GSM]. Finally, (iii) is a consequence of (i) and of the relation $P^*([\widetilde{D}_{g-1}]) = \delta_0^{'}$, see [GSM]. $\square$

Assume now that g is an even integer and let $\tilde{\pi} : \widetilde{\mathcal{C}} \to \overline{\mathcal{R}}_g$ be the universal curve of genus $2g-1$, that is, $\widetilde{\mathcal{C}} = \overline{\mathcal{M}}_{2g-1,1} \times_{\overline{\mathcal{M}}_{2g-1}} \overline{\mathcal{R}}_g$, and $\pi : \overline{\mathcal{C}} \to \overline{\mathcal{R}}_g$ the universal curve of genus g, that is, $\overline{\mathcal{C}} = \overline{\mathcal{M}}_{g,1} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{R}}_g$. There is a degree two map $f : \widetilde{\mathcal{C}} \to \overline{\mathcal{C}}$ unramified in codimension one and an involution $\iota : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$, such that $f \circ \iota = f$. Note that $\omega_{\tilde{\pi}} = f^*(\omega_\pi)$.

We consider the global Abel-Prym map $\mathfrak{ap} : \widetilde{\mathcal{C}}^{g-1} \dashrightarrow \widetilde{\mathcal{Y}}_g$, defined by

$$\mathfrak{ap}\Big(\widetilde{C}/C,\, x_1, \ldots, x_{g-1}\Big) := \Big(\widetilde{C}/C,\, \mathcal{O}_{\widetilde{C}}\big(x_1 - \iota(x_1) + \cdots + x_{g-2} - \iota(x_{g-2}) + 2x_{g-1} - 2\iota(x_{g-1})\big)\Big).$$

Remark 4.2. We recall that if $\widetilde{C} \to C$ is an étale double cover and $\iota : \widetilde{C} \to \widetilde{C}$ the induced involution, then the Prym variety $P(\widetilde{C}/C) \subset \mathrm{Pic}^0(\widetilde{C})$ can be realized as the locus of line bundles $\mathcal{O}_{\widetilde{C}}(E - \iota(E))$, where E is a divisor on $\widetilde{C}$ having *even* degree, see [B3]. Furthermore, for a general point $[\widetilde{C} \to C] \in \overline{\mathcal{R}}_g$, where $g \geq 3$, and for an integer $1 \leq n \leq g-1$, the difference map $\widetilde{C}_n \to \mathrm{Pic}^0(\widetilde{C})$ given by $E \mapsto \mathcal{O}_{\widetilde{C}}(E - \iota(E))$ is generically finite. In particular, for even g, the locus

$$Z_{g-2}(\widetilde{C}/C) := \Big\{\mathcal{O}_{\widetilde{C}}(E - \iota(E)) : E \in \widetilde{C}_{g-2}\Big\}$$

is a divisor inside $P(\widetilde{C}/C)$. We refer to $Z_{g-2}(\widetilde{C}/C)$ as the *top difference Prym variety*.

One computes the pull-back of the universal theta divisor under the Abel-Prym map. Recall that $\psi_{x_1}, \ldots, \psi_{x_{g-1}} \in CH^1(\widetilde{\mathcal{C}}^{g-1})$ are the cotangent classes corresponding to the marked points on the curves of genus $2g-1$.

Proposition 4.3. *For even g, if $\mu = \varphi \circ \mathfrak{ap} : \widetilde{\mathcal{C}}^{g-1} \to \widetilde{\mathcal{R}}_g$ denotes the projection map, one has*

$$\mathfrak{ap}^*(\theta_{\mathrm{pr}}) = \frac{1}{2}\sum_{j=1}^{g-2} \psi_{x_j} + 2\psi_{x_{g-1}} + 0 \cdot \Big(\lambda + \mu^*(\delta_0^{'} + \delta_0^{''} + \delta_0^{\mathrm{ram}})\Big) - \cdots \in CH^1(\widetilde{\mathcal{C}}^{g-1}).$$

Proof. We factor the map $\mathfrak{ap} : \widetilde{\mathcal{C}}^{g-1} \dashrightarrow \widetilde{\mathcal{Y}}_g$ as $\mathfrak{ap} = \mathfrak{aj} \circ \Delta$, where $\Delta : \widetilde{\mathcal{C}}^{g-1} \to \widetilde{\mathcal{C}}^{2g-2}$ is defined by $(x_1, \ldots, x_{g-1}) \mapsto (x_1, \ldots, x_{g-1}, \iota(x_1), \ldots, \iota(x_{g-1}))$ and $\mathfrak{aj} : \widetilde{\mathcal{C}}^{2g-2} \dashrightarrow \overline{\mathfrak{Pic}}^0_{2g-1}$ is the difference Abel-Jacobi map between the first and the last $g-1$ marked points on each curve into the universal Jacobian of degree zero over $\overline{\mathcal{M}}_{2g-1}$ respectively . There is a generically injective rational map $\widetilde{\mathcal{Y}}_g \dashrightarrow \overline{\mathfrak{Pic}}^0_{2g-1}$, which globalizes the usual inclusion $P(\widetilde{C}/C) \subset \mathrm{Pic}^0(\widetilde{C})$ valid for each Prym curve $[\widetilde{C} \to C] \in \mathcal{R}_g$. Using [GZ] Theorem 6, one computes the pull-back $\mathfrak{aj}^*(\theta_{2g-1}) \in CH^1(\widetilde{\mathcal{C}}^{2g-2})$ of the universal theta divisor θ_{2g-1} on $\overline{\mathfrak{Pic}}^0_{2g-1}$ trivialized along the zero section. Remarkably, the coefficient of λ, as well as that of the $\delta_0^{'}, \delta_0^{''}$ and δ_0^{ram} classes in this expression, are all zero. This is then pulled-back to $\widetilde{\mathcal{C}}^{g-1}$ keeping in mind that the pull-back of θ_{2g-1} to $\widetilde{\mathcal{Y}}_g$ is equal to $2\theta_{\mathrm{pr}}$. Using the formulas $\Delta^*(\psi_{x_j}) = \Delta^*(\psi_{y_j}) = \psi_{x_j}$, and $\Delta^*(\delta_{0:x_i y_j}) = \delta_{0:x_i x_j}$, as well as $\Delta^*(\delta_{0:y_i y_j}) = \delta_{0:x_i x_j}$, we conclude. $\square$

Remark 4.4. The other boundary coefficients of $\mathfrak{ap}^*(\theta_{\mathrm{pr}}) \in CH^1(\widetilde{\mathcal{C}}^{g-1})$ can be determined explicitly, but play no role in our future considerations.

Remark 4.5. Restricting ourselves to even g, we consider the restricted (non-dominant) Abel-Prym map $\mathfrak{ap}_{g-2}:\widetilde{\mathcal{C}}^{g-2}\dashrightarrow\widetilde{\mathcal{Y}}_g$ given by

$$\mathfrak{ap}_{g-2}\big(\widetilde{C}/C,\ x_1,\ldots,x_{g-2}\big):=\Big(\widetilde{C}/C,\ \mathcal{O}_{\widetilde{C}}\big((x_1-\iota(x_1)+\cdots+x_{g-2}-\iota(x_{g-2})\big)\Big),$$

and obtain the formula: $\mathfrak{ap}_{g-2}^*(\theta_{\mathrm{pr}})=\frac{1}{2}\sum_{j=1}^{g-2}\psi_{x_j}+0\cdot\big(\lambda+\mu^*(\delta_0^{'}+\delta_0^{''}+\delta_0^{\mathrm{ram}})\big)-\cdots.$

The image of $\mathfrak{ap}_{g-2}$ is a divisor $\mathcal{Z}_{g-2}$ on $\widetilde{\mathcal{Y}}_g$ characterized by the property

$$(\mathcal{Z}_{g-2})_{|P(\widetilde{C}/C)}=Z_{g-2}(\widetilde{C}/C),$$

for each $[\widetilde{C}\to C]\in\mathcal{R}_g$. In other words, $\mathcal{Z}_{g-2}$ is the divisor cutting out on each Prym variety the top difference variety. A similar difference variety inside the universal Jacobian over $\overline{\mathcal{M}}_g$ has been studied in [FV]. Specializing to the case $g=6$, the locus

$$\mathcal{U}_4:=\overline{(j\circ\chi)(\mathcal{Z}_4)}\subset\overline{\mathcal{A}}_6$$

is a codimension two cycle on $\overline{\mathcal{A}}_6$, which will appear as an obstruction for an effective divisor on $\overline{\mathcal{A}}_6$ to have small slope.

We use these considerations to bound from below the slope of $\overline{\mathcal{A}}_6$.

Proof of Theorem 0.4. We have seen that the boundary divisor D_6 of $\overline{\mathcal{A}}_6$ is filled-up by rational curves $h:\mathbf{P}^1\to D_6$ constructed in Theorem 3.4 by pushing-forward the sweeping rational curve $i:\mathbf{P}^1\to\widetilde{\mathcal{C}}^5$ of discriminants of a pencil of conic bundles. In particular, $\gamma:=h_*(\mathbf{P}^1)\in NE_1(\overline{\mathcal{A}}_6)$ is an effective class that intersects every non-boundary effective divisor on $\overline{\mathcal{A}}_6$ non-negatively. We compute using Propositions 4.1 and 4.3:

$$\gamma\cdot\lambda_1=i_*(\mathbf{P}^1)\cdot\mu^*\Big(\lambda-\frac{1}{4}\delta_0^{\mathrm{ram}}\Big)=6\cdot 9-\frac{3\cdot 32}{4}=30,\quad\text{and}$$

$$\gamma\cdot[D_6]=-i_*(\mathbf{P}^1)\cdot\Big(\sum_{j=1}^{4}\psi_{x_j}+4\psi_{x_5}\Big)+i_*(\mathbf{P}^1)\cdot\mu^*(\delta_0^{'})=-8\cdot 9+3\cdot 77=159.$$

We obtain the bound $s(\overline{\mathcal{A}}_6)\geq\frac{\gamma\cdot[D_6]}{\gamma\cdot\lambda_1}=\frac{53}{10}$. □

For effective divisors on $\overline{\mathcal{A}}_6$ transversal to $\mathcal{U}_4$, we obtain a better slope bound:

Theorem 4.6. *If E is an effective divisor on $\overline{\mathcal{A}}_6$ not containing the universal codimension two Prym difference variety $\mathcal{U}_4\subset\overline{\mathcal{A}}_6$, then $s(E)\geq\frac{13}{2}$.*

Proof. We consider the family $(q_{\mathbf{P}^1}:\mathcal{Z}\to\mathbf{P}^1,\ \tau_1,\ldots,\tau_4:\mathbf{P}^1\to\mathcal{Z})$ obtained from the construction explained in Theorem 3.4, where we retain only the first four sections. We obtain an induced moduli map $i_4:\mathbf{P}^1\to\widetilde{\mathcal{C}}^4$. Pushing i_4 forward via the Abel-Prym map, we obtain a curve $h_4:\mathbf{P}^1\to\mathcal{U}_4\subset\overline{\mathcal{A}}_6$, which fills-up the locus $\mathcal{U}_4$. Thus $\gamma_4:=(h_4)_*(\mathbf{P}^1)\in NE_1(\overline{\mathcal{A}}_6)$ is an effective class which intersects non-negatively any effective divisor on $\overline{\mathcal{A}}_6$ not containing $\mathcal{U}_4$. We compute using Theorems 3.3 and 3.4:

$$\gamma_4\cdot\lambda_1=\gamma\cdot\lambda_1=30\quad\text{and}\quad\gamma_4\cdot[D_6]=-4\cdot 9+3\cdot 77=195.$$

□

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