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Higher ramification and varieties of secant divisors on the generic curve

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Abstract

For a smooth projective curve, the cycles of e-secant k-planes are among the most studied objects in classical enumerative geometry, and there are well-known formulas due to Castelnuovo, Cayley and MacDonald concerning them. Despite various attempts, surprisingly little is known about the enumerative validity of such formulas. The aim of this paper is to clarify this problem in the case of the generic curve C of given genus. We determine precisely under which conditions the cycle of e-secant k-planes is non-empty, and we compute its dimension. We also precisely determine the dimension of the variety of linear series on C carrying e-secant k-planes.

Introduction

For a smooth projective curve C of genus g, we denote by C_e the eth symmetric product of C and by $G^r_d(C)$ the variety of linear series of type $\mathfrak{g}^r_d$ on C, that is,

$$G^r_d(C) := \{(L, V) : L \in \mathrm{Pic}^d(C), V \in G(r+1, H^0(L))\}.$$

The main result of the Brill–Noether theory states that if $[C] \in \mathcal{M}_g$ is a general curve then $G^r_d(C)$ is a smooth variety of dimension equal to $\rho(g, r, d) := g - (r+1)(g - d + r)$. For a linear series $l = (L, V) \in G^r_d(C)$ and an effective divisor $D \in C_e$, using the natural inclusion $H^0(L \otimes \mathcal{O}_C(-D)) \subset H^0(L)$ we can define a new linear series

$$l(-D) := \big(L \otimes \mathcal{O}_C(-D), V \cap H^0(L \otimes \mathcal{O}_C(-D))\big).$$

We fix integers $0 \leqslant f < e$ and introduce the determinantal cycle

$$V^{e-f}_e(l) := \{D \in C_e : \dim l(-D) \geqslant r - e + f\}$$

of effective divisors of degree e that impose at most $e - f$ independent conditions on l. If l is very ample and we view $C \overset{l}{\hookrightarrow} \mathbf{P}^r$ as an embedded curve, then $V^{e-f}_e(l)$ parametrizes e-secant $(e-f-1)$-planes to C. Each irreducible component of $V^{e-f}_e(l)$ has dimension at least $e - f(r+1-e+f)$. The cycles $V^{e-f}_e(l)$ have been extensively studied in classical enumerative geometry. The virtual class $[V^{e-f}_e(l)]^{\mathrm{virt}} \in A^{f(r+1-e+f)}(C_e)$ has been computed by MacDonald; its expression is tremendously complicated and is thus of limited practical use (see [**1**, Chapter VIII]). One case when we have a manageable formula is for $e = 2r-2$ and $f = r-1$, when $[V^{r-1}_{2r-2}(l)]^{\mathrm{virt}}$ computes the (virtual) number of $(r-2)$-planes in $\mathbf{P}^r$ that are $(2r-2)$-secant to C (cf. [**2**]).

Surprisingly little is known about the validity of these classical enumerative formulas (see [**13**] and [**15**] for partial results in the case of curves in $\mathbf{P}^3$). The aim of this paper is to clarify this problem for a general curve $[C] \in \mathcal{M}_g$. For every linear series $l \in G^r_d(C)$ we determine precisely under which conditions the cycle $V^{e-f}_e(l)$ is non-empty and has the expected dimension. Then

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having fixed $[C] \in \mathcal{M}_g$, we determine the dimension of the family of linear series $l \in G^r_d(C)$ with an e-secant $(e-f-1)$-plane. For our first result, we use degeneration techniques together with a few facts about the ample cone of the moduli space $\overline{\mathcal{M}}_{0,g}$ to prove the following theorem.

THEOREM 0.1. *Let $[C] \in \mathcal{M}_g$ be a general curve, and fix non-negative integers $0 \leqslant f < e$, r and d, such that $r - e + f \geqslant 0$. Then we see that*

$$\dim\{l \in G^r_d(C) : V^{e-f}_e(l) \neq \emptyset\} \leqslant \rho(g, r, d) - f(r + 1 - e + f) + e.$$

In particular, if $\rho(g, r, d) - f(r + 1 - e + f) + e < 0$, then $V^{e-f}_e(l) = \emptyset$, for every $l \in G^r_d(C)$.

More precisely, in Section 2 we prove the following dimensionality estimate:

$$\dim\{(D, l) \in C_e \times G^r_d(C) : D \in V^{e-f}_e(l)\} \leqslant \rho(g, r, d) - f(r + 1 - e + f) + e,$$

which obviously implies Theorem 0.1. This result generalizes the Brill–Noether theorem. Indeed, when $l = K_C$, then $V^{e-f}_e(K_C) = C^f_e := \{D \in C_e : h^0(\mathcal{O}_C(D)) \geqslant f + 1\}$. Since the fibres of the Abel–Jacobi map $C^f_e \to W^f_e(C)$ are at least f-dimensional, clearly $G^f_e(C) \neq \emptyset$ implies that $\dim C^f_e \geqslant f$. Our result reads as $G^f_e(C) = \emptyset$ when $\rho(g, f, e) < 0$, which is the non-existence part of the classical Brill–Noether theorem (cf. [**8**]). More generally, we have the following result in the case $\rho(g, r, d) = 0$.

COROLLARY 0.2. *Suppose that $\rho(g, r, d) = 0$ and that $e < f(r + 1 - e + f)$. Then for a general curve $[C] \in \mathcal{M}_g$ we see that $V^{e-f}_e(l) = \emptyset$ for every $l \in G^r_d(C)$; that is, no linear series of type $\mathfrak{g}^r_d$ on C has any e-secant $(e - f - 1)$-planes.*

An immediate consequence of Theorem 0.1 is a proof of the following conjecture of Coppens and Martens (cf. [**6**, Theorem 3.3.1] for a proof in the case $f = 1$).

COROLLARY 0.3. *Let $[C] \in \mathcal{M}_g$ be a general curve, and fix integers $0 \leqslant f < e$, d and r such that $r - e + f \geqslant 0$. Let l be a general linear series of type $\mathfrak{g}^r_d$ belonging to an irreducible component of $G^r_d(C)$. Assuming that $V^{e-f}_e(l)$ is not empty, we see that $e - f(r + 1 - e + f) \geqslant 0$. Moreover $V^{e-f}_e(l)$ is equidimensional and* $\dim V^{e-f}_e(l) = e - f(r + 1 - e + f)$.

We note that when $f = 1$, Theorem 0.1 concerns the higher-order very ampleness of linear series on a general curve. We recall that a linear series $l \in G^r_d(C)$ is said to be $(e - 1)$-*very ample* if $\dim l(-p_1 - \ldots - p_e) = r - e$, for any choice of (not necessarily distinct) e points $p_1, \ldots, p_e \in C$. Thus 0-very ampleness is equivalent to generation by global sections and 1-very ampleness reduces to the classical notion of very ampleness.

COROLLARY 0.4. *Let $[C] \in \mathcal{M}_g$ be a general curve, and let e, r, d be non-negative integers such that $\rho(g, r, d) + 2e - 2 - r < 0$. Then every linear series $l \in G^r_d(C)$ is $(e - 1)$-very ample.*

Theorem 0.1 does not address the issue of existence of linear series with e-secant $(e - f - 1)$-planes. We prove the following existence result for secant planes corresponding to linear series $\mathfrak{g}^r_d$ on an *arbitrary* smooth curve of genus g.

Theorem 0.5. *Let $[C] \in \mathcal{M}_g$ be an arbitrary smooth curve and fix integers $0 \leqslant f < e \leqslant g$, d and r, such that $f(r+1-e+f) \geqslant e$, $d \geqslant 2e-f-1$, $g-d+r \geqslant 0$,*

$$\rho(g,r,d) - f(r+1-e+f) + e \geqslant 0 \quad \text{and} \quad \rho(g, r-e+f, d-e) \geqslant 0.$$

Assume moreover that we are in one of the following situations:
(i) $2f \leqslant e-1$,
(ii) $e = 2r-2$ *and* $f = r-1$,
(iii) $e < 2(r+1-e+f)$ *or*
(iv) $\rho(g,r,d) \geqslant f(r+1-e+f) - (g-d+r)$.
Then there exists a linear series $l \in G^r_d(C)$ such that $V_e^{e-f}(l) \neq \emptyset$. Moreover, one has the following dimensionality statement:

$$\dim\{(D,l) \in C_e \times G^r_d(C) : D \in V_e^{e-f}(l)\} = \rho(g,r,d) - f(r+1-e+f) + e.$$

The inequalities $\rho(g, r-e+f, d-e) \geqslant 0$ and $\rho(g,r,d) + e - f(r+1-e+f) \geqslant 0$ are obvious necessary conditions for the existence of $l \in G^r_d(C)$ with $V_e^{e-f}(l) \neq \emptyset$ on a general curve $[C] \in \mathcal{M}_g$. To give an example, an elliptic quartic curve $C \subset \mathbf{P}^3$ has no 3-secant lines even though $\rho(g,r,d) + e - f(r+1-e+f) > 0$ (note that $e = 3$ and $f = 1$ in this case). Theorem 0.5 is stated in the range $f(r+1-e+f) \geqslant e$, corresponding to the case when linear series $l \in G^r_d(C)$ with $V_e^{e-f}(l) \neq \emptyset$ are expected to be special in the Brill–Noether cycle $G^r_d(C)$. It is clear though that the methods of this paper can be applied to the case $e \geqslant f(r+1-e+f)$ as well. In that range, however, when one expects existence of e-secant $(e-f-1)$-planes for every $l \in G^r_d(C)$, there are nearly optimal existence results obtained by using positivity for Chern classes of certain vector bundles in the style of [**12**]: for every curve $[C] \in \mathcal{M}_g$ and $l \in G^r_d(C)$, assuming that $d \geqslant 2e-1$ and $e - f(r+1-e+f) \geqslant r-e+f$, one knows that $V_e^{e-f}(l) \neq \emptyset$ (cf. [**5**, Theorem 1.2]). For $l \in G^r_d(C)$ such that $g-d+r \leqslant 1$ (for example, when l is non-special), if we keep the assumption that $e - f(r+1-e+f) \geqslant 0$, it is known that $V_e^{e-f}(l) \neq \emptyset$ if and only if $\rho(g, r-e+f, d-e) \geqslant 0$ (cf. [**1**, p. 356]). This appears to be the only case when MacDonald's formula displays some positivity features that can be used to derive existence results on $V_e^{e-f}(l)$. In the case $l = K_C$, one recovers of course the existence theorem from the classical Brill–Noether theory. We finally mention that Theorem 0.5 holds independently of the assumptions (i)–(iii), whenever a certain transversality condition (18) concerning a general curve $[Y,p] \in \mathcal{M}_{e,1}$ is satisfied (see Section 3 for details). Theorem 0.5 is then proved by verifying this condition (18) in each of the cases (i)–(iii).

We now specialize to the case when $e = f(r+1-e+f)$, which is covered by Theorem 0.5. One can write $r = (u-1)(f+1)$ and $e = uf$ for some $u \geqslant 1$, and we obtain the following result concerning the classical problem of existence of uf-secant secant $(uf-f-1)$-planes to curves in $\mathbf{P}^r$.

Corollary 0.6. *Let C be a smooth curve of genus g. We fix integers $d, u, f \geqslant 2$ and assume that the inequalities $g \geqslant uf, d \geqslant 2uf - f - 1$, $\rho(g, uf+u-f-1, d) \geqslant 0$ and $\rho(g, u-1, d-uf) \geqslant 0$ hold. Then there exists an embedding $C \subset \mathbf{P}^{(u-1)(f+1)}$ with $\deg(C) = d$, such that C has a uf-secant $(uf-f-1)$-plane. If, moreover, $[C] \in \mathcal{M}_g$ is general in moduli, then the embedded curve $C \overset{l}{\hookrightarrow} \mathbf{P}^{(u-1)(f+1)}$ corresponding to a general linear series $l \in G_d^{(u-1)(f+1)}(C)$ has only a finite number of uf-secant $(uf-f-1)$-planes.*

If $[C] \in \mathcal{M}_g$ is suitably general then we can prove that the Cayley–Castelnuovo numbers predicting the number of $(2r-2)$-secant $(r-2)$-planes of a curve in $C \subset \mathbf{P}^r$ have a precise enumerative meaning.

THEOREM 0.7. *Let $[C] \in \mathcal{M}_g$ be a general curve. We fix integers $d, r \geqslant 3$ such that $d \geqslant 3r - 2$, $\rho(g, r, d) \geqslant \emptyset$ and $\rho(g, 1, d - 2r + 2) \geqslant 0$. Then if $C \overset{l}{\hookrightarrow} \mathbf{P}^r$ is an embedding corresponding to a general linear series $l \in G^r_d(C)$, then C has only finitely many $(2r-2)$-secant $(r-2)$-planes. Their number (counted with multiplicities) is*

$$C(d, g, r) = \sum_{i=0}^{r-1} \frac{(-1)^i}{r-i} \binom{d-r-i+1}{r-1-i} \binom{d-r-i}{r-1-i} \binom{g}{i}.$$

A modern proof of the formula for $C(d, g, r)$ is due to MacDonald and appears in [**1**, Chapter VIII]. The original formula is due to Castelnuovo (cf. [**2**]). When $r = 3$, we recover Cayley's formula for the number of 4-secant lines of a smooth space curve $C \subset \mathbf{P}^3$ of degree d (cf. [**3**]):

$$C(d, g, 3) = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{g}{2}(d^2 - 7d + 13 - g).$$

To make a historical remark, there have been various attempts to rigorously justify the so-called *functional method* that Cayley (1863), Castelnuovo (1889) and Severi (1900) used to derive their enumerative formulas and to determine their range of applicability (see [**15**, **19**]). For instance, Cayley's formula is shown to hold for an arbitrary smooth curve in $\mathbf{P}^3$, provided that $C(d, g, 3)$ is defined as the degree of a certain 0-cycle $\mathrm{Sec}_4(C)$ in $\mathbf{G}(1, 3)$ (cf. [**16**]). The drawback of this approach is that it becomes very difficult to determine when this newly defined invariant is really enumerative. For instance, Le Barz only shows that this happens for very special curves in $\mathbf{P}^3$ (rational curves and generic complete intersections), and one of the aims of this paper is to establish the validity of such formulas for curves that are general with respect to moduli.

The second topic that we study concerns ramification points of powers of linear series on curves. This question appeared first in a particular case in [**11**]. We recall that for a pointed curve $[C, p] \in \mathcal{M}_{g,1}$ and a linear series $l = (L, V) \in G^r_d(C)$, the *vanishing sequence* of l at p

$$a^l(p) : a^l_0(p) < \ldots < a^l_r(p) \leqslant d$$

is obtained by ordering the set $\{\mathrm{ord}_p(\sigma)\}_{\sigma \in V}$. The *weight* of p with respect to l is defined as $w^l(p) := \sum_{i=0}^r (a^l_i(p) - i)$. One says that p is a *ramification point* of l if $w^l(p) \geqslant 1$ and we denote by $R(l)$ the finite set of ramification points of l. If $[C, p] \in \mathcal{M}_{g,1}$ and $\overline{\alpha} : 0 \leqslant \alpha_0 \leqslant \ldots \leqslant \alpha_r \leqslant d - r$ is a Schubert index of type (r, d), then the cycle

$$G^r_d(C, p, \overline{\alpha}) := \{l \in G^r_d(C) : a^l_i(p) \geqslant \alpha_i + i \text{ for } i = 0 \ldots r\}$$

can be realized as a generalized determinantal variety inside $G^r_d(C)$ having virtual dimension $\rho(g, r, d, \overline{\alpha}) := \rho(g, r, d) - \sum_{j=0}^r \alpha_j$. For a general pointed curve $[C, p] \in \mathcal{M}_{g,1}$, it is known that the virtual dimension equals the actual dimension; that is,

$$\dim\, G^r_d(C, p, \overline{\alpha}) = \rho(g, r, d, \overline{\alpha})$$

(cf. [**9**, Theorem 1.1]).

We address the following question: suppose that $l = (L, V) \in G^r_d(C)$ is a linear series with a prescribed ramification sequence $\overline{\alpha}$ at a fixed point $p \in C$. Is then p a ramification point of any of the powers $L^{\otimes n}$ for $n \geqslant 2$? If so, can we describe the sequence $a^{L^{\otimes n}}(p)$? One certainly expects that under suitable genericity assumptions on C and L, the points in $\bigcup_{n \geqslant 1} R(L^{\otimes n})$ should be uniformly distributed on C. For example, it is known that for every C and $L \in \mathrm{Pic}^d(C)$, the set $\bigcup_{n \geqslant 1} R(L^{\otimes n})$ is dense in C with respect to the classical topology (cf. [**17**]). Silverman and Voloch showed that for any $L \in \mathrm{Pic}^d(C)$ there exist finitely many points $p \in C$ such that the set $\{n \geqslant 1 : p \in R(L^{\otimes n})\}$ is infinite (cf. [**18**]).

We prove that on a generic pointed curve $[C,p]$, a linear series (L,V) and its multiples $L^{\otimes n}$ share no ramification points; that is, $R(l)$ and $R(L^{\otimes n})$ are as transverse as they can be expected to be and, moreover, the vanishing sequence $a^{L^{\otimes n}}(p)$ is close to being minimal.

THEOREM 0.8. *We fix a general pointed curve* $[C,p] \in \mathcal{M}_{g,1}$, *integers* $r,d \geqslant 1, n \geqslant 3$ *and a Schubert index* $\overline{\alpha} : 0 \leqslant \alpha_0 \leqslant \ldots \leqslant \alpha_r \leqslant d-r$. *We also set* $m := [(n+1)/2]$. *Then for every linear series* $l = (L,V) \in G^r_d(C,p,\overline{\alpha})$ *and every positive integer*

$$a < nd - \rho(g,r,d,\overline{\alpha}) - g - \left[\frac{g}{m}\right],$$

we find that $h^0(C, L^{\otimes n}(-ap)) = h^0(C, L^{\otimes n}) - a = nd + 1 - g - a$. *Inother words,* $a_i^{L^{\otimes n}}(p) = i$ *for* $0 \leqslant i \leqslant a-1$.

In the case $n = 2$, when we compare $R(l)$ and $R(L^{\otimes 2})$, our results are sharper.

THEOREM 0.9. *We fix a general pointed curve* $[C,p] \in \mathcal{M}_{g,1}$, *integers* $r,d \geqslant 1$ *and a Schubert index* $\overline{\alpha} : \alpha_0 \leqslant \ldots \leqslant \alpha_r \leqslant d-r$. *Then for every* $(L,V) \in G^r_d(C,p,\overline{\alpha})$ *and every positive integer*

$$a < \max\left\{2d + 2 - 2g - \rho(g,r,d,\overline{\alpha}) + \left[\frac{g-1}{2}\right],\ 2d + 2 - 2g - 2\rho(g,r,d,\overline{\alpha}) + 2\left[\frac{g}{3}\right]\right\},$$

we see that $h^0(C, L^{\otimes 2}(-ap)) = h^0(C, L^{\otimes 2}) - a = 2d + 1 - g - a$.

Comparing the bounds on a given in Theorems 0.8 and 0.9 with the obvious necessary condition $a \leqslant nd - g + 1$ that comes from the Riemann–Roch theorem, we see that our results are essentially optimal for relatively small values of $\rho(g,r,d,\overline{\alpha})$ when the linear series $(L,V) \in G^r_d(C,p,\overline{\alpha})$ have a strong geometric characterization. On the other hand if, for instance, $\rho(g,r,d,\overline{\alpha}) = g$, then $L \in \mathrm{Pic}^d(C)$ and $p \in C$ are arbitrary, and one cannot expect to prove a uniform result about the vanishing of $H^1(C, L^{\otimes n}(-ap))$.

Theorems 0.8 and 0.9 concern line bundles L with prescribed ramification at a given point $p \in C$. Such bundles are, of course, very special in $\mathrm{Pic}^d(C)$. If, instead, we try to describe $\bigcup_{n \geqslant 1} R(L^{\otimes n})$ for a general line bundle $L \in \mathrm{Pic}^d(C)$, the answer turns out to be particularly simple. We give a short proof of the following result.

THEOREM 0.10. *Let* C *be a smooth curve of genus* g, *and let* $L \in \mathrm{Pic}^{\mathrm{d}}(\mathrm{C})$ *be a very general line bundle.*

(1) *All the ramification points of the powers* $L^{\otimes n}$ *are ordinary; that is,* $w^{L^{\otimes n}}(p) \leqslant 1$ *for all* $p \in C$ *and* $n \geqslant 1$.

(2) $R(L^{\otimes a}) \cap R(L^{\otimes b}) = \emptyset$ *for* $a \neq b$; *that is, a point* $p \in C$ *can be a ramification point for at most a single power of* L.

1. *Ramification points of multiples of linear series*

In this section we use the technique of limit linear series to prove Theorems 0.8 and 0.9. We start by fixing a Schubert index $\overline{\alpha} : 0 \leqslant \alpha_0 \leqslant \ldots \leqslant \alpha_r \leqslant d-r$ and two integers $a \geqslant 0, n \geqslant 2$. We also set $m := [(n+1)/2]$.

We assume that for every $[C,p] \in \mathcal{M}_{g,1}$ there exists a linear series $l = (L,V) \in G^r_d(C,p,\overline{\alpha})$ such that $H^0(K_C \otimes L^{\otimes(-n)} \otimes \mathcal{O}_C(ap)) \neq 0$. By a degeneration argument we are going to show that this implies the inequalities

$$a \geqslant nd - g - \rho(g,r,d,\overline{\alpha}) - \left[\frac{g}{m}\right], \quad \text{when } n \geqslant 3, \tag{1}$$

$$a \geqslant 2d + 2 - 2g - \rho(g,r,d,\overline{\alpha}) + \left[\frac{g-1}{2}\right], \tag{2}$$

and

$$a \geqslant 2d + 2 - 2g - 2\rho(g,r,d,\overline{\alpha}) + 2\left[\frac{g}{3}\right], \quad \text{when } n = 2. \tag{3}$$

This will prove both Theorems 0.8 and 0.9.

We degenerate $[C,p]$ to a stable curve $[X_0 := E_0 \bigcup_{p_1} E_1 \bigcup_{p_2} \ldots \bigcup_{p_{g-1}} E_{g-1}, p_0]$, where E_i is a general elliptic curve, $p_i, p_{i+1} \in E_i$ are points such that $p_{i+1} - p_i \in \mathrm{Pic}^0(E_i)$ is not a torsion class and, moreover, $E_i \cap E_{i+1} = \{p_{i+1}\}$ for $0 \leqslant i \leqslant g-2$. Thus X_0 is a string of g elliptic curves, and the marked point p_0 specializes to a general point lying on the first component E_0. We also consider a 1-dimensional family $\pi : \mathcal{X} \to B$ together with a section $\sigma : B \to \mathcal{X}$, such that $B = \mathrm{Spec}(R)$ with R being a discrete valuation ring having uniformizing parameter t. We assume that $\mathcal{X}$ is a smooth surface and that there exists an isomorphism between X_0 and $\pi^{-1}(0)$. Under this isomorphism we also assume that $\sigma(0) = p_0 \in X_0$. Here $0 \in B$ is the point corresponding to the maximal ideal of R, and we denote by η and $\overline{\eta}$ the generic point and the geometric generic point of B, respectively. By assumption, there exists a linear series $l_{\overline{\eta}} = (L_{\overline{\eta}}, V_{\overline{\eta}}) \in G^r_d(X_{\overline{\eta}}, \sigma(\overline{\eta}), \overline{\alpha})$, such that $H^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}} \otimes L_{X_{\overline{\eta}}}^{\otimes(-n)} \otimes \mathcal{O}_{X_{\overline{\eta}}}(a\sigma(\overline{\eta}))) \neq 0$. By possibly blowing up $\mathcal{X}$ at the nodes of X_0 and thus replacing the central fibre by a curve X obtained from X_0 by inserting chains of smooth rational curves at the points $p_1, \ldots, p_{g-1}$, we may assume that $l_{\overline{\eta}}$ comes from a linear series $l_\eta = (L_\eta, V_\eta) \in G^r_d(X_\eta, \sigma(\eta), \overline{\alpha})$ on the generic fibre X_η.

We denote by $l_{E_i} = (L_{E_i}, V_{E_i}) \in G^r_d(E_i)$ the E_i-aspect of the limit linear series on X induced by l_η: precisely, if $\mathcal{L}$ is a line bundle on $\mathcal{X}$ extending L_η, then $L_{E_i} \in \mathrm{Pic}^d(E_i)$ is the restriction to E_i of the unique twist $\mathcal{L}_{E_i}$ of $\mathcal{L}$ along components of $\pi^{-1}(0)$ such that $\deg_Z(\mathcal{L}_{i|Z}) = 0$ for any irreducible component $Z \neq E_i$ of $\pi^{-1}(0)$ (see also [**8**, p. 348]). Since we gave ourselves the freedom of blowing up $\mathcal{X}$ at the nodes of $\pi^{-1}(0)$, we can also assume that $\{l_{E_i}\}_{i=0}^{g-1}$ constitutes a limit $\mathfrak{g}^r_d$ on X_0 that is obtained from a refined limit $\mathfrak{g}^r_d$ on X by retaining only the aspects of the elliptic components of X. The compatibility relations between the vanishing orders of the aspects l_{E_i} imply the following inequality between Brill–Noether numbers:

$$\rho(g,r,d,\overline{\alpha}) \geqslant \rho(l_{E_0}, p_0, p_1) + \rho(l_{E_1}, p_1, p_2) + \ldots + \rho(l_{E_{g-2}}, p_{g-2}, p_{g-1}) + \rho(l_{E_{g-1}}, p_{g-1}), \tag{4}$$

where $\rho(l_{E_i}, p_i, p_{i+1}) := \rho(1,r,d) - w^{l_{E_i}}(p_i) - w^{l_{E_i}}(p_{i+1})$. By assumption, there exists a non-zero section $\rho_\eta \in H^0\big(X_\eta, \omega_{X_\eta} \otimes \mathcal{L}_\eta^{\otimes(-n)} \otimes \mathcal{O}_{X_\eta}(a\sigma(\eta))\big)$. This implies that if we denote by $\tilde{\mathcal{L}}_i$ the unique line bundle on the surface $\mathcal{X}$ such that: (1) $\tilde{\mathcal{L}}_{i|X_\eta} = L_\eta$, and (2) $\deg_Z(\omega_X \otimes \tilde{\mathcal{L}}_i^{\otimes(-n)} \otimes \mathcal{O}_X(ap_0)) = 0$, for every component Z of X such that $Z \neq E_i$, then $H^0(E_i, \omega_X \otimes \tilde{\mathcal{L}}_i^{\otimes(-n)} \otimes \mathcal{O}_X(ap_0) \otimes \mathcal{O}_{E_i}) \neq 0$. We set

$$\mathcal{M}_i := \omega_\pi \otimes \tilde{\mathcal{L}}_i^{\otimes(-n)} \otimes \mathcal{O}_{\mathcal{X}}(a\ \sigma(B)) \in \mathrm{Pic}(\mathcal{X}).$$

Then $\mathcal{M}_{i|E_i} = \mathcal{O}_{E_i}\big((a+2i)\cdot p_i + (2g-2-2i)\cdot p_{i+1} \otimes L_{E_i}^{\otimes(-n)}\big)$ for all $0 \leqslant i \leqslant g-1$. For each such i we denote by n_i the smallest integer such that $\tilde{\rho}_i := t^{n_i}\rho_\eta \in \pi_*(\mathcal{M}_i)$ and we set

$$\rho_i := \tilde{\rho}_{i|E_i} \in H^0(E_i, \mathcal{M}_{i|E_i}).$$

Thus $0 \neq \rho_i \in H^0(E_i, \mathcal{O}_{E_i}((a+2i)\cdot p_i + (2g-2-2i)\cdot p_{i+1} \otimes L_{E_i}^{\otimes(-n)}))$ and in a way similar to [**8**, Proposition 2.2] we can prove that

$$\mathrm{ord}_{p_i}(\rho_i) + \mathrm{ord}_{p_i}(\rho_{i-1}) \geqslant 2g - 2 - nd + a = \deg(\mathcal{M}_{i|E_i}). \tag{5}$$

One also has the inequalities $\mathrm{ord}_{p_i}(\rho_i) + \mathrm{ord}_{p_{i+1}}(\rho_i) \leqslant 2g-2-nd+a$ (and similar inequalities when passing through the rational components of X), from which it follows that one can write down a non-decreasing sequence of vanishing orders

$$0 \leqslant \mathrm{ord}_{p_0}(\rho_0) \leqslant \mathrm{ord}_{p_1}(\rho_1) \leqslant \ldots \leqslant \mathrm{ord}_{p_i}(\rho_i) \leqslant \ldots \leqslant \mathrm{ord}_{p_{g-1}}(\rho_{g-1}). \tag{6}$$

Since ρ_{g-1} is a non-zero section of a line bundle of degree $2g-2-nd+a$ on E_{g-1}, it is required that $\mathrm{ord}_{p_{g-1}}(\rho_{g-1}) \leqslant 2g-2-nd+a$. This inequality will eventually lead to the bound on the constant a.

Let us suppose now that we have fixed one of the elliptic components of X, say E_i, such that $\rho(l_{E_i}, p_i, p_{i+1}) = 0$. By counting dimensions, we see that for every $0 \leqslant j \leqslant r$ there exists a section $u_j \in V_{E_i}$ such that $\mathrm{div}(u_j) \geqslant a_j^{l_{E_i}}(p_i)\cdot p_i + a_{r-j}^{l_{E_i}}(p_{i+1})\cdot p_{i+1}$. In particular, we have $a_j^{l_{E_i}}(p_i) + a_{r-j}^{l_{E_i}}(p_{i+1}) \leqslant d$. Since $p_{i+1} - p_i \in \mathrm{Pic}^0(E_i)$ is not a torsion class, it follows that the equality $a_j^{l_{E_i}}(p_i) + a_{r-j}^{l_{E_i}}(p_{i+1}) = d$ can hold for at most one value $0 \leqslant j \leqslant r$. Because $\rho(l_{E_i}, p_i, p_{i+1}) = 0$, this implies that

$$a_j^{l_{E_i}}(p_i) + a_{r-j}^{l_{E_i}}(p_{i+1}) \geqslant d-1 \quad \text{for all } 0 \leqslant j \leqslant r,$$

and there exists precisely one such index j such that $a_j^{l_{E_i}}(p_i) + a_{r-j}^{l_{E_i}}(p_{i+1}) = d$. In this case we get that $\mathrm{div}(u_j) = a_j^{l_{E_i}}(p_i)\cdot p_i + a_{r-j}^{l_{E_i}}(p_{i+1})\cdot p_{i+1}$, and for degree reasons we must have $L_{E_i} = \mathcal{O}_{E_i}(a_j^{l_{E_i}}(p_i)\cdot p_i + a_{r-j}^{l_{E_i}}(p_{i+1})\cdot p_{i+1}) \in \mathrm{Pic}^d(E_i)$.

To summarize, if $\rho(l_{E_i}, p_i, p_{i+1}) = 0$, then the vanishing sequence $a^{l_{E_{i+1}}}(p_{i+1})$ of the E_{i+1}-aspect of the limit $\mathfrak{g}^r_d$ on X is obtained from the vanishing sequence $a^{l_{E_i}}(p_i)$ by raising all entries by 1, except one single entry which remains unchanged. Thus, $a_j^{l_{E_i}}(p_i) = a_j^{l_{E_{i+1}}}(p_{i+1})$ for one index $0 \leqslant j \leqslant r$ and $a_k^{l_{E_{i+1}}}(p_{i+1}) = a_k^{l_{E_i}}(p_i) + 1$ for $k \neq j$.

We now study what happens to the non-decreasing sequence (6) as we pass through a component E_i with $\rho(l_{E_i}, p_i, p_{i+1}) = 0$. Assume that $\mathrm{ord}_{p_i}(\rho_i) = \mathrm{ord}_{p_{i+1}}(\rho_{i+1}) := b$. This implies that $\mathrm{ord}_{p_{i+1}}(\rho_i) = 2g-2-nd+a-b$ and

$$L_{E_i}^{\otimes n} = \mathcal{O}_{E_i}((a+2i-b)\cdot p_i + (nd-a+b-2i)\cdot p_{i+1}) \in \mathrm{Pic}^{nd}(E_i).$$

Because $\rho(l_{E_i}, p_i, p_{i+1}) = 0$, as we have seen, L_{E_i} can be represented by an effective divisor that is supported only at p_i and p_{i+1}. Precisely, we can write that $L_{E_i} = \mathcal{O}_{E_i}\big(a_j^{l_{E_i}}(p_i)\cdot p_i + a_{r-j}^{l_{E_i}}(p_{i+1})\cdot p_{i+1}\big)$ for a unique $0 \leqslant j \leqslant r$. Since L_{E_i} cannot admit two different representations by effective divisors supported only at p_i and p_{i+1}, we must have

$$L_{E_i} = \mathcal{O}_{E_i}\left(\frac{a+2i-b}{n}\cdot p_i + \frac{nd-a+b-2i}{n}\cdot p_{i+1}\right). \tag{7}$$

In particular, we find that $(a+2i-b)/n \in \mathbb{Z}$ and $a_j^{l_{E_i}}(p_i) = (a+2i-b)/n$.

We consider a connected subcurve $Y \subset X$ containing $m+1$ elliptic components E_i and we measure the increase in (6) as we pass through the components of Y.

Lemma 1.1. *We fix $m := [(n+1)/2]$ and integers i and b such that $bm \leqslant i \leqslant g-1$. We have $R(i) := \#\{0 \leqslant l \leqslant i-1 : \rho(l_{E_l}, p_l, p_{l+1}) \geqslant 1\}$. Then the following inequality holds:*

$$\mathrm{ord}_{p_i}(\rho_i) + R(i) \geqslant b(m-1).$$

Proof. We proceed by induction on b. For $b = 0$ there is nothing to prove. We set $b \geqslant 1$ and $i := (b-1)m$, and we assume that $\mathrm{ord}_{p_i}(\rho_i) + R(i) \geqslant (b-1)(m-1)$. We are going to prove that the following inequality holds:

$$\mathrm{ord}_{p_{i+m}}(\rho_{i+m}) - \mathrm{ord}_{p_i}(\rho_i) + R(i+m) - R(i) \geqslant m-1. \tag{8}$$

Assume that this is not the case. Then there exist integers $0 \leqslant l < j \leqslant m-1$ such that the following relations hold:

(i) $\rho(l_{E_{i+l}}, p_{i+l}, p_{i+l+1}) = \rho(l_{E_{i+j}}, p_{i+j}, p_{i+j+1}) = 0$ and

(ii) $\mathrm{ord}_{p_{i+l}}(\rho_{i+l}) = \mathrm{ord}_{p_{i+l+1}}(\rho_{i+l+1}) := b, \mathrm{ord}_{p_{i+j}}(\rho_{i+j}) = \mathrm{ord}_{p_{i+j+1}}(\rho_{i+j+1}) := c.$

Using (7) this implies that

$$L_{E_{i+l}} = \mathcal{O}_{E_{i+l}}\left(\frac{a+2i+2l-b}{n} \cdot p_{i+l} + \frac{nd-a+b-2i-2l}{n} \cdot p_{i+l+1}\right),$$

and

$$L_{E_{i+j}} = \mathcal{O}_{E_{i+j}}\left(\frac{a+2i+2j-c}{n} \cdot p_{i+j} + \frac{nd-a+c-2i-2j}{n} \cdot p_{i+j+1}\right).$$

In particular, $(2j-2l-c+b)/n \in \mathbb{Z}$, and hence we can write $c = b - kn + 2(j-l)$ for some $k \in \mathbb{Z}$. If $k \geqslant 1$, since $c \geqslant b$ we obtain that $m-1 \geqslant j-l \geqslant n/2$, which is a contradiction. Therefore we must require that $k \leqslant 0$, and this holds for every pair (j,l) satisfying (i) and (ii). We now choose the pair $0 \leqslant l < j \leqslant m-1$ satisfying (i) and (ii) and for which the difference $j-l$ is maximal.

For each integer $0 \leqslant e \leqslant l-1$ we see that either $\rho(l_{E_{i+e}}, p_{i+e}, p_{i+e+1}) \geqslant 1$ or $\mathrm{ord}_{p_{i+e+1}}(\rho_{i+e+1}) > \mathrm{ord}_{p_{i+e}}(\rho_{i+e})$. This fact leads to the inequality

$$\mathrm{ord}_{p_{i+l}}(\rho_{i+l}) - \mathrm{ord}_{p_i}(\rho_i) + R(i+l) - R(i) \geqslant l. \tag{9}$$

Similarly, by studying the subcurve of Y containing $E_{i+j+1}, \ldots, E_{i+m-1}$, we find that

$$\mathrm{ord}_{p_{i+m}}(\rho_{i+m}) - \mathrm{ord}_{p_{i+j+1}}(\rho_{i+j+1}) + R(i+m) - R(i+j+1) \geqslant m-j-1. \tag{10}$$

Finally, we look at the subcurve of X containing $E_{i+l}, \ldots, E_{i+j}$ and we can write

$$\mathrm{ord}_{p_{i+j}}(\rho_{i+j}) - \mathrm{ord}_{p_{i+l}}(\rho_{i+l}) + R(i+j+1) - R(i+l) \geqslant c-b \geqslant 2(j-l) \geqslant j-l+1. \tag{11}$$

By adding (9)–(11) together we obtain (8), which proves the lemma. □

When $n=2$ we have a slightly better estimate than in the general case.

Lemma 1.2 ($n=2$). (1) *Let i be an integer such that $2b \leqslant i \leqslant g-1$. Then $\mathrm{ord}_{p_i}(\rho_i) + R(i) \geqslant b$.*

(2) *We fix $0 \leqslant i \leqslant g-4$, and let Y be a connected subcurve of X containing precisely three elliptic curves E_i, E_{i+1} and E_{i+2}. If $R(i+3) = R(i)$, that is,*

$$\rho(l_{E_i}, p_i, p_{i+1}) = \rho(l_{E_{i+1}}, p_{i+1}, p_{i+2}) = \rho(l_{E_{i+2}}, p_{i+2}, p_{i+3}) = 0,$$

then we have the inequality $\mathrm{ord}_{p_{i+3}}(\rho_{i+3}) \geqslant \mathrm{ord}_{p_i}(\rho_i) + 2$.

Proof. We only prove (2), the remaining statement being analogous to Lemma 1.1. We may assume that $\mathrm{ord}_{p_i}(\rho_i) = \mathrm{ord}_{p_{i+1}}(\rho_{i+1}) := b$. Hence $(a+2i-b)/2 \in \mathbb{Z}$ and there exists an index $0 \leqslant j \leqslant r$ such that

$$a_j^{l_{E_i}}(p_i) = a_j^{l_{E_{i+1}}}(p_{i+1}) = \tfrac{1}{2}(a+2i-b), \quad \text{while } a_k^{l_{E_{i+1}}}(p_{i+1}) = a_k^{l_{E_i}}(p_i) + 1 \text{ for } k \neq j.$$

If $\mathrm{ord}_{p_{i+2}}(\rho_{i+2}) = \mathrm{ord}_{p_{i+1}}(\rho_{i+1}) = b$, then (7) implies that $(a+2i+2-b)/2$ is an entry in the vanishing sequence $a^{l_{E_{i+1}}}(p_{i+1})$. However, this is impossible, because $(a+2i-b)/2$ was an entry in the sequence $a^{l_{E_i}}(p_i)$, and hence it is required that $\mathrm{ord}_{p_{i+2}}(\rho_{i+2}) \geqslant b+1$. Next, if $\mathrm{ord}_{p_{i+3}}(\rho_{i+3}) = b+1$, then this implies that $\mathrm{ord}_{p_{i+3}}(\rho_{i+3}) = \mathrm{ord}_{p_{i+2}}(\rho_{i+2}) = b+1$, and hence again $\big(a+2(i+2)-(b+1)\big)/2 \in \mathbb{Z}$, which is not possible for parity reasons. Thus we must require that $\mathrm{ord}_{p_{i+3}}(\rho_{i+3}) \geqslant b+2$. □

Proof of Theorem 0.8. We complete the proof of our result in the case $n \geqslant 3$. We write $g = bm + c$ with $0 \leqslant c \leqslant m-1$ and we set $i := bm$. From Lemma 1.1 we obtain that $\mathrm{ord}_{p_i}(\rho_i) + R(i) \geqslant b(m-1)$. Using the reasoning of Lemma 1.1 for the connected subcurve of X which contains $E_i, E_{i+1}, \ldots, E_{i+c-1} = E_{g-1}$, we get that

$$\mathrm{ord}_{p_{g-1}}(\rho_{g-1}) - \mathrm{ord}_{p_i}(\rho_i) + R(g-1) - R(i) \geqslant c-2. \tag{12}$$

Using (12), together with the inequality $R(g-1) \leqslant \rho(g,r,d,\overline{\alpha})$, we can write that $\deg(K_C \otimes L^{\otimes(-n)} \otimes \mathcal{O}_C(ap)) = 2g-2-nd+a \geqslant \mathrm{ord}_{p_{g-1}}(\rho_{g-1}) \geqslant g - \left[\frac{g}{m}\right] - \rho(g,r,d,\overline{\alpha}) - 2$, which finishes the proof of Theorem 0.8. $\square$

Proof of Theorem 0.9. From Lemma 1.2(1), we obtain that

$$\mathrm{ord}_{p_{g-1}}(\rho_{g-1}) + R(g-1) \geqslant [(g-1)/2].$$

Since $R(g-1) \leqslant \rho(g,r,d,\overline{\alpha})$, this leads to the inequality $a \geqslant 2d+2-2g+[(g-1)/2] - \rho(g,r,d,\overline{\alpha})$. To prove (3) we divide X into $e := [g/3]+1$ connected subcurves $Y_1, \ldots, Y_e$ such that $Y_1, \ldots, Y_{e-1}$ each contain three elliptic components, $\#(Y_i \cap Y_{i+1}) = 1$ for all $1 \leqslant i \leqslant e-2$ and $Y_e := \overline{(\bigcup_{i=1}^{e-1} Y_i)^c}$. The curves Y_i fall into two categories: those for which there exists an elliptic component $E_l \subset Y_i$ such that $\rho(l_{E_l}, p_l, p_{l+1}) \geqslant 1$ (and there are at most $\rho(g,r,d,\overline{\alpha})$ such Y_i), and those for which $\rho(l_{E_l}, p_l, p_{l+1}) = 0$ for each elliptic component $E_l \subset Y_i$. Lemma 1.2 part (2) gives that $\mathrm{ord}_{p_{g-1}}(\rho_{g-1}) \geqslant 2([g/3] - \rho(g,r,d,\overline{\alpha}))$. This proves part (2) and finishes the proof of Theorem 0.9. $\square$

Remark 1.3. It is natural to ask how close to being optimal the bounds are that we obtained above. For $\rho(g,r,d,\overline{\alpha})$ relatively small, when any $L \in G^r_d(C,p,\overline{\alpha})$ has a strong geometric characterization, the inequalities (1)–(3) are in fact optimal. To see an example, we set $g=3$, $r=3$, $d=6$ and $\rho(g,r,d,\overline{\alpha}) = 0$. Thus we look at instances of $\mathfrak{g}^3_6$ on a general $[C,p] \in \mathcal{M}_{3,1}$ having ramification at p equal to $(0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant 3)$, where $\sum_{i=0}^3 \alpha_i = 3$. Theorem 0.9 gives us that $H^0(K_C \otimes L^{\otimes(-2)} \otimes \mathcal{O}_C(a \cdot p)) = 0$ for every integer $a \leqslant 9$. We show that this is optimal by noting that when $a = 10$ and $\overline{\alpha} = (0,0,1,2)$, we have

$$H^0(K_C \otimes L^{\otimes(-2)} \otimes \mathcal{O}_C(10p)) \neq 0 \quad \text{ for every } L \in G^3_6(C,p,\overline{\alpha}).$$

Indeed, any such linear series is of the form $L = K_C \otimes A^\vee \otimes \mathcal{O}_C(5p) \in W^3_6(C)$, where $A \in W^1_3(C)$ is such that $h^0(A(-2p)) \geqslant 1$. A non-hyperelliptic curve of genus 3 has two such cases of $\mathfrak{g}^1_3$. Precisely, if $z, t \in C$ are the two points that the tangent line at p to $C \overset{|K_C|}{\hookrightarrow} \mathbf{P}^2$ meets C again, then $A = \mathcal{O}_C(2p+z)$ or $A = \mathcal{O}_C(2p+t)$. Say, we choose $A = \mathcal{O}_C(2p+z)$. By direct calculation we obtain that $L^{\otimes 2} \otimes \mathcal{O}_C(-10p) = K_C^{\otimes 2} \otimes A^{\otimes(-2)} = \mathcal{O}_C(2t)$, and hence $h^0(K_C \otimes L^{\otimes(-2)} \otimes \mathcal{O}_C(10p)) = 1$.

2. *Varieties of secant planes to the general curve*

We fix a smooth curve $[C] \in \mathcal{M}_g$ and two integers $0 \leqslant f < e$. In this section we study the varieties $V_e^{e-f}(l)$ of e-secant $(e-f-1)$-planes corresponding to a linear series $l \in G^r_d(C)$. We first define the correspondence

$$\Sigma_C := \{(D,l) \in C_e \times G^r_d(C) : \dim l(-D) \geqslant r-e+f\},$$

and denote by $\pi_1 : \Sigma_C \to C_e$ and $\pi_2 : \Sigma_C \to G^r_d(C)$ the two projections. We assume that $\Sigma_C \neq \emptyset$ for the general curve $[C] \in \mathcal{M}_g$. Under this assumption, we show that

$$\dim(\Sigma_C) \leqslant \rho(g,r,d) - f(r+1-e+f) + e. \tag{13}$$

(We recall that the dimension of a scheme is the maximum of the dimensions of its irreducible components.) Since Σ_C is a determinantal subvariety of $C_e \times G^r_d(C)$, it follows that for a general $[C] \in \mathcal{M}_g$, if non-empty, the scheme Σ_C is equidimensional and $\dim(\Sigma_C) = \rho(g,r,d) - f(r+1-e+f)+e$. Note that this result does not establish the non-emptiness of Σ_C, which is an issue that we will deal with in Section 3. In any event, (13) implies the dimensional estimate

$$\dim\{l \in G^r_d(C) : V^{e-f}_e(l) \neq \emptyset\} \leqslant \rho(g,r,d) - f(r+1-e+f)+e.$$

This will prove Theorem 0.1 as well as Corollaries 0.3 and 0.4.

We start by setting some notation. We denote by $j : \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}_g$ the 'flag' map obtained by attaching to each stable curve $[R, x_1, \ldots, x_g] \in \overline{\mathcal{M}}_{0,g}$ fixed elliptic tails $E_1, \ldots, E_g$ at the points $x_1, \ldots, x_g$, respectively. Thus $j([R, x_1, \ldots, x_g]) := [\tilde{R}] = [R \bigcup_{x_1} E_1 \cup \ldots \bigcup_{x_g} E_g]$ and for such a curve, we denote by $p_R : \tilde{R} \to R$ the projection onto R; that is, $p_R(E_i) = \{x_i\}$ for $1 \leqslant i \leqslant g$. We denote by $\overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1}$ the universal curve and by $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the morphism forgetting the $(n+1)$st marked point. We write $\pi_e : \overline{\mathcal{C}}^e_{g,n} \to \overline{\mathcal{M}}_{g,n}$ for the e-fold fibre product of $\overline{\mathcal{C}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$, and we introduce a map $\chi : \overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}^e_g \to \overline{\mathcal{C}}^e_{0,g}$ which collapses the elliptic tails. Thus χ is defined by

$$\chi\big([R, x_1, \ldots, x_g], (y_1, \ldots, y_e)\big) := \big([R, x_1, \ldots, x_g], p_R(y_1), \ldots, p_R(y_e)\big),$$

for points $y_1, \ldots, y_e \in \tilde{R}$. Let $W \subset \overline{\mathcal{C}}^e_g$ be the closure of the locus

$$\{[C, y_1, \ldots, y_e] \in \mathcal{C}^e_g : \exists l \in G^r_d(C) \text{ with } \dim\, l(-y_1 - \ldots - y_e) \geqslant r - e + f\}.$$

By assumption $\pi_e(W) = \overline{\mathcal{M}}_g$, and we define the locus $U := \chi\big(W \cap (\overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}^e_g)\big)$. Then $\pi_e(U) = \overline{\mathcal{M}}_{0,g}$, and we denote by $e-m$ the minimal fibre dimension of the map $\pi_{e|U} : U \to \overline{\mathcal{M}}_{0,g}$. Thus $0 \leqslant m \leqslant e$ and $\dim(U \cap \pi_e^{-1}[R, x_1, \ldots, x_g]) \geqslant e - m$, for every $[R, x_1, \ldots, x_g]$, with equality for a general point $[R, x_1, \ldots, x_g] \in \overline{\mathcal{M}}_{0,g}$.

We recall that for every choice of four marked points $\{i,j,k,l\} \subset \{1, \ldots, g\}$, one has a fibration $\pi_{ijkl} : \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}_{0,4}$ obtained by forgetting the marked points with labels in the set $\{i,j,k,l\}^c$ and stabilizing the resulting rational curve. If we single out the first three marked points x_1, x_2, x_3 as being $0, 1$ and ∞, then we can obtain a birational map $\pi_{123} = (\pi_{1234}, \ldots, \pi_{123i}, \ldots, \pi_{123g}) : \overline{\mathcal{M}}_{0,g} \to \overline{\mathcal{M}}^{g-3}_{0,4} = (\mathbf{P}^1)^{g-3}$ defined by

$$\pi_{123}([R, x_1, \ldots, x_g]) := \big([R, x_1, x_2, x_3, x_4], [R, x_1, x_2, x_3, x_5], \ldots, [R, x_1, x_2, x_3, x_g]\big).$$

The map π_{123} expresses $\overline{\mathcal{M}}_{0,g}$ as a blow-up of $(\mathbf{P}^1)^{g-3}$ such that all exceptional divisors of π_{123} are boundary divisors of $\overline{\mathcal{M}}_{0,g}$ (cf. [**14**]). In a similar manner, one has a birational map $f : \overline{\mathcal{C}}^e_{0,g} \to \overline{\mathcal{M}}^{g-3+e}_{0,4} = (\mathbf{P}^1)^{g-3+e}$ defined by

$$\begin{aligned} &f\big([R, x_1, \ldots, x_g], y_1, \ldots, y_e\big) \\ &\quad := \big([R, x_1, x_2, x_3, x_4], \ldots, [R, x_1, x_2, x_3, x_g], [R, x_1, x_2, x_3, y_1], \ldots, [R, x_1, x_2, x_3, y_e]\big). \end{aligned}$$

For simplicity, sometimes we write $f([R, x_1, \ldots, x_g], y_1, \ldots, y_e) = (x_4, \ldots, x_g, y_1, \ldots, y_e)$. The maps f and π_{123} fit in a commutative diagram, where $p_1 : (\mathbf{P}^1)^{g-3+e} \to (\mathbf{P}^1)^{g-3}$ is the projection on the first $g-3$ factors.

$$\begin{tikzcd} \overline{\mathcal{C}}^e_{0,g} \arrow[r, "f"] \arrow[d, "\pi_e"] & (\mathbf{P}^1)^{g-3+e} = \overline{\mathcal{M}}^{g-3+e}_{0,4} \arrow[d, "p_1"] \\ \overline{\mathcal{M}}_{0,g} \arrow[r, "\pi_{123}"] & (\mathbf{P}^1)^{g-3} = \overline{\mathcal{M}}^{g-3}_{0,4} \end{tikzcd}$$

Finally, for $2 \leqslant k \leqslant e$ we define the diagonal loci $\Delta_k \subset (\mathbf{P}^1)^{g-3+e}$ as consisting of those points $(x_4, \ldots, x_g, y_1, \ldots, y_e)$ for which at least k of the points $y_1, \ldots, y_e$ coincide. We need the following result concerning existence of sublinear limit linear series of a fixed limit $\mathfrak{g}^r_d$, having prescribed vanishing sequence at a given point.

Lemma 2.1. *Let X be a curve of compact type and $Y \subset X$ an irreducible component, and let $p \in Y$ be a smooth point of X. Assume that l is a (refined) limit $\mathfrak{g}^r_d$ on X and let $(a_0 < a_1 < \ldots < a_r)$ be the vanishing sequence $a^l(p)$. We fix a subsequence $(a_{j_0} < a_{j_1} < \ldots < a_{j_b})$ of $a^l(p)$, where $0 \leqslant b \leqslant r$. Then there exists a limit $\mathfrak{g}^b_d$ on X, say $l' \subset l$, such that $a^{l'}(p) = (a_{j_0}, \ldots, a_{j_b})$.*

Proof. Let us denote by $l := \{l_Z = (L_Z, V_Z)\}_{Z \subset X}$ the original limit $\mathfrak{g}^r_d$ on X. For each integer $0 \leqslant k \leqslant b$ there exists a section $\sigma_{j_k} \in V_Y$ such that $\mathrm{ord}_p(\sigma_{j_k}) = a_{j_k}$. We consider the subspace $W_Y := \langle \sigma_{j_0}, \ldots, \sigma_{j_b} \rangle \subset V_Y$. Since $\#\{\mathrm{ord}_p(\sigma)\}_{\sigma \in W_Y} = b+1$, we obtain that $\dim(W_Y) = b+1$ and we set $l'_Y := (L_Y, W_Y) \in G^b_d(Y)$. Suppose now that Z is a component of X meeting Y in a point q. We denote by $(c_{j_0} < c_{j_1} < \ldots < c_{j_b})$ the vanishing sequence $a^{l'_Y}(q)$. Let $(e_{j_0} < e_{j_1} < \ldots < e_{j_b})$ be the complementary sequence; that is, $e_{j_k} = d - c_{j_{b-k}}$ for each $0 \leqslant k \leqslant b$. Then we can choose a section $\tau_k \in V_Z$ such that $\mathrm{ord}_q(\tau_k) = e_{j_k}$. We define $W_Z := \langle \tau_0, \ldots, \tau_b \rangle \subset V_Z$. Because all the entries $(e_{j_k})_{k=0}^b$ are distinct, we get that $\dim(W_Z) = b+1$ and then set $l'_Z := (L_Z, W_Z) \in G^b_d(Z)$. We continue inductively, and for each irreducible component $Z' \subset X$ we obtain an aspect $l'_{Z'} = (L_{Z'}, W_{Z'}) \in G^b_d(Z')$. The collection $\{l'_Z\}_{Z \subset X}$ is the desired limit $\mathfrak{g}^b_d$ on X. ☐

Next we explain how the assumption that for every $[C] \in \mathcal{M}_g$ there exists a linear series $l \in G^r_d(C)$ with $V^{e-f}_e(l) \neq \emptyset$ can be used to construct a flag curve $\tilde{R} \in j(\overline{\mathcal{M}}_{0,g})$ such that all the e points coming from the limit of an effective divisor $D \in V^{e-f}_e(l)$ specialize to a connected subcurve of $\tilde{R}$ having arithmetic genus at most $\min\{g, e\}$.

Proposition 2.2. *Let $U \subset \overline{\mathcal{C}}^e_{0,g}$ be an irreducible component of the closure of the locus of limits of e-secant divisors with respect to linear series $\mathfrak{g}^r_d$ on flag curves from $\overline{\mathcal{M}}_g$. Assuming that $\dim(U) = g - 3 + e - m$ with $0 \leqslant m \leqslant e$, there exists a point $([R, x_1, \ldots, x_g], \tilde{y}_1, \ldots, \tilde{y}_e) \in W \cap (\overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}^e_g)$ corresponding to a genus g flag curve*

$$\tilde{R} = R \bigcup_{x_1} E_1 \cup \ldots \bigcup_{x_g} E_g \text{ and points } \tilde{y}_1, \ldots, \tilde{y}_e \in \tilde{R},$$

such that either:

(i) *$\tilde{y}_1 = \ldots = \tilde{y}_e \in R - \{x_1, \ldots, x_g\}$, or else*

(ii) *all the points $\tilde{y}_1, \ldots, \tilde{y}_e$ lie on a connected subcurve $Y \subset \tilde{R}$ satisfying $p_a(Y) \leqslant \min\{m, g\}$ and $\#(Y \cap \overline{(\tilde{R} - Y)}) \leqslant 1$.*

Proof. We start by noting that if $m = 0$ then $U = \overline{\mathcal{C}}^e_{0,g}$ and possibility (i) is satisfied. Thus we may assume that $m \geqslant 1$. First, we claim that $\dim f(U) = \dim U = g - 3 + e - m$. Indeed, since $\pi_e(U) = \overline{\mathcal{M}}_{0,g}$ it follows that $p_1(f(U)) = (\mathbf{P}^1)^{g-3}$ and we choose a general point $t = (x_4, \ldots, x_g) \in (\mathbf{P}^1 - \{0, 1, \infty\})^{g-3}$ such that $x_i \neq x_j$ for $i \neq j$. Then $\pi_e^{-1}(t) = (\mathbf{P}^1)^e$ and $f_{|\pi_e^{-1}(t)}$ is an isomorphism onto its image, and hence $f_{|U}$ is birational onto its image as well. Obviously, when $m \geqslant g$ we can take $Y = \tilde{R}$. From now on we shall assume that $1 \leqslant m \leqslant g-1$.

Let us assume first that $f(U) \cap \Delta_e \neq \emptyset$. Then $\dim\big(f(U) \cap \Delta_e\big) \geqslant g - m - 2$. For dimensional reasons, there must exist a point $z = (x_4, \ldots, x_g, y_1, \ldots, y_1) \in f(U) \cap \Delta_e$ such that either: (i) at least $g - m - 3$ of the points x_j with $4 \leqslant j \leqslant g$ are mutually distinct and belong to the set $\mathbf{P}^1 - \{0, 1, \infty, y_1\}$ and $y_1 \in \mathbf{P}^1 - \{0, 1, \infty\}$, or (ii) at least $g - m - 2$ of the x_j $(4 \leqslant j \leqslant g)$ are mutually distinct and belong to the set $\mathbf{P}^1 - \{0, 1, \infty, y_1\}$ and then $y_1 \in \mathbf{P}^1$ may or may not be equal to one of the points $0, 1$ or ∞. Suppose that we are in situation (i), the remaining case being similar. We fix a point $([R, x_1, \dots, x_g], y_1, \dots, y_e) \in f^{-1}(z)$, and hence $y_1, \dots, y_e \in R$. If $Z \subset R$ denotes the minimal connected subcurve of R containing all the points $y_1, \dots, y_e$, then $x_1, x_2, x_3 \in R - Z$, unless $y_1 = \dots = y_e$. (In the latter case either $y_1 \in R - \{x_1, \dots, x_g\}$, which corresponds to the situation when all the points $\tilde{y}_i = y_i$ specialize to the same smooth point of $\tilde{R}$ lying on the rational spine, or else, if $y_1 = x_j$ for some $4 \leqslant j \leqslant g$, then we can find a connected subcurve of $\tilde{R}$ of genus 1 containing $\tilde{y}_1, \dots, \tilde{y}_e$, where $p_R(\tilde{y}_i) = y_i$ for $1 \leqslant i \leqslant e$.) Since at least $g - m = 3 + (g - m - 3)$ of the points $x_1, \dots, x_g$ lie on Z^c, it follows that $\tilde{y}_1, \dots, \tilde{y}_e$ lie on a connected subcurve of $\tilde{R}$ of genus at most m, which completes the proof in this case.

We are left with the possibility $f(U) \cap \Delta_e = \emptyset$ and we denote by $k \leqslant e - 1$ the largest integer for which $f(U) \cap \Delta_k \neq \emptyset$ and by L an irreducible component of $f(U) \cap \Delta_k$. Since by definition $f(U) \cap \Delta_{k+1} = \emptyset$, it follows that there exists a point $t_0 = (p_1, \dots, p_e) \in (\mathbf{P}^1)^e$ such that $L \subset (\mathbf{P}^1)^{g-3} \times \{t_0\}$. In particular, the projection map $p_{1|L} : L \to p_1(L)$ is $1:1$ and then $\dim p_1(L) = \dim(L) \geqslant g - m + (e - k - 2) \geqslant g - m$, unless $k = e - 1$, when $\dim p_1(L) \geqslant g - m - 1$. In the first case it follows that there exists a point $(x_4, \dots, x_g, p_1, \dots, p_e) \in f(U) \cap \Delta_k$ such that at least $g - m$ of the points $x_4, \dots, x_g$ are equal to a fixed point $r \in \mathbf{P}^1 - \{p_1, \dots, p_e\}$. In the second case, that is, when $k = e - 1$, since $\#\{p_i\}_{i=1}^e = 2$, one of the points $0, 1$ or ∞, say 0, does not appear among the p_i. Then we can find a point $(x_4, \dots, x_g, p_1, \dots, p_e) \in f(U) \cap \Delta_{e-1}$ with at least $g - m$ of the x_j equal to 0.

The conclusion in both cases is that there exists a point $\big([R, x_1, \dots, x_g], y_1, \dots, y_e\big) \in W \cap (\overline{\mathcal{M}}_{0,g} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{C}}_g^e)$ corresponding to the flag curve $\tilde{R} = R \bigcup_{x_1} E_1 \cup \dots \bigcup_{x_g} E_g$, such that the points $y_1, \dots, y_e$ lie on a connected subcurve $Y \subset \tilde{R}$, where $\#(Y \cap (\overline{\tilde{R} - Y})) \leqslant 1$ and $p_a(Y) \leqslant m \leqslant e$. □

Proof of Theorem 0.1. We choose $\tilde{R} = R \bigcup_{x_1} E_1 \cup \dots \bigcup_{x_g} E_g$ as above and denote by $Y \subset \tilde{R}$ a connected subcurve onto which the points $y_1, \dots, y_e$ specialize. We know that either: (a) $p_a(Y) = m \leqslant \min\{e, g\}$, or (b) $y_1 = \dots = y_e \in R - \{x_1, \dots, x_g\}$.

We first deal with case (a) and dispose of (b) at the end using [**9**]. If $m < g$ we set $Z := \overline{\tilde{R} - Y}$ and $\{p\} := Y \cap Z$, and we denote by Y' and Z' the components of Y and Z, respectively, containing the point p. When $m = g$, necessarily $e \geqslant g$ and $Y := \tilde{R}, Z = \emptyset$ and $p \in \tilde{R}$ is a general (smooth) point. By assumption, $[\tilde{R}, y_1, \dots, y_e] \in W$, and hence there exists a proper flat morphism $\phi : \mathcal{X} \to B$ satisfying the following properties.

- $\mathcal{X}$ is a smooth surface, B is a smooth affine curve, $0 \in B$ is a point such that $\phi^{-1}(0)$ is a curve stably equivalent to $\tilde{R}$ and $X_t = \phi^{-1}(t)$ is a smooth projective curve of genus g for $t \neq 0$. Moreover, there are e sections $\sigma_i : B \to \mathcal{X}$ of ϕ satisfying the condition $\sigma_i(0) = y_i \in \phi^{-1}(0)_{reg}$ for all $1 \leqslant i \leqslant e$.
- If $X_\eta := \mathcal{X} - \phi^{-1}(0)$, then there exist a line bundle $L_\eta \in \mathrm{Pic}(X_\eta)$ of relative degree d and a subvector bundle $V_\eta \subset \phi_*(L_\eta)$ having rank $r + 1$, such that for $t \neq 0$ we have

$$\dim V_t \cap H^0\left(X_t, L_t\left(-\sum_{j=1}^{e} \sigma_j(t)\right)\right) = r + 1 - e + f.$$

After possibly making a finite base change and resolving the resulting singularities, the pair (L_η, V_η) induces a (refined) limit $\mathfrak{g}^r_d$ on $\tilde{R}$, which we denote by $\mathfrak{l}$. The vector bundle $V_\eta \cap \phi_*\big(L_\eta \otimes \mathcal{O}_{X_\eta}(-\sum_{j=1}^e \sigma_j(B - \{0\}))\big)$ induces a limit linear series $\mathfrak{g}^{r-e+f}_{d-e}$ on $\phi^{-1}(0)$, which we denote by $\mathfrak{m}$. For a component A of $\phi^{-1}(0)$, if $(L_A, V_A) \in G^r_d(A)$ denotes the A-aspect of $\mathfrak{l}$, then there exists a unique *effective* divisor $D_A \in A_e$ supported only at the points from $(A \cap \bigcup_{j=1}^e \sigma_j(B)) \cup (A \cap \overline{\phi^{-1}(0) - A})$ such that the A-aspect of $\mathfrak{m}$ is of the form

$$\mathfrak{m}_A = \big(M_A := L_A \otimes \mathcal{O}_A(-D_A),\ W_A \subset V_A \cap H^0(M_A)\big) \in G^{r-e+f}_{d-e}(A).$$

The collection $\mathfrak{m}_Y := \{\mathfrak{m}_A\}_{A\subset Y}$ forms a limit $\mathfrak{g}^{r-e+f}_{d-e}$ on Y. We denote by $(a_0 < \ldots < a_r)$ the vanishing sequence of $\mathfrak{l}_{Y'}$ at p, thus $\{a_i\}_{i=0}^r = \{\mathrm{ord}_p(\sigma)\}_{\sigma\in V_{Y'}}$ and we denote by $(b_0 < \ldots < b_r)$ the vanishing sequence $a^{\mathfrak{l}_{Z'}}(p)$. By ordering the set $\{\mathrm{ord}_p(\sigma)\}_{\sigma\in W_{Y'}}$ we obtain a subsequence $(a_{i_0} < \ldots < a_{i_{r-e+f}})$ of $a^{\mathfrak{l}_{Y'}}(p)$. When we order the entries in $\{a_i\}_{i=0}^r - \{a_{i_k}\}_{k=0}^{r-e+f}$ we obtain a new sequence $(a_{j_0} < a_{j_1} < \ldots < a_{j_{e-f-1}})$. Using Lemma 2.1, we find that there exists a limit linear series $\mathfrak{l}'_Y$ of type $\mathfrak{g}^{e-f-1}_d$ on Y with the property that $a^{\mathfrak{l}'_Y}(p) = (a_{j_0}, a_{j_1}, \ldots, a_{j_{e-f-1}})$.

Let us assume first that we are in the situation $m < g$; hence $Z \neq \emptyset$. The point $p \in Y$ lies on a rational component that implies the following inequality corresponding to Y (see also [**9**, Theorem 1.1]):

$$V_1 := \rho(m, e-f-1, d) - \sum_{k=0}^{e-f-1} a_{j_k} + \binom{e-f}{2} \geqslant 0. \tag{14}$$

Applying the same principle for the limit linear series $\mathfrak{m}_Y$ on Y, we find that the adjusted Brill–Noether number with respect to the point p is non-negative:

$$V_2 := \rho(m, r-e+f, d-e) - \sum_{k=0}^{r-e+f} a_{i_k} + \binom{r+1-e+f}{2} \geqslant 0. \tag{15}$$

Next we turn our attention to Z and use the fact that the point $p \in Z$ does not lie on an elliptic component; hence $[Z, p]$ satisfies the 'strong' pointed Brill–Noether theorem:

$$V_3 := \rho(g-m, r, d) - \sum_{k=0}^{r} b_k + \binom{r+1}{2} \geqslant 0. \tag{16}$$

If we add (14)–(16) together and use the fact that $\sum_{k=0}^r b_k + \sum_{k=0}^{r-e+f} a_{i_k} + \sum_{k=0}^{e-f-1} a_{j_k} = (r+1)d$, then we obtain the inequality

$$\rho(g, r, d) - f(r+1-e+f) + e \geqslant e - m \geqslant 0.$$

The case $m = g$, when $Y = \tilde{R}$, is similar but simpler. We add together (14) and (15) (now there is no (16)) and we write the following inequalities:

$$\begin{aligned}\rho(g,r,d) + e - f(r+1-e+f) &= \left(\rho(g, r-e+f, d-e) - \sum_{k=0}^{r-e+f} a_{i_k} + \binom{r+1-e+f}{2}\right)\\ &\quad + \left(\rho(g, e-f-1, d) - \sum_{k=0}^{e-f-1} a_{j_k} + \binom{e-f}{2}\right) + \sum_{k=0}^{r-e+f} a_{i_k}\\ &\quad + \sum_{k=0}^{e-f-1} a_{j_k} - \binom{r+1}{2} + e - g \geqslant e - g\\ &\geqslant 0,\end{aligned}$$

since $\sum_{k=0}^{r-e+f} a_{i_k} + \sum_{k=0}^{e-f-1} a_{j_k} \geqslant \binom{r+1}{2}$. Thus we obtain the same numerical conclusion as in the case $m < g$.

Assume now that we are in the case (b) when $y_1 = \ldots = y_e \in R - \{x_1, \ldots, x_g\}$. Then, reasoning as above, we find a limit $\mathfrak{g}^r_d$ on $\tilde{R}$ having a vanishing sequence at y_1 at least $(0, 1, \ldots, e-f-1, e, e+1, \ldots, r+f-1, r+f)$. Using once more [**9**, Theorem 1.1], we obtain the inequality

$$\rho(g, r, d) + e - f(r+1-e+f) \geqslant \rho(g, r, d) - f(r+1-e+f) \geqslant 0.$$

Using the semicontinuity of the dimension of the fibres, it follows that for a general curve $[C] \in \mathcal{M}_g$, if $\pi_1 : \Sigma_C \to C_e$ is the first projection, then the minimal fibre dimension of π_1 cannot exceed the dimension of the space of pairs of limit linear series $\mathfrak{l} \supset \mathfrak{m}$ consisting of a $\mathfrak{g}^r_d \supset \mathfrak{g}^{r-e+f}_{d-e}$ on the flag curve $\phi^{-1}(0)$ such that $\mathfrak{m} = \mathfrak{l}(-D_e)$, where D_e is a degree e effective divisor on $\phi^{-1}(0)$ with the property that $\mathrm{supp}(D_e) \subset Y \cap \phi^{-1}(0)_{\mathrm{reg}}$. Since the map $(\mathfrak{l} \supset \mathfrak{m}, \mathfrak{m}_Y, \mathfrak{l}'_Y) \mapsto (\mathfrak{m}_Y, \mathfrak{l}'_Y, \mathfrak{l}_Z) \in \tilde{G}^{r-e+f}_{d-e}(Y) \times \tilde{G}^{e-f-1}_d(Y) \times \tilde{G}^r_d(Z)$ is injective, it follows that for a general divisor $D_{\mathrm{gen}} \in \pi_1(\Sigma_C)$ we have the estimate

$$\dim \pi_1^{-1}(D_{\mathrm{gen}}) \leqslant V_1 + V_2 + V_3 = \rho(g, r, d) - f(r + 1 - e + f) + m,$$

and hence $\dim(\Sigma_C) = \dim \pi_1^{-1}(D_{\mathrm{gen}}) + e - m \leqslant \rho(g, r, d) - f(r + 1 - e + f) + e$. This finishes the proof of Theorem 0.1. $\square$

3. *Existence of linear series with secant planes*

We turn our attention to showing the existence of linear series that possess e-secant $(e - f - 1)$-planes. The strategy we pursue is to construct limit linear series $\mathfrak{g}^r_d$ on a curve of compact type $[Y \bigcup_p Z] \in \overline{\mathcal{M}}_g$, where (Y, p) and (Z, p) are suitably general smooth pointed curves of genus e and $g - e$, respectively. These limit $\mathfrak{g}^r_d$ will carry a sublinear series $\mathfrak{g}^{r-e+f}_{d-e} = \mathfrak{g}^r_d(-D_e)$, where D_e is a degree e effective divisor on Y. As in the proof of Theorem 0.1, such $\mathfrak{g}^r_d$ are determined by their Z-aspect and by a pair of linear series $(\mathfrak{g}^{r-e+f}_{d-e}, \mathfrak{g}^{e-f-1}_d)$ on Y. We determine the dimension of the space of such pairs, which will enable us to show that the original pair $(\mathfrak{g}^{r-e+f}_{d-e}, \mathfrak{g}^{e-f-1}_d)$ on $Y \bigcup_p Z$ can be smoothed to every smooth curve of genus g. This will complete the proof of Theorem 0.5.

We start by choosing two general pointed curves $[Y, p] \in \mathcal{M}_{e,1}$ and $[Z, p] \in \mathcal{M}_{g-e,1}$ such that both (Y, p) and (Z, p) satisfy the Brill–Noether theorem with prescribed ramification (cf. [**9**, Theorem 1.1 and Proposition 1.2]). If $\overline{\alpha} : 0 \leqslant \alpha_0 \leqslant \ldots \leqslant \alpha_r \leqslant d - r$ is a Schubert index of type (r, d), then (Y, p) possesses a $\mathfrak{g}^r_d$ with ramification sequence at least $\overline{\alpha}$ at the point p, if and only if

$$\sum_{i=0}^{r} \max\{\alpha_i + g(Y) - d + r, 0\} \leqslant g(Y). \tag{17}$$

In the case where this inequality is satisfied, $\dim G^r_d(Y, p, \overline{\alpha}) = \rho(g, r, d, \overline{\alpha})$ (one obviously has a similar statement for $[Z, p]$).

We denote by $\pi : \mathcal{X} \to (T, 0)$ the versal deformation space of the stable curve $\pi^{-1}(0) = X_0 := Y \bigcup_p Z$. Let $\Delta \subset T$ be the boundary divisor corresponding to singular curves, and write $\pi^{-1}(\Delta) = \Delta_e + \Delta_{g-e}$, where Δ_e and Δ_{g-e} are the divisors corresponding to the marked points lying on the components of genus e and $g - e$, respectively. We consider the e-fold fibre product $\mathcal{U} := (\mathcal{X} - \Delta_{g-e}) \times_T \ldots \times_T (\mathcal{X} - \Delta_{g-e})$, the projection $\phi : \mathcal{U} \to T$ and the induced curve $p_2 : \mathcal{X} \times_T \mathcal{U} \to \mathcal{U}$. Then we introduce the stack of limit linear series of type $\mathfrak{g}^r_d$ over $\mathcal{U}$:

$$\sigma : \widetilde{\mathfrak{G}}^r_d(\mathcal{X} \times_T \mathcal{U}/\mathcal{U}) \longrightarrow \mathcal{U}, \quad \text{where } \widetilde{\mathfrak{G}}^r_d(p_2) = \widetilde{\mathfrak{G}}^r_d(\mathcal{X} \times_T \mathcal{U}/\mathcal{U}) = \widetilde{\mathfrak{G}}^r_d(\pi) \times_T \mathcal{U},$$

and we write $\tau := \phi \circ \sigma : \widetilde{\mathfrak{G}}^r_d(p_2) \to T$ (see [**8**, Theorem 3.4], for details on the construction of $\widetilde{\mathfrak{G}}^r_d(\pi)$). The fibre $\tau^{-1}(t)$ corresponding to a point $t \in \Delta$ (in which case one can write $\pi^{-1}(t) = Y_t \cup Z_t$, with $g(Y_t) = e, g(Z_t) = g - e$), parametrizes limit $\mathfrak{g}^r_d$ on $Y_t \cup Z_t$ together with e-tuples $(x_1, \ldots, x_e) \in (Y_t - Y_t \cap Z_t)^e$. Let us denote by $\mathcal{L}_Y$ a degree d Poincaré bundle on $\pi_2 : \mathcal{X} \times_T \widetilde{\mathfrak{G}}^r_d(p_2) \to \widetilde{\mathfrak{G}}^r_d(p_2)$ characterized by the property that its restriction to curves of type $Y_t \cup Z_t$ are line bundles of bidegree $(d, 0)$. We also write $\mathcal{V}_Y \subset (\pi_2)_*(\mathcal{L}_Y)$ for the rank $r + 1$ tautological bundle with fibres that correspond to the global sections of the genus e-aspect of each limit $\mathfrak{g}^r_d$. Finally, for $1 \leqslant j \leqslant e$, we denote by $D_j \subset \mathcal{X} \times_T \widetilde{\mathfrak{G}}^r_d(p_2)$ the diagonal divisor corresponding to pulling back the diagonal under the map $\mathcal{X} \times_T \widetilde{\mathfrak{G}}^r_d(p_2) \to \mathcal{X} \times_T \mathcal{X}$ which projects onto the jth factor; that is, $(x, l, x_1, \ldots, x_e) \mapsto (x, x_j)$, where $x, x_1, \ldots, x_e \in \pi^{-1}(t)$.

There exists an evaluation vector bundle morphism over $\widetilde{\mathfrak{G}}^r_d(p_2)$

$$\chi : \mathcal{V}_Y \longrightarrow (\pi_2)_*(\mathcal{L}_Y \otimes \mathcal{O}_{\sum_{j=1}^e D_j}),$$

and we denote by $\mathcal{H}$ the rank $e-f$ degeneracy locus of the map χ. Set-theoretically, $\mathcal{H}$ consists of those points $(t, l, x_1, \dots, x_e)$ with $\phi(x_1, \dots, x_e) = t \in T$ and $l \in \widetilde{G}^r_d(\pi^{-1}(t))$, satisfying the condition that $\dim l(-x_1 - \ldots - x_e) \geqslant r+1-e+f$. The dimension of every irreducible component of $\mathcal{H}$ is at least $\rho(g,r,d) + \dim T + e - f(r+1-e+f)$.

In order to show that $\tau : \mathcal{H} \to T$ is dominant, it suffices to prove that $\tau^{-1}(0)$ has at least one irreducible component of dimension $\rho(g,r,d) + e - f(r+1-e+f)$. This, in fact, will prove the stronger statement that $\Sigma_C \neq \emptyset$ for *every* $[C] \in \mathcal{M}_g$. Indeed, even though $\tau : \widetilde{\mathfrak{G}}^r_d(p_2) \to T$ is not a proper morphism, the restriction $\tau_{\tau^{-1}(T-\Delta)} : \tau^{-1}(T-\Delta) \to T-\Delta$ is proper, and hence there exists an irreducible component of $\mathcal{H}$ which maps onto $T-\Delta$. Since $\pi : \mathcal{X} \to (T, 0)$ can be chosen in such a way that there exists a point $t \in T$ with $\pi^{-1}(t) \cong C$, this proves our contention. We set the integer

$$\alpha_0 := \left[\frac{\rho(e, r-e+f, d-e)}{r+1-e+f}\right] = \left[\frac{e}{r+1-e+f}\right] + d - r - f - e;$$

thus we can write $\rho(e, r-e+f, d-e) = \alpha_0 \cdot (r+1-e+f) + c$, where $0 \leqslant c \leqslant r-e+f$. Then there exists a unique Schubert index of type $(r-e+f, d-e)$,

$$\overline{\alpha} : 0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_{r-e+f} \leqslant d-r-f,$$

with $\alpha_{r-e+f} - \alpha_0 \leqslant 1$, such that $\sum_{j=0}^{r-e+f} \alpha_j = \rho(e, r-e+f, d-e)$. We have $\alpha_j = \alpha_0$ for $0 \leqslant j \leqslant r-e+f-c$ and $\alpha_j = \alpha_0 + 1$ for $r-e+f-c+1 \leqslant j \leqslant r-e+f$. Note that since $\alpha_0 + g(Y) - (d-e) + r - e + f = [e/(r+1-e+f)] \geqslant 0$, condition (17) is verified and the variety $G^{r-e+f}_{d-e}(Y, p, \overline{\alpha})$ is non-empty of dimension $\rho(e, r-e+f, d-e) - \sum_{j=0}^{r-e+f} \alpha_j = 0$.

Next we set $\beta_0 := [e/(e-f)]$ and write $e = \beta_0 \cdot (e-f) + \tilde{c}$, where $0 \leqslant \tilde{c} \leqslant e-f-1$. Then there exists a unique Schubert index of type $(e-f-1, 2e-f-1)$,

$$\overline{\beta} : 0 \leqslant \beta_0 \leqslant \beta_1 \leqslant \ldots \leqslant \beta_{e-f-1} \leqslant e,$$

such that $\beta_{e-f+1} - \beta_0 \leqslant 1$ and $\sum_{j=0}^{e-f-1} \beta_j = e$. Precisely,

$$\beta_j = \beta_0 \quad \text{for } 0 \leqslant j \leqslant e-f-\tilde{c}-1 \quad \text{and} \quad \beta_j = \beta_0 + 1 \quad \text{for } e-f-\tilde{c} \leqslant j \leqslant e-f-1.$$

By (17), the variety $G^{e-f-1}_{2e-f-1}(Y, p, \overline{\beta})$ is non-empty and is of dimension $e - \sum_{j=0}^{e-f-1} \beta_j = 0$.

First, we are going to prove Theorem 0.5 under the assumption that there exist two linear series $(A, W_A) \in G^{r-e+f}_{d-e}(Y, p, \overline{\alpha})$ and $(L, W_L) \in G^{e-f-1}_{2e-f-1}(Y, p, \overline{\beta})$ satisfying the condition

$$H^0\big(Y, L \otimes A^\vee \otimes \mathcal{O}_Y((d+f-2e) \cdot p)\big) = 0. \tag{18}$$

Note that $\deg\big(L \otimes A^\vee \otimes \mathcal{O}_Y((d+f-2e) \cdot p\big) = g(Y) - 1$, and (18) states that a suitable translate of at least one of the finitely many line bundles of type $L \otimes A^\vee$ lies outside the theta divisor of Y.

Remark 3.1. Condition (18) is a subtle statement concerning $[Y, p]$. It is not true that (18) holds for *every* choice of $(A, W_A) \in G^{r-e+f}_{d-e}(Y, p, \overline{\alpha})$ and $(L, W_L) \in G^{e-f-1}_{2e-f-1}(Y, p, \overline{\beta})$. For instance, in the case $e = 2r-2$ and $f = r-1$, corresponding to $(2r-2)$-secant $(r-2)$-planes which every curve $Y \subset \mathbf{P}^r$ is expected to possess in finite number, we obtain that $A = B \otimes \mathcal{O}_Y((d-3r+2) \cdot p)$, where $B \in W^1_r(Y)$ and $L \otimes \mathcal{O}_Y(-2p) \in W^{r-2}_{3r-6}(Y)$. By Riemann–Roch, we can write that $L = K_Y \otimes \mathcal{O}_Y(2 \cdot p) \otimes \tilde{B}^\vee$, where $\tilde{B} \in W^1_r(Y)$ and then (18) translates into the vanishing statement $H^0(Y, B \otimes \tilde{B} \otimes \mathcal{O}_Y(-3 \cdot p)) = 0$. The curve Y has $(2r-2)!/r!(r-1)!$ pencils $\mathfrak{g}^1_r$. If we choose $B \neq \tilde{B} \in W^1_r(Y)$, then $h^0(Y, B \otimes \tilde{B}) \geqslant 4$ and (18) has no chance of being satisfied. If $B = \tilde{B}$, then the Gieseker–Petri theorem implies that the map $H^0(Y, B) \otimes$

$H^0(Y, K_Y \otimes B^\vee) \to H^0(Y, K_Y)$ is an isomorphism, whence $h^0(Y, B^{\otimes 2}) = 3$. Choosing $p \in Y$ outside the set of ramification points of the finitely many line bundles $B^{\otimes 2}$, where $B \in W^1_r(Y)$, we obtain that $H^0(B^{\otimes 2} \otimes \mathcal{O}_Y(-3 \cdot p)) = 0$. Therefore in this case, condition (18) is equivalent to the Gieseker–Petri theorem.

We shall study when (18) is actually satisfied. We note that by the Riemann–Roch theorem, (18) also implies that $h^0\big(Y, L \otimes A^\vee \otimes \mathcal{O}_Y((d+f-2e+1)\cdot p)\big) = 1$. Assuming that $(A, W_A) \in G^{r-e+f}_{d-e}(Y, p, \overline{\alpha})$ and $(L, W_L) \in G^{e-f-1}_{2e-f-1}(Y, p, \overline{\beta})$ satisfy (18), it follows from Riemann–Roch that there exists a unique effective divisor of degree e

$$D \in |L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p) \otimes A^\vee|,$$

and moreover $p \notin \mathrm{supp}(D)$. We introduce the space of sections

$$V_Y := W_A + W_L \subset H^0\big(Y, L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p)\big),$$

where we view

$$W_A \subset H^0\big(L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p - D)\big)$$

and

$$W_L \subset H^0(L) \subset H^0\big(L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p)\big).$$

We claim that $\dim(V_Y) = r+1$, and hence $\mathfrak{l}_Y = (L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p), V_Y) \in G^r_d(Y)$. Moreover, $\mathfrak{l}_Y$ has the following vanishing sequence at p:

$$\begin{aligned} a^{\mathfrak{l}_Y}(p) = (&\alpha_0, \ldots, \alpha_{r-e+f} + r - e + f, \beta_0 + d - 2e + f + 1, \\ &\beta_1 + d - 2e + f + 2, \ldots, \beta_{e-f-1} + d - e). \end{aligned} \tag{19}$$

Indeed, our original assumption that $f(r+1-e+f) \geqslant e$ is equivalent with the inequality $\alpha_{r-e+f} + r - e + f < d - 2e + f + 1$, which shows that sequence (19) contains $r+1$ distinct entries. Since $p \notin \mathrm{supp}(D)$, we obtain that the vanishing orders of the sections from $W_A \subset H^0(L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p))$ are precisely

$$\alpha_0, \alpha_1 + 1, \ldots, \alpha_{r-e+f} + r - e + f,$$

while those of the sections from $W_L \subset H^0(L \otimes \mathcal{O}_Y((d-2e+f+1)\cdot p))$ are precisely

$$\begin{aligned} &\beta_0 + d - 2e + f + 1, \beta_1 + d - 2e + f + 2, \ldots, \beta_{e-f-1} + e - f - 1 + d - 2e + f + 1 \\ &\quad = \beta_{e-f-1} + d - e. \end{aligned}$$

We have found $r+1$ sections from V_Y having distinct vanishing orders at the point p, and hence $\dim(V_Y) = r+1$. Moreover, $a^{\mathfrak{l}_Y}(p)$ is equal to sequence (19).

Next we choose a linear series $\mathfrak{l}_Z \in G^r_d(Z, p)$ such that $\{\mathfrak{l}_Y, \mathfrak{l}_Z\}$ is a refined limit $\mathfrak{g}^r_d$. Then the ramification sequence of $\mathfrak{l}_Z$ at the point p must be equal to

$$\begin{aligned} \alpha^{\mathfrak{l}_Z}(p) = \overline{\gamma} := (&e - \beta_{e-f-1}, e - \beta_{e-f-2}, \ldots, e - \beta_0, d - r \\ &- \alpha_{r-e+f}, \ldots, d - r - \alpha_1, d - r - \alpha_0). \end{aligned}$$

We claim that condition (17) is satisfied for Z and that the variety $G^r_d(Z, p, \overline{\gamma})$ is non-empty and is of dimension $\rho(g-e, r, d, \overline{\gamma}) = \rho(g, r, d) + e - f(r+1-e+f)$. For this to happen, one has to check that the following inequality holds:

$$\sum_{j=0}^{r} \max\left\{\alpha_j^{\mathfrak{l}_Z}(p) + g - e - d + r, 0\right\} \leqslant g - e. \tag{20}$$

There are two things to notice. First, by direct computation we have

$$\alpha^{\mathfrak{l}_Z}_{e-f}(p)+g-e-d+r=g-e-\alpha_{r-e+f}=(g-d+r)+\Big[f-\frac{e}{r+1-e+f}\Big]\geqslant 0,$$

and hence $\alpha^{\mathfrak{l}_Z}_j(p)+g-e-d+r\geqslant 0$ for all $e-f\leqslant j\leqslant r$. Second, since $0\leqslant \beta_{e-f-1}-\beta_0\leqslant 1$, in order to estimate the sum of the first $e-f$ terms in the sum (20), there are two cases to consider. Either $\alpha^{\mathfrak{l}_Z}_0(p)+g-e-d+r\geqslant 0$, in which case we find that

$$\begin{aligned}\sum_{j=0}^{r}\max\{\alpha^{\mathfrak{l}_Z}_j(p)+g-e-d+r,0\}&=\sum_{j=0}^{r}(\alpha^{\mathfrak{l}_Z}_j(p)+g-e-d+r)=g-e-\rho(g-e,r,d,\overline{\gamma})\\&=g-e-\big(\rho(g,r,d)+e-f(r+1-e+f)\big)\\&\leqslant g-e.\end{aligned}$$

Otherwise, if $\alpha^{\mathfrak{l}_Z}_0(p)+g-e-d+r\leqslant -1$, then also $\alpha^{\mathfrak{l}_Z}_j(p)+g-e-d+r\leqslant 0$ for $0\leqslant j\leqslant e-f-1$, and the left-hand side of (20) equals

$$\begin{aligned}\sum_{j=e-f}^{r}(\alpha^{\mathfrak{l}_Z}_j(p)+g-e-d+r)&=(r+1-e+f)(g-e)-\sum_{i=0}^{r-e+f}\alpha_i\\&=g-e-\rho(g,r-e+f,d-e)\\&\leqslant g-e.\end{aligned}$$

In both cases inequality (17) is satisfied, which proves our claim.

Since the chosen $(A,W_A)\in G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})$ and $(L,W_L)\in G^{e-f-1}_{2e-f-1}(Y,p,\overline{\beta})$ are isolated points in their corresponding varieties of linear series on Y, it follows that limit $\mathfrak{g}^r_d$ on X_0, constructed in the way we have just described, fill up a component of $\tau^{-1}(0)\subset\mathcal{H}$.

Indeed, suppose that $(\mathfrak{n}_Y,\mathfrak{n}_Z,\tilde{D})\in\mathcal{H}$ is a point lying in the same irreducible component of $\tau^{-1}(0)$ as $(\mathfrak{l}_Y,\mathfrak{l}_Z,D)$. Here, $\mathfrak{n}_Y\in G^r_d(Y),\mathfrak{n}_Z\in G^r_d(Z,p,\overline{\gamma})$, and $\tilde{D}\in Y_e$ is a divisor such that $p\notin\mathrm{supp}(\tilde{D})$. Then $a^{\mathfrak{n}_Y}(p)=a^{\mathfrak{l}_Y}(p)$, which is given by (19); therefore $\mathfrak{n}_Y(-(d-2e+f+1)\cdot p)\in G^{2e-f-1}_{e-f-1}(Y,p,\overline{\beta})$, which is a reduced 0-dimensional variety. This implies that $\mathfrak{n}_Y(-(d-2e+f+1)\cdot p)=(L,W_L)$. Next, we consider the linear series $\mathfrak{n}_Y(-\tilde{D})\in G^{r-e+f}_{d-e}(Y)$. Since $p\notin\mathrm{supp}(\tilde{D})$, the vanishing sequence of this linear series is a subsequence of length $r+1-e+f$ of $a^{\mathfrak{l}_Y}(p)$. Necessarily, $\alpha^{\mathfrak{n}_Y(-\tilde{D})}(p)\geqslant\overline{\alpha}$ and because $\rho(e,r-e+f,d-e,\overline{\alpha})=0$, we must have $\mathfrak{n}_Y(-\tilde{D})\in G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})$, which is a discrete set, and hence $\mathfrak{n}_Y(-\tilde{D})=(A,W_A)$ and $\tilde{D}=D\in Y_e$. This shows that $\mathfrak{n}_Y=\mathfrak{l}_Y$ and every point of this component of $\tau^{-1}(0)$ is determined by the $\mathfrak{n}_Z$. The dimension of this component is thus equal to

$$\begin{aligned}&\rho(e,r-e+f,d-e,\overline{\alpha})+\rho(g-e,r,d,\overline{\gamma})+\rho(e,e-f-1,2e-f-1,\overline{\beta})\\&\quad=\rho(g,r,d)-f(r+1-e+f)+e,\end{aligned}$$

which finishes the proof of Theorem 0.5, subject to proving assumption (18).

REMARK 3.2. A slight variation of the argument described above enables us to prove Theorem 0.5 even in some cases when we cannot establish (18). We start with a linear series $(A,W_A)\in G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})$ and assume that the following condition holds:

$$H^0\big(Y,\mathcal{O}_Y((d-1)\cdot p)\otimes A^\vee\big)=0. \tag{21}$$

There exists a unique divisor $D\in|\mathcal{O}_Y(d\cdot p)\otimes A^\vee)|$, and (21) guarantees that $p\notin\mathrm{supp}(D)$. We define the space of sections

$$V_Y:=H^0(\mathcal{O}_Y(2e-f-1)\cdot p)+W_A\subset H^0(\mathcal{O}_Y(d\cdot p)),\quad\text{where } W_A\subset H^0(\mathcal{O}_Y(d\cdot p-D)).$$

Reasoning along the same lines as in the previous case, since $p \notin \mathrm{supp}(D)$ we find that $\dim(V_Y) = r+1$, and hence $\mathfrak{l}_Y = (\mathcal{O}_Y(d\cdot p), V_Y) \in G^r_d(Y)$. Moreover, we can check that

$$a^{\mathfrak{l}_Y}(p) = (\alpha_0, \alpha_1+1, \ldots, \alpha_{r-e+f}+r-e+f, d-2e+f+1, d-2e+f+2, \ldots, d-e-1, d).$$

As in the previous situation, we choose a linear series $\mathfrak{l}_Z \in G^r_d(Z,p)$ such that $\{\mathfrak{l}_Y, \mathfrak{l}_Z\}$ is a refined limit $\mathfrak{g}^r_d$. Thus we must have the following ramification sequence at p:

$$\alpha^{\mathfrak{l}_Z}(p) = \overline{\gamma} := (0, e, \ldots, e, d-r-\alpha_{r-e+f}, \ldots, d-r-\alpha_1, d-r-\alpha_0).$$

Condition (17) which guarantees the existence of $\mathfrak{l}_Z$ is satisfied if and only if

$$\rho(g,r,d) \geqslant f(r+1-e+f) - (g-d+r), \quad \text{in the case } g-d+r < e$$

and

$$\rho(g,r,d) \geqslant f(r+1-e+f) - e, \quad \text{in the case } g-d+r \geqslant e.$$

Since we are always working under the hypothesis that $\rho(g,r,d) - f(r+1-e+f) + e \geqslant 0$, we see that the previous condition holds whenever $g-d+r \geqslant e$, and that, in general, $\mathfrak{l}_Z \in G^r_d(Z,p,\overline{\gamma})$ exists if and only if

$$\rho(g,r,d) \geqslant f(r+1-e+f) - (g-d+r). \tag{22}$$

Assuming (22), the variety $G^r_d(Z,p,\overline{\gamma})$ is non-empty of dimension $\rho(g-e,d,r,\overline{\gamma}) = \rho(g,r,d) - f(r+1-e+f)+e$. The same argument as before shows that limit $\mathfrak{g}^r_d$ on X_0, constructed in such a way, fill up a component of $\tau^{-1}(0) \subset \mathcal{H}$ of expected dimension $\rho(g,r,d) - f(r+1-e+f)+e$, which completes the proof.

Now we complete the proof of Theorem 0.5 by discussing under which assumptions we can establish (18).

Proof of Theorem 0.5. We retain the notation introduced above and show that there exist two linear series $(A, W_A) \in G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})$ and $(L, W_L) \in G^{e-f-1}_{2e-f-1}(Y,p,\overline{\beta})$ satisfying (18) whenever one of the following conditions is satisfied:

(i) $2f \leqslant e-1$,
(ii) $e = 2r-2$ and $f = r-1$,
(iii) $e < 2(r+1-e+f)$.

As we have already explained, (18) in case (ii) is a consequence of the Gieseker–Petri theorem.

We now treat case (i) when $\beta_0 = 1$ and $\tilde{c} = f \leqslant e-f-1$. By Riemann–Roch we find that $L = K_Y \otimes \mathcal{O}_Y((e-2f+2)\cdot p) \otimes B^\vee$, where $B \in W^1_{e-f+1}(Y)$ is a pencil such that $h^0\big(Y, B\otimes \mathcal{O}_Y(-(e-2f+1)\cdot p)\big) \geqslant 1$ (There are finitely many such $B \in W^1_{e-f+1}(Y)$ for a generic choice of $[Y,p] \in \mathcal{M}_{e,1}$). Applying the base-point-free pencil trick, (18) is equivalent to the injectivity of the multiplication map

$$\mu_{B,M} : H^0(Y,B) \otimes H^0(Y,M) \longrightarrow H^0(Y, B\otimes M),$$

where $M := K_Y \otimes A^\vee \otimes \mathcal{O}_Y((d-f-e+2)\cdot p) \in W^{e-f}_{2e-f}(Y)$ is a complete linear series with vanishing sequence at p equal to

$$a^M(p) = (0,1,\ldots,e-f-a-1, e-f-a+c, r-a+2, r-a+3, \ldots, r, r+1). \tag{23}$$

Here we have set $a := [e/(r+1-e+f)]$, and hence we can write $e = a\cdot(r+1-e+f)+c$, where $0 \leqslant c \leqslant r-e+f$. By assumption we have $e-2a > c$ and clearly $\rho(M, \alpha^M(p)) = 0$; that is, there are finitely many $M \in W^{e-f}_{2e-f}(Y)$ satisfying (23).

To prove that $\mu_{B,M}$ is injective, we degenerate $[Y,p] \in \mathcal{M}_{e,1}$ to a particular stable curve: $[Y_0,p_0] := [E_0 \bigcup_{p_1} E_1 \cup \ldots \cup E_{e-2a-1} \bigcup_{p_{e-2a}} T, p_0]$, where $E_0, \ldots, E_{e-2a-1}$ are elliptic curves, $[T = E_{e-2a}, p_{e-2a}] \in \mathcal{M}_{2a,1}$ is a Petri general smooth pointed curve and the points $p_i, p_{i+1} \in E_i$

are such that $p_{i+1}-p_i\in \mathrm{Pic}^0(E_i)$ is not a torsion class for $0\leqslant i\leqslant e-2a-1$. Note that p_0 lies on the first component E_0. By contradiction, we assume that $\mu_{B,M}$ is not injective for every $[Y,p]\in \mathcal{M}_{e,1}$ and for each of the finitely many linear series $M\in W^{e-f}_{2e-f}(Y)$ satisfying (23) and each $B\in G^1_{e-f+1}\big(Y,p,(0,e-2f)\big)$.

We construct a limit $\mathfrak{g}^{e-f}_{2e-f}$ on $[Y_0,p_0]$, say $\mathfrak{m}=\{(M_{E_i},V_i)\in G^{e-f}_{2e-f}(E_i)\}_{i=0}^{e-2a}$, which satisfies condition (23) with respect to p_0, by specifying the vanishing sequences $a^{\mathfrak{m}_{E_i}}(p_i)$ for $0\leqslant i\leqslant e-2a$. For $0\leqslant i\leqslant c-1$, the sequence $a^{\mathfrak{m}_{E_{i+1}}}(p_{i+1})$ is obtained from $a^{\mathfrak{m}_{E_i}}(p_i)$ by raising all entries by 1, except for the term

$$a^{\mathfrak{m}_{E_{i+1}}}_{e-f-a}(p_{i+1})=a^{\mathfrak{m}_{E_i}}_{e-f-a}(p_i)=e-f-a+c.$$

After c steps we arrive at the following vanishing sequence on E_c with respect to p_c:

$$\begin{aligned}a^{\mathfrak{m}_{E_c}}(p_c)=(c,c+1,\dots,e-f-a+c-1,e-f-a+c,r-a+2\\ +c,r-a+3+c,\dots,r+c+1).\end{aligned}$$

For an index $c\leqslant i\leqslant e-2a-1$, which we write as $i=c+a\cdot\beta+j$, with $0\leqslant j\leqslant a-1$ and $0\leqslant \beta\leqslant r-2-e+f$, we choose $a^{\mathfrak{m}_{E_{i+1}}}(p_{i+1})$ to be obtained from $a^{\mathfrak{m}_{E_i}}(p_i)$ by raising all entries by 1, except for the term

$$a^{\mathfrak{m}_{E_{i+1}}}_{e-f-a+j+1}(p_{i+1})=a^{\mathfrak{m}_{E_i}}_{e-f-a+j+1}(p_i)=r-a+2+c+(a-1)\cdot\beta+2j.$$

In this way $\mathfrak{m}\in \tilde{G}^{e-f}_{2e-f}(Y_0)$ becomes a (refined) limit linear series which smooths to a complete linear series $M\in G^{e-f}_{2e-f}(Y)$ on every smooth pointed curve $[Y,p]\in\mathcal{M}_{e,1}$ such that the ramification condition (23) with respect to p is satisfied.

Next we construct a limit $\mathfrak{g}^1_{e-f+1}$ on $[Y_0,p_0]$, say $\mathfrak{b}=\{(B_{E_i},W_i)\in G^1_{e-f+1}(E_i)\}_{i=0}^{e-2a}$ such that $a^{\mathfrak{b}}(p_0)=(0,e-2f+1)$. For $0\leqslant i\leqslant e-2f$ we set $a^{\mathfrak{b}_{E_i}}(p_i)=(i,e-2f+1)$. For an index of type $i=e-2f+2k-1$, where $0\leqslant k\leqslant f-a$, we choose $a^{\mathfrak{b}_{E_i}}(p_i)=(e-2f+k-1,e-2f+k+1)$. If $i=e-2f+2k$, we choose the sequence $a^{\mathfrak{b}_{E_i}}(p_i)=(e-2f+k,e-2f+k+1)$. It is clear that each sequence $a^{\mathfrak{b}_{E_i}}(p_i)$ is obtained from $a^{\mathfrak{b}_{E_{i-1}}}(p_{i-1})$ by raising one entry by 1 while keeping the other fixed, and hence $\mathfrak{b}$ is a limit $\mathfrak{g}^1_{e-f+1}$ that smooths to a pencil $B\in G^1_{e-f+1}(Y,p,(0,e-2f))$ on every nearby smooth curve $[Y,p]$. For each $0\leqslant i\leqslant e-2a-1$, there exists a section (unique up to scaling) $\sigma_i\in W_i$ such that $\mathrm{ord}_{p_i}(\sigma_i)+\mathrm{ord}_{p_{i+1}}(\sigma_i)=\deg(B_{E_i})$. We denote by $\sigma_i^c\in W_i$ a complementary section such that $\{\mathrm{ord}_{p_i}(\sigma_i),\mathrm{ord}_{p_i}(\sigma_i^c)\}=\{a_0^{\mathfrak{b}_{E_i}}(p_i),a_1^{\mathfrak{b}_{E_i}}(p_i)\}$.

Using the set-up developed in [**7**] and [**10**] for studying degenerations of multiplication maps, we find that the assumption that $\mu_{B,M}$ is not injective implies the existence elements $\rho_i\neq 0,\rho_i\in \mathrm{Ker}\{W_i\otimes V_i\to H^0(E_i,B_{E_i}\otimes M_{E_i})\}$, for each $i\leqslant e-2a$, $i\geqslant 0$, satisfying the property that $\mathrm{ord}_{p_{i+1}}(\rho_{i+1})\geqslant \mathrm{ord}_{p_i}(\rho_i)+1$, for all i (see, for example [**10**, Section 4], for an explanation of how to obtain the ρ_i). Moreover, if $\mathrm{ord}_{p_{i+1}}(\rho_{i+1})=\mathrm{ord}_{p_i}(\rho_i)+1$, then if $\tau_i\in V_i$ is the section (unique up to scaling) such that $\mathrm{ord}_{p_i}(\tau_i)+\mathrm{ord}_{p_{i+1}}(\tau_i)=\deg(M_{E_i})$, then we must have

$$\mathrm{ord}_{p_i}(\rho_i)=\mathrm{ord}_{p_i}(\tau_i)+\mathrm{ord}_{p_i}(\sigma_i^c)=\mathrm{ord}_{p_i}(\sigma_i)+\mathrm{ord}_{p_i}(\tau_i'),$$

where $\tau_i'\in V_i$ is another section such that $\mathrm{ord}_{p_i}(\tau_i')\neq \mathrm{ord}_{p_i}(\tau_i)$. In particular, since we have explicitly described all the sequences $a^{\mathfrak{b}_{E_i}}(p_i)$ and $a^{\mathfrak{m}_{E_i}}(p_i)$, the assumption that $\mathrm{ord}_{p_{i+1}}(\rho_{i+1})\leqslant \mathrm{ord}_{p_i}(\rho_i)+1$ uniquely determines $\mathrm{ord}_{p_i}(\rho_i)$.

Since $a^{\mathfrak{b}_{E_0}}(p_0)=(0,e-2f+1)$ and $\mu_{B_{E_0},M_{E_0}}(\rho_0)=0$, the non-zero section ρ_0 must involve both sections σ_0 and σ_0^c, and then clearly $\mathrm{ord}_{p_0}(\rho_0)\geqslant e-2f+1$. We prove inductively that for all integers $i\leqslant e-2a$, $i\geqslant 0$, we have the inequality

$$\mathrm{ord}_{p_i}(\rho_i)\geqslant e-2f+1+2i. \tag{24}$$

Assuming (24) for $i \leqslant e-2a-1$, since $\mathrm{ord}_{p_{i+1}}(\rho_{i+1}) \geqslant \mathrm{ord}_{p_i}(\rho_i)+1$, the only way (24) can fail for $i+1$ is when $\mathrm{ord}_{p_i}(\rho_i) = e-2f+2i+1$ and $\mathrm{ord}_{p_{i+1}}(\rho_{i+1}) = \mathrm{ord}_{p_i}(\rho_i)+1$. As explained above, this implies that $\mathrm{ord}_{p_i}(\rho_i) = \mathrm{ord}_{p_i}(\tau_i) + \mathrm{ord}_{p_i}(\sigma_i^c)$.

Writing $i = c + a\cdot\beta + j$ as above, $\mathrm{ord}_{p_i}(\tau_i) = r-a+2+c+(a-1)\cdot\beta+2j$ if $i \geqslant c$, while $\mathrm{ord}_{p_i}(\tau_i) = e-f-a+c$, for $0 \leqslant i \leqslant c-1$. We deal only with the case $i \geqslant c$, the case $0 \leqslant i \leqslant c-1$ being analogous. To determine $\mathrm{ord}_{p_i}(\sigma_i^c)$ we must distinguish between two cases. When $i = e-2f+2k-1$ with $k \geqslant 1$, then $\mathrm{ord}_{p_i}(\sigma_i^c) = e-2f+k-1$. Otherwise, we write $i = e-2f+2k$, in which case $\mathrm{ord}_{p_i}(\sigma_i^c) = e-2f+k+1$. Suppose that we are in the former case. Then we obtain the equality

$$e-2f+2i+1 = \mathrm{ord}_{p_i}(\rho_i) = \big(r-a+2+c+(a-1)\cdot\beta+2j\big) + (e-2f+k-1),$$

which ultimately leads to the relation $(a+2)(r-e+f-\beta) = a-j-1$. However, $j \leqslant a-1$ and $\beta \leqslant r-e+f-1$, and hence we have reached a contradiction. The case when one can write $i = e-2f+2k$ is dealt with similarly. All in all, we may assume that we have proved the inequality $\mathrm{ord}_{p_{e-2a}}(\rho_{e-2a}) \geqslant e-2f+1+2(e-2a)$. We note that on the curve $[T,q] = [E_{e-2a}, p_{e-2a}]$ we have $a^{\mathfrak{b}_T}(p_{e-2a}) = (e-f-a, e-f-a+1)$, while

$$a^{\mathfrak{m}_T}(p_{e-2a}) = (e-2a, e-2a+1, \ldots, 2e-f-3a, 2e-f-3a+3, \ldots, 2e-f-2a+2).$$

Equivalently $\mathfrak{b}_T = |B| + (e-f-a)\cdot q$, where $B \in W^1_{a+1}(T)$, while $\mathfrak{m}_T = (e-2a)\cdot q + |N|$, where $N \in \mathrm{Pic}^{e-f+2a}(T)$ has the property that $h^0\big(T, N(-(e-f-a+3)\cdot q)\big) \geqslant a$. Remembering that $\mathrm{ord}_q(\rho_{e-2a}) \geqslant (e-2f+1)+2(e-2a)$, after subtracting the base locus supported at q, we find an element

$$0 \neq \rho_T \in \mathrm{Ker}\{H^0(B)\otimes H^0(N) \longrightarrow H^0(B\otimes N)\}$$

such that $\mathrm{ord}_q(\rho_T) \geqslant e-f-a+1$. Equivalently, the multiplication map

$$\mu_{B,N} : H^0(B)\otimes H^0\big(N(-(e-f-a+3)\cdot q)\big) \longrightarrow H^0\big(B\otimes N(-(e-f-a+3)\cdot q)\big)$$

is not injective. By using Riemann–Roch we find that $N(-(e-f-a+3)\cdot q) = K_T \otimes \tilde{B}^\vee$, where $\tilde{B} \in W^1_{a+1}(T)$. Choosing $\tilde{B} = B \in W^1_{a+1}(T)$, we notice that $\mu_{B,N}$ can be identified with the Petri map $H^0(B)\otimes H^0(K_T\otimes B^\vee) \to H^0(K_T)$, which is injective because $[T] \in \mathcal{M}_{2a}$ was chosen to be Petri general. Thus we have reached a contradiction by reducing (18) to the Gieseker–Petri theorem, which completes the proof in the case (i).

Next we turn to case (iii) when $[e/(r+1-e+f)] < 2$. Since the argument is similar to that for (i), we only outline the main steps. If $e \leqslant r-e+f$, that is, when $\alpha_0 = d-r-f-e$, we can easily determine a linear series $(A, W_A) \in G^{r-e+f}_{d-e}(Y, p, \overline{\alpha})$. Precisely, one can see that $A = K_Y \otimes \mathcal{O}_Y((d-3e+2)\cdot p)$ and

$$|W_A| = (d-r-f-e)\cdot p + |K_Y\otimes\mathcal{O}_Y\big((r+f-2e+2)\cdot p\big)|.$$

In this case we find that $|G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})| = 1$. Condition (18) translates into saying that for a generic $(L, W_L) \in G^{e-f-1}_{2e-f-1}(Y, p, \overline{\beta})$ we have the vanishing statement

$$H^0\big(Y, L\otimes K_Y^\vee((e+f-2)\cdot p)\big) = 0 \iff H^0\big(Y, K_Y^{\otimes 2}\otimes L^\vee(-(e+f-2)\cdot p)\big) = 0. \qquad (25)$$

One can prove (25) by degenerating Y to a generic string of elliptic curves and we skip the details. Finally, if $[e/(r+1-e+f)] = 1$, then $c = 2e-r-f-1$ and condition (18) boils down to showing that one can find a pencil $B \in G^1_{e-c+1}\big(Y, p, (0, r-e+f-c+1)\big)$ and a linear series $L \in G^{e-f-1}_{2e-f-1}(Y, p, \overline{\beta})$, such that the multiplication map

$$H^0(B)\otimes H^0\big(K_Y^{\otimes 2}\otimes L^\vee(-(2e-4-r)\cdot p)\big) \longrightarrow H^0\big(K_Y^{\otimes 2}\otimes B\otimes L^\vee(-(2e-4-r)\cdot p)\big)$$

is injective. This situation is handled along the lines of (i) and we omit the details here. $\square$

Finally, we prove Theorem 0.5 assuming that condition (22) is satisfied. This case is not covered by cases (i)–(iii) above.

Proposition 3.3. *Let $[Y,p]\in \mathcal{M}_{e,1}$ be a general pointed curve. Then there exists a linear series $(A,W_A)\in G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})$ such that $H^0\big(Y,\mathcal{O}_Y((d-1)\cdot p\otimes A^{\vee})\big)=0$.*

Proof. By contradiction, we assume that $H^0(\mathcal{O}_Y((d-1)\cdot p)\otimes A^{\vee})\neq 0$ for every $[Y,p]\in \mathcal{M}_{e,1}$ and for every linear series $(A,V_A)\in G^{r-e+f}_{d-e}(Y,p,\overline{\alpha})$. We let $[Y,p]$ degenerate to the stable curve $[Y_0:=E_0\bigcup_{p_1}E_1\bigcup_{p_2}\ldots\bigcup_{p_{e-3}}E_{e-3}\bigcup_{p_{e-2}}B,p_0]$, where $E_0,\ldots,E_{e-3}$ are elliptic curves, the points $p_i,p_{i+1}\in E_i$ are such that $p_i-p_{i+1}\in \mathrm{Pic}^0(E_i)$ is not a torsion class, and $[B,p_{e-2}]\in\mathcal{M}_{2,1}$ is such that $p_{e-2}\in B$ is not a Weierstrass point. For all integers $0\leqslant i\leqslant e-3$ we find that there exist sections

$$0\neq \tau_i\in H^0\big(\mathcal{O}_{E_i}((d-1)\cdot p_i)\otimes A^{\vee}_{E_i}\big)\quad\text{and}\quad 0\neq\tau_B=\tau_{e-2}\in H^0\big(\mathcal{O}_B((d-1)\cdot p_{e-2})\otimes A_B^{\vee}\big)$$

such that

$$0\leqslant \mathrm{ord}_{p_0}(\tau_0)\leqslant \mathrm{ord}_{p_1}(\tau_1)\leqslant\ldots\leqslant \mathrm{ord}_{p_{e-3}}(\tau_{e-3})\leqslant \mathrm{ord}_{p_{e-2}}(\tau_B).$$

Moreover, we see that $\mathrm{ord}_{p_i}(\tau_i)\geqslant i$ for $0\leqslant i\leqslant e-2$. In particular, $\mathrm{ord}_{p_{e-2}}(\tau_B)\geqslant e-2$. Since $\rho(e,r-e+f,d-e,\overline{\alpha})=0$, limits $\mathfrak{g}^{r-e+f}_{d-e}$ on $E_0\cup\ldots\cup E_{e-3}\cup B$ are smoothable to every curve of genus g. These finitely many limits $\mathfrak{g}^{r-e+f}_{d-e}$ are in bijective correspondence with possibilities of choosing the vanishing sequences $\{a^{l_{E_i}}(p_i)\}_{0\leqslant i\leqslant e-3}$ and $a^{l_B}(p_{e-2})$ in such a way that for all $0\leqslant i\leqslant e-3$, the sequence $a^{l_{E_{i+1}}}(p_{i+1})$ is obtained from $a^{l_{E_i}}(p_i)$ by raising all entries by 1 except a single entry which remains unchanged. To complete the proof it suffices to exhibit a single limit $\mathfrak{g}^{r-e+f}_{d-e}$ on $E_0\cup\ldots\cup E_{e-3}\cup B$ having the property that if (A_B,V_B) denotes its B-aspect, then $H^0(\mathcal{O}_B((d-e+1)\cdot p_{e-2})\otimes A_B^{\vee})=0$.

We describe such a $\mathfrak{g}^{r-e+f}_{d-e}$ explicitly by specifying the sequences $\{\alpha^{l_{E_i}}(p_i)\}_{0\leqslant i\leqslant e-3}$ and $\alpha^{l_B}(p_{e-2})$. Clearly, $\alpha^{l_{E_0}}(p_0)$ equals $(\alpha_0,\ldots,\alpha_0,\alpha^{l_{E_0}}_{r-e+f+1-c}(p_0)=\alpha_0+1,\ldots,\alpha_0+1)$. For $1\leqslant i\leqslant c$, $\alpha^{l_{E_i}}(p_i)$ is obtained from $\alpha^{l_{E_{i-1}}}(p_{i-1})$ by increasing all entries by 1, except for $\alpha^{l_{E_i}}_{r-e+f+i-c}(p_i)=\alpha^{l_{E_{i-1}}}_{r-e+f+i-c}(p_{i-1})$. Thus $\alpha^{l_{E_c}}(p_c)=(\alpha_0+c,\ldots,\alpha_0+c)$. Next, for an index i such that $c+\beta(r+1-e+f)<i\leqslant c+(\beta+1)(r+1-e+f)$, where $0\leqslant\beta\leqslant[e/(r+1-e+f)]$, if we write $i\equiv j+c \bmod r+1-e+f$, with $1\leqslant j\leqslant r-e+f$, then the sequence $\alpha^{l_{E_i}}(p_i)$ is obtained from $\alpha^{l_{E_{i-1}}}(p_{i-1})$ by raising all entries by 1, except for $\alpha^{l_{E_i}}_{j-1}(p_i)=\alpha^{l_{E_{i-1}}}_{j-1}(p_{i-1})$. Switching from ramification to vanishing sequences we obtain

$$a^{l_B}(p_{e-2})=(d-r-f-2,d-r-f-3,\ldots,d-e-5,d-e-4,d-e-2,d-e-1),$$

that is, $A_B=\mathcal{O}_B((d-e-2)\cdot p_{e-2})\otimes\mathfrak{g}^1_2$, and then

$$H^0(\mathcal{O}_B((d-e+1)\cdot p_{e-2})\otimes A_B^{\vee})=H^0(\mathcal{O}_B(3\cdot p_{e-2})\otimes(\mathfrak{g}^1_2)^{\vee})=0.$$

This contradicts the fact that $\mathrm{ord}_{p_{e-2}}(\tau_B)\geqslant e-2$, which completes the proof. $\square$

4. *Higher ramification points of a general line bundle*

In this section we prove Theorem 0.10. We fix an arbitrary smooth curve C of genus g and for $n\geqslant 1$ we denote by $[n]_C:\mathrm{Pic}^d(C)\to\mathrm{Pic}^{nd}(C)$ the multiplication by n map, $[n]_C(L):=L^{\otimes n}$. It is an immediate consequence of Riemann–Roch that for a general $L\in\mathrm{Pic}^d(C)$, we have $h^0(L^{\otimes n})=\max\{nd+1-g,0\}$.

First we show that for a very general $L\in\mathrm{Pic}^d(C)$ we have $w^{L^{\otimes n}}(p)\leqslant 1$ for all $p\in C$ and $n\geqslant 1$. Indeed, let us assume that $w^{L^{\otimes n}}(p)\geqslant 2$, where n is chosen such that $nd\geqslant g$, so that

$h^0(C, L^{\otimes n}) = nd + 1 - g$. Then there are two possibilities:

(i) $h^0\big(C, L^{\otimes n}(-(nd+2-g)\cdot p)\big) \geqslant 1$ or

(ii) $h^0\big(C, L^{\otimes n}(-(nd-g)\cdot p)\big) \geqslant 2$.

In case (i) we consider the map $C \times C_{g-2} \to \mathrm{Pic}^{nd}(C)$, $(p, E) \mapsto \mathcal{O}_C\big((nd+2-g)\cdot p + E\big)$, and we denote by Σ_n its image, which is a divisor on $\mathrm{Pic}^{nd}(C)$. Then (i) is equivalent to $L \in [n]_C^*(\Sigma_n)$, which is a divisorial condition on $\mathrm{Pic}^d(C)$ for each n.

In case (ii) we look at the map $C \times C_g^1 \to \mathrm{Pic}^{nd}(C), (p, E) \mapsto \mathcal{O}_C\big((nd-g)\cdot p + E\big)$, and we denote by V_n its image. Since C_g^1 is generically a $\mathbf{P}^1$-bundle over C_{g-2}, it follows that V_n is a divisor on $\mathrm{Pic}^{nd}(C)$ and then possibility (ii) is equivalent to $L \in [n]_C^*(V_n)$. Thus we see that for $L \in \mathrm{Pic}^d(C) - \bigcup_{n\geqslant 1}[n]_C^*(\Sigma_n + V_n)$ all the ramification points of all powers $L^{\otimes n}$ with $n \geqslant 1$ are ordinary. This proves the first part of Theorem 0.10. To prove the second part we start with the following.

Proposition 4.1. *We fix a point $p \in C$ and integers n and d such that $nd \geqslant g$. Then the locus*

$$D_n := \{L \in \mathrm{Pic}^d(C) : h^0\big(C, L^{\otimes n}(-(nd+1-g)\cdot p)\big) \geqslant 1\}$$

is an irreducible divisor on $\mathrm{Pic}^d(C)$ and $[D_n] = n^2\theta$.

Proof. We set $a := \max\{0, 2g-1-nd\}$ and define two vector bundles $\mathcal{E}_n$ and $\mathcal{F}_n$ on $\mathrm{Pic}^d(C)$ of the same rank and having fibres $\mathcal{E}_n(L) = H^0(C, L^{\otimes n} \otimes \mathcal{O}_C(a\cdot p))$ and $\mathcal{F}_n(L) = H^0(C, L^{\otimes n} \otimes \mathcal{O}_{(a+nd+1-g)\cdot p}(a\cdot p))$ over each point $L \in \mathrm{Pic}^d(C)$. Then D_n is the degeneracy locus of the morphism $\mathcal{E}_n \to \mathcal{F}_n$ obtained by evaluation sections of $L^{\otimes n} \otimes \mathcal{O}_C(a\cdot p)$ along $(a+nd+1-g)\cdot p$. The Picard bundle $\mathcal{E}_n$ is negative (that is, $\mathcal{E}_n^\vee$ is ample), because $\mathcal{E}_n$ is the pull-back under the finite map $[n]_C$ of a negative bundle on $\mathrm{Pic}^d(C)$ (cf. [**1**, p. 310]). Moreover, $\mathcal{F}_n$ is algebraically equivalent to a trivial bundle, and hence $\mathcal{E}_n^\vee \otimes \mathcal{F}_n$ is ample too. Applying the Fulton–Lazarsfeld connectedness theorem (see [**12**] or [**1**, p. 311]), we conclude that D_n is connected. Since D_n is also smooth in codimension 2 we obtain that D_n must be irreducible. Finally, $[D_n] = c_1(\mathcal{F}_n - \mathcal{E}_n) = [n]_C^*(\theta) = n^2\theta$. $\square$

End of the proof of Theorem 0.10. We fix integers $1 \leqslant a < b$ and consider the variety $\Sigma_{ab} := \{(p, L) \in C \times \mathrm{Pic}^d(C) : p \in R(L^{\otimes a}) \cap R(L^{\otimes b})\}$ and we denote by $\phi_1 : \Sigma_{ab} \to C$ and $\phi_2 : \Sigma_{ab} \to \mathrm{Pic}^d(C)$ the two projections. For a fixed $p \in C$, the fibre $\phi_1^{-1}(p)$ is identified with the intersection of the two irreducible divisors D_a and D_b. Since $[D_a] \neq [D_b]$ for $a \neq b$, it follows that $D_a \cap D_b$ is of pure codimension 2 inside $\mathrm{Pic}^d(C)$, and therefore $\dim(\Sigma_{ab}) = g - 1$. We obtain that a line bundle $L \in \mathrm{Pic}^d(C) - \bigcup_{a<b}\phi_2(\Sigma_{ab})$ will enjoy the property that $R(L^{\otimes a}) \cap R(L^{\otimes b}) = \emptyset$ for $a < b$. $\square$

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