# SINGULARITIES OF THE MODULI SPACE OF LEVEL CURVES

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Abstract. We describe the singular locus of the compactification of the moduli space $\mathcal{R}_{g,\ell}$ of curves of genus $g$ paired with an $\ell$-torsion point in their Jacobian. Generalising previous work for $\ell \leq 2$, we also describe the sublocus of noncanonical singularities for any positive integer $\ell$. For $\ell \leq 6$ and $\ell \neq 5$, this allows us to provide a lifting result on pluricanonical forms playing an essential role in the computation of the Kodaira dimension of $\mathcal{R}_{g,\ell}$: every pluricanonical form on the smooth locus of the moduli space extends to a desingularisation of the compactified moduli space.

The modular curve $X_1(\ell) := \mathcal{H}/\Gamma_1(\ell)$ classifying elliptic curves together with an $\ell$-torsion point in their Jacobian is among the most studied objects in arithmetic geometry. In a series of recent papers, the birational geometry of its higher genus generalisations and their variants (*e.g.* theta characteristics) has been systematically studied and proved to be, in many cases such as $\ell = 2$, better understandable than that of the underlying moduli space of curves $\mathcal{M}_g$. As an example, we refer to the complete computation of the Kodaira dimension of all components of the moduli of theta characteristics ($L^{\otimes 2} \cong \omega$), see [20, 11, 13, 14].

In this paper, for $g \geq 2$ and for all positive levels $\ell$, we classify the singularities of the moduli space $\mathcal{R}_{g,\ell}$ parametrizing level-$\ell$ curves, *i.e.* triples $(C, L, \phi)$ where $C$ is a smooth curve equipped with a line bundle $L$ and a trivialisation $\phi\colon L^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}$. Since the Kodaira dimension of $\mathcal{R}_{g,\ell}$ is defined as the Kodaira dimension of an arbitrary resolution of singularities of its completion $\overline{\mathcal{R}}_{g,\ell}$, the first step toward the birational classification of $\mathcal{R}_{g,\ell}$ is the study of the singular locus $\mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$. More precisely one needs to determine the sublocus $\mathrm{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell}) \subseteq \mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$ of noncanonical singularities.

For $\ell = 2$, this analysis has been carried out by the second author and Ludwig in [12] using Cornalba's compactification in terms of quasistable curves [5] of $\overline{\mathcal{R}}_{g,2}$. Clearly, we can leave out the case $\ell = 1$, which coincides with Deligne and Mumford's functor of stable curves $\overline{\mathcal{M}}_g = \overline{\mathcal{R}}_{g,1}$. The passage to all higher levels presents a new feature from Abramovich and Vistoli's theory of stable maps to stacks: the points of the compactification cannot be interpreted in terms of $\ell$-torsion line bundles on a scheme-theoretic curve, but rather on a stack-theoretic curve. Instead of the above triples $(C, L \in \mathrm{Pic}(C), \phi\colon L^{\otimes \ell} \xrightarrow{\sim} \mathcal{O})$, we simply consider their stack-theoretic analogues

$$(\mathsf{C}, \mathsf{L} \in \mathrm{Pic}(\mathsf{C}), \phi\colon \mathsf{L}^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}) \in \overline{\mathcal{R}}_{g,\ell},$$

where $\mathsf{C}$ is a one-dimensional stack, whose nodes may have nontrivial stabilisers, and where $\mathsf{L} \to \mathsf{C}$ is a line bundle whose fibres are faithful representations, see Definition 1.7. This yields a compactification which is represented by a smooth Deligne–Mumford stack.

Research of the first author partially supported by the ANR grant ThéorieGW.
Research of the second author partially supported by the Sonderforschungsbereich 647 Raum-Zeit-Materie.

In analogy with the moduli space of stable curves $\overline{\mathcal{M}}_g$, the boundary locus $\overline{\mathcal{R}}_{g,\ell} \setminus \mathcal{R}_{g,\ell}$ can be described in terms of the combinatorics of the standard dual graph $\Gamma$ whose vertices correspond to irreducible components of the curve and whose edges correspond to nodes of the curve. In §1.3-4, we revisit this well known description by emphasising the natural role of an extra *multiplicity* datum enriching the graph. Indeed, the stack-theoretic structure of the underlying curve $\mathsf{C}$ and the line bundle $\mathsf{L} \to \mathsf{C}$ are determined, locally at a node, by assigning to each oriented edge $e$ a character $\chi_e \colon \boldsymbol{\mu}_\ell \to \mathbb{C}^\times$. Hence, to each point of the boundary we attach a dual graph $\Gamma$ and a $\mathbb{Z}/\ell$-valued 1-cochain $M \colon e \mapsto \chi_e$ in $C^1(\Gamma; \mathbb{Z}/\ell)$ which we refer to as the *multiplicity of the level curve*. (Proposition 1.11 recalls that a multiplicity co-chain arises at the boundary if and only if it lies in the kernel of $\partial \colon C^1(\Gamma; \mathbb{Z}/\ell) \to C^0(\Gamma; \mathbb{Z}/\ell)$.)

In order to describe the singular locus of $\overline{\mathcal{R}}_{g,\ell}$, we lift to the moduli of level curves a result of Harris and Mumford [15] showing that, for $g \geq 4$, the local structure $\mathrm{Def}(C)/\mathrm{Aut}(C)$ of $\overline{\mathcal{M}}_g$ is singular if and only if $C$ is equipped with an automorphism which is *not* the product of "*elliptic tail involutions*" (ETI for short):

$$\mathrm{Sing}(\overline{\mathcal{M}}_g) = N_1 := \{C \mid \mathrm{Aut}(C) \ni \alpha \textit{ not } \text{a product of ETI}\}.$$

By definition, an ETI operates nontrivially on the curve $C$ only at a genus-1 component $E$ which meets the rest of the curve at exactly one node $n$; its restriction to the "tail" $(E, n)$ is the canonical involution. These automorphisms are the only nontrivial automorphisms of curves (and also of level curves) which do not yield singularities: their action on the moduli is simply a quasireflection. An example of a point of $N_1$ is given by choosing a tail $(E, n)$ with $\mathrm{Aut}(E, n) \cong \mathbb{Z}/6$. This type of curves fill-up a codimension-2 locus within a codimension-2 locus within $\overline{\mathcal{M}}_g$ and a sublocus $T_1 \subset N_1$ which plays a remarkable role in this paper. Indeed, the order-6 automorphism $\alpha$ spanning $\mathrm{Aut}(E, n)$ and fixing $C \setminus E$ is clearly not a product of ETI and, most important, yields a noncanonical singularity. This can be checked by the Reid–Shepherd-Barron–Tai criterion: $\alpha$ operates on the regular space $\mathrm{Def}(C)/\langle \mathrm{ETI} \rangle$ as $(\frac{1}{3}, \frac{1}{3}, 0, \ldots, 0) := \mathrm{Diag}(\xi_3, \xi_3, 1, \ldots, 1)$ and most importantly modding out $\alpha$ yields a noncanonical singularity simply because the *age* $\frac{1}{3} + \frac{1}{3} + 0 + \ldots + 0$ of $\alpha$ is less than 1. Harris and Mumford show that these special tailed curves are the only possible curves carrying a *junior* (that is aged less than 1) automorphism; this amounts to the following statement.

$$\mathrm{Sing}_{\mathrm{nc}}(\overline{\mathcal{M}}_g) = T_1 := \{C \mid C \supset E,\ C \cap \overline{C \setminus E} = \{n\},\ \mathrm{Aut}(E, n) \cong \mathbb{Z}/6\}.$$

The generalisation of this statement to level-$\ell$ curves is subtle due to a new phenomenon: stack-theoretic curves $\mathsf{C}$ may be equipped with *ghost automorphisms* $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ which fix all geometric points of $\mathsf{C}$ and yet operate nontrivially on the stack $\mathsf{C}$. The group $\mathrm{Aut}_C(\mathsf{C})$ has been completely determined by Abramovich, Corti, and Vistoli [1]; here, we describe the ghosts of level structures $(\mathsf{C}, \mathsf{L}, \phi)$

$$\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \{\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C}) \mid \mathsf{a}^*\mathsf{L} \cong \mathsf{L}\}.$$

The loci $N_1$ and $T_1$ naturally lift to $N_\ell$ and $T_\ell$ within $\overline{\mathcal{R}}_{g,\ell}$. For the definition of $N_\ell$, no modification is needed (the notion of ETI naturally generalises to stack-theoretic curves, Definition 2.12). The locus $T_\ell$ is given by imposing the extra condition that the line bundle be trivial on the genus-1 tail (see Definition **??**). For general values of $\ell$, we have proper inclusions $N_\ell \subsetneq \mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$ and $T_\ell \subsetneq \mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$. In order to obtain $\mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell})$ one needs to

include also the entire locus of level curves with a nontrivial ghosts (*haunted* level curves)

$$H_\ell = \{(\mathsf{C}, \mathsf{L}, \phi) \mid \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \neq 1\}.$$

Similarly, in order to obtain $\mathrm{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell})$ one needs to take the union of $T_\ell$ and of the locus of level curves haunted by a *junior ghost*

$$J_\ell = \{(\mathsf{C}, \mathsf{L}, \phi) \mid \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) \ni \mathsf{a},\ \mathrm{age}(\mathsf{a}) < 1\},$$

where, as above, the age refers to the action on the regular space $\mathrm{Def}(C)/\langle \mathrm{ETI} \rangle$.

Assembling Theorems 2.27, 2.40 and 2.43, we summarise the above claims and provide the desired extension of pluricanonical forms for $\ell \leq 6$ and $\ell \neq 5$.

**Theorem.** *Let $g \geq 4$. We have*

$$\mathrm{Sing}(\overline{\mathcal{R}}_{g,\ell}) = N_\ell \cup H_\ell \qquad \textit{and} \qquad \mathrm{Sing}_{\mathrm{nc}}(\overline{\mathcal{R}}_{g,\ell}) = T_\ell \cup J_\ell.$$

*Furthermore, the locus $J_\ell$ is empty if and only if $5 \neq \ell \leq 6$; therefore, under such condition, we have*

$$\Gamma\big((\overline{\mathcal{R}}_{g,\ell})^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_{g,\ell}}^{\otimes q}\big) \cong \Gamma\big(\widehat{\mathcal{R}}_{g,\ell}, K_{\widehat{\mathcal{R}}_{g,\ell}}^{\otimes q}\big) \tag{1}$$

*for any desingularisation $\widehat{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{R}}_{g,\ell}$ and for all integers $q \geq 0$.*

The case $\ell = 1$ is proven by Harris and Mumford in [15]. The case $\ell = 2$ is proven by the second author in collaboration with Ludwig [12] (following work of Ludwig, [20]). The above formulation presents the isomorphism (1) as a consequence of $J_\ell = \varnothing$ (and Harris and Mumford's work on the locus $T_1$). However, the question of whether (1) holds in the remaining cases (for $\ell = 5$ or $\ell > 6$) remains open. In this direction, one can exploit the complete computation of the group $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ carried out in §2.4.

To this effect we conclude this introduction by providing a simple combinatorial device detecting the presence of ghosts. Write $\ell$ as $\prod_{p|\ell} p^{e_p}$, where $p$ denotes a prime divisor of $\ell$ and $e_p$ the $p$-adic valuation of $\ell$. Fix a level curve $(\mathsf{C}, \mathsf{L}, \phi)$, its dual graph $\Gamma$ and the multiplicity $M\colon e \mapsto \chi_e$. Consider the sequence of subgraphs

$$\varnothing \subseteq \Delta_p^{e_p} \subseteq \ldots \subseteq \Delta_p^k := \{e \mid \chi_e \in (p^k) \text{ in } \mathbb{Z}/(p^{e_p})\} \subseteq \ldots \subseteq \Delta_p^1 \subseteq \Delta_p^0 = \Gamma, \tag{2}$$

where $\chi_e \in \mathbb{Z}/\ell$ is regarded as an element of $\mathbb{Z}/(p^{e_p})$. The respective contractions yield

$$\Gamma \to \Gamma_p^{e_p} \to \ldots \to \Gamma_p^k \to \ldots \to \Gamma_p^1 \to \bullet. \tag{3}$$

Then all the ghost automorphisms are trivial, *i.e.* $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = 1$, if and only if, $\Gamma_p^{e_p}$ are bouquet (connected graphs with a single vertex), all $p$. Lemma 2.21 provides an explicit description of the group structure of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$. In particular, we get the number of ghosts.

**Corollary.** *We have $\#\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \frac{1}{\ell} \prod_{p|\ell} p^{V_p}$, where $V_p$ is the total number of vertices appearing in the graphs $\Gamma_p^j$ for $1 \leq j \leq e_p$.*

Note that, if $\Gamma_p^j$ is a bouquet for all $p$ and $j$, then $\#\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = \frac{1}{\ell} \prod_{p|\ell} p^{e_p} = 1$. See Example 2.23 for a simple demonstration. In Remark 2.24 the above formula is used to match Caporaso, Casagrande, and Cornalba's computation [6] of the length of the fibre of the moduli of level curves over the moduli of stable curves.

The above combinatorics plays an essential role in detecting when $J_\ell$ is empty and can be used to further analyse the irreducible components of $J_\ell$ in view of generalisations of (1). The above description leads to the claim that junior ghosts (hence noncanonical singularities of the form $\mathrm{Def}\,/\,\mathrm{Aut}_C(\mathsf{C},\mathsf{L},\phi)$) can be completely ruled out for $5 \neq \ell \leq 6$ and are relatively rare in general: their appearance is due to the presence of age-delay edges which we describe in the proof of the No-Ghost Lemma 2.40.

The computation of the Kodaira dimension of $\overline{\mathcal{R}}_{g,\ell}$ for $\ell \leq 6$ and $\ell \neq 5$ can be carried out without further study of resolutions of noncanonical singularities; for instance, in [9], we show the following statement.

**Theorem** ([9, Thm. 0.2])**.** *$\mathcal{R}_{g,3}$ is a variety of general type for $g \geq 12$. Furthermore, the Kodaira dimension of $\mathcal{R}_{11,3}$ is at least 19.*

**Structure of the paper.** In Section 1 we introduce moduli of smooth level curves, their compactification, the relevant combinatorics and the boundary locus of the compactified moduli space. In Section 2 we study the local structure of the moduli space, we develop the suitable machinery for the computation of the ghost automorphism group and we deduce the theorem stated above.

**Acknowledgements.** We are grateful to Roland Bacher for several illuminating conversations that put on the right track the computation of the group of ghost automorphisms in the case where $\ell$ is composite. The first author wishes to thank the Mathematics Department of the Humboldt Universität zu Berlin where this work started.

## 1. Level curves

**Terminology: coarsening and local pictures.** The interplay between stacks and their coarse spaces is crucial in this paper. Any stack $\mathsf{X}$ of Deligne–Mumford (DM) type admits an algebraic space $X$ and a morphism $\epsilon_{\mathsf{X}}\colon \mathsf{X} \to X$ universal with respect to morphisms from $\mathsf{X}$ to algebraic spaces, [19]. We regard this operation as a functor. The *coarsening* of any DM stack $\mathsf{X}$ is the algebraic space $X$ (also called coarse space). The *coarsening of a morphism* $\mathsf{f}\colon \mathsf{X} \to \mathsf{Y}$ between DM stacks is the corresponding morphism $f\colon X \to Y$. Two examples will be often used. (1) Consider the quotient DM stack $\mathsf{C} = [\mathbb{P}^1/(\mathbb{Z}/k)]$ with $\xi_k$ acting as $z \mapsto \xi_k z$ ($k \geq 2$); the coarsening $C$ of $\mathsf{C}$ is the (smooth) quotient scheme $C = \mathbb{P}^1/(\mathbb{Z}/k)$. (2) The coarsening of the proper, smooth, $3g-3$-dimensional DM stack $\overline{\mathsf{M}}_g$ of stable curves of genus $g \geq 2$ is the $3g-3$-dimensional projective scheme $\overline{\mathcal{M}}_g$.

When we refer to the local picture of $\mathsf{X}$ at the geometric point $\mathsf{p}$, we mean the strict Henselisation of $\mathsf{X}$ at $\mathsf{p}$. Hence, the local picture of $\overline{\mathsf{M}}_g$ at $[C]_{\cong}$ is the quotient stack $[\mathrm{Def}(C)/\,\mathrm{Aut}(C)]$ (see §2.1.1).

### 1.1. Smooth level curves.

We set up $\mathsf{R}_{g,\ell}$, the space $\mathcal{R}_{g,\ell}$, and the compactification problem.

1.1.1. *The moduli stack of level smooth curves.* The integers $g \geq 2$ and $\ell \geq 1$ denote the *genus* and the *level* (we do not consider smooth curves with infinite automorphism groups).

**Definition 1.1.** The stack $\mathsf{R}_{g,\ell}$ is the category of level-$\ell$ curves $(C, L, \phi)$ where $C$ is a smooth genus-$g$ curve (over a base scheme $B$), $L$ is a line bundle on $C$, $\phi$ is an isomorphism

$\phi\colon L^{\otimes\ell} \to \mathcal{O}_C$. The order of the isomorphism class of $L$ in $\mathrm{Pic}(C)$ is exactly $\ell$. A morphism from a family $(C \to B, L, \phi)$ to a family $(C' \to B', L', \phi')$ is given by a pair $(s, \rho)$ where $s\colon (C'/B') \to (C''/B'')$ is a morphism of curves and $\rho$ is an isomorphism of line bundles $s^*L'' \to L'$ satisfying $\phi' \circ \rho^{\otimes\ell} = s^*\phi''$.

The category $\mathsf{R}_{g,\ell}$ is a DM stack. Its points have finite stabilisers and we have a coarsening $\mathcal{R}_{g,\ell}$ and a morphism $\mathsf{R}_{g,\ell} \to \mathcal{R}_{g,\ell}$. The forgetful functor $\mathsf{f}\colon \mathsf{R}_{g,\ell} \to \mathsf{M}_g$ to the category of smooth genus-$g$ curves is an étale, connected cover, and indeed a finite morphism of stacks. Finiteness can be regarded as a consequence of the fact that every fibre (pullback of $\mathsf{f}$ via a geometric point) consists of $\Phi_{2g}(\ell)$ points, with

$$\Phi_n(\ell) = \ell^n \prod_{p\in\mathfrak{P}} \left(1 - \frac{1}{p^n}\right) \qquad (\Phi_n(\ell) = \ell^n - 1 \text{ if } \ell \text{ is prime}),$$

where $\mathfrak{P}$ is the set of prime factors of $\ell$. Each of such points of the fibre is isomorphic to the stack $\mathsf{B}(\mathbb{Z}/\ell) = [\mathrm{Spec}\,\mathbb{C}/(\mathbb{Z}/\ell)]$. This happens because each point has *quasitrivial* automorphisms acting on $C$ as the identity (*i.e.*, $s$ equals $\mathrm{id}_C$), and scaling the fibres of $L$ by an $\ell$-th root of unity $\xi_\ell$ (that is, $\rho\colon z \mapsto \xi_\ell z$). Since $\mathsf{B}(\mathbb{Z}/\ell)$ has degree $1/\ell$ over $\mathrm{Spec}\,\mathbb{C}$, we get

$$\deg(\mathsf{f}\colon \mathsf{R}_{g,\ell} \to \mathsf{M}_g) = \frac{\Phi_{2g}(\ell)}{\ell} \qquad \left( = \frac{\ell^{2g}-1}{\ell} \text{ if } \ell \text{ is prime}\right).$$

When we pass to the coarsening $f\colon \mathcal{R}_{g,\ell} \to \mathcal{M}_g$ the automorphisms are forgotten. The morphism $f$ is still a finite connected cover, but it may well be ramified.

The stack $\mathsf{R}_{g,\ell}$ is not compact. If we allow triples $(C_{\mathrm{st}}, L, \phi)$, where $C_{\mathrm{st}}$ is a stable genus-$g$ curve, and keep the rest of the definition unchanged we obtain an étale cover of $\overline{\mathsf{M}}_g$. Properness fails: the fibre is not constant, as can be easily checked for a one-nodal irreducible curve.

1.2. **Twisted level curves.** The compactification becomes straightforward once we use the analogue of nodal curves in the context of DM stacks.

1.2.1. *Twisted curves.* We point out that a less restrictive definition occurs in the literature, where no stability condition on $C$ is preimposed (see for instance [21]).

**Definition 1.2.** A twisted curve $\mathsf{C}$ is a DM stack whose smooth locus is represented by a scheme, whose singularities are nodes and whose coarse space $C$ is a stable curve.

**Remark 1.3.** Explicitly, the stabilisers are trivial on smooth points, whereas, locally at a node, the stabiliser $G$ is automatically isomorphic to a group of roots of unity and the local picture of the curve at the node is given by $\{xy = 0\} \subset \mathbb{C}^2$ with a primitive $r$th root of unity $\xi_r$ acting as $\xi_r \cdot (x, y) = (\xi_r x, \xi_r^a y)$.

1.2.2. *Balanced twisted curves, twisted curves that can be smoothed.* By definition, smooth twisted curves are simply smooth scheme-theoretic curves. Nodal scheme-theoretic curve can be smoothed; we impose a condition insuring that this is the case also for twisted curves.

**Definition 1.4.** A *balanced* twisted curve is a twisted curve for which the stabilisers at the nodes $\{xy = 0\} \subset \mathbb{C}^2$ act with determinant 1 (*i.e.*, $\xi_r$ acts as $\xi_r \cdot (x, y) = (\xi_r x, \xi_r^{-1} y)$).

1.2.3. *Faithful line bundles.* A line bundle $\mathsf{L}$ on a twisted curve $\mathsf{C}$ may be pulled back from the coarse space $C$ or from an intermediate twisted curve fitting in a sequence of morphisms $\mathsf{C} \to \mathsf{C}' \to C$ (with $\mathsf{C}' \neq \mathsf{C}$ and $C' = C$). The following condition rules out this possibility.

**Definition 1.5.** A *faithful* line bundle on a twisted curve is a line bundle $\mathsf{L} \to \mathsf{C}$ for which the associated morphism $\mathsf{C} \to \mathsf{B}(\mathbb{C}^\times)$ is representable.

**Remark 1.6.** Let us phrase the condition explicitly. We may write the local picture of $\mathsf{L} \to \mathsf{C}$ at a node $\mathsf{n}$ of $\mathsf{C}$ as the projection $\mathbb{C} \times \{xy = 0\} \longrightarrow \{xy = 0\}$, with the primitive $r$th root $\xi_r$ acting as $\xi_r \cdot (x, y) = (\xi_r x, \xi_r^{-1} y)$ on $\{xy = 0\}$ and as $\xi_r \cdot (t, x, y) = (\xi_r^m t, \xi_r x, \xi_r^{-1} y)$ on $\mathbb{C} \times \{xy = 0\}$ for a suitable index $m$ (modulo $r$). The index $m \in \mathbb{Z}/r$ is uniquely determined as soon as we assign a privileged choice of a branch of the node on which $\xi_r$ acts by multiplication as $x \mapsto \xi_r x$ (the action on the remaining branch is opposite $y \mapsto \xi_r^{-1} y$). In this setting, we may restate faithfulness as follows

$$\mathsf{L} \text{ is faithful at } \mathsf{n} \iff \text{the representation } \mathsf{L}\,|_{\mathsf{n}} \text{ is faithful} \iff \gcd(m, r) = 1.$$

Notice that if we switch the roles of the two branches, then $m$ changes sign modulo $r$. Faithfulness does not depend on the sign of $m$ and on the choice of the branch.

1.2.4. *Twisted curves and their level structures.* Once the suitable notion of twisted curve (balancedness condition) and of line bundle (faithfulness condition) are given, level-$\ell$ structures are defined as for smooth curves. This is the main advantage of the twisted curve approach.

**Definition 1.7.** A *level-$\ell$ twisted curve* $(\mathsf{C} \to B, \mathsf{L}, \phi)$ consists of a balanced twisted curve $\mathsf{C}$ of genus $g$ over a base scheme $B$, a faithful line bundle $\mathsf{L}$, and an isomorphism $\phi\colon \mathsf{L}^{\otimes \ell} \to \mathcal{O}_{\mathsf{C}}$. The order of $\mathsf{L}$ is exactly $\ell$ in $\mathrm{Pic}(\mathsf{C})$.

The category of stable level-$\ell$ curves forms a smooth DM stack $\overline{\mathsf{R}}_{g,\ell}$ of dimension $3g-3$, with a finite forgetful morphism over the stack of stable curves $\mathsf{f}\colon \overline{\mathsf{R}}_{g,\ell} \to \overline{\mathsf{M}}_g$ of degree $\deg(\mathsf{f}) = \Phi_{2g}(\ell)/\ell$ (or, simply, $(\ell^{2g} - 1)/\ell$ when $\ell$ is prime). This definition is given implicitly in [3] by Abramovich and Vistoli (level-$\ell$ curves correspond to a connected component of the moduli stack of stable maps to $\mathsf{B}(\mathbb{Z}/\ell)$). The forgetful morphism $\mathsf{f}$ is ramified as we illustrate in §1.4. See also work of the first author [7] for a slightly modified version, which preserves the étaleness of the forgetful morphism from level-$\ell$ smooth curves.

1.2.5. *Local indices.* Consider the local picture from Remark 1.6 of a level-$\ell$ curve at a node:

$$\xi_r \cdot (t, x, y) = (\xi_r^m t, \xi_r x, \xi_r^{-1} y).$$

Notice that $\mathsf{L}^{\otimes \ell} \cong \mathcal{O}$ implies $(\xi_r^m)^\ell = 1$; that is $\ell m \in r\mathbb{Z}$ with $r \geq 1$ and $m \in \{0, \dots, r-1\}$. Faithfulness implies $\gcd(r, m) = 1$; hence $r \mid \ell$. In the rest of the paper, we often use a single *multiplicity index* $M = m\ell/r$ to encode the local indices $r$ and $m$:

$$\begin{aligned} r(M) = \tfrac{\ell}{\gcd(M,\ell)},\ m(M) = \tfrac{M}{\gcd(M,\ell)} \quad & (M \in \{0, \dots, \ell-1\}), \\ M(r, m) = m\ell/r \quad & (r \mid \ell,\ m \in \{0, \dots, r-1\},\ \gcd(r, m) = 1). \end{aligned} \tag{4}$$

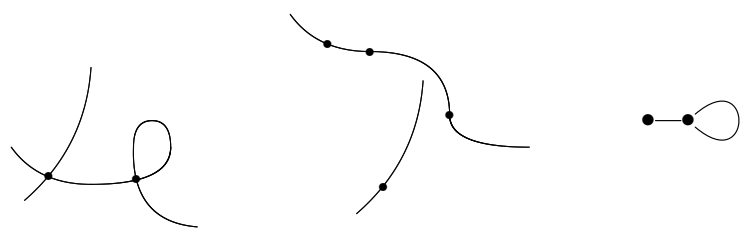
The first interesting example is $\ell = 3$. In this case, $M$ equals $m$ and, once we choose a privileged branch at a node, there are three possible local pictures: $(M = 0)$ (*i.e.* $(m, r) = (0, 1)$) trivial stabiliser; $(M = 1)$ (*i.e.* $(m, r) = (1, 3)$) non trivial action $\xi_3 \cdot (t, x, y) = (\xi_3 t, \xi_3 x, \xi_3^2 y)$;

$(M = 2)$ (*i.e.* $(m, r) = (2, 3)$) non trivial action $\xi_3 \cdot (t, x, y) = (\xi_3^2 t, \xi_3 x, \xi_3^2 y)$. Notice that if we interchange the roles of the branches at a node, then $M$ changes sign modulo 3. Therefore, we may summarise this analysis by saying that the nodes of level-3 twisted curves are either trivial ($M = 0$) or nontrivial ($M = 1$) and, in this case, equipped with a distinguished choice of a branch (the branch with action $x \mapsto \xi_3 x$).

1.3. **Dual graphs of twisted curves and multiplicity of level curves.** The dual graph of a twisted curve is simply the dual graph of the coarse curve.

1.3.1. *Dual graphs.* Dual graphs arising from the standard construction recalled below are connected nonoriented graphs, possibly containing multiple edges (edges linking the same two vertices) and loops (edges starting and ending at the same vertex).

**Definition 1.8.** Consider a twisted curve $\mathsf{C}$ and its normalisation $\mathsf{nor}\colon \mathsf{C}' \to \mathsf{C}$. The vertex set $V$ of the dual graph is the set of connected components of $\mathsf{C}'$. The edge set $E$ of the dual graph is the set of nodes of $\mathsf{C}$. The two sets $V$ and $E$ determine a graph as follows: a node identifies the connected components of $\mathsf{C}'$ where its preimages lie, in this way an edge links two (possibly equal) vertices.



twisted curve $\mathsf{C}$ normalisation $\mathsf{C}' = \bigsqcup \mathsf{C}_j$ dual graph $\Gamma$

FIGURE 1. Normalisation and dual graph of a twisted curve

1.3.2. *Cochains.* Each node of $\mathsf{C}$ has two branches. Let $\mathbb{E}$ be the set of branches of each node of $\mathsf{C}$. The cardinality of $\mathbb{E}$ is twice that of $E$; there is a 2-to-1 projection $\mathbb{E} \to E$ and an involution $e \mapsto \overline{e}$ of $\mathbb{E}$. On $\mathbb{E}$ we can define a function $\mathbb{E} \to V$, noted $e \mapsto e_+ \in V$, assigning to each oriented edge the vertex $v = e_+$ corresponding to the connected component $\mathsf{C}'_v$ of $\mathsf{C}'$ where the chosen branch lies. We get $e \mapsto e_-$ by applying $(\ )_+$ after the involution. If $e_+ = e_-$ we have a loop (Figure 1): $\overline{e} \neq e$ in $\mathbb{E}$ map to the same vertex via $e \mapsto e_+$.

We define the group of 1-cochains and 0-cochains of the dual graph as follows. We define $C^0(\Gamma; \mathbb{Z}/\ell)$ as the set of $\mathbb{Z}/\ell$-valued functions on $V$

$$C^0(\Gamma; \mathbb{Z}/\ell) = \{a\colon V \to \mathbb{Z}/\ell\} = \bigoplus\nolimits_{v \in V} \mathbb{Z}/\ell.$$

We define 1-cochains as antisymmetric $\mathbb{Z}/\ell$-valued functions on $\mathbb{E}$

$$C^1(\Gamma; \mathbb{Z}/\ell) = \{b\colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\overline{e}) = -b(e)\},$$

where $\overline{e}$ and $e$ are oriented edges with opposite orientations. After assigning an orientation for each edge $e \in E$, we may identify $C^1(\Gamma; \mathbb{Z}/\ell)$ to $\bigoplus_{e \in E} \mathbb{Z}/\ell$, but we prefer working with $\mathbb{E}$.

The space of 0-cochains and 1-cochains $C^0(\Gamma;\mathbb{Z}/\ell)$ and $C^1(\Gamma;\mathbb{Z}/\ell)$ are equipped with bilinear $\mathbb{Z}/\ell$-valued forms

$$\langle a_1, a_2\rangle = \sum\nolimits_{v\in V} a_1(v)a_2(v) \qquad \langle b_1, b_2\rangle = \frac{1}{2}\sum\nolimits_{e\in\mathbb{E}} b_1(e)b_2(e) \tag{5}$$

with $a_1, a_2 \in C^0$ and $b_1, b_2 \in C^1$. The exterior differential is

$$\begin{aligned} \delta\colon C^0(\Gamma;\mathbb{Z}/\ell) &\to C^1(\Gamma;\mathbb{Z}/\ell), \\ a &\mapsto \delta a, \qquad\qquad \text{with} \quad \delta a(e) = a(e_+) - a(e_-). \end{aligned}$$

The adjoint operator with respect to $\langle\ ,\ \rangle$ is given by

$$\begin{aligned} \partial\colon C^1(\Gamma;\mathbb{Z}/\ell) &\to C^0(\Gamma;\mathbb{Z}/\ell), \\ b &\mapsto \partial b, \qquad\qquad \text{with} \ \ \partial b(v) = \sum_{\substack{e\in\mathbb{E}\\ e_+=v}} b(e). \end{aligned}$$

**Remark 1.9** (cuts and circuits)**.** The image $\mathrm{im}(\delta)$ is freely generated by $\#V-1$ cuts (see [4, Ch. 4]),

$$\mathrm{im}(\delta) \cong (\mathbb{Z}/\ell)^{\oplus(\#V-1)}. \tag{6}$$

We recall that a cut is determined by a proper nonempty subset $W$ of the vertex set $V$ of $\Gamma$: the sets $W$ and $V\setminus W$ form a partition of $V$. Cuts are 1-cochains $b\colon \mathbb{E}\to\mathbb{Z}/\ell$ in $C^1(\Gamma;\mathbb{Z}/\ell)$ equal to 1 on the (nonempty) set $H_W$ of edges having only one end on $W$ and oriented from $W$ to $V\setminus W$, equal to $-1$ on $\overline{H}_W = \{\overline{e} \mid e\in H_W\}$, and vanishing elsewhere. By construction, $H_W$ and $\overline{H}_W$ contain no loops.

The kernel $\ker\partial$ is freely generated by $b_1 = 1-\chi(\Gamma) = 1-\#V+\#E$ circuits

$$\ker\partial \cong (\mathbb{Z}/\ell)^{\oplus(1-\#V+\#E)}.$$

We recall that a circuit within a graph is a sequence of $n$ oriented edges $e_0,\dots,e_{n-1}\in\mathbb{E}$, overlying $n$ distinct nonoriented edges in $E$, and such that the head of $e_i$ is also the tail of $e_{i+1}$ for $0\le i<n-1$ and the head of $e_{n-1}$ is the tail of $e_0$. In this way a circuit identifies $n$ distinct vertices $v_i = (e_i)_-$ for $0\le i<n$. Here, we treat circuits as 1-cochains regarding their characteristic function (given by 1 on $e_i$, $-1$ on $\overline{e}_i$ and 0 elsewhere) as an element of $C^1(\Gamma;\mathbb{Z}/\ell)$. Circuits formed by a single oriented edge will be called loops.

We recall that $\mathrm{im}\,\delta$ is the orthogonal complement of $\ker\partial$

$$\mathrm{im}\,\delta = (\ker\partial)^{\perp}_{\langle\ ,\ \rangle} \tag{7}$$

with respect to the pairing $\langle\ ,\ \rangle$ from (5). We derive a simple criterion to decide when $b$ in $C^1(\Gamma;\mathbb{Z}/\ell)$ belongs to $\mathrm{im}\,\delta$:

$$b\in C^1(\Gamma;\mathbb{Z}/\ell) \text{ is in } \mathrm{im}\,\delta \iff \langle b, K\rangle = 0 \text{ for all circuits } K \text{ of } \Gamma. \tag{8}$$

1.3.3. *Line bundles whose normalisation is trivial are classified by* $\mathrm{im}\,\delta$. Recall the exact sequences

$$(\mathbb{C}^\times)^{\#V} \to (\mathbb{C}^\times)^{\#E} \to \mathrm{Pic}(\mathsf{C}) \xrightarrow{\mathsf{nor}^*} \mathrm{Pic}(\mathsf{C}'), \tag{9}$$

$$C^0(\Gamma;\mathbb{Z}/\ell) \xrightarrow{\delta} C^1(\Gamma;\mathbb{Z}/\ell) \xrightarrow{\tau} \mathrm{Pic}(\mathsf{C})[\ell] \xrightarrow{\mathsf{nor}^*} \mathrm{Pic}(\mathsf{C}')[\ell]. \tag{10}$$

Let us state explicitly the definition of the homomorphism $\tau$ on the 1-cochain $e \mapsto \chi_{e_0}(e)$ vanishing on all edges except $e_0$ and $\overline{e}_0$ where it equals 1 and $-1$ respectively. The line bundle $\tau(\chi_{e_0})$ is the locally free sheaf of regular functions $f$ on the normalisation of $\mathsf{C}$ at the node $\mathsf{n}$ satisfying $f(\mathsf{x}) = \xi_\ell f(\mathsf{y})$ for $\mathsf{x}$ and $\mathsf{y}$ pre-images of $\mathsf{n}$, with $\mathsf{x}$ lying on the branch corresponding to $e_0$ and $\mathsf{y}$ lying on the remaining branch. (Clearly, we have $\tau(\chi_{\overline{e}_0}) = \tau(\chi_{e_0})^\vee$.)

1.3.4. *Multiplicity and* $\ker\partial$. Since oriented edges are in one-to-one correspondence with branches of nodes of $C$, using §1.2.5, we define the multiplicity cochain.

**Definition 1.10** (multiplicity 1-cochain of stable level-$\ell$ curves)**.** Consider a stable level-$\ell$ curve $(\mathsf{C}, \mathsf{L}, \phi)$. To each oriented edge $e$, we can attach the multiplicity $M(e)$ of $(\mathsf{C}, \mathsf{L}, \phi)$ at the node (with its prescribed branch). The function $M\colon e \mapsto M(e)$ satisfies $M(\overline{e}) = -M(e)$ for all $e \in \mathbb{E}$: $M \in C^1(\Gamma; \mathbb{Z}/\ell)$.

**Proposition 1.11.** *Let $C$ be a stable curve and consider the set of stable level-$\ell$ curves $(\mathsf{C}, \mathsf{L}, \phi)$ with coarsening $C$. Consider the dual graph of $C$ and the differential $\partial$. Then the multiplicity cochain $M$ takes values in* $\ker\partial$ *and is surjective.*

*Proof.* All level-$\ell$ structures overlying $C$ maybe regarded as elements in $\mathrm{Pic}(\widetilde{\mathsf{C}})[\ell]$, for a canonical balanced twisted curve $\widetilde{\mathsf{C}}$ with $\mathbb{Z}/\ell$-stabilisers at all nodes (in $\mathrm{Pic}(\widetilde{\mathsf{C}})[\ell]$ we do not impose faithfulness). The multiplicity cochain lifts to a homomorphism $M\colon \mathrm{Pic}(\widetilde{\mathsf{C}})[\ell] \to C^1(\Gamma; \mathbb{Z}/\ell)$. The claim follows from the exact sequence $1 \to \mathrm{Pic}(C)[\ell] \to \mathrm{Pic}(\widetilde{\mathsf{C}})[\ell] \to C^1(\Gamma; \mathbb{Z}/\ell) \to C^0(\Gamma; \mathbb{Z}/\ell)$ (see [7, Cor. 3.1]) and from the existence of an order-$\ell$ element in $\mathrm{Pic}(C)[\ell]$ (for $g \geq 2$). □

**Example 1.12.** Consider a two-component twisted curve obtained as the union of two smooth one-dimensional stacks $\mathsf{X}$ and $\mathsf{Y}$ meeting transversely at 2 nodes. For each node, let us measure the multiplicities with respect to the branch lying in $\mathsf{X}$. Proposition 1.11 says that the multiplicities $M_1$ and $M_2$ should add up to 0 (modulo $\ell$). Let us examine in greater detail the case $\ell = 3$, $M_1 = 1$ and $M_2 = 2$. Over $\mathsf{X}$ the third root $\mathsf{L}$ of $\mathcal{O}$ is given by a divisor $D'$ of degree 0 (a root of $\mathcal{O}_{\mathsf{X}}$) with rational coefficients of the form $D' = \lfloor D' \rfloor + [\mathsf{x}_1]/3 + 2[\mathsf{x}_2]/3$, where $\mathsf{x}_1\colon \operatorname{Spec}\mathbb{C} \to \mathsf{X}$ and $\mathsf{x}_2\colon \operatorname{Spec}\mathbb{C} \to \mathsf{X}$ are the geometric points lifting $\mathsf{n}_1$ and $\mathsf{n}_2$ to $\mathsf{X}$. Conversely $\mathsf{L}|_{\mathsf{Y}}$ can be expressed as the degree-0 line bundle $\mathcal{O}(D'')$ with $D'' = \lfloor D'' \rfloor + 2[\mathsf{y}_1]/3 + [\mathsf{y}_2]/3$, where, again, $\mathsf{y}_1$ and $\mathsf{y}_2$ lift $\mathsf{n}_1$ and $\mathsf{n}_2$ to $\mathsf{Y}$.

The multiplicity 1-cochain encodes much of the relevant topological information characterising a level curve. In what follows, we describe some natural invariants of 1-cochains.

1.3.5. *The support and its characteristic function.* For any 1-cochain $c\colon \mathbb{E} \to \mathbb{Z}/\ell$ we consider the characteristic function of the support of $c$ taking values in the extended set $\mathbb{Z} \cup \{\infty\}$ (we use the standard conventions $a < \infty$ and $a + \infty = \infty$ for $a \in \mathbb{Z}$).

$$\nu_c(e) = \begin{cases} \infty & \text{if } c(e) = 0 \in \mathbb{Z}/\ell \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Proposition 1.11 implies $\nu_c(e) = \infty$ for any separating edge.

We now present a natural subcomplex $C^\bullet_\nu(\Gamma; \mathbb{Z}/\ell)$ of $C^\bullet(\Gamma; \mathbb{Z}/\ell)$ attached to a given symmetric characteristic function $\nu\colon \mathbb{E} \to \{0, \infty\}$; *i.e.* to any subset of $E$. In §2.4 we generalise

this construction by allowing, instead of characteristic functions, more general functions arising as the truncated valuations of $M$, see (28). When $\ell$ is prime we recover the above defined function $\nu_c$.

1.3.6. *The contracted graph* $\Gamma(\nu)$. We define precisely the graphs obtained by iterated edge-contractions of $\Gamma$ mentioned in the introduction. Let us consider any symmetric characteristic function $\nu\colon \mathbb{E} \to \{0, \infty\}$ (since $\nu$ is symmetric it descends to $E$ and we sometimes abuse the notation by regarding it as a function on $E$). We attach to $\Gamma$ a new graph $\Gamma(\nu)$ whose sets of vertices and edges $(\overline{V}, \overline{E})$ are obtained from $(V, E)$

(i) by setting $E(\nu) = \{e \mid \nu(e) = 0\}$,
(ii) by modding out $V$ by the relations $e_+ \sim_\nu e_-$ if $\nu(e) = \infty$, *i.e.* $V(\nu) = V/\sim_\nu$.

In the new graph, the set of vertices of the edge $e \in E(\nu)$ is the set of vertices of $e \in E$ in $V$ modulo the relation $\sim_\nu$. In simple terms, $\Gamma(\nu)$ is the contraction of all edges where $v > 0$. We refer to $\Gamma(\nu)$ as a contraction of $\Gamma$ and, conversely, to $\Gamma$ as a blowup of $\Gamma(\nu)$ (the graph obtained from an iterated edge-contraction is a "minor" of the initial graph, but we do not use this terminology here.)

1.3.7. *The complex* $C^\bullet_\nu(\Gamma; \mathbb{Z}/\ell)$. The inclusion $i\colon E(\nu) \hookrightarrow E$ and the projection $p\colon V \twoheadrightarrow V(\nu)$ yield homomorphisms $p_*\colon C^0(\Gamma; \mathbb{Z}/\ell) \twoheadrightarrow C^0(\Gamma(\nu); \mathbb{Z}/\ell)$ and $i^*\colon C^1(\Gamma; \mathbb{Z}/\ell) \twoheadrightarrow C^1(\Gamma(\nu); \mathbb{Z}/\ell)$ and the *contraction homomorphism* between complexes with differentials given by $\partial$

$$\mathcal{C}\colon (C^\bullet(\Gamma; \mathbb{Z}/\ell), \partial) \twoheadrightarrow (C^\bullet(\Gamma(\nu); \mathbb{Z}/\ell), \partial). \tag{12}$$

Conversely, the homomorphisms $p^*\colon C^0(\Gamma(\nu); \mathbb{Z}/\ell) \hookrightarrow C^0(\Gamma; \mathbb{Z}/\ell)$ and $i_*\colon C^1(\Gamma(\nu); \mathbb{Z}/\ell) \hookrightarrow C^1(\Gamma; \mathbb{Z}/\ell)$ yield the *blowup homomorphism* between complexes with differential $\delta$

$$\mathcal{B}\colon (C^\bullet(\Gamma(\nu); \mathbb{Z}/\ell), \delta) \hookrightarrow (C^\bullet(\Gamma; \mathbb{Z}/\ell), \delta). \tag{13}$$

The subcomplex $\mathcal{B}(C^\bullet(\Gamma(\nu); \mathbb{Z}/\ell), \delta)$ consists of the 0-cochains $a \in C^0(\Gamma; \mathbb{Z}/\ell)$ and the 1-cochains $b \in C^1(\Gamma; \mathbb{Z}/\ell)$ satisfying $a(e_+) = a(e_-)$ and $b(e) = 0$ if $\nu(e) = \infty$. Within $(C^\bullet(\Gamma; \mathbb{Z}/\ell), \delta)$ we denote such a subcomplex by

$$C^\bullet_\nu(\Gamma; \mathbb{Z}/\ell) \subseteq C^\bullet(\Gamma; \mathbb{Z}/\ell)$$

In fact, we have

$$\mathcal{B}(\mathrm{im}(\delta)) = \mathrm{im}(\delta) \cap C^1_\nu(\Gamma; \mathbb{Z}/\ell). \tag{14}$$

The inclusion from left to right follows from (13). Conversely, $b = \delta(a)$ is in $C^1_\nu(\Gamma; \mathbb{Z}/\ell)$ only if, for any contracted edge $e$, we have $a(e_+) = a(e_-)$; that is only if $a$ lies in $C^0_\nu(\Gamma; \mathbb{Z}/\ell))$. Passing to the adjoint operator we also get

$$\mathcal{C}(\ker \partial) = \ker \partial. \tag{15}$$

Summarising, the contraction of a circuit is a circuit and the blowup of a cut is a cut.

1.4. **The boundary locus.** We describe $\overline{\mathsf{R}}_{g,\ell} \setminus \mathsf{R}_{g,\ell}$ by classifying one-nodal level curves.

1.4.1. *Reducible one-nodal curves.* Consider the union $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{C}_2$ of two smooth stack-theoretic curves $\mathsf{C}_1$ and $\mathsf{C}_2$ of genus $i$ and $g-i$ meeting transversally at a point. Proposition 1.11 implies that the node has multiplicity zero or, in other words, trivial stabiliser. Hence, we have $\mathsf{C} = C$; *i.e.* $\mathsf{C}$ is an ordinary stable curve of compact type $C = C_1 \cup C_2$. The line bundle $\mathsf{L} = L$ on $C$ is determined by the choice of two line bundles $L_1$ and $L_2$ satisfying $L_1^{\otimes \ell} \cong \mathcal{O}_{C_1}$ and $L_2^{\otimes \ell} \cong \mathcal{O}_{C_1}$ respectively. There are three possibilities:

$$\text{(i) } L_1 \cong \mathcal{O},\ L_2 \ncong \mathcal{O}; \qquad \text{(ii) } L_1 \ncong \mathcal{O},\ L_2 \cong \mathcal{O} \qquad \text{(iii) } L_1, L_2 \ncong \mathcal{O}$$

(since $\mathsf{L} \ncong \mathcal{O}$, the possibility that both line bundles are trivial is excluded). If $0 < i < g/2$, these three cases characterise three loci in the moduli space whose closures are the divisors $\Delta_{g-i}, \Delta_i$ and $\Delta_{i:g-i}$ respectively. We write $\delta_{g-i}, \delta_i$ and $\delta_{i:g-i}$ for the corresponding $\mathbb{Q}$-divisors defined by the same conditions in the moduli stack. The morphism $\mathsf{f}$ is not ramified along these divisors. We have that

$$\mathsf{f}^*(\delta_i^{\text{stable}}) = \delta_{g-i} + \delta_i + \delta_{i:g-i} \tag{16}$$

where $\delta_i^{\text{stable}}$ is the $\mathbb{Q}$-divisor class in $\overline{\mathsf{M}}_g$ defined by stable curves with at least one node separating the curve into two components of genus $i$ and $g-i$.

If $i = g/2$ the same classification reduces to two divisors: the closure of the locus of one-nodal level curves for which only one line bundle among $L_1$ and $L_2$ is trivial yields $\Delta_{g/2}$, the closure of the locus classifying curve where both $L_1$ and $L_2$ are nontrivial yields $\Delta_{g/2:g/2}$.

1.4.2. *Irreducible one-nodal curves.* If $\mathsf{C}$ is irreducible and has one node, then the node is of nonseparating type: the normalisation $\mathsf{nor}\colon \mathsf{C}' \to \mathsf{C}$ is given by a connected curve. There are three possibilities:

$$\text{(i) } M = 0 \text{ and } \mathsf{nor}^*\mathsf{L} \ncong \mathcal{O}; \qquad \text{(ii) } M = 0 \text{ and } \mathsf{nor}^*\mathsf{L} \cong \mathcal{O}; \qquad \text{(iii) } M \neq 0$$

The closures of the loci of level curves satisfying the three conditions above determine three divisors denoted by $\Delta_0', \Delta_0'', \Delta_0^{\text{ram}}$ in the moduli space. We write again $\delta_0', \delta_0'', \delta_0^{\text{ram}}$ for the corresponding classes of divisors defined by the same conditions in the moduli stack. The morphism $\mathsf{f}$ is not ramified along $\delta_0'$ and $\delta_0''$. When $\ell$ is prime, $\mathsf{f}$ is ramified with order $\ell$ along $\delta_0^{\text{ram}}$. Precisely, we have that, cf. [9]

$$\mathsf{f}^*(\delta_0^{\text{stable}}) = \delta_0' + \delta_0'' + l\delta_0^{\text{ram}}. \tag{17}$$

In general, $\delta_0^{\text{ram}}$ can be decomposed into several components depending on the value of the multiplicity index $M$; we refer to §1.4.4 for the study of the order of the ramification.

This calls for an analysis of the irreducible components of the boundary divisors $\delta_0', \delta_0'', \delta_0^{\text{ram}}$ as well as for the previous divisors $\delta_i, \delta_{i:g-i}$, for $1 \leq i \leq g/2$. We carry it out in the last part of this section (§1.4.3 and §1.4.4) as a nice application of the combinatorial invariants of level curves illustrated above. On the other hand, the present description of the boundary locus is sufficient for the entire Section 2 and may already be already regarded as a decomposition into irreducible components of the boundary for $\ell = 3$ (see Remarks 1.15 and 1.14). Therefore, it is worthwhile to illustrate it further by an example, which will play an important role in the rest of the paper: the case of level structures on elliptic-tailed curves.

**Example 1.13** (two level-$\ell$ structures on the elliptic-tailed curve)**.** We provide examples of two distinct twisted level curves, one representing a point of $\Delta_1 \cap \Delta_0^{\text{ram}}$, and the other representing a point in $\Delta_1 \cap \Delta_0''$. Consider the stack-theoretic quotient $\mathsf{E}$ of $\widetilde{E} = \mathbb{P}^1/(0 \equiv \infty)$

by $\mathbb{Z}/\ell$ spanned by $z \mapsto \xi_\ell z$. Now let $\mathsf{C}$ be a twisted curve containing, as a subcurve, a copy of such a genus-1 stack-theoretic curve $\mathsf{E}$. We assume $\mathsf{E} \cap \overline{\mathsf{C} \setminus E} = \{n\}$, where $n$ is a separating node with trivial stabiliser (see Proposition 1.11).

Level-$\ell$ structures in $\Delta_1$ can be defined on $\mathsf{C}$ by extending trivially on $\overline{\mathsf{C} \setminus \mathsf{E}}$ nontrivial $\ell$th roots of $\mathcal{O}$ on $\mathsf{E}$. To this effect, we can exploit $\mathsf{p}\colon \widetilde{E} \to \mathsf{E}$, which is an étale $(\mathbb{Z}/\ell)$-cyclic cover of $\mathsf{E}$. We have $\mathsf{p}_*\mathcal{O} = \mathcal{O} \oplus \mathsf{L}_{\mathrm{ram}} \oplus \mathsf{L}_{\mathrm{ram}}^{\otimes 2} \oplus \ldots \oplus \mathsf{L}_{\mathrm{ram}}^{\otimes \ell - 1}$, with $\phi_{\mathrm{ram}}\colon \mathsf{L}_{\mathrm{ram}}^{\otimes \ell} \cong \mathcal{O}$. Then $\mathsf{L}_{\mathrm{ram}} \to \mathsf{C}$ yields an object $(\mathsf{C}, \mathsf{L}_{\mathrm{ram}}, \phi_{\mathrm{ram}})$ in $\Delta_1 \cap \Delta_0^{\mathrm{ram}}$ because the multiplicity of $\mathsf{L}_{\mathrm{ram}}$ at the nonseparating node is $\neq 0$ (1 or $l-1$ depending on the chosen branch).

The projection to the coarse space $\epsilon_{\mathsf{E}}\colon \mathsf{E} \to E$ yields another nontrivial line bundle in $\mathrm{Pic}(\mathsf{C})[\ell]$. On $\mathsf{E}$ this is simply the pullback $\mathsf{L}_{\text{ét}}$ of the line bundle of regular functions $f$ on the normalisation $E' \cong \mathbb{P}^1$ satisfying $f(\infty) = \xi_\ell f(0)$. The line bundle $\mathsf{L}_{\text{ét}} \to \mathsf{C}$ yields a point in $\Delta_1 \cap \Delta_0''$ (the multiplicity at the nonseparating node is 0 and $\mathsf{L}$ is trivial on the normalisation by construction).

1.4.3. *The closure of the locus of reducible one-nodal curves: irreducible components.* We provide a decomposition into irreducible components of the divisor defined above as the closure of the substack of reducible level-$\ell$ one-nodal curves. It is convenient to reformulate the problem in $\overline{\mathsf{M}}_g$: we study the divisor

$$\mathsf{D}_{\mathrm{red}}^{\mathrm{stable}} = \sum\nolimits_{1 \le i \le g/2} \delta_i^{\mathrm{stable}} \tag{18}$$

of stable curves with at least one separating node. We do so, by analysing its two-folded étale cover $\widetilde{\mathsf{D}}_{\mathrm{red}}^{\mathrm{stable}}$ classifying stable curves alongside with a separating node and a branch of the node. We have the natural decomposition $\widetilde{\mathsf{D}}_{\mathrm{red}}^{\mathrm{stable}} = \bigsqcup_{i=1}^{g-1} \widetilde{\mathsf{D}}_i^{\mathrm{stable}}$ where $\widetilde{\mathsf{D}}_i^{\mathrm{stable}}$ classifies objects where the chosen branch lies in the genus-$i$ connected component $\mathsf{Z}$ of the normalisation of the separating node. Then, for $i = 1, \ldots, g-1$, we write $\mathsf{D}_i^{\mathrm{stable}}$ for the pushforward in $\overline{\mathsf{M}}_g$ of the cycle $\widetilde{\mathsf{D}}_i^{\mathrm{stable}}$ via the map forgetting the branch; for $i \neq g/2$, the forgetful map from $\widetilde{\mathsf{D}}_i^{\mathrm{stable}}$ has degree 1 and we have $\mathsf{D}_i^{\mathrm{stable}} = \mathsf{D}_{g-i}^{\mathrm{stable}}$, for $i = g/2$ the forgetful map $\widetilde{\mathsf{D}}_{g/2}^{\mathrm{stable}}$ is a 2-folded cover. In this way, we reformulate (18) as follows

$$\mathsf{D}_{\mathrm{red}}^{\mathrm{stable}} = \frac{1}{2} \sum\nolimits_{g=1}^{g-1} \mathsf{D}_i^{\mathrm{stable}}.$$

For level curves, consider the stack $\widetilde{\mathsf{D}}_{\mathrm{red}}$ classifying stable level-$\ell$ curves alongside with a separating node and a branch of the node. Hence, we get the decomposition of $\widetilde{\mathsf{D}}_{\mathrm{red}}$ into connected components and the corresponding decomposition of $\mathsf{D}_{\mathrm{red}}$ into irreducible components

$$\widetilde{\mathsf{D}}_{\mathrm{red}} = \bigsqcup\nolimits_{d_1, d_2, i} \widetilde{\mathsf{D}}_i^{d_1, d_2} \qquad \text{and} \qquad \mathsf{D}_{\mathrm{red}} = \frac{1}{2} \sum\nolimits_{d_1, d_2, i} \mathsf{D}_i^{d_1, d_2},$$

where $d_1$ and $d_2$ are divisors of $\ell$ whose least common multiple equals $\ell$, $i$ ranges between 1 and $g-1$, and the loci $\widetilde{\mathsf{D}}_i^{d_1, d_2}$ and $\mathsf{D}_i^{d_1, d_2}$ are defined as follows. The stack $\widetilde{\mathsf{D}}_i^{d_1, d_2}$ is the full subcategory of objects where the data of the chosen branch and of the genus-$i$ connected component $\mathsf{Z}$ of the normalisation of the separating node satisfy

(i) the branch lies in $\mathsf{Z}$
(ii) the order of $\mathsf{L}$ on $\mathsf{Z}$ equals $d_1$,
(iii) the order of $\mathsf{L}$ on $\overline{\mathsf{C} \setminus \mathsf{Z}}$ equals $d_2$.

The divisor $\mathsf{D}_i^{d_1,d_2}$ is the pushforward of the cycle $\widetilde{\mathsf{D}}_i^{d_1,d_2}$ via the forgetful functor forgetting the choice of the branch. Since the stack-theoretic structure of one-nodal $\ell$-level curves of compact type is trivial, there is no ramification of $\mathsf{f}$ along $\mathsf{D}_{\mathrm{red}}$: we have $\mathsf{D}_{\mathrm{red}} = \mathsf{f}^*\mathsf{D}_{\mathrm{red}}^{\mathrm{stable}}$.

**Remark 1.14.** For $\ell$ prime, we notice that $\delta_i$, $\delta_{g-i}$ and $\delta_{i:g-i}$ are precisely the divisors $\mathsf{D}_i^{\ell,1}$, $\mathsf{D}_i^{1,\ell}$ and $\mathsf{D}_i^{\ell,\ell}$. For $i \neq g/2$, they are the three irreducible components of $\mathsf{f}^*\delta_i^{\mathrm{stable}}$ of degrees $(\ell^{2i}-1)/l$, $(\ell^{2g-2i}-1)/l$ and $(\ell^{2g-2i}-1)(\ell^{2i}-1)/l$ over $\delta_i^{\mathrm{stable}}$ (we check that they add up to $\deg(\mathsf{f}) = (\ell^{2g}-1)/\ell$).

1.4.4. *The closure of the locus of irreducible one-nodal curves: irreducible components.* We study the divisor $\delta_0^{\mathrm{stable}}$ of stable curves with at least one separating node. In analogy with §1.4.3 we use the notation $\mathsf{D}_{\mathrm{irr}}^{\mathrm{stable}} = \delta_0^{\mathrm{stable}}$ and we analyze its two-folded étale connected cover $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{\mathrm{stable}}$ classifying stable curves alongside with a nonseparating node and a branch of the node. Consider the stack $\widetilde{\mathsf{D}}_{\mathrm{irr}}$ classifying stable level-$\ell$ curves $(\mathsf{C}, \mathsf{L}, \phi)$ equipped with a prescribed choice of a nonseparating node and of a branch of such node: this yields a notation $\mathsf{x}$ and $\mathsf{y}$ for the points lifting the node to the normalisation $\mathrm{nor}\colon \mathsf{C}' \to \mathsf{C}$ of the nonseparating node. On $\widetilde{\mathsf{D}}_{\mathrm{irr}}$, we can define the data $M, d, h$

- the multiplicity $M \in \mathbb{Z}/\ell$,
- the order $d$ (dividing $\ell$ and multiple of $\ell/\gcd(M,\ell)$) of $\mathrm{nor}^*\mathsf{L}$ on $\mathsf{C}'$,
- the datum $h \in d\mathbb{Z}/\ell\mathbb{Z}$ satisfying $f(\mathsf{x}) = \xi_\ell^h f(\mathsf{y})$ for the sections $f$ of $(\mathrm{nor}^*\mathsf{L})^{\otimes d} \cong \mathcal{O}$.

In this way, for $M, d, h$ satisfying the above conditions, we obtain a decomposition into connected components and the desired decomposition into irreducible components

$$\widetilde{\mathsf{D}}_{\mathrm{irr}} = \bigsqcup\nolimits_{M,d,h} \widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,d,h} \qquad \text{and} \qquad \mathsf{D}_{\mathrm{irr}} = \frac{1}{2}\sum\nolimits_{M,d,h} \mathsf{D}_{\mathrm{irr}}^{M,d,h},$$

where $\mathsf{D}_{\mathrm{irr}}^{M,d,h}$ are the pushforwards in $\overline{\mathsf{R}}_{g,\ell}$ of the cycles $\widetilde{\mathsf{D}}_{\mathrm{irr}}^{M,d,h}$ via the morphism forgetting the prescribed branch (note that, if $M, h \in \{0, \ell/2\}$, this forgetful morphism is a 2-to-1 cover) The order of the ramification of the morphism $\mathsf{f}$ along $\mathsf{D}_{\mathrm{irr}}^{M,d,h}$ equals the order $\ell/\gcd(M,\ell)$ of $M$ in $\mathbb{Z}/\ell$.

**Remark 1.15.** If $\ell = 3$ the divisors $\delta_0'$, $\delta_0''$ and $\delta_0^{\mathrm{ram}}$ equal $\mathsf{D}_{\mathrm{irr}}^{0,1,1}$, $\mathsf{D}_{\mathrm{irr}}^{0,3,0}$, $\mathsf{D}_{\mathrm{irr}}^{1,3,0}$ and coincide with the irreducible components of $\mathsf{D}_{\mathrm{irr}}$. As substacks over $\delta_0^{\mathrm{stable}} = \mathsf{D}_{\mathrm{irr}}^{\mathrm{stable}}$ they have respectively degree 1/3 times $3(3^{2g-2}-1)$, 2 and $2(3^{2g-2})$; using (17), we count the degree of $\delta_0^{\mathrm{ram}}$ over $\delta_0$ with multiplicity 3 and we obtain again $\deg(\mathsf{f}) = (3^{2g}-1)/3$.

## 2. The singularities of the moduli space of level curves

In this section we assume $g \geq 4$; this is a standard condition in the study of the singularity locus of the coarse moduli space of curves essentially motivated by Harris and Mumford's work [15] (see Remark 2.11 and Proposition 2.13 and also the role played by this condition in the proof of Theorem 2.40).

At the point represented by $(\mathsf{C}, \mathsf{L}, \phi)$, the local pictures of $\overline{\mathsf{R}}_{g,\ell}$ and of $\overline{\mathcal{R}}_{g,\ell}$ are given by $[\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)]$ and $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)$. We relate these local pictures to $[\mathrm{Def}(C)/\operatorname{Aut}(C)]$ and $\mathrm{Def}(C)/\operatorname{Aut}(C)$, the local pictures of $\overline{\mathsf{M}}_g$ and $\overline{M}_g$ at $C$.

2.1. **Deformation spaces and automorphism groups.** The space $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ can be expressed in terms of $\mathrm{Def}(C)$.

2.1.1. *Deformations of $C$.* For the stable curve $C$, we have

$$\mathrm{Def}(C)=\Big(\bigoplus\nolimits_{e\in E}\mathbb{C}_{t_e}\Big)\oplus\Big(\bigoplus\nolimits_{v\in V}H^1(C'_v,T(-D_v))\Big),$$

where, by $C'_v\subset C$, we denote the connected component of the normalisation of $C$ attached to $v$, and, by $D_v$, we denote the divisor formed by the inverse images of the nodes of $C$ under the normalisation map (the group $H^1(C'_v,T(-D_v))$ parameterises deformations of the pair $(C'_v,D_v)$). The parameter $t_e$ may be interpreted geometrically as the base parameter of the one-dimensional family smoothing the node labelled by $e$.

2.1.2. *Deformations of $(\mathsf{C},\mathsf{L},\phi)$.* The versal deformation space $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ is given by

$$\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)=\Big(\bigoplus\nolimits_{e\in E}\mathbb{C}_{\tau_e}\Big)\oplus\Big(\bigoplus\nolimits_{v\in V}H^1(C'_v,T(-D_v))\Big),\tag{19}$$

an extension of $\mathrm{Def}(C)$, where the map $\mathrm{Def}((\mathsf{C},\mathsf{L},\phi))\to\mathrm{Def}(C)$ is the identity on the second summand and an $r(e)$th power on the first summand (see §1.2.5 and §1.3.4)

2.1.3. *Automorphisms of $(\mathsf{C},\mathsf{L},\phi)$.* An automorphism of a level curve $(\mathsf{C},\mathsf{L},\phi)$ is given by $(\mathsf{s},\rho)$ where $\mathsf{s}$ is an isomorphism of $\mathsf{C}$, and $\rho$ is an isomorphism of line bundles $\mathsf{s}^*\mathsf{L}\to\mathsf{L}$ satisfying $\phi\circ\rho^{\otimes\ell}=\mathsf{s}^*\rho$. We write

$$\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)=\{(\mathsf{s},\rho)\mid\rho\in\mathrm{Aut}(\mathsf{C}),\ \rho\colon\mathsf{s}^*\mathsf{L}\xrightarrow{\cong}\mathsf{L},\ \phi\circ\rho^{\otimes\ell}=\mathsf{s}^*\rho\}.$$

On the other hand, we consider

$$\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)=\{\mathsf{s}\in\mathrm{Aut}(\mathsf{C})\mid\mathsf{s}^*\mathsf{L}\cong\mathsf{L}\}.$$

It is easy to see that for each element $\mathsf{s}\in\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)$ there exists $(\mathsf{s},\rho)\in\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$. Two pairs of this form differ by a power of a quasitrivial automorphism of the form $(\mathrm{id}_{\mathsf{C}},\xi_\ell)$ operating by scaling the fibres. We have the following exact sequence

$$0\to\mathbb{Z}/\ell\to\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)\to\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)\to 0.$$

As already mentioned, quasitrivial isomorphisms act trivially on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$. Therefore, it is natural to study the action of $\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ by focusing on $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)=\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)/(\mathbb{Z}/\ell)$.

The coarsening $\mathsf{s}\mapsto s$, induces a group homomorphism

$$\mathfrak{coarse}\colon\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)\to\mathrm{Aut}(C).$$

The kernel and the image are natural geometric objects of independent interest. We denote them by $\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ and $\mathrm{Aut}'(C)$ and we refer to them as the group of ghost automorphisms and the group of automorphisms of $C$ lifting to $(\mathsf{C},\mathsf{L},\phi)$

$$1\to\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)\to\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)\to\mathrm{Aut}'(C)\to 1.\tag{20}$$

2.1.4. *Ghosts automorphisms.* The kernel of $\mathfrak{coarse}$ is the group of ghosts automorphisms: automorphisms $\mathsf{s}$ of $\mathsf{C}$ fixing at the same time the underlying curve $C$ and the isomorphism class of the overlying line bundle $\mathsf{L}$; we write

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi) := \ker(\mathfrak{coarse}).$$

It is worth pointing out that an automorphism of a stack $\mathsf{X}$ may well be nontrivial and, at the same time, operate as the identity on the coarse space $X$. In our case, stabilisers are isolated and we may treat this issue locally. Consider $\mathsf{U} = [\{xy = 0\}/(\mathbb{Z}/r)]$ the quotient stack where $\xi_r$ acts on $(x, y)$ as $(\xi_r x, \xi_r^{-1} y)$. All automorphisms $(x, y) \mapsto (\xi_r^b x, \xi_r^a y)$ induce the identity on the quotient space. The automorphisms fixing the coarsening $U$ up to natural transformations (the 2-isomorphisms $(x, y) \mapsto (\xi_r^i x, \xi_r^{-i} y)$) form a group $\mathrm{Aut}_U(\mathsf{U}) \cong \mathbb{Z}/r$ generated by $(x, y) \mapsto (\xi_r x, y)$. In this way, the automorphisms of a twisted curve $\mathsf{C}$ with order-$r$ stabilisers at $k$ nodes which fix $C$ are freely generated by $k$ automorphisms each one operating as $(x, y) \mapsto (\xi_r x, y)$ at a node, [1, §7]. Note that no branch has been privileged: via the natural transformation $(x, y) \mapsto (\xi_r x, \xi_r^{-1} y)$, the automorphism $(x, y) \mapsto (\xi_r x, y)$ is 2-isomorphic to $(x, y) \mapsto (x, \xi_r y)$. This explains the canonical identification of [1, §7]

$$\mathrm{Aut}_C(\mathsf{C}) \cong \bigoplus\nolimits_{e\in E} \mathbb{Z}/r(e). \tag{21}$$

2.1.5. *Automorphisms of $C$ lifting to $(\mathsf{C},\mathsf{L},\phi)$.* The image of $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)$ via $\mathfrak{coarse}$ is the group of automorphisms $s$ of $C$, which can be obtained as the coarsening of a morphism $\mathsf{s}$ of $\mathsf{C}$ satisfying $\mathsf{s}^*\mathsf{L} \cong \mathsf{L}$. Clearly, this group differs in general from $\mathrm{Aut}(C)$; notice for instance that automorphisms of the coarse curve $C$ that do not preserve the order of the overlying stabiliser of $\mathsf{C}$ cannot be lifted to $\mathsf{C}$. More precisely we have the obvious inclusion

$$\mathrm{Aut}'(C) := \mathrm{im}(\mathfrak{coarse}) \subset \{s \in \mathrm{Aut}(C) \mid s_\Gamma^* M = M\}$$

where $s_\Gamma$ is the dual graph automorphism induced by $s$. The condition $s_\Gamma^* M = M$ is restrictive in general (except, of course, when $M$ vanishes), but it does not guarantee the existence of an automorphism $\mathsf{s}$ lifting $s$. For a simple counterexample, consider a point of the divisor $\Delta_{g/2}$ from §1.4 lying over the isomorphism class in $\Delta_{g/2}^{\mathrm{stable}}$ of two isomorphic 1-pointed genus-$g/2$ curves meeting transversely at their marked point; here the involution of the underlying stable curve respects the multiplicity cochain, but does not lift to the level structure. We also point out that in general, even when a lift $\mathsf{s}$ exists, there may well be no canonical choice for $\mathsf{s}$. Lifting a morphism that maps a $\mathsf{B}(\mathbb{Z}/k)$-node to another $\mathsf{B}(\mathbb{Z}/k)$-node amounts to extracting a $k$th root of the identifications between local parameters on both branches (there may no distinguished choice, although all choices can be identified via a ghost isomorphism, up to natural transformation).

**Example 2.1.** We conclude this subsection with the study of automorphisms of the genus-one curve $\mathsf{E} = [\widetilde{E}/(\mathbb{Z}/\ell)]$, stack quotient of a nodal cubic $\mathbb{P}^1/(0 \equiv \infty)$, from Example 1.13. Although the group of automorphisms of $\mathsf{E}$ and of $E = \widetilde{E}/(\mathbb{Z}/\ell)$ is not finite ($E$ is not stable), the study of this case is relevant to the study of stable level curves containing, as a subcurve, a copy of $\mathsf{E}$ meeting the rest of the curve at one separating node $n$ (the orbit $(\mathbb{Z}/\ell) \cdot 1$) with trivial stabiliser by Proposition 1.11. To this effect, it is crucial to study the finite group of automorphisms of $\mathsf{E}$ that fix $n$

$$\mathrm{Aut}(\mathsf{E}, n) = \{\mathsf{s} \in \mathrm{Aut}(E) \mid \mathsf{s}(n) = n\}.$$

The exact sequence $1 \to \mathrm{Aut}_E(\mathsf{E}, n) \to \mathrm{Aut}(\mathsf{E}, n) \to \mathrm{Aut}(E, n)$ reads

$$1 \to \mathbb{Z}/\ell \to \mathrm{Aut}(\mathsf{E}, n) \xrightarrow{\mathfrak{coarse}} \mathbb{Z}/2.$$

Here, $\mathbb{Z}/\ell$ is spanned by the automorphism $\mathsf{g}$ with coarsening $g = \mathrm{id}$ and local picture $(x, y) \mapsto (\xi_\ell x, y)$ at the node. On the other hand, $\mathbb{Z}/2$ is spanned by the unique involution $i$ fixing $n$ and the node, and interchanging the branches at the node. In this special case, $\mathfrak{coarse}$ is surjective and the involution $i$ admits a distinguished lift $\mathsf{i} \in \mathrm{Aut}[\widetilde{E}/(\mathbb{Z}/\ell)]$ as follows. At the level of $\widetilde{E}$, consider the unique involution of $\widetilde{E}$ fixing the node of $\widetilde{E}$ and the point 1 and exchanging the branches of the node. At the level of the group $\mathbb{Z}/\ell$, consider the passage to the inverse. We obtain $\mathsf{i}\colon [\widetilde{E}/(\mathbb{Z}/\ell)] \to [\widetilde{E}/(\mathbb{Z}/\ell)]$ and we have the exact sequence[1]

$$0 \to \mathbb{Z}/\ell \to \mathrm{Aut}(\mathsf{E}, n) \to \mathbb{Z}/2 \to 0.$$

Consider the automorphisms of an explicitly defined level-2 curve. Let $\mathsf{C}$ be a twisted curve, union of $\mathsf{E} = [\widetilde{E}/(\mathbb{Z}/2)]$ with a smooth $(g-1)$-curve $X$ with $\mathrm{Aut}(X) = \{\mathrm{id}_X\}$. The curves $\mathsf{E}$ and $X$ meet transversely at $n$ and the coarse spaces form a genus-$g$ stable curve $C$. Hence, by construction, the above short exact sequence reads $0 \to \mathrm{Aut}_C(\mathsf{C}) \to \mathrm{Aut}(\mathsf{C}) \to \mathrm{Aut}(C) \to 0$. Let $(\mathsf{C} = X \cup \mathsf{E}, \mathsf{L} = \mathcal{O} \cup (\mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\text{ét}}), \mathsf{s})$ be the unique level-2 curve obtained by glueing over $n$ the fibre of $\mathcal{O}_X$ and that of $\mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\text{ét}}$ from Example 1.13. By construction $\mathsf{i}^*$ operates trivially on both $\mathsf{L}_{\mathrm{ram}}$ and $\mathsf{L}_{\text{ét}}$; therefore in this example $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \mathsf{s}) \to \mathrm{Aut}(C)$ is surjective and $\mathrm{Aut}'(C) = \mathrm{Aut}(C)$. On the other hand $\mathsf{g}^*$ acts trivially on $\mathsf{L}_{\text{ét}}$ but nontrivially on $\mathsf{L}_{\mathrm{ram}}$

$$\mathsf{g}^*\mathsf{L}_{\mathrm{ram}} = \mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\text{ét}}$$

(this relation can be shown directly, but we refer to (26) for a general rule). We deduce that $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial: there are no ghost automorphisms. This is a consequence of the more general No-Ghosts Lemma 2.10. The sequence (20) reads $0 \to 0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$ and $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \mathbb{Z}/2$ operates nontrivially only on the parameter $\tau_n = t_n$ appearing in (19) and corresponding to the family smoothing the node $n$ (the local picture is $\tau_n \mapsto -\tau_n$ because $\mathsf{i}$ operates trivially on the $y$-branch lying on $X$, operates by a change of sing on the $x$-branch lying on the component $\mathsf{E}$, and $\tau_n$ equals $xy$). In other words $\mathsf{i}$ fixes a hyperplane of $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$; *i.e.* $\mathsf{i}$ is a quasireflection.

2.2. **Dual graph and ghost automorphisms when the level is prime.** Ghost automorphisms of the level curve $(\mathsf{C}, \mathsf{L}, \phi)$ can be described in terms of the dual graph $\Gamma$ of $\mathsf{C}$.

2.2.1. *Setup.* Consider the characteristic function $\nu = \nu_M$ of the support of the multiplicity $M$ of $(\mathsf{C}, \mathsf{L}, \phi)$ and the corresponding contraction $\Gamma \to \Gamma(\nu)$ (the condition $\nu > 0$, or $\nu = \infty$, holds if and only if $M = 0$ and singles out contracted edges, see (11)). Recall $(C^\bullet_\nu(\Gamma; \mathbb{Z}/\ell), \delta)$

$$C^0_\nu(\Gamma; \mathbb{Z}/\ell) = \{a\colon V \to \mathbb{Z}/\ell \mid a(e_+) = a(e_-) \text{ if } \nu(e) > 0\}, \tag{22}$$

$$C^1_\nu(\Gamma; \mathbb{Z}/\ell) = \{b\colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\overline{e}) = -b(e), \text{ and } b(e) = 0 \text{ if } \nu(e) > 0\}. \tag{23}$$

By (14) we have the following identification via $\mathcal{B}$

$$\mathrm{im}\left(\delta\colon C^0(\Gamma(\nu); \mathbb{Z}/\ell) \to C^1(\Gamma(\nu); \mathbb{Z}/\ell)\right) \cong C^1_\nu(\Gamma; \mathbb{Z}/\ell) \cap \mathrm{im}\,\delta.$$

[1]One can observe explicitly that $\mathrm{Aut}(\mathsf{E}, n)$ is the direct product $(\mathbb{Z}/\ell) \times (\mathbb{Z}/2)$; *i.e.* the involution $\mathsf{i}$ commutes with the ghost $\mathsf{g}$ defined locally at the node as $(x, y) \mapsto (\xi_\ell x, y)$. We only need to check $\mathsf{g} \circ \mathsf{i} = \mathsf{i} \circ \mathsf{g}$ at a local picture $[\{xy = 0\}/(\mathbb{Z}/\ell)]$ at the node of $[\widetilde{E}/(\mathbb{Z}/\ell)]$. There, the morphism $\mathsf{i}$ may be described as the map interchanging the branches $(x, y) \mapsto (y, x)$ and $\mathsf{i} \circ \mathsf{g}\colon (x, y) \mapsto (\xi_\ell y, x)$ equals $\mathsf{g} \circ \mathsf{i}\colon (x, y) \mapsto (y, \xi_\ell x)$ up to the natural transformation $(x, y) \mapsto (\xi_\ell x, \xi_\ell^{-1} y)$.

2.2.2. *Automorphisms of* $\mathsf{C}$ *via* $\Gamma$ *and* $\nu$. It is natural to define the group of symmetric $\mathbb{Z}/\ell$-valued functions vanishing on the set of edges with zero multiplicity

$$S_\nu(\Gamma;\mathbb{Z}/\ell) = \{b\colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\overline{e}) = b(e), \text{ and } b(e) = 0 \text{ for } \nu(e) > 0\},$$

canonically isomorphic to $\bigoplus_{e|\nu(e)>0} \mathbb{Z}/\ell$. As mentioned in (21), the group $\mathrm{Aut}_C(\mathsf{C})$ is easy to describe by [1, §7]. For $\ell$ prime, there is a canonical isomorphism

$$\mathrm{Aut}_C(\mathsf{C}) \cong S_\nu(\Gamma;\mathbb{Z}/\ell), \tag{24}$$

where $e \mapsto \mathsf{a}(e) \in \mathbb{Z}/\ell$ corresponds to $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C})$ acting at the node attached to $e \in E$ as

$$(x,y) \mapsto (\xi_\ell^{\mathsf{a}(e)}x, y) \equiv (x, \xi_\ell^{\mathsf{a}(e)}y). \tag{25}$$

2.2.3. *Ghost automorphisms via* $\Gamma$ *and* $\nu$. We characterise ghost automorphisms of the level structure $(\mathsf{C},\mathsf{L},\phi)$. We may regard $S_\nu(\Gamma;\mathbb{Z}/\ell)$ as a ring, via the ring structure of $\mathbb{Z}/\ell$. We notice that $C^1_\nu(\Gamma;\mathbb{Z}/\ell)$ is a module over $S_\nu(\Gamma;\mathbb{Z}/\ell)$:

$$S_\nu(\Gamma;\mathbb{Z}/\ell) \times C^1_\nu(\Gamma;\mathbb{Z}/\ell) \to C^1_\nu(\Gamma;\mathbb{Z}/\ell); \qquad (\mathsf{a}, f) \mapsto \mathsf{a}f.$$

Let us write $\{xy = 0\}$ for the local picture at a chosen node attached to the oriented edge $e$ (as already observed the choice of the notation $(x,y)$ yields $e \in \mathbb{E}$). Then, consider the pullback via the automorphism $\mathsf{a}\colon (x,y) \mapsto (\xi_\ell x, y)$ of the line bundle $\mathsf{L}$ defined by the action $\xi_\ell \cdot (x,y,t) = (\xi_\ell x, \xi_\ell^{-1}y, \xi_\ell t)$ on $\{xy=0\}\times\mathbb{C}$ locally at the chosen node and trivial elsewhere. This definition of $\mathsf{L}$ makes sense because the quotient is canonically trivialised off the node by the invariant sections $xt^{-1}$ on one branch and by $yt$ on the other branch. Pulling back via $\mathsf{a}$ changes the trivialisation only at one branch; in other words, it is equivalent to tensoring by $\tau(\chi_e)$. For any $\mathsf{a} \in \mathrm{Aut}_C(\mathsf{C}) = S_\nu(\Gamma;\mathbb{Z}/\ell)$, we have (see also [7])

$$\mathsf{a}^*\mathsf{L} \cong \mathsf{L} \otimes \tau(\mathsf{a}M). \tag{26}$$

The above statement implies (via (10)) that $\mathsf{a}$ is a ghost if and only if $\mathsf{a}M$ lies in $\ker\tau = \mathrm{im}\,\delta$. This completely justifies the following notation.

**Definition 2.2.** Set $G_\nu(\Gamma;\mathbb{Z}/\ell) = C^1_\nu(\Gamma;\mathbb{Z}/\ell) \cap \mathrm{im}\,\delta$.

**Remark 2.3.** Via the contraction $\Gamma \to \Gamma(\nu)$ we get the alternative presentation

$$G_\nu(\Gamma;\mathbb{Z}/\ell) = \mathrm{im}\left(\delta\colon C^0(\Gamma(\nu);\mathbb{Z}/\ell) \to C^1(\Gamma(\nu);\mathbb{Z}/\ell)\right) \tag{27}$$

yielding the isomorphism $G_\nu(\Gamma;\mathbb{Z}/\ell) = (\mathbb{Z}/\ell)^{\oplus(\#V(\nu)-1)}$.

**Proposition 2.4.** *For $\ell$ prime, let $(\mathsf{C},\mathsf{L},\phi)$ be a stable level-$\ell$ curve. We have a canonical identification*

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi) \cong G_\nu(\Gamma;\mathbb{Z}/\ell).$$

*A 1-cochain $b\colon e \mapsto b(e)$ of $G_\nu(\Gamma;\mathbb{Z}/\ell)$ corresponds, to the symmetric function*

$$\mathsf{a}\colon e \mapsto \begin{cases} [M(e)^{-1}]_\ell b(e) = \mathsf{a}(e) & \text{for } M(e) \neq 0, \\ 0 & \text{if } M(e) = 0, \end{cases}$$

*(where $[M(e)^{-1}]_\ell$ is the inverse of $M(e)$ in $\mathbb{Z}/\ell$). Then $\mathsf{a} \in S_\nu(\Gamma;\mathbb{Z}/\ell)$ satisfies $\mathsf{a}M \in \mathrm{im}\,\delta$ and acts on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ as $\mathbb{C}_{\tau_e} \to \mathbb{C}_{\tau_e}$; $\tau_e \mapsto \xi_\ell^{\mathsf{a}(e)}\tau_e$ (see (19)).* □

**Remark 2.5.** As an easy consequence of the above analysis a ghost automorphisms $\mathsf{a} \in \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ fixes every irreducible component $\mathsf{Z} \subseteq \mathsf{C}$. Indeed, the restriction of $\mathsf{a}$ may operate nontrivially only at the nodes of $\mathsf{Z}$. These are represented by loops in the dual graph. Indeed $G_\nu(\Gamma; \mathbb{Z}/\ell)$ is not supported on the loops (cuts are supported off the loops)

**Example 2.6.** Let us assume $\ell = 3$. Consider the case where the dual graph is formed by a single circuit $K$ consisting of $n$ edges. In this case $\ker \partial \cong \mathbb{Z}/3 = \langle K \rangle$. There are two possibilities: $M = 0$, where $\mathrm{Aut}_C(\mathsf{C}, \mathsf{L}, \phi) = 0$, and $M \neq 0$, where $C^1_\nu(\Gamma; \mathbb{Z}/3) = C^1(\Gamma; \mathbb{Z}/3)$ and the group of ghosts $G_\nu(\Gamma; \mathbb{Z}/\ell)$ is isomorphic to $\mathrm{im}\, \delta \cong (\mathbb{Z}/\ell)^{\oplus(\#V-1)} = (\mathbb{Z}/3)^{\oplus 2}$. The elements of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ are the functions $\mathsf{a} \in S_\nu(\Gamma; \mathbb{Z}/3)$ such that $\langle \mathsf{a}M, K \rangle = 0$.

(i) Assume $n = 3$. In this case $M$ lies in $\mathrm{im}\, \delta$ and we get an element of $G_\nu(\Gamma; \mathbb{Z}/3)$ by taking $\mathsf{a} \colon \mathbb{E} \to \mathbb{Z}/3$ constant. If $\mathsf{a} = 1$ we get the ghost operates as $(x, y) \mapsto (\xi_3 x, y)$ at all nodes and acts on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as $(\xi_3 \mathbb{I}_3) \oplus \mathrm{id}$ (see (19)).
(ii) Assume $n = 2$, let $e_1$ and $e_2$ be the two edges. Here $M \notin \mathrm{im}\, \delta$. We define a symmetric function $\mathsf{a} \colon \mathbb{E} \to \mathbb{Z}/\ell$ mapping one edge to 1 ($e_1, \overline{e}_1 \mapsto 1$) and the other to $2 \in \mathbb{Z}/3$ ($e_2, \overline{e}_2 \mapsto 2$); then $\mathsf{a}M$ is a cut, lies in $\mathrm{im}\, \delta$ and acts on (19) as $\mathrm{Diag}(\xi_3, \xi_3^2) \oplus \mathrm{id}$.
(iii) If the circuit has a single edge, then $\mathrm{im}\, \delta = (0)$. There are no nontrivial ghosts.

**Example 2.7.** The argument at point (iii) shows that the level structure $\mathcal{O} \cup (\mathsf{L}_{\mathrm{ram}} \otimes \mathsf{L}_{\text{ét}})$ of Example 2.1 has no nontrivial ghosts. Indeed, the dual graph in that case has two vertices $v_X$ and $v_{\mathsf{E}}$ corresponding to $X$ and $\mathsf{E}$, one edge $e_n$ connecting them and corresponding to the node $n$ and a second edge $e_{\mathrm{loop}}$ with both ends on $v_{\mathsf{E}}$. The multiplicity is supported on this last vertex, and the vertex set $V(\nu)$ of the graph $\Gamma(\nu)$ obtained by contracting all edges with vanishing multiplicity reduces to a single vertex. We have $\mathrm{im}(\delta \colon C^0(\Gamma(\nu); \mathbb{Z}/\ell) \to C^1(\Gamma(\nu); \mathbb{Z}/\ell)) \cong (\mathbb{Z}/\ell)^{\oplus \# V(\nu) - 1} = 0$. Notice that this argument holds for any tree-like graph (that is a graph that becomes a tree once the loops are removed), see Corollary 2.14.

**Example 2.8.** Consider a dual graph with two vertices $v_1, v_2$ and three edges, each of them linking the two vertices to each other. As a multiplicity cochain we chose $e \mapsto M(e)$ equal to 1 on the oriented edges of $\mathbb{E}$ oriented from $v_1$ to $v_2$. For $\ell = 3$, it is easy to check that $M$ belongs to the kernel and is indeed the sum of two different two-edged circuits. The cochain $M$ lies also in $\mathrm{im}\, \delta$ (it is the $\mathbb{Z}/3$-valued cut attached to the proper nonempty subset $H = \{v_1\}$). Therefore $\mathsf{a} = 1 \in S_\nu(\Gamma; \mathbb{Z}/3)$ satisfies $\mathsf{a}M \in \mathrm{im}\, \delta$ and acts on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as $\xi_3 \mathbb{I}_3 \oplus \mathrm{id}$.

2.3. **The singular points of the moduli space.** Notice that, in all the above examples of ghost automorphisms $\mathsf{g} \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$, the fixed space $\{v \in \mathrm{Def}((\mathsf{C}, \mathsf{L}, \phi)) \mid \mathsf{g} \cdot v = v\}$ is never a hyperplane. An automorphism of an affine space whose fixed space is exactly an hyperplane is called a *quasireflection*. A general property of nontrivial ghosts is that they never act as quasireflections. Let us recall that this is crucial for classifying singularities.

**Fact 2.9.** *The scheme-theoretic quotient* $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ *is singular if and only if* $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ *is spanned by elements acting as the identity or as quasireflections (see* [22]*).*

2.3.1. *Nontrivial ghosts are not quasireflections.* Here is a consequence of Proposition 2.4.

**Lemma 2.10.** *If* $\mathsf{a} \in \underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ *fixes a hyperplane of* $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$*, then* $\mathsf{a} = \mathrm{id}_{\mathsf{C}}$.

*Proof.* Let $b$ be a nontrivial ghost $G_\nu(\Gamma; \mathbb{Z}/\ell)$; *i.e.* a 1-cochain of $C^1_\nu(\Gamma; \mathbb{Z}/\ell)$ lying in $\mathrm{im}\, \delta$. Then there exists a (nonseparating) edge $e$ with $\nu(e) = 0$ and $b(e) \neq 0$. In this case, there

is circuit $K$ passing through it satisfying $\langle b, K\rangle = 0$ by (8). The support of $b$ contains an oriented edge $e'$ which differs from $e$ regardless of its orientation. Proposition 2.4 claims that the unique automorphism $\mathsf{a}$ such that $M\mathsf{a} = b$ acts nontrivially on $\mathbb{C}_{\tau_e}$ and $\mathbb{C}_{\tau_{e'}}$. □

**Remark 2.11** ($\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ operates faithfully on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$)**.** Under the assumption $g \geq 4$, any nontrivial automorphism $a \in \mathrm{Aut}(C)$ acts nontrivially on $\mathrm{Def}(C)$, see [15]. Then, the faithfulness of $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ follows from that of $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ and from the above lemma.

2.3.2. *Elliptic tail involutions.* In [15, Thm.2, §2], Harris and Mumford prove that an automorphism $a \in \mathrm{Aut}(C)$ is a quasireflection of $\mathrm{Def}(C)$ if and only if $a$ is an *elliptic tail involution* (ETI): the curve $C$ contains a genus-1 subcurve $E$ meeting the rest of the curve at a single point $n$ and $a$ is the identity on $\overline{C \setminus E}$ and is the nontrivial canonical involution $i$ of $\mathrm{Aut}(E, n)$. This involution is canonically identified both if $E$ is elliptic or rational: it is the hyperelliptic involution in the first case whereas, in the second case, it is the unique involution fixing the point $n$ and the node of $E \cong \mathbb{P}^1/(0 \equiv \infty)$ and interchanging the branches of such node. We need to generalise to twisted curves the notion of ETI. Because all separating nodes of stable level-$\ell$ curves have trivial stabilisers, a genus-1 subcurve $\mathsf{E}$ meeting the rest of the curve at a single point $n$ is either a scheme $(E, n)$ or is isomorphic to the pointed stack-theoretic curve $(\mathsf{E}, n)$ of Example 1.13. In both cases, these tails are equipped with a canonical involution $\mathsf{i}$.

**Definition 2.12** (elliptic tail and ETI)**.** Let $(\mathsf{C}, \mathsf{L}, \phi)$ be a stable level-$\ell$ curve. An elliptic tail is a genus-one subcurve $\mathsf{E}$ meeting the rest of the curve $\mathsf{C}$ at a single point. An elliptic tail involution of $(\mathsf{C}, \mathsf{L}, \phi)$ is an automorphism of $\mathsf{C}$ such that the restriction to $\overline{\mathsf{C} \setminus \mathsf{E}}$ is the identity and the restriction to $\mathsf{E}$ equals the canonical involution $\mathsf{i}$ and satisfies $\mathsf{i}^*(\mathsf{L}|_{\mathsf{E}}) = \mathsf{L}|_{\mathsf{E}}$ .

**Proposition 2.13.** *Consider a stable genus-$g$ level-$\ell$ curve with $g \geq 4$. An automorphism $\mathsf{s} \in \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ acts as a quasireflection on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ if and only if it is an ETI.*

*Proof.* Let $\mathsf{s}$ be an automorphism of $(\mathsf{C}, \mathsf{L}, \phi)$ acting as a quasireflection on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$. Then, its coarsening $s$ acts either as the identity or as a quasireflection on $\mathrm{Def}(C)$. We rule out $s = \mathrm{id}_C$: in this case $\mathsf{s}$ would be a ghost, and, by Lemma 2.10, there is no ghost acting as quasireflection. Then, by [15], $s$ operates as an ETI on $C$. If the elliptic tail is represented by a scheme, then $\mathsf{s}$ is an ETI (using Lemma 2.10 on $\overline{C \setminus \mathsf{E}}$). Otherwise, the elliptic tail is the curve $\mathsf{E}$ of Example 1.13 and we need to check that $\mathsf{i}$ is the only automorphism lifting the ETI $i$ and operating as a quasireflection on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$. By (20) the remaining automorphisms are of the form $\mathsf{i} \circ \mathsf{g}^n$ with $\mathsf{g}^n \neq \mathrm{id}$ (using the notation of Example 1.13); due to Proposition 2.4, the automorphism $\mathsf{g}^n$ acts nontrivially on $\mathrm{Def}(\mathsf{C})$ and $\mathsf{i} \circ \mathsf{g}^n$ is not a quasireflection. □

2.3.3. *No-Ghosts.* By Remark 1.9 and Proposition 2.4, $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial if and only if the multiplicity graph $\Gamma(\nu)$ has only one vertex. We call such graphs *bouquets*.

**Corollary 2.14.** *Let $\ell$ be prime. The group of ghost automorphisms $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial if and only if $\Gamma(\nu)$ is a bouquet.* □

Combining Corollary 2.14 and Proposition 2.13 we get the following result.

**Theorem 2.15.** *Let $\ell$ be prime and assume $g \geq 4$. The following conditions are equivalent.*

*(i) The point of $\overline{\mathcal{R}}_{g,\ell}$ representing $(\mathsf{C}, \mathsf{L}, \phi)$ is smooth.*
*(ii) The group $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is spanned by ETIs of $\mathsf{C}$.*

*(iii) The graph $\Gamma(\nu)$ is a bouquet and $\mathrm{Aut}'(C)$ is spanned by ETIs of $C$.*

*Proof.* The point representing $(\mathsf{C}, \mathsf{L}, \phi)$ is smooth if and only $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ is generated by elements operating on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as the identity or as quasireflections. Nontrivial elements of $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ never operate as the identity, see Remark 2.11. By Proposition 2.13 elements operating as quasireflections are precisely the ETIs of $\mathsf{C}$; hence (i) $\Leftrightarrow$ (ii). Now (iii) implies $\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = 0$ and $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Aut}'(C)$ generated by ETIs of $C$; we deduce (ii) because the ETIs generating $\mathrm{Aut}'(C)$ lift canonically to ETIs generating $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$. Conversely, (ii) holds only if there are no nontrivial ghosts ($\underline{\mathrm{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi) = 0$) because any nontrivial composition of ETIs has a nontrivial coarsening. Hence, $\Gamma(\nu)$ is a bouquet, $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Aut}'(C)$, and the coarsening of the ETIs spanning $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ are ETIs spanning $\mathrm{Aut}'(C)$. $\square$

2.4. **Generalisation to the case of level curves of composite level.** The generalisation of the above statement requires a modification of the condition "$\Gamma(\nu)$ is a bouquet" in part (iii); we introduce a new set of contractions. We are grateful to Roland Bacher for several ideas that helped us a great deal in finding the correct setup for this section.

2.4.1. *The truncated valuation of $\mathbb{Z}/p^n$.* For any prime $p$ we recall that the ring $\mathbb{Z}/p^n$ is a truncated valuation ring in the sense of [10, §1.1]. We recall the definition, which applies to any local ring $R$ whose maximal ideal $\mathfrak{m}$ is generated by a nilpotent element. We set the valuation $\mathrm{val}_{\mathfrak{m}} \colon R \to \mathbb{Z} \cup \{\infty\}$, $x \mapsto \sup\{i \mid x \in \mathfrak{m}^i\}$, taking values in $\mathbb{Z} \cap [0, \mathrm{length}(R) - 1]$ on $R \setminus \{0\}$ and satisfying $\mathrm{val}_{\mathfrak{m}}(0) = \infty$ (if $R = \mathbb{Z}/p^n$, then $\mathfrak{m} = (p)$ and $\mathrm{length}(R) = n$).

2.4.2. *The vector-valued function $\boldsymbol{\nu}$.* Consider the prime factorisation of $\ell$

$$\ell = \prod_{p \in \mathfrak{P}} p^{e_p},$$

where $\mathfrak{P}$ is the set of prime factors of $\ell$ (and $e_p$ is the $p$-adic valuation of $\ell$). Then, the following vector-valued function $\boldsymbol{\nu}_M$, or simply $\boldsymbol{\nu}$, encodes the truncated valuations of $M(e) \mod p^{e_p}$ in $\mathbb{Z}/p^{e_p}$ for all $p \in \mathfrak{P}$.

$$e \mapsto \boldsymbol{\nu}(e) = \big(\nu_p(e)\big)_{p \in \mathfrak{P}} \qquad \text{where} \quad \nu_p(e) := \mathrm{val}_{(p)}\big(M(e) \mod p^{e_p}\big). \tag{28}$$

Notice that, when $\ell$ is prime, we recover the characteristic function $\nu_M$ of the support of $M$

$$\mathrm{val}_{(p)}(M(e)) = \nu_M(e).$$

2.4.3. *Contractions.* For each $p \in \mathfrak{P}$, the coordinate $\nu_p$ of $\boldsymbol{\nu} = (\nu_p)_{p \in \mathfrak{P}}$ yields a filtration

$$\varnothing \subseteq \{\nu_p \geq e_p\}_E \subseteq \{\nu_p \geq e_p - 1\}_E \subseteq \ldots \subseteq \{\nu_p \geq k\}_E \subseteq \ldots \subseteq \{\nu_p \geq 1\}_E \subseteq \{\nu_p \geq 0\}_E = E.$$

To each of the above edge subsets we can naturally associate a subgraph (the vertex set is formed by the heads and the tails of the chosen edges):

$$\varnothing \subseteq \Delta(\nu_p^{e_p}) \subseteq \Delta(\nu_p^{e_p - 1}) \subseteq \cdots \subseteq \Delta(\nu_p^k) \subseteq \cdots \subseteq \Delta(\nu_p^1) \subseteq \Delta(\nu_p^0) = \Gamma. \tag{29}$$

The respective contractions $\Gamma(\nu_p^k)$ of $\{\nu_p \geq k\}_E$ fit in the sequence of contractions

$$\Gamma \longrightarrow \Gamma(\nu_p^{e_p}) \longrightarrow \Gamma(\nu_p^{e_p - 1}) \longrightarrow \ldots \longrightarrow \Gamma(\nu_p^k) \longrightarrow \ldots \longrightarrow \Gamma(\nu_p^1) \longrightarrow \Gamma(\nu_p^0), \tag{30}$$

where the graph $\Gamma(\nu_p^0)$ is the null graph ($\Gamma$ is connected). The sets of vertices $V(\nu_p^k)$ fit in

$$V \twoheadrightarrow V(\nu_p^{e_p}) \twoheadrightarrow V(\nu_p^{e_p-1}) \twoheadrightarrow \ldots \twoheadrightarrow V(\nu_p^k) \twoheadrightarrow \ldots \twoheadrightarrow V(\nu_p^1) \twoheadrightarrow V(\nu_p^0) = \{\bullet\}. \tag{31}$$

The sets of edges $E(\nu_p^k)$ are related by the reversed inclusions

$$E \supseteq E(\nu_p^{e_p}) \supseteq E(\nu_p^{e_p-1}) \supseteq \ldots \supseteq E(\nu_p^k) \supseteq \ldots \supseteq E(\nu_p^1) \supseteq E(\nu_p^0) = \varnothing. \tag{32}$$

In the introduction, for brevity, we used the notation $\Delta_p^k$ and $\Gamma_p^k$ the graphs $\Delta(\nu_p^k)$ and $\Gamma(\nu_p^k)$. Contracting $\{\nu_p \geq k\}_E$ makes sense for any $k$ in $\mathbb{Z} \cup \{\infty\}$; for $k \geq e_p$ we get $\Gamma(\nu_p^k) = \Gamma(\nu_p^{e_p})$, for $k \leq 0$ we get the null graph $\Gamma(\nu_p^k) = \Gamma(\nu_p^0)$. For $k \in \{0, \ldots, e_p\}$, the following holds.

**Definition 2.16** (the graph $\Gamma(\nu_p^k)$)**.** For $p$ prime dividing $\ell$ and $k \in \{0, \ldots, e_p\}$, the map $\Gamma \to \Gamma(\nu_p^k)$ is given by contracting the edges $e$ for which $p^k$ divides $M(e) \in \mathbb{Z}/p^{e_p}$.

2.4.4. *The subcomplex $C_{\boldsymbol{\nu}}^\bullet(\Gamma; \mathbb{Z}/\ell)$.* Let us point out that, for

$$\overline{\nu}_p(e) = \min(e_p, \nu_p(e))$$

we have $\gcd(M(e), \ell) = \prod_{p \in \mathfrak{P}} p^{\overline{\nu}_p(e)} (= \ell/r(e))$. We systematically use

$$\bigoplus\nolimits_{p \in \mathfrak{P}} \mathbb{Z}/p^{e_p - \overline{\nu}_p(e)} \xrightarrow[=]{\times p^{\overline{\nu}_p(e)}} \bigoplus\nolimits_{p \in \mathfrak{P}} p^{\overline{\nu}_p(e)} \mathbb{Z}/p^{e_p} \hookrightarrow \bigoplus\nolimits_{p \in \mathfrak{P}} \mathbb{Z}/p^{e_p} \cong \mathbb{Z}/\ell. \tag{33}$$

Now, we generalise the above mentioned subcomplex $C_\nu^\bullet(\Gamma; \mathbb{Z}/\ell)$ (see (22) and (23)). Set

$$C_{\boldsymbol{\nu}}^0(\Gamma; \mathbb{Z}/\ell) = \left\{ a \colon V \to \mathbb{Z}/\ell \mid a(e_+) - a(e_-) \in \bigoplus\nolimits_{p \in \mathfrak{P}} \mathbb{Z}/p^{e_p - \overline{\nu}_p(e)} = \bigoplus\nolimits_{p \in \mathfrak{P}} M(e)\mathbb{Z}/\ell \right\}$$

$$C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell) = \left\{ b \colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\overline{e}) = -b(e) \text{ and } b(e) \in \bigoplus\nolimits_{p \in \mathfrak{P}} \mathbb{Z}/p^{e_p - \overline{\nu}_p(e)} = \bigoplus\nolimits_{p \in \mathfrak{P}} M(e)\mathbb{Z}/\ell \right\}.$$

By restricting $\delta$ we get the differential

$$C_{\boldsymbol{\nu}}^0(\Gamma; \mathbb{Z}/\ell) \xrightarrow{\delta(\boldsymbol{\nu})} C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell);$$

and $(C_{\boldsymbol{\nu}}^\bullet(\Gamma; \mathbb{Z}/\ell), \delta)$ is a subcomplex of $(C^\bullet(\Gamma; \mathbb{Z}/\ell), \delta)$. Definition 2.2 extends word for word.

**Definition 2.17.** Set $G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell) = C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell) \cap \operatorname{im} \delta$

By construction $G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell)$ equals $\operatorname{im}(\delta(\boldsymbol{\nu}))$ via the inclusion $C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell) \subset C^1(\Gamma; \mathbb{Z}/\ell)$. The following theorem proves that, with this setup, $G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell)$ is again isomorphic to the group of ghost automorphisms. First, we introduce the generalised group of symmetric functions

$$S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell) = \left\{ b \colon \mathbb{E} \to \mathbb{Z}/\ell \mid b(\overline{e}) = b(e) \text{ and } b(e) \in \bigoplus\nolimits_{p \in \mathfrak{P}} \mathbb{Z}/p^{e_p - \overline{\nu}_p(e)} \right\}.$$

The ring structure of $\mathbb{Z}/r(e)$ endows

$$S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell) = \bigoplus\nolimits_{e \in E} \mathbb{Z}/r(e)$$

with a ring structure and $C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell)$ with a $S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell)$-module structure

$$S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell) \times C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell) \to C_{\boldsymbol{\nu}}^1(\Gamma; \mathbb{Z}/\ell); \qquad (\mathsf{a}, f) \mapsto \mathsf{a}f. \tag{34}$$

**Theorem 2.18.** *Let $(\mathsf{C}, \mathsf{L}, \phi)$ be a stable level curve of level $\ell \in \mathbb{N}^\times$; write $M$ for its multiplicity and $\boldsymbol{\nu}$ for the corresponding vector-valued function* (28)*. We have the following statements.*

*(i) There is a canonical isomorphism $\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi) = S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell)$. The above local description of $\mathsf{a} \in S_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/\ell)$ holds without changes if we write $\mathsf{a}$ as a $\mathbb{Z}/\ell$-valued function.*

*(ii) Let* $\mathsf{a} \in S_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$*; then, we have (using* (34)*)* $\mathsf{a}^*\mathsf{L} = \mathsf{L} \otimes \tau(\mathsf{a}M)$.

*(iii) We have*

$$\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi) \cong G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell).$$

*The* 1*-cochain* $b\colon e \mapsto b(e) \in \mathbb{Z}/r(e)$ *in* $G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$ *identifies the ghost* $\mathsf{a}\colon e \mapsto \mathsf{a}(e)$

$$\mathsf{a}(e) = [m(e)^{-1}]_{r(e)} b(e) \qquad (\textit{in } \mathbb{Z}/r(e)),$$

*where all terms in the equations are meant modulo* $r(e)$ *and* $[m(e)^{-1}]_{r(e)}$ *denotes the inverse of* $m(e)$ *in* $\mathbb{Z}/r(e)$*. If we express* $\mathsf{a}(e) \in \mathbb{Z}/r(e)$ *as an element of* $\mathbb{Z}/\ell$ *via* $\widetilde{a}(e) = \frac{\ell}{r(e)} a(e)$ *then the ghost* $\mathsf{a}$ *operates on* $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ *as*

$$\Big\{\bigoplus\nolimits_e \Big(\tau_e \mapsto \xi_\ell^{\widetilde{\mathsf{a}}(e)} \tau_e\Big)\Big\} \oplus \mathrm{id}\,.$$

*Proof.* Since $S_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell) = \bigoplus_{e\in E}\mathbb{Z}/r(e)$, we recover the group $\mathrm{Aut}_C(\mathsf{C})$ of [1, §7]. Point (ii) yields (iii) immediately and is a direct consequence of [7, Prop. 2.18] as before. □

**Remark 2.19.** Every ghost restricts to the identity on the irreducible components of $\mathsf{C}$.

2.4.5. *Computing the group* $G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$*.* When $\ell$ is prime, the group of ghosts is a free $\mathbb{Z}/\ell$-module and Remark 2.3 allows to compute its rank over $\mathbb{Z}/\ell$: the number of vertices of the contracted graph minus 1. In general, when $\ell$ is composite, the group of ghosts is not free on $\mathbb{Z}/\ell$. By generalising Remark 2.3, we provide an explicit formula for its elementary divisors.

**Remark 2.20.** Once an orientation $E \to \mathbb{E}$ is specified, $C^1_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$ may be written as $\bigoplus_{e\in E}\bigoplus_{p\in\mathfrak{P}} \mathbb{Z}/p^{e_p-\overline{\nu}_p(e)}$. We may invert the order of the direct sums

$$\bigoplus\nolimits_{p\in\mathfrak{P}} \Big(\bigoplus\nolimits_{e\in E} p^{\overline{\nu}_p(e)}\mathbb{Z}/p^{e_p}\Big).$$

and characterise the term between brackets as a sum over $\mathfrak{P}$ of the subgroups

$$\sum\nolimits_{1\le k\le e_p} \mathcal{B}C^1(\Gamma(\nu_p^k);\mathbb{Z}/p^{e_p-k+1}) \subseteq C^1(\Gamma;\mathbb{Z}/\ell).$$

We deduce from this characterisation the following identity which does not involve any fixed orientation $E \to \mathbb{E}$ and holds both for 1-cochains and for 0-cochains. We have

$$C^i_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell) = \bigoplus\nolimits_{p\in\mathfrak{P}} \sum\nolimits_{1\le k\le e_p} \mathcal{B}C^i(\Gamma(\nu_p^k);\mathbb{Z}/p^{e_p-k+1}) \qquad i=0,1. \tag{35}$$

Moreover, we immediately get an explicit computation of the groups $C^1_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$: because $\mathcal{B}C^0(\Gamma(\nu);\mathbb{Z}/p^h) \cong (\mathbb{Z}/p^h)^{\oplus \#V(\nu)}$ and $\mathcal{B}C^1(\Gamma(\nu);\mathbb{Z}/p^h) \cong (\mathbb{Z}/p^h)^{\oplus \#E(\nu)}$, we have

$$C^i_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell) \cong \bigoplus\nolimits_{p\in\mathfrak{P}} \bigoplus\nolimits_{k=1}^{e_p} (\mathbb{Z}/p^k)^{\oplus \eta^i(\nu_p^k)},$$

where, using the Kronecker delta, we can compute $\eta^i(\nu_p^k)$ from (31) and (32)

$$\eta^i(\nu_p^k) := \begin{cases} \#V(\nu_p^{e_p-k+1}) - \delta_{k,e_p}\#V(\nu_p^{e_p-k}) & i=0,\\ \#E(\nu_p^{e_p-k+1}) - \delta_{k,e_p}\#E(\nu_p^{e_p-k}) & i=1.\end{cases}$$

The following lemma, embodying the corollary stated in the introduction, follows.

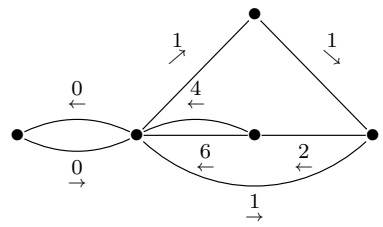



FIGURE 2. A dual graph $\Gamma$ of a level-8 curve with multiplicities

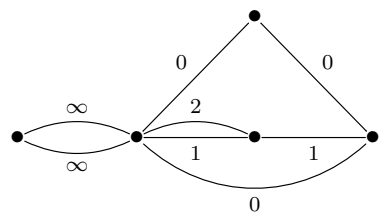



FIGURE 3. The truncated valuation associated to (2) in $\mathbb{Z}/8$.

**Lemma 2.21.** *We have*

$$G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell) = \bigoplus\nolimits_{p\in\mathfrak{P}} \sum\nolimits_{1\le k\le e_p} \operatorname{im}\delta(\nu_p^k),$$

*where* $\delta(\nu_p^k)$ *is* $C^0(\Gamma(\nu_p^k);\mathbb{Z}/p^{e_p-k+1}) \to C^1(\Gamma(\nu_p^k);\mathbb{Z}/p^{e_p-k+1})$. *More explicitly, set*

$$\alpha_p^k := \#V(\nu_p^{e_p-k+1}) - \#V(\nu_p^{e_p-k});$$

*then, the group* $G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$ *decomposes as*

$$G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell) \cong \bigoplus\nolimits_{p\in\mathfrak{P}} \bigoplus\nolimits_{k=1}^{e_p} (\mathbb{Z}/p^k)^{\oplus\alpha_p^k}$$

*and has order* $\frac{1}{\ell}\prod_{p|\ell} p^{\#V_p}$ *for* $V_p = \sqcup_{k=1}^{e_p} V(\nu_p^k)$. □

**Remark 2.22.** For $\ell$ prime, we recover (6): (31) reads $V(\nu) \to \{\bullet\}$, we get $\alpha_p^1 = \#V(\nu) - 1$. (Note that Kronecker delta does not occur in the formula for the elementary divisors $\alpha_p^k$.)

**Example 2.23.** We consider the dual graph $\Gamma$ in Figure 2 of a level-8 curve. The multiplicities assigned to each oriented edge define a cocycle $M \in \ker\partial$. We have $\mathfrak{P} = \{2\}$. In Figure 3, we write next to each edge $e$ the value of $\nu_2(e)$. Then, in Figure 4, we show the corresponding contractions. We observe that, in this case, at each step the number of vertices decreases by 1. Therefore, by Lemma 2.21, we compute $\alpha_2^k = 1$ for $k = 1, 2, 3$. We finally obtain 64 ghosts

$$G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/8) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$$

that can be spanned by ghosts of order 2, 4 and 8 corresponding to the $\mathbb{Z}/8$-valued symmetric functions in $S_{\boldsymbol{\nu}}(\Gamma,\mathbb{Z}/8)$ displayed in Fig. 5. We check the corollary stated in the introduction: there are 9 vertices in $\Gamma(\nu_2^3), \Gamma(\nu_2^2)$, and $\Gamma(\nu_2^1)$ and there are $2^9/8$ (*i.e.* 64) ghosts.

**Remark 2.24.** In [6, §4.1, after Lem. 4.1.1], the authors compute the length of the fibre of the forgetful morphism from moduli of $\ell$th roots to moduli of curves. More precisely, they consider moduli spaces of $\ell$th roots of a line bundle $N$ defined over a family of curves $C \to B$. They provide a proper moduli space $\overline{S}^\ell(N)$ over $B$, which is possibly not normal

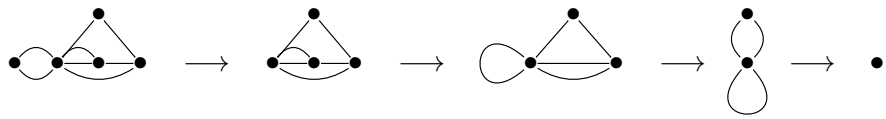


FIGURE 4. The contractions $\Gamma \to \Gamma(\nu_2^3) \to \Gamma(\nu_2^2) \to \Gamma(\nu_2^1) \to \bullet$; we have $\#V_3 = 9$.

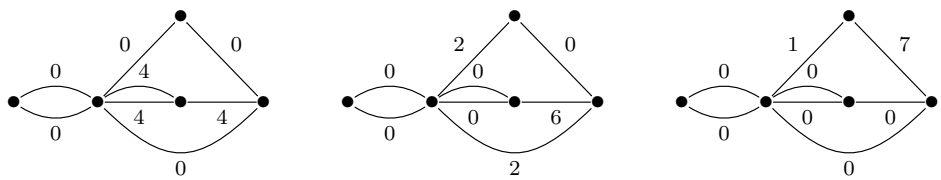



FIGURE 5. Three generators of order $2, 4$ and $8$ of the group of ghosts.

(see discussion after [6, Thm. 4.2.3]). They check that the length of each scheme-theoretic, zero-dimensional, fibre $\overline{S}^\ell(N)_b$ over $b \in B$ is equal to $\ell^{2g}$, all $b \in B$.

It is instructive to do the same for the present moduli functors. We assume $N = \mathcal{O}$; the general case is similar but cumbersome to treat in the present setup. We consider, for any $g \geq 2$ and $\ell \geq 1$, the moduli spaces $\mathcal{T}_{g,\ell} = \sqcup_{d|\ell}\mathcal{R}_{g,d}$ and the compactifications $\overline{\mathcal{T}}_{g,\ell} = \sqcup_{d|\ell}\overline{\mathcal{R}}_{g,d}$, which, as a byproduct of the twisted curves approach, are normalisations of the moduli spaces $\overline{S}^\ell(\mathcal{O})$. The normalisation is nontrivial when $\ell$ is not prime (see [9, §1.2-3]) and, in these cases, the number of connected components of the fibres is higher than in the case of $\overline{S}^\ell(\mathcal{O})$. This explains why, instead of a single graph contraction noted "$\Gamma \to \Sigma_X$" in [6] (analogous to $\Gamma \to \Gamma(\nu_\ell^1)$ when $\ell$ is prime), we have to consider all the above contractions $\Gamma(\nu_p^k)$.

Fix a stable curve $C$ with dual graph $\Gamma$, that is a geometric point $b \to \overline{\mathcal{M}}_g$. We check $\mathrm{length}(F_b) = \ell^{2g}$ for the scheme-theoretic fibre $F_b$, base change of $\overline{\mathcal{T}}_{g,\ell} \to \overline{\mathcal{M}}_g$ to $b$. Each connected component of $F_b$ is a, possibly nonreduced, zero-dimensional scheme corresponding to an isomorphism class of a triple $(\mathsf{C}, \mathsf{L}, \phi\colon \mathsf{L}^{\otimes\ell} \to \mathcal{O})$ with a multiplicity $M$ and corresponding characteristic functions $\boldsymbol{\nu} = (\nu_p)$. By Theorem 2.18, its length is

$$\#\operatorname{Aut}_C(\mathsf{C})/\#\underline{\operatorname{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \left(\prod\nolimits_{p|\ell}\prod\nolimits_{k=1}^{e_p} p^{\#E(\nu_p^k)}/p^{\#V(\nu_p^k)-1}\right) = \prod\nolimits_{p|\ell}\prod\nolimits_{k=1}^{e_p} p^{b_1(\Gamma(\nu_p^k))}.$$

By Proposition 1.11, the number of connected components is

$$\sum_{\substack{M\in\ker\partial\\ \boldsymbol{\nu}_M=(\nu_p)}} \ell^{2p_g(C)}\prod\nolimits_{p|\ell}\prod\nolimits_{k=1}^{e_p} p^{b_1(\Delta(\nu_p^k))}.$$

Since $\Gamma(\nu_p^k)$ is given by collapsing the subgraph $\Delta(\nu_p^k)$, we get $b_1(\Delta(\nu_p^k)) + b_1(\Gamma(\nu_p^k)) = b_1(\Gamma)$;

$$\mathrm{length}(F_b) = \sum_{\substack{M\in\ker\partial\\ \boldsymbol{\nu}_M=(\nu_p)}} \ell^{2p_g(C)}\prod\nolimits_{p|\ell}\prod\nolimits_{k=1}^{e_p} p^{b_1(\Delta(\nu_p^k))+b_1(\Gamma(\nu_p^k))} = \sum_{\substack{M\in\ker\partial\\ \boldsymbol{\nu}_M=(\nu_p)}} \ell^{2p_g(C)}\ell^{b_1(\Gamma)} = \ell^{2g}.$$

2.4.6. *No-Ghosts.* Thm. 2.18 and Lem. 2.21 imply a no (nontrivial) ghost criterion.

**Corollary 2.25.** *Let $\ell$ be any positive integer. The group $\underline{\operatorname{Aut}}_C(\mathsf{C}, \mathsf{L}, \phi)$ is trivial if and only if for any prime factor $p$ of $\ell$ the graph $\Gamma(\nu_p^{e_p})$ is a bouquet.* □

**Remark 2.26.** In analogy with the case where $\ell$ is prime, one may consider the condition "the contraction $\Gamma'$ of $\{e \mid \ell \text{ divides } M(e)\}$ is a bouquet", which clearly implies the above no-ghosts condition. The converse is false: for $\ell = 6$, consider $\Gamma$ with vertices $v_1, v_2$, edges $e_1, e_2, e_3$ going from $v_1$ to $v_2$, set $M(e_i) = i$.

Lem. 2.10, Prop. 2.13 generalise *verbatim*, and, by Cor. 2.25, the same holds for Thm. 2.15 once we replace "$\Gamma(\nu)$ is a bouquet" by "$\Gamma(\nu_p^{e_p})$ is a bouquet for any prime $p \mid \ell$". We can also state the generalisation as follows.

**Theorem 2.27.** *Let $g \geq 4$ and let $\ell \geq 1$. The point representing $(\mathsf{C}, \mathsf{L}, \phi)$ in $\overline{\mathcal{R}}_{g,\ell}$ is smooth if and only if the group $\mathrm{Aut}'(C)$ is generated by ETIs of $C$ and the graphs $\Gamma(\nu_p^{e_p})$ (obtained by contracting the edges $e$ for which $M(e) \in \mathbb{Z}/p^{e_p}$ vanishes) are bouquets for any prime $p \mid \ell$.*

2.4.7. *Automorphism group of level structures over stack-theoretic elliptic tails.* We describe the action of the automorphism group of the stack-theoretic elliptic tail of Exa. 1.13 on Pic. The order of $\mathsf{L}\,|_{\mathsf{E}}\,\mathsf{E}$ is $l$, a divisor of $\ell$. The 1-pointed 1-nodal twisted genus-1 curve $(\mathsf{E}, n)$ is given by the stack-theoretic quotient of $\widetilde{E} = \mathbb{P}^1/(0 \equiv \infty)$ by $\mathbb{Z}/r$ acting as $z \mapsto \xi_r z$ as in Exa. 1.13, for $r$ dividing $l$. We consider the $l$-torsion line bundle $\mathsf{L}_{\mathrm{ram}}$ on $\mathsf{E}$ fitting in $\mathsf{p}_*\mathcal{O} = \mathcal{O} \oplus \mathsf{L}_{\mathrm{ram}} \oplus \ldots \oplus \mathsf{L}_{\mathrm{ram}}^{\otimes r-1}$, with $\phi_{\mathrm{ram}} \colon \mathsf{L}_{\mathrm{ram}}^{\otimes l} \to \mathcal{O}$ given by the $(l/r)$th tensor power of the isomorphism $\mathsf{L}_{\mathrm{ram}}^{\otimes r} \cong \mathcal{O}$. We also consider the $l$-torsion line bundle $\mathsf{L}_{\text{ét}}$, pull-back via $\epsilon_{\mathsf{E}} \colon \mathsf{E} \to E$ of the sheaf of regular functions $f$ on the normalisation satisfying $f(\infty) = \xi_l f(0)$.

**Proposition 2.28.** *We have*

$$\mathrm{Pic}(\mathsf{E})[l] = \langle \mathsf{L}_{\text{ét}}, \mathsf{L}_{\mathrm{ram}} \rangle \cong (\mathbb{Z}/l) \oplus (\mathbb{Z}/r) \tag{36}$$

*and, for $\mathsf{g} \in \mathrm{Aut}_{(E,n)}(\mathsf{E}, n)$ with local picture $(x, y) \mapsto (\xi_r x, y)$ at $n$,*

$$\mathrm{Aut}(\mathsf{E}, n) = \{\mathsf{a} \in \mathrm{Aut}(\mathsf{E}) \mid \mathsf{a}(n) = n\} = \langle \mathsf{i}, \mathsf{g} \rangle \cong (\mathbb{Z}/2) \oplus (\mathbb{Z}/r).$$

*Then, $\mathsf{i}$ and $\mathsf{g}$ operate on $\mathrm{Pic}(\mathsf{E})[l]$ as $(k_1, k_2) \to (-k_1, -k_2)$ and $(k_1, k_2) \mapsto (k_1 + k_2, k_2)$, with respect to (36). In other words, the action of $(a_1, a_2) = \mathsf{i}^{a_2} \circ \mathsf{g}^{a_2} \in \mathrm{Aut}(\mathsf{E}, n)$ on the line bundle $(k_1, k_2) = (\mathsf{L}_{\text{ét}})^{\otimes k_1} \otimes \mathsf{L}_{\mathrm{ram}}^{\otimes k_2}$ in $\mathrm{Pic}(\mathsf{E})[l]$ yields*

$$(a_1, a_2) \cdot (k_1, k_2) = ((-1)^{a_1} k_1 + a_2 k_2, (-1)^{a_1} k_2), \tag{37}$$

*where $a_2 k_2$ is the product in $\mathbb{Z}/r$ and the sum $(-1)^{a_1} k_1 + a_2 k_2$ is taken within $\mathbb{Z}/l$ via the canonical embedding $\mathbb{Z}/r = (l/r)\mathbb{Z}/l \subseteq \mathbb{Z}/l$.* □

In view of the study of ghost automorphisms of level-$l$ curves we consider a faithful order-$l$ line bundle $\mathsf{L}$ on $\mathsf{E}$; in other words, we consider $\mathsf{L} = (\mathsf{L}_{\text{ét}})^{\otimes k_1} \otimes \mathsf{L}_{\mathrm{ram}}^{\otimes k_2}$ with $0 \leq k_1 < l$, $0 \leq k_2 < r$, $k_2$ of order $r$ in $\mathbb{Z}/r$ (faithfulness) and $(k_1, k_2)$ of order $l$ in $\mathbb{Z}/l \oplus \mathbb{Z}/r$.

**Corollary 2.29.** *The complete list of nontrivial automorphisms of the form $\mathsf{i}^{a_2} \circ \mathsf{g}^{a_2}$ (with $0 \leq a_1 < 2$, $0 \leq a_2 < r$) fixing the faithful order-$l$ line bundle of the form $\mathsf{L} = (\mathsf{L}_{\text{ét}})^{\otimes k_1} \otimes \mathsf{L}_{\mathrm{ram}}^{\otimes k_2}$ (with $0 \leq k_1 < l$, $0 \leq k_2 < r$) is as follows*

*(i) $l = 1$, $r = 1$, $(k_1, k_2) = (0, 0)$, and $(a_1, a_2) = (1, 0)$;*
*(ii) $l = 2$, $r = 1$, $(k_1, k_2) = (1, 0)$, and $(a_1, a_2) = (1, 0)$;*
*(iii) $l = 2$, $r = 2$, $(k_1, k_2) = (0, 1)$ or $(1, 1)$, and $(a_1, a_2) = (1, 0)$;*
*(iv) $l = 4$, $r = 2$, $(k_1, k_2) = (1, 1)$ or $(3, 1)$, and $(a_1, a_2) = (1, 1)$.*

*Proof.* Solve the equations $(-1)^{a_1} k_1 + a_2 k_2 = k_1$ and $(-1)^{a_1} k_2 = k_2$. □

**Remark 2.30.** The automorphisms (i), (ii) and (iii) coincide with the canonical involution $\mathsf{i}$, which in case (i) fixes $\mathcal{O}$, in case (ii) fixes $\mathsf{L}_{\text{ét}}$, and in case (iii) fixes $\mathsf{L}_{\text{ram}}$ and $\mathsf{L}_{\text{ram}} \otimes \mathsf{L}_{\text{ét}}$. If $(\mathsf{E}, n)$ is the elliptic tail of a stable level-$\ell$ curve $(\mathsf{C}, \mathsf{L}, \phi)$, then the ETI fixing $\overline{\mathsf{C} \setminus \mathsf{E}}$ and yielding $\mathsf{i}$ on $\mathsf{E}$ operates on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ as the quasireflection $(-\mathbb{I}_1) \oplus \mathrm{id}$.

**Remark 2.31.** The involution (iv) $= \mathsf{i} \circ \mathsf{g}$ fixes the level-4 structure $\mathsf{L}_{(\mathsf{E},n)} = \mathsf{L}_{\text{ram}} \otimes \mathsf{L}_{\text{ét}}$

$$(\mathsf{i} \circ \mathsf{g})^*(\mathsf{L}_{(\mathsf{E},n)}) = \mathsf{i}^*(\mathsf{g}^*\mathsf{L}_{\text{ram}} \otimes \mathsf{g}^*\mathsf{L}_{\text{ét}}) \overset{(37)}{=} \mathsf{i}^*(\mathsf{L}_{\text{ram}} \otimes \mathsf{L}_{\text{ét}}^{\otimes 2} \otimes \mathsf{L}_{\text{ét}}) = \mathsf{i}^*(\mathsf{L}_{\text{ram}}^{\vee} \otimes \mathsf{L}_{\text{ét}}^{\vee}) = \mathsf{L}_{(\mathsf{E},n)}.$$

Assume $(\mathsf{E}, n)$ is the elliptic tail of a stable level-$\ell$ curve $(\mathsf{C}, \mathsf{L}, \phi)$ with $\mathsf{L}\,|_{\mathsf{E}} = \mathsf{L}_{(\mathsf{E},n)}$. Then, the natural extension of (iv) fixing $\mathsf{C} \setminus \mathsf{E}$ is not an ETI and, by Prop. 2.13, does not act as a quasireflection. It yields $(\tau_1, \tau_2) \mapsto (-\tau_1, -\tau_2)$ on the parameters $\tau_1$ and $\tau_2$ smoothing $n$ and deforming the tail. Note also that in this case the canonical involution $\mathsf{i}$ does not fix $\mathsf{L}$.

2.5. **Noncanonical singularities.** The age invariant detects noncanonical singularities.

2.5.1. *Age, the Reid–Shepherd-Barron–Tai criterion.* Assume $V = \mathbb{C}^m$ and let $G$ be a finite group operating on $V$ via $G \to \mathrm{GL}(V)$. We define the notion of junior group elements $g \in G$.

**Definition 2.32** (junior group elements)**.** Consider $G$ operating on $V = \mathbb{C}^m$ and let $g \in G$ be an element of order $n$. The matrix corresponding to the action of $g$ on $V$ is conjugate to a diagonal matrix $\mathrm{Diag}(\xi_n^{a_1}, \ldots, \xi_n^{a_m})$, where $\xi_n$ is an $n$th root of unit and we have $0 \le a_i \le n$ for $i = 1, \ldots, m$. The age of $g$ is

$$\mathrm{age}_V(g) := \frac{a_1}{n} + \ldots + \frac{a_m}{n} \in \mathbb{Q}_{\ge 0}$$

Following [18], we say that $g \in G$ operating nontrivially on $V$ is *senior on* $V$ if $\mathrm{age}_V(g) \ge 1$, and is *junior on* $V$ if $0 < \mathrm{age}_V(g) < 1$.

Assume that the point at the origin of $V$ modulo $G \in \mathrm{GL}(V)$ is singular. Such a singularity is *canonical* if and only if any pluricanonical form on the smooth locus extends to any desingularisation of $V/G$. In other words, for all $q \in \mathbb{Z}$ sufficiently high and divisible, we have

$$\Gamma((V/G)^{\text{reg}}, \omega^{\otimes q}) = \Gamma(\widehat{V/G}, \omega^{\otimes q}) \qquad \text{for any desingularisation } \widehat{V/G} \to V/G.$$

**Theorem 2.33** (Reid–Shepherd-Barron–Tai criterion [23, 26, 24])**.** *Let us assume that $G$ operates on $V$ without quasireflections. The scheme-theoretic quotient $V/G$ has a noncanonical singularity at the origin if and only if there exists an element $g \in G$ which is junior on $V$.*

2.5.2. *The computation of the age of an automorphism on* $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\langle \mathrm{QR} \rangle$. We mod out $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ by the group $\langle \mathrm{QR} \rangle$ of automorphisms spanned by quasireflections; *i.e.*, by Proposition 2.13 this amounts to modding out the ETIs restricting to the identity on the entire curve except from an elliptic tail component $\mathsf{E}$ where the canonical involution $\mathsf{i}$ fixes $\mathsf{L}\,|_{\mathsf{E}}$. These involutions operate on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ simply by changing the sign of the parameter $\tau_e$ smoothing the node where $\mathsf{E}$ meets the rest of the curve. We refer to $\mathsf{E}$ as a quasireflection elliptic tail component or, simply, *quasireflection tail* (QR tail). We refer to the node joining the quasireflection tail to the rest of the curve as a *quasireflection node* (QR nodes) and we identify in this way a partition $\mathrm{Sing}(\mathsf{C}) = \mathrm{Sing}_{\mathrm{QR}}(\mathsf{C}) \sqcup \mathrm{Sing}_{\mathrm{nonQR}}(\mathsf{C})$ and a partition

$E = E_{\mathrm{QR}} \sqcup E_{\mathrm{nonQR}}$. Equation (19) yields

$$\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\langle \mathrm{QR}\rangle = \Big(\bigoplus_{e\in E}\mathbb{C}_{\overline{\tau}_e}\Big)\oplus\Big(\bigoplus_{v\in V}H^1(C'_v,T(-D_v))\Big),\ \text{with } \overline{\tau}_e = \begin{cases}\tau_e^2 & \text{for } e\in E_{\mathrm{QR}};\\ \tau_e & \text{for } e\in E\setminus E_{\mathrm{QR}}.\end{cases}$$

The action of $\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ on $(\mathsf{C},\mathsf{L},\phi)$ descends to an action without quasireflections on the above space.

**Corollary 2.34.** *The point at the origin of* $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ *is a noncanonical singularity if and only if there exists an element* $\mathsf{a}\in\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ *which is junior on* $\mathrm{Def}/\langle\mathrm{QR}\rangle$.

For any $\mathsf{a}\in\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$, we simply write $\mathrm{age}(\mathsf{a})$ for the age on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\langle\mathrm{QR}\rangle$. If $\mathsf{a}$ acts nontrivially on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)/\langle\mathrm{QR}\rangle$, it is junior if $\mathrm{age}(\mathsf{a})<1$ and senior if $\mathrm{age}(\mathsf{a})\geq 1$.

**Example 2.35.** The stack-theoretic ETI of Rem. 2.30 acts trivially on $\mathrm{Def}/\langle\mathrm{QR}\rangle$.

**Example 2.36.** The automorphism $\mathsf{a}$ extending $\mathsf{i}\circ\mathsf{g}$ in Rem. 2.31 operates on $\mathbb{C}_{\tau_1}\oplus\mathbb{C}_{\tau_2}\oplus\mathbb{C}^{3g-1}$ as $(-\mathbb{I}_2)\oplus\mathrm{id}$. We also pointed out that $\mathsf{i}$ is not an automorphism of $(\mathsf{C},\mathsf{L},\phi)$. We have $\mathrm{Def}/\langle\mathrm{QR}\rangle = \mathbb{C}_{\tau_1}\oplus\mathbb{C}_{\tau_2}\oplus\mathbb{C}^{3g-1}$ and $\mathrm{age}(\mathsf{a}) = 1/2+1/2 = 1$.

2.5.3. *The computation of the age of a ghost.* Using Proposition 2.4 and Theorem 2.18 we can easily compute the age of a ghost automorphism $\mathsf{a}\in\underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$ attached to $b\in G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/\ell)$. Notice that, since elliptic tail nodes are scheme-theoretic, $\mathsf{a}$ acts trivially on their smoothing parameter. When $\ell$ is prime we have

$$\mathrm{age}(\mathsf{a}) = \sum_{e\in E}\left\langle\frac{\mathsf{a}(e)}{\ell}\right\rangle = \sum_{e\in E}\left\langle\frac{[M(e)^{-1}]_\ell b(e)}{\ell}\right\rangle \qquad (\ell \text{ prime}), \tag{38}$$

where the terms at the numerators are integer representatives of $\mathsf{a}(e), b(e)$ and $[M(e)^{-1}]_\ell$ in $\mathbb{Z}/\ell$ (each summand in the above expression is clearly independent of the choice of the representatives modulo $\ell$). For composite $\ell$, Thm. 2.18, (3) yields

$$\mathrm{age}(\mathsf{a}) = \sum_{e\in E}\left\langle\frac{\mathsf{a}(e)}{\ell}\right\rangle = \sum_{e\in E}\left\langle\frac{[m(e)^{-1}]_{r(e)}b(e)}{r(e)}\right\rangle = \sum_{e\in E}\left\langle\frac{[m(e)^{-1}]_{r(e)}\widetilde{b}(e)}{\ell}\right\rangle, \tag{39}$$

where $\widetilde{b}(e)\in\mathbb{Z}/\ell$ is the image of $b(e)\in\mathbb{Z}/r(e)$ via the identification $\widetilde{b}(e) = (\ell/r(e))b(e) = \gcd(\ell, M(e))b(e)$. Again the above definition does not depend on the choices of the integer representatives of $\mathsf{a}(e)\in\mathbb{Z}/\ell$ and of $[m(e)^{-1}]_{r(e)}, b(e)\in\mathbb{Z}/r(e)$.

In Example 2.23, we presented three ghost automorphisms, the corresponding symmetric functions $e\mapsto\mathsf{a}(e)$ on the set of oriented edges are given in Figure 5. Equation 39 allows us to compute their age. According to (39), the order-2 automorphism has age 3/2, the order-4 automorphisms has age 5/4, whereas the order-8 automorphism has age 1. Hence, in all these three cases the ghosts are senior. However, ghost automorphisms operating as junior ghosts actually occur, we provide some examples, which will play a role in the proof of Thm. 2.40.

**Example 2.37.** Let $\ell = 5$. Consider a level curve whose dual graph has multiplicity $M$, pictured in the first diagram of Fig. 6; write $\boldsymbol{\nu}$ for the characteristic function of the support of $M$. Here we have $\boldsymbol{\nu} = 0$. In the second and third diagram we specify the symmetric function $\mathsf{a}\in S_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/5)$ and the corresponding 1-cochain $b\in G_{\boldsymbol{\nu}}(\Gamma;\mathbb{Z}/5)$. Using (38) we get $\mathrm{age}(\mathsf{a}) = 1/4+1/4+1/4+1/4 = 4/5$.

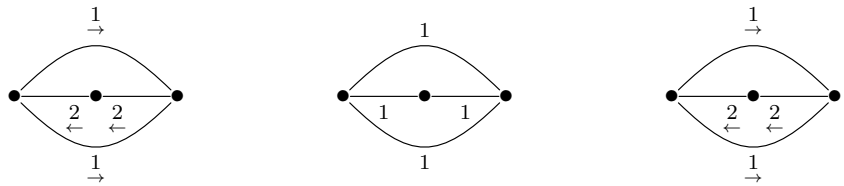



FIGURE 6. The multiplicity cochain $M$, the symmetric function $e \mapsto \mathsf{a}(e)$ and the cochain $e \mapsto b(e)$.

**Example 2.38.** Let $\ell = 8$. We adopt the notation $M$ and $\boldsymbol{\nu}$ as above. This time $\boldsymbol{\nu}$ is the vector-valued function attached to $M$. Again, the second and third diagrams specify the symmetric function $\mathsf{a} \in S_\nu(\Gamma; \mathbb{Z}/8)$ and the corresponding 1-cochain $b \in G_{\boldsymbol{\nu}}(\Gamma; \mathbb{Z}/8)$. More precisely, we have written next to each edge the values of $\widetilde{\mathsf{a}}$ and $\widetilde{b}$ in $\mathbb{Z}/8$; *e.g.*, "2" appearing in the second diagram represents the order-4 element $2 \mod 8$ in $\mathbb{Z}/8$. Using (39) we get $\mathrm{age}(\mathsf{a}) = 1/8 + 1/8 + 1/8 + 1/8 + 2/8 = 3/4$.

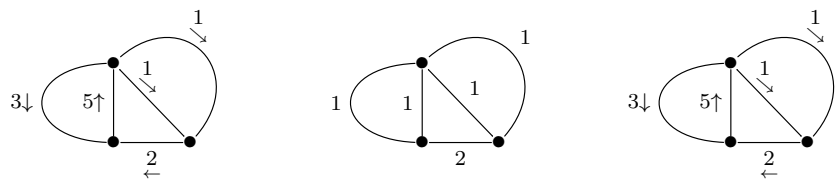



FIGURE 7. The multiplicity cochain $M$, the $\mathbb{Z}/8$-valued symmetric function $e \mapsto \widetilde{\mathsf{a}}(e) = \mathsf{a}(e)\gcd(8, M(e))$ and the cochain $e \mapsto \widetilde{b}(e) = b(e)\gcd(8, M(e))$.

**Example 2.39.** Let $\ell = 12$. In view of the proof of Theorem 2.40 we slightly generalise Example 2.38. We refer to Figure 8, where we adopt the established conventions. Using (39) we get $\mathrm{age}(\mathsf{a}) = 1/12 + 1/12 + 1/12 + 1/12 + 2/12 + 2/12 = 2/3$.

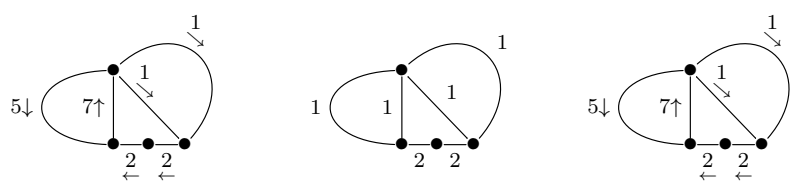



FIGURE 8. The multiplicity cochain $M$, the $\mathbb{Z}/12$-valued symmetric function $e \mapsto \widetilde{\mathsf{a}}(e) = \mathsf{a}(e)\gcd(12, M(e))$ and the cochain $e \mapsto \widetilde{b}(e) = b(e)\gcd(12, M(e))$.

2.5.4. *The locus of noncanonical singularities of $\overline{R}_{g,\ell}$.* We may apply the above criterion as follows. Within Deligne and Mumford's moduli stack $\overline{\mathsf{M}}_g$ of stable curves, consider the locus

$$\overline{\mathsf{M}}_g^\circ = \{\mathsf{C} \mid \mathrm{Aut}(C) = 0\}_{/\cong}$$

of stable curves with trivial automorphism group. This is a stack that can be represented by a smooth scheme. We study the overlying stack

$$\overline{\mathsf{R}}_{g,\ell}^\circ = \{(\mathsf{C}, \mathsf{L}, \phi) \mid \mathrm{Aut}(C) = 0\}_{/\cong}$$

of stable level curves $(\mathsf{C}, \mathsf{L}, \phi)$ such that the coarsening $C$ of $\mathsf{C}$ has trivial automorphism group $\mathrm{Aut}(C)$. The scheme $\overline{R}^{\circ}_{g,\ell}$ coarsely representing $\overline{\mathsf{R}}^{\circ}_{g,\ell}$ may well have singularities; this happens as soon as $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi) = \mathrm{Aut}_C(\mathsf{C}, \mathsf{L}, \phi)$ is nontrivial (by Lemma 2.10, nontrivial ghosts cannot operate as the identity or as quasireflections; hence singular points in $\overline{R}^{\circ}_{g,\ell}$ are characterised by the presence of nontrivial ghosts). Furthermore, since the action of $\mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ satisfies the hypotheses of the Reid–Shepherd-Barron–Tai criterion (Theorem 2.33), noncanonical singular points are characterised by the presence of junior nontrivial ghosts in the sense of Definition 2.32. As a consequence of the following theorem, *for level $\ell \leq 6$ and $l \neq 5$, all singularities of the scheme $\overline{R}^{\circ}_{g,\ell}$ are canonical.* We provide a statement which applies to the entire category $\overline{\mathsf{R}}_{g,\ell}$.

**Theorem 2.40** (No-Junior-Ghosts Theorem)**.** *For $g \geq 4$ and $\ell \geq 1$ consider the stack of level-$\ell$ genus-$g$ curves $\overline{\mathsf{R}}_{g,\ell}$. In $\overline{\mathsf{R}}_{g,\ell}$, every nontrivial ghost automorphism is senior if and only if $\ell \leq 6$ and $\ell \neq 5$.*

*Proof.* Proving the "only if"-part of the statement for a given level $\ell$ and genus $g$ amounts to exhibiting a dual graph $\Gamma$ attached to an object of $\overline{\mathsf{R}}_{g,\ell}$ with a multiplicity $M \in \ker \partial$ and a symmetric function $\mathsf{a}\colon E \to \mathbb{Z}/\ell$ defining a junior ghost. Notice that if there exists such a triple $(\Gamma, M, \mathsf{a})$ for $\overline{\mathsf{R}}_{g,\ell}$, then we can exhibit a triple $(\Gamma, (\ell'/\ell) \times M, (\ell'/\ell) \times \mathsf{a})$ for $\overline{\mathsf{R}}_{g,\ell'}$ for any multiple $\ell'$ of $\ell$ (here, Proposition 1.11 has been implicitly used). Examples 2.37, 2.38, 2.39 actually occur for $g \geq 4$ and exhibit junior ghosts for positive levels $\ell \in 5\mathbb{Z} \cup 8\mathbb{Z} \cup 12\mathbb{Z}$. By halving a single straight edge in Figure 6 ($\bullet\!-\!\bullet \to \bullet\!-\!\bullet\!-\!\bullet$), we can immediately generalise Example 2.37 from $\ell = 5$ to $\ell = 7$; iterating this procedure, for all odd levels $\ell \geq 5$ and for their multiples, we exhibit junior ghosts. The "only if" part is proven: in order for junior ghost not to occur, $\ell$ should be a positive integer of the form $2^a \times 3^b$ with $a \in \mathbb{N}$ and $b = 0, 1$ (*i.e.* not a multiple of an odd integer $\geq 5$), with $a < 3$ (*i.e* not a multiple of 8) and with $a < 2$ if $b = 1$ (*i.e* not a multiple of 12).

The "if"-part of the statement claims that there is no junior ghost $\mathsf{a}$ in $G_{\boldsymbol{\nu}}(M; \mathbb{Z}/\ell)$ for any stable graph $\Gamma$ with $\boldsymbol{\nu} = \boldsymbol{\nu}_M$ attached $M \in \ker \partial$. Recall the conditions

(i) $\mathrm{age}(\mathsf{a}) < 1$ (junior);
(ii) $M = \sum_{i \in I} K_i$, where $I$ is a finite set of circuits ($M \in \ker \partial$);
(iii) $\langle \mathsf{a}M, K \rangle = 0$ for all circuits ($\mathsf{a}M \in \mathrm{im}\, \delta$).

(see (34) for the definition of $\mathsf{a}M$). Let us say that an edge $e$ is *active* if $\mathsf{a}(e) \neq 0$ and that a *circuit is active* if it contains an active edge (note that $e$ is active if and only if $M(e) \neq 0$ and $(\mathsf{a}M)(e) \neq 0$). For $\ell = 2$, by condition (iii) above, any active circuit contains a positive even number of active edges. Then we have $\sum_E \mathsf{a}(e)/2 \geq 1$, contradicting condition (i) and yielding the desired no-junior-ghosts claim. Notice that, for the same argument to hold, we simply need to make sure that there is no oriented edge $e \in \mathbb{E}$ satisfying

$$\langle \mathsf{a}(e)/l \rangle < \langle \mathsf{a}M(e)/l \rangle. \tag{40}$$

Indeed such oriented edges yield low age contribution; we call them *age-delay* edges.

For $\ell = 3$, using $\mathsf{a}(e) \in M(e)\mathbb{Z}/\ell$, an age-delay edge $e$ occurs if and only if $e \in \mathbb{E}$ satisfies $(M(e), \mathsf{a}(e)) = (2, 1)$. By the $\ell = 2$ argument we may assume from now on that any active circuit contains an age-delay edge. Conditions (i) and (iii) imply that any active circuit contains exactly two active edges $e$ and $e'$, with $(M(e), \mathsf{a}(e)) = (2, 1)$ and $(M(e'), \mathsf{a}(e')) =$

$(1,1)$. Condition (i) does not allow more active edges. All circuits through $e$ go through $e'$. Hence, condition (ii) implies $M(e) = M(e')$, a contradiction.

For $\ell = 4$, there is, again, only one sort of age-delay edge $e \in \mathbb{E}$ satisfying $(M(e), \mathsf{a}(e)) = (\ell - 1, 1)$. Furthermore, any active circuit necessarily contains such an age-delay edge $e$ and exactly one more active edge $e'$ satisfying $(M(e'), \mathsf{a}(e')) = (1,1)$. We may define a graph $\Omega$ whose vertex set $V(\Omega)$ is the set of active edges in $E$ (we forget their orientation). Two vertices in $V(\Omega)$ are linked by an edge $E(\Omega)$ if they lie on a circuit $K$ in $\Gamma$. Since all active edges lie on at least one active circuit (by $M(e) \neq 0$ and condition (ii)) their multiplicity $M(e)$ equals 1 or $\ell - 1$ depending on their orientation; we choose a canonical orientation $e \in \mathbb{E}$ by imposing $M(e) = 1$. In this way $M$ descends to a $\mathbb{Z}/\ell$-valued 0-cochain $\mu \in C^0(\Omega; \mathbb{Z}/\ell)$ constantly equal to 1. Furthermore, one can rephrase condition (ii) in terms of $\partial_\Omega \colon C^1(\Omega; \mathbb{Z}/\ell) \to C^0(\Omega; \mathbb{Z}/\ell)$: by construction, (ii) may be regarded as saying $\mu \in \operatorname{im} \partial_\Omega$. This happens only if $\sum_{v \in V(\Omega)} \mu(v) = 0 \in \mathbb{Z}/\ell$, which implies the existence of at least $\ell$ active circuits contradicting condition (i).

For $\ell = 6$, we have two possible sorts of age-delay edges $e \in \mathbb{E}$: $(M(e), \mathsf{a}(e)) = (5,1)$ or $(M(e), \mathsf{a}(e)) = (4,2)$. By (iii), the set $A_K$ of active edges of an active circuit $K$ equals:

(i) $A_K = \{e_1, e_2\}$ with $(M, \mathsf{a})$ taking values $(5,1)$, $(1,1)$;
(ii) $A_K = \{e_1, e_2\}$ with $(M, \mathsf{a})$ equal to $(4,2)$ and $(2,2)$;
(iii) $A_K = \{e_1, e_2, e_3\}$ with $(M, \mathsf{a})$ equal to $(4,2)$, $(1,1)$ and $(1,1)$;
(iv) $A_K = \{e_1, e_2, e_3\}$ with $(M, \mathsf{a})$ equal to $(5,1)$, $(5,1)$ and $(2,2)$;
(v) $A_K = \{e_1, e_2, e_3, e_4\}$ with $(M, \mathsf{a})$ with $(5,1)$, $(5,1)$, $(1,1)$ and $(1,1)$.

Take any circuit $K$ as above of different type from (i); it contributes $2/3$ to $\operatorname{age}(\mathsf{a})$ and, since $M$ is not constant on it, $\Gamma$ contains exactly one more active edge $e_5 \notin A_K$ with $\mathsf{a}(e_5) = 1$ (more than one active edge would contradict condition (i)). For any circuit $H$ going through $e_5$, consider the set $A_H$ of its active edges; then, the multiplicity $M$ should be constant on $A_K \setminus A_H$ and $A_K \cap A_H$. This cannot happen if $K$ is not of type (i). Then, assuming $K$ is of type (i), the argument used for $\ell = 4$ yields the desired no-junior-ghost claim without modifications except setting $\ell = 6$. □

**Definition 2.41.** A level-$\ell$ curve $(\mathsf{C}, \mathsf{L}, \phi)$ is a J-curve if $\operatorname{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ contains a junior ghost.

The points representing J-curves are noncanonical singularities by definition. Noncanonical singularities may occur even if the level curves has no junior ghost automorphisms and regardless of the level $\ell$. Indeed this is the case of level curves of type T (or simply T-curves) which we now illustrate. T-curves represent a codimension-1 locus within the divisor $\Delta_{g-1}$; *i.e.* a codimension-2 locus in $\overline{\mathsf{R}}_{g,\ell}$.

**Definition 2.42.** A stable level-$\ell$ curve $(\mathsf{C}, \mathsf{L}, \phi)$ is a T-curve if

- $\mathsf{C}$ contains an elliptic tail (that is $\subset E \subset \mathsf{C}$ with $\mathsf{C} \cap \overline{\mathsf{C} \setminus E} = \{n\}$) (Tail-condition);
- $E$ admits an order-3 automorphism (that is $\operatorname{Aut}(E, n) \cong \mathbb{Z}/6$) (Three-condition);
- the restriction of $\mathsf{L}$ is trivial; *i.e.* $(\mathsf{C}, \mathsf{L}, \phi) \in \Delta_{g-1}$ (Triviality-condition).

**Theorem 2.43.** *The point representing $(\mathsf{C}, \mathsf{L}, \phi)$ in $\overline{\mathsf{R}}_{g,\ell}$ is a noncanonical singularity if and only if $(\mathsf{C}, \mathsf{L}, \phi)$ is a T-curve or a J-curve.*

*Proof.* For the "if"-part of the statement we only need to show that a T-curve $(\mathsf{C}, \mathsf{L}, \phi)$ has a junior automorphism. Let us define the automorphism $\mathsf{a}_{1/6} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$, whose restriction to $\overline{\mathsf{C} \setminus E}$ is the identity and whose restriction to $E$ generates $\mathrm{Aut}(E, n) \cong \mathbb{Z}/6$ and operates on the local parameter of $E$ at $n$ as $z \mapsto \xi_6 z$. The coordinates $\tau_1$ and $\tau_2$ correspond to the direction smoothing the node $n$ and to the direction preserving the node and varying along $\Delta_{g-1}$. The action of $\mathsf{a}_{1/6}$ on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ is given by $\mathrm{Diag}(\xi_6, \xi_6^2, 1, \dots, 1)$, where the first coordinates are $\tau_1$ and $\tau_2$. The action of $\mathsf{a}_{1/6}$ on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)/\langle \mathrm{QR} \rangle$ is given by $\mathrm{Diag}(\xi_6^2, \xi_6^2, 1, \dots, 1)$, where the first coordinates are $\overline{\tau}_1 = \tau_1^2$ and $\overline{\tau}_2 = \tau_2$ (see §2.5.2). The age of $\mathsf{a}_{1/6}$ on $\mathrm{Def}\,/\langle \mathrm{QR} \rangle$ is $1/3 + 1/3 = 2/3 < 1$, and $\mathrm{Def}\,/\,\mathrm{Aut}$ has a noncanonical singularity. Notice that all the junior automorphisms operating as $z \mapsto \lambda z$ on the tail (and fixing the rest of the curve) are of the form $\mathsf{a}_{1/6}$ up to ETI.

The "only if"-part reduces to the following proposition.

**Proposition 2.44.** *Let $(\mathsf{C}, \mathsf{L}, \phi)$ be a stable level-$\ell$ curve which is not a J-curve and has a junior automorphism $\mathsf{a}$, then it is a T-curve and the isomorphism $\mathsf{a}$ coincides, up to ETIs, with the above isomorphism $\mathsf{a}_{1/6}$.*

Preliminary 1. As in [15, p.33] we begin by slightly simplifying the problem by adding a further condition to the hypotheses. A level curve $(\mathsf{C}, \mathsf{L}, \phi)$ representing a noncanonical singularity in $\overline{\mathcal{R}}_{g,\ell}$ is $(\star)$-*smoothable* if the following conditions are satisfied.

(a) There is a junior automorphism $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ and $m$ nodes $\mathsf{n}_0, \dots, \mathsf{n}_{m-1}$ lying in $\mathrm{Sing}(\mathsf{C}) \setminus \{\text{QR nodes}\}$ labeled by $j \in \mathbb{Z}/m$ so that $\mathsf{a}(\mathsf{n}_j) = \mathsf{n}_{j+1}$.
(b) We have $\prod_{i=0}^{m-1} c_j = 1$, where $c_j$ are the complex nonvanishing constants satisfying $\mathsf{a}^* \tau_{j+1} = c_j \tau_j$ for all $j \in \mathbb{Z}/m$, and $\tau_j$ is the parameter smoothing $\mathsf{n}_j$.

By [15, p.33], if $(\mathsf{C}, \mathsf{L}, \phi)$ is $(\star)$-smoothable, then the data $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ can be deformed to $\mathsf{a}' \in \mathrm{Aut}(\mathsf{C}', \mathsf{L}', \phi')$ in such a way that the $m$ nodes above are smoothed and the age of the action on $\mathrm{Def}\,/\langle \mathrm{QR} \rangle$ is preserved. In [20, Prop.3.6], Ludwig proves a generalisation applying to moduli of roots of any line bundle; in particular we can use this fact for stable level-$\ell$ curves. Hence, by iterating such deformations, within the locus of noncanonical singular points in $\overline{\mathcal{R}}_{g,\ell}$, we can smooth any $(\star)$-smoothable curve to a curve which is no more $(\star)$-smoothable; we refer to this condition as $(\star)$-*rigidity*. The loci of T-curves and of J-curves are closed: in the above deformation, if $(\mathsf{C}', \mathsf{L}', \phi')$ is a J-curve (a T-curve), then $(\mathsf{C}, \mathsf{L}, \phi)$ is a J-curve (a T-curve). Proposition 2.44 can be shown under the following assumption.

**Assumption 2.45.** In the proof of Proposition 2.44 we assume that $(\mathsf{C}, \mathsf{L}, \phi)$ is $(\star)$-rigid.

Preliminary 2. Set $\mathfrak{ord}(\mathsf{a}) = \mathrm{ord}(a)$; this is also the least integer for which $\mathsf{a}^m$ is a ghost and is a divisor of $\mathrm{ord}(\mathsf{a})$. We can provide lower bounds for the age of $\mathsf{a}$.

**Lemma 2.46.** *Consider a stable level-$\ell$ curve $(\mathsf{C}, \mathsf{L}, \phi)$.*

0. *For any automorphism $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$, we have $\mathrm{age}(\mathsf{a}) \geq (\#E - N)/2$, where $N$ is the number of cycles of the permutation of $E$ induced by $\mathsf{a}$.*

*We can improve the lower bound in the following situations.*

1. *Assume that* $(\mathsf{C}, \mathsf{L}, \phi)$ *is a noncanonical* $(\star)$*-rigid singularity in* $\overline{\mathcal{R}}_{g,\ell}$*; then, for any subcurve* $\mathsf{Z}$ *such that* $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$ *and for any length-*$k$ *cycle of the induced permutation of* $\mathrm{Sing}_{\mathrm{nonQR}}(\mathsf{C}) \cap \mathrm{Sing}(\mathsf{Z})$*, we have* $\mathrm{age}(\mathsf{a}) \geq k/\mathrm{ord}(\mathsf{a}|_{\mathsf{Z}}) + (\#E - N)/2$.
2. *If* $\mathsf{a}^{\mathfrak{ord}(\mathsf{a})}$ *is a senior ghost, then we have* $\mathrm{age}(\mathsf{a}) \geq 1/\mathfrak{ord}(\mathsf{a}) + (\#E - N)/2$.

*Proof.* We can express the action of $\mathsf{a}$ on $\bigoplus_{e \in E} \mathbb{C}_{\overline{\tau}_e}$ (see §2.5.2 for the notation $\overline{\tau}_e$) in terms of a block-diagonal matrix $H$ whose blocks $H_1, \ldots, H_N$ are of the form

$$H_i = D_i P_i = \mathrm{Diag}(\exp(2\pi \mathtt{i} q_0^{(i)}), \ldots, \exp(2\pi \mathtt{i} q_{n_i-1}^{(i)}))P_i,$$

where $P_i$ is the permutation matrix attached to the cycle permutation operating on $\mathbb{Z}/n_i$ as $\sigma(j) = (j+1)$. Note that $n_i$ divides the order of the permutation of $E$ induced by $\mathsf{a}$. Then, since the characteristic polynomial of $H_i$ is $x^{n_i} - \det D_i$, we have

$$\mathrm{age}(\mathsf{a}) \geq \sum_{i=1}^{N} \left( \left\langle \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle + \frac{n_i - 1}{2} \right) = \sum_{i=1}^{N} \left\langle \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle + \frac{\#E - N}{2},$$

where the right hand side is of the form $A + \frac{\#E-N}{2}$ with $A \geq 0$ (see also [20, Prop. 3.7]). At point (1), there is a $k \times k$-block $H = H_{i_0}$ with $D = D_{i_0} = \mathrm{Diag}(\exp(2\pi \mathtt{i} q_0), \ldots, \exp(2\pi \mathtt{i} q_{k-1}))$, $H^k = \exp(2\pi \mathtt{i} q)\mathbb{I}$, and $q = \langle \sum_{j=0}^{k-1} q_j \rangle \neq 0$ (see condition (b) defining $(\star)$-smoothability). Since, for $w = \mathrm{ord}(\mathsf{a}|_{\mathsf{Z}})$ we have $H^w = \mathrm{id}$, we have $(w/k)q \in \mathbb{Z}$; hence, we have $A \geq q \geq k/w$ as required.

In case (2) we are assuming that $\mathsf{a}^m$ is senior for $m = \mathfrak{ord}(\mathsf{a})$. Notice that $m/n_i$ is integer for all $i$. We want to show $A \geq 1/m$. Assume $A < 1/m$; then, for all $i$, we preliminarily notice

$$\left\langle \frac{m}{n_i} \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle = \frac{m}{n_i} \left\langle \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle. \tag{41}$$

This happens because we have

$$\frac{m}{n_i} \left\langle \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle \leq \sum_{i=1}^{N} m \left\langle \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle = mA < 1.$$

On the other hand $\mathsf{a}^m$ is a ghost automorphism and operates on $\bigoplus_{e \in E} \mathbb{C}_{\overline{\tau}_e}$ as the diagonal matrix $H^m$ with $n_i$ eigenvalues equal to $(\det D_i)^{m/n_i}$ for $i = 1, \ldots, N$; using (41), we get

$$\mathrm{age}(\mathsf{a}^m) = \mathrm{age}(H^m) = \sum_{i=1}^{N} n_i \left\langle \sum_{j=0}^{n_i-1} \frac{m}{n_i} q_j^{(i)} \right\rangle = \sum_{i=1}^{N} m \left\langle \sum_{j=0}^{n_i-1} q_j^{(i)} \right\rangle < 1.$$

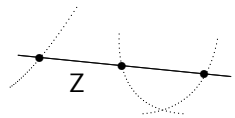
contradicting the assumption that $\mathsf{a}^m$ is senior. □

Step 1: the automorphism $\mathsf{a}$ fixes all nodes except, possibly, from a single transposition of two nodes. Indeed, by [15, p.34] (embodied in the first part of Lemma 2.46), each node transposition contributes $1/2$ to $\mathrm{age}(\mathsf{a})$.

Step 2: for each irreducible component $\mathsf{Z}$ we have $\mathsf{a}(\mathsf{Z}) = \mathsf{Z}$. Harris and Mumford's argument excludes[2] the condition $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ apart from one situation which we now state precisely.

[2]The space parametrizing the deformations of a hypothetical component $\mathsf{Z}$ for which $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ (alongside with its special points) should have dimension $d = 0$ or $1$ and in this second case we must have $\mathsf{a}(\mathsf{a}(\mathsf{Z})) = \mathsf{Z}$, *i.e.*

Case (a) of p.35 in [15] concerns a smooth, rational, irreducible component $\mathsf{Z}$ meeting the rest




of the curve at three special points; in the present setup we should of course allow nontrivial stabilisers at these three points. Then [15] relies on the following claim in the special case $\mathsf{a} = a \in \mathrm{Aut}(C)$. We state a generalised version, which is due to Ludwig, see [20, end of Proof of Prop. 3.8].

**Lemma 2.47.** *Assume the coarsening of* $\mathsf{a} \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$ *operates locally at a scheme-theoretic node* $\mathrm{Spec}\,\mathbb{C}[x,y]/(xy)$ *of* $C$ *as* $(x,y) \mapsto (y,x)$. *Then* $\mathsf{a}$ *fixes the parameter smoothing the node in* $\mathrm{Def}(C)$ *and operates on the parameter* $\tau$ *smoothing the node in* $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$ *as either* $\tau \mapsto \tau$ *or* $\tau \mapsto -\tau$. *In the first case the curve is not* $(\star)$*-rigid.* □

It is worthwhile to sketch the proof since Ludwig uses the different setup of quasistable curves (which is equivalent in this case). If $\mathsf{a} = a$ we have $xy = t = \tau = \tau$, hence $\tau \mapsto \tau$. Otherwise note that the multiplicity at the oriented edge $e$ corresponding to the above node satisfies $M(e) = -M(\overline{e}) = -M(e)$; hence $M(e) = \ell/2$ and the action on $\tau_e$ is $\tau_e \mapsto \tau_e$ or $\tau_e \mapsto -\tau_e$.

Now, let us assume $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ and apply the fact that $\mathsf{a}$ is junior and that $(\mathsf{C}, \mathsf{L}, \phi)$ is $(\star)$-rigid. The three special points of $\mathsf{Z}$ are nodes of $\mathsf{C}$. If they are fixed they have two branches, one in $\mathsf{Z}$ and one in $\mathsf{a}(\mathsf{Z})$. Since the coarsening $Z$ of $\mathsf{Z}$ is a projective line these fixed nodes satisfy the condition of the above lemma and, by $(\star)$-rigidity, yield age contribution $1/2$. Recall that each non-fixed node also contributes $1/2$. The age is at least 1 (with one pair of nodes exchanged and the remaining node is fixed). So, the argument of [15] holds true: $\mathsf{a}(\mathsf{Z}) \neq \mathsf{Z}$ is ruled out.

Step 3: classification of the irreducible components. For any irreducible component $\mathsf{Z}$ of $\mathsf{C}$ let us set up the notation for the rest of the proof. We write $\mathsf{N} \to \mathsf{Z}$ for its normalisation, $\mathsf{D} \subset \mathsf{N}$ for the divisor representing special points lifting the nodes of $\mathsf{C}$, $\mathsf{r}$ for the restriction $\mathsf{a}\,|_{\mathsf{Z}}$, and $\mathsf{r}_{\mathsf{N}} \in \mathrm{Aut}(\mathsf{N})$ for the lift to $\mathsf{N}$. Coarsening yields $N \to Z$, $D \subset N$, $r \in \mathrm{Aut}(Z)$ and $r_N \in \mathrm{Aut}(N)$. Since all components are fixed, we establish a list of possible cases by recalling the classification [15, Prop. p.28] of nontrivial automorphisms $r_N$ of a smooth scheme-theoretic curve $N$ paired with a divisor $D \subset N$ operating on $H^1(N, T(-D))$ with age less then 1:

(I) $N$ rational with $r_N \colon z \mapsto \xi_n z$ for $n = 2, 4$,
(II) $N$ elliptic with $r_N$ of order $2, 3, 4$, or $6$,
(III) $N$ hyperelliptic of genus 2 or 3 with $r_N$ the hyperelliptic involution,
(IV) $N$ of genus-2 with an involution $r_N$ such that $N/\langle r_N \rangle$ is an elliptic curve.

Step 4. Classification of the irreducible components $\mathsf{Z}$ satisfying the following extra condition: $\mathsf{a}$ fixes all nodes of $\mathsf{Z} \cap \mathrm{Sing}(\mathsf{C})$. We keep the above notation $\mathsf{N}, \mathsf{D}, \mathsf{r}, \mathsf{r}_{\mathsf{N}}, N, D, r, r_N$.

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the cycle of irreducible components obtained by applying the automorphism $\mathsf{a}$ iteratively starting from $\mathsf{Z}$ must have length $m = 2$ (indeed, via an age estimate analogue to Lemma 2.46, one can prove age $\geq d(m-1)/2$). The case $d = 1$ corresponding to (c), (d) and (e) in [15, p. 35] is ruled out by the authors as well as the case named (b) where $\mathsf{Z}$ is a singular elliptic tail, because it yields $g(\mathsf{C}) \leq 3$.

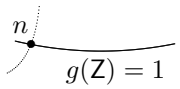
First, case (I) does not occur. Indeed, we argue as in [15, case (b), p.37]: since the nodes of $\mathsf{Z}\cap\mathrm{Sing}(\mathsf{C})$ are fixed, the special points of $\mathsf{N}$ are either fixed or of order 2 with respect to $\mathsf{r_N}$. We deduce that $r_N$ necessarily operates on the coarse space as $z \mapsto \xi_2 z$. Then, using the stability condition, it is easy to show that there is at least one pair of points with opposite coordinate on $N$ mapping to a node $n$ of $Z$. By Lemma 2.47 and $(\star)$-rigidity this yields age contribution $1/2$ and the nontriviality of the stabiliser over $n$; we deduce (using $\mathrm{age}(a) < 1$) that there is exactly one node of $Z$ whose preimages in $N$ are interchanged by $r_N$. The remaining nodes lying in $Z$ are contained in the images of the two fixed points of $z \mapsto \xi_2 z$. Note that there cannot be two such nodes, otherwise the action on $H^1(N, T(-D))$ gives extra contribution of at least $1/2$, because the order-2 automorphism does not deform to the general four-pointed rational curve. Therefore, the only possibility is that $\mathsf{Z}$ is a stack-theoretic elliptic tail as in Definition 2.12. Since $\mathsf{a}$ operates by changing the sign of the parameter deforming the elliptic tail, we are necessarily in the situation (iv) of Corollary 2.29 and, by Remark 2.31, we have $\mathrm{age}(\mathsf{a}) \geq 1$, a contradiction.

By a simple age computation[3] [15, p.39, case (e)] rules out, without changes, the genus-2 curve of case (IV).

Second, case (II) occurs only if $r_N$ fixes at least one point. Assume, by way of contradiction, that $r_N$ is a translation $z \mapsto z + t_0$ on $N = \mathbb{C}/\text{lattice}$. Since the translation does not allow fixed points it should allow two-points orbits. In this way $r_N$ is a translation of order-2. This implies that $\mathsf{C} = \mathsf{Z}$; *i.e.* $\mathsf{C}$ is irreducible. Since $g(\mathsf{C}) \geq 4$, then there are at least three nodes satisfying the conditions of Lemma 2.47. Applying $(\star)$-rigidity we get age contribution $3/2$ and we can conclude as in [15] that $r_N$ fixes at least one point; then we can use Harris and Mumford's list of cases "(c2)-(c5)" at [15, p. 37-39] specifying the configuration of the elliptic component and their age contribution. We summarise this in (i) and (ii), below.

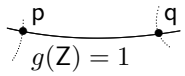
We can reproduce Harris and Mumford's list of possible irreducible components $\mathsf{Z}$ for which the restriction $\mathsf{r} = \mathsf{a}|_{\mathsf{Z}}$ does not satisfy $r = \mathrm{id}_Z$ and fixes all points of $\mathsf{Z} \cap \mathrm{Sing}(\mathsf{C})$.

(i) $\mathsf{Z}$ is a scheme-theoretic elliptic tail $\mathsf{r}$ is the ETI (age contribution 0) or an automorphism of order $3, 4$ or $6$ of a smooth elliptic tail $\mathsf{Z} = Z$ meeting the rest of the curve at $n$ acting on $H^1(Z, T(n))$ with age $1/3, 1/2$ or $1/3$ (see figures at pages 38 and 39 in [15]).



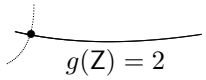

ord $= 3, 4, 6,$ age contribution $= \frac{1}{3}, \frac{1}{2}, \frac{1}{3}$.

(ii) $\mathsf{Z}$ is a smooth genus-1 component ($\mathsf{Z} = Z$) meeting the rest of the curve at two points $\mathsf{p}$ and $\mathsf{q}$; the action on $H^1(Z, T(p+q)$ has order 2 or 4 and age $1/2$ or $3/4$ (see figures at pages 38 and 39 in [15]).




ord $= 2, 4,$ age contribution $= \frac{1}{2}, \frac{3}{4}$.

(iii) $\mathsf{Z}$ is an hyperelliptic tail of genus $g = 2$; the restriction $\mathsf{r}$ is the hyperelliptic involution contributing $1/2$ to $\mathrm{age}(\mathsf{a})$ (see case (d) of [15, p. 39]).

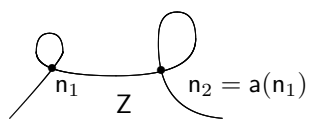



ord $= 2,$ age contribution $= \frac{1}{2}$.

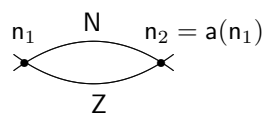
[3]The dimension of the $(-1)$-eigenspace of $r_N$ on $H^1(N, T(P))$ (where $P$ is a fixed point of $r_N$) is 2.

Step 5. We now argue that $\mathsf{a} \in \mathrm{Aut}(\mathsf{C})$ fixes all nodes. To this effect, by Step 1, we need to rule out the cases where $\mathsf{a}$ transposes a pair of nodes $(\mathsf{n}_1,\mathsf{n}_2)$. Since the node transposition contributes 1/2 to age($\mathsf{a}$) we can exclude the presence in the curve of components of the form (ii) and (iii). We can assume that $\mathsf{a}$ operates as the identity on the elliptic tails (i). If this is not the case, we can simply modify $\mathsf{a}$ by restricting to $\mathsf{B} = \overline{\mathsf{C} \setminus \{\text{elliptic tails}\}}$ and by trivially extending to $\mathsf{C}$; the resulting automorphism has lower age but it is still nontrivial because it exchanges two nodes; hence it is a nontrivial junior automorphism, which we will refer to as $\mathsf{a}$ in this step.

We now see that $\mathsf{n}_2 = \mathsf{a}(\mathsf{n}_1)$ yields a contradiction; since all irreducible components are (globally) fixed by Step 2, we reduce to the following cases.




case (a)




case (b)

(a) All the branches of $\mathsf{n}_1$ and of $\mathsf{n}_2 = \mathsf{a}(\mathsf{n}_1)$ lie in the same irreducible component $\mathsf{Z}$. Then, Lemma 2.46 (1), yields age contribution $2/n + 1/2$, where $n = \mathrm{ord}(\mathsf{r})$ and fits in the conditions required by 2.46. We observe that $\mathrm{ord}(\mathsf{r}) = \mathrm{ord}(r)$ because every ghost is the identity on the irreducible components of $\mathsf{C}$ (see Remark 2.19). The age contribution coincides with that used in [15, p 36-37] in order to rule out this case.

(b) There is a component $\mathsf{Z}$ containing exactly one branch for each node $\mathsf{n}_1$ and $\mathsf{n}_2$. Then let $\mathsf{H}$ be the second component through $\mathsf{n}_1$ and $\mathsf{n}_2$. Then, by Lemma 2.46 (2), the age of $\mathsf{a}$ is at least $1/n + 1/2$ where $n$ is the order of the coarsening of $\mathsf{a}|_{\mathsf{Z}\cup\mathsf{H}}$. According to the list of cases (I-IV), $n$ can be $2, 4, 6$, or 12. Since the lower bound $1/n + 1/2$ is smaller than the lower bound $2/n+1/2$ found in [15, p. 36-37] we should study more carefully the case $n = 4$. The same issue arises in [20, Proof of Prop.3.10], where Ludwig notices that, for $n = 4$, there is extra age contribution of 1/4. Indeed, either $Z$ or $H$ is an elliptic curve on which the coarsening $a$ operates, locally at a point $p \neq n_1, n_2$, as $z \mapsto \exp(2\pi i/4)$. This yields extra age contribution 1/4. (Ludwig also checks that the arguments of Harris and Mumford allow to conclude for $n = 6$ and 12 because of the respective extra age contributions 1/3 and 1/2 that they find in these two cases. The argument fits equally well here.)

Step 6. We are left with the problem of patching together the few curves of genus 1 and 2 listed in (i), (ii), and (iii) with lots of identity components; *i.e.* components where the coarsening of $\mathsf{a}$ restricts to the identity. We do it by following [15, p.39], see also [20, Propositions 3.12-15]. In case there is a component of type (iii), the second component $\mathsf{H}$ through the node separating $\mathsf{Z}$ from the rest of the curve cannot be of type (iii) (each component of type (iii) adds 1/2 to the age($\mathsf{a}$)). By the same argument we should rule out $\mathsf{H}$ of type (ii). On the other hand $\mathsf{H}$ cannot be an elliptic tail because $g(\mathsf{C}) \geq 4$. Finally $\mathsf{H}$ cannot be an identity component because, this yields a 1/2-age contribution due to the parameter smoothing the node $\mathsf{H} \cap \mathsf{Z}$. As a consequence, case (iii) is impossible.

Let us assume that there is a component $\mathsf{Z}$ of type (ii), that is a so-called elliptic ladder. Since such components contribute at least 1/2 to age($\mathsf{a}$) we assume there is exactly one such case. We argue as in Step 5 where we have replaced $\mathsf{a}$ by another junior automorphism

operating as the identity on the elliptic tails. In this way, we have $n = \mathfrak{ord}(\mathsf{r}) = \mathfrak{ord}(\mathsf{a})$. If $\mathsf{a}^{\mathfrak{ord}(\mathsf{a})}$ is a nontrivial ghost, then it is senior, because $(\mathsf{C}, \mathsf{L}, \phi)$ is not a J-curve; by Lemma 2.46,(2) we have $\mathrm{age}(\mathsf{a}) \geq 1/n$. The same inequality holds, by Lemma 2.46 (1), if $\mathsf{a}^{\mathfrak{ord}(\mathsf{a})}$ is trivial, *i.e.* if $\mathfrak{ord}(\mathsf{a}) = \mathrm{ord}(\mathsf{a})$ (since $g \geq 4$ there is at least one fixed node in $\mathsf{C} \setminus \{\text{elliptic tails}\}$). Now, for $n = 2$, the total age contribution is $1/2+1/2$ and, for $n = 4$, the total age contribution if $3/4 + 1/4$. We may rule out this case.

Now the coarsening of $\mathsf{a}$ is the identity on all components that are not elliptic tails. In fact $\mathsf{a}$ is actually the identity on all such components; if this were not the case, we could replace $\mathsf{a}$ by a junior ghost automorphism of $(\mathsf{C}, \mathsf{L}, \phi)$ contradicting the assumption that $(\mathsf{C}, \mathsf{L}, \phi)$ is not a J-curve. So, $\mathsf{a}$ is the identity everywhere except from some scheme-theoretic elliptic tails. We can now go through the study of elliptic tails (i) and add the age contribution from the parameter smoothing the QR node where the tail meets the rest of the curve. As in [15] and [20] we conclude that $\mathsf{a}$ has order 6 and should operate on the elliptic tail precisely as prescribed by the statement of Proposition 2.44. □

By definition, noncanonical singularities are local obstructions to the extension of pluricanonical forms. On the other hand Harris and Mumford show that noncanonical singularities at T-curves do not pose a global obstruction: pluricanonical forms extend across the locus $\mathsf{T}$ of level curves of type T as soon as they are globally defined off of $\mathsf{T}$. Their statement can immediately adapted to level curves (the argument is spelled out in [12, Thm. 6.1] and [20, Thm 4.1] and relies on the fact that the morphism forgetting the level structure is not ramified along $\delta_{g-1}$). The precise statement is as follows.

**Corollary 2.48.** *We fix $g \geq 4$ and $5 \neq \ell \leq 6$. Let $\widehat{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{R}}_{g,\ell}$ be any desingularisation. Then every pluricanonical form defined on the smooth locus $(\overline{\mathcal{R}}_{g,\ell})^{\mathrm{reg}}$ of $\overline{\mathcal{R}}_{g,\ell}$ extends holomorphically to $\widehat{\mathcal{R}}_{g,\ell}$, that is, for all integers $q \geq 0$ we have isomorphisms*

$$\Gamma\big((\overline{\mathcal{R}}_{g,\ell})^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_{g,\ell}}^{\otimes q}\big) \cong \Gamma\big(\widehat{\mathcal{R}}_{g,\ell}, K_{\widehat{\mathcal{R}}_{g,\ell}}^{\otimes q}\big).$$

□

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