$\overline{\mathcal{M}}_{22}$ IS OF GENERAL TYPE

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1. INTRODUCTION

The aim of this paper is to prove the following result:

Theorem: *The moduli space of curves of genus* 22 *is of general type.*

We recall that it is a famous result due to Harris, Mumford and Eisenbud that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$ (cf. [HM], [H1], [EH]). On the other hand, a classical theorem due to Severi says $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$ (see [AC1] for a modern exposition). Similar results for higher g were proved in the 1980's: Sernesi showed that $\overline{\mathcal{M}}_{12}$ is unirational (cf. [Se]), then Chang and Ran settled the unirationality of $\overline{\mathcal{M}}_{11}$ and $\overline{\mathcal{M}}_{13}$ and gave a different proof of Sernesi's result in genus 12 (cf. [CR1]). One also knows that $\overline{\mathcal{M}}_{15}$ and $\overline{\mathcal{M}}_{16}$ have negative Kodaira dimension (cf. [CR2]). The case of $\overline{\mathcal{M}}_{14}$ remained open for a long time until recently when Verra proved that $\overline{\mathcal{M}}_{14}$ is also unirational (cf. [V]). The highest genus not entirely covered by [EH] is $g = 23$. One knows that the Kodaira dimension of $\overline{\mathcal{M}}_{23}$ is ≥ 2 (cf. [F1]).

Closely related to the problem of determining the Kodaira dimension of $\overline{\mathcal{M}}_g$ is the Harris-Morrison Slope Conjecture which asserts that the slope of every effective divisor on $\overline{\mathcal{M}}_g$ is $\geq 6 + 12/(g+1)$ -this quantity being the slope of every Brill-Noether divisor $\overline{\mathcal{M}}_{g,d}^r$ of curves $[C] \in \mathcal{M}_g$ carrying a $\mathfrak{g}^r_d$ when $g - (r+1)(g-d+r) = -1$ (cf. [HMo], [EH]). Recalling that the canonical divisor on $\overline{\mathcal{M}}_g$ is given by the formula $K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]}$, the Slope Conjecture trivially implies that $\kappa(\overline{\mathcal{M}}_g) = -\infty$ for all $g \leq 22$. However, the Slope Conjecture fails for a large number of genera (see [FP], [F2], [F3], [Kh]) in the sense that there are examples of effective divisors on $\overline{\mathcal{M}}_g$ of slope $< 6 + 12/(g+1)$. This raised the prospect of constructing an effective divisor D on $\overline{\mathcal{M}}_g$ for some $g \leq 23$, having slope $s(D) < 13/2 = s(K_{\overline{\mathcal{M}}_g})$. In [F2] we came very close to succeeding in this when we showed that on $\overline{\mathcal{M}}_{22}$, the slope of the closure of the divisor $\mathcal{Z}_{22,2}$ consisting of curves $[C] \in \mathcal{M}_{22}$ for which there exists a linear series $L \in W^{10}_{30}(C)$ such that $C \overset{|L|}{\hookrightarrow} \mathbf{P}^{10}$ fails to satisfy the Green-Lazarsfeld property (N_2), is equal to $s(\overline{\mathcal{Z}}_{22,2}) = 1665/256 = 6.503\ldots < 6 + 12/23$.

In this paper we construct an effective divisor on $\overline{\mathcal{M}}_{22}$ of slope $< 13/2$ and prove the following result:

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Theorem 1.1. *The following locus of smooth curves of genus* 22

$$\mathfrak{D}_{22} := \{[C] \in \mathcal{M}_{22} : \exists L \in W^6_{25}(C) \text{ with } \mathrm{Sym}^2 H^0(C,L) \to H^0(C, L^{\otimes 2}) \text{ not injective}\}$$

is a divisor on $\mathcal{M}_{22}$ *and the class of its compactification on* $\overline{\mathcal{M}}_{22}$ *is given by the formula:*

$$\overline{\mathfrak{D}}_{22} \equiv 132822768\Big(\frac{17121}{2636}\lambda - \delta_0 - \frac{14511}{2636}\delta_0 - \sum_{j=2}^{11} b_j \delta_j\Big),$$

where $b_j > 1$ *for* $2 \leq j \leq 11$. *It follows that* $s(\overline{\mathfrak{D}}_{22}) = 17121/2636 = 6.49506\ldots$, *hence* $\overline{\mathcal{M}}_{22}$ *is of general type.*

2. THE DIVISOR $\mathfrak{D}_{22}$

In this section we construct two tautological vector bundles over the Severi variety of plane curves of genus 22 and degree 17 and define the divisor $\mathfrak{D}_{22}$ as the image of the first degeneration locus of a natural map between these bundles.

We denote by $\mathcal{M}^0_{22}$ the open substack of $\mathcal{M}_{22}$ consisting of curves $[C] \in \mathcal{M}_{22}$ such that $W^6_{24}(C) = \emptyset$ and $W^7_{25}(C) = \emptyset$. Standard results in Brill-Noether theory guarantee that codim$(\mathcal{M}_{22} - \mathcal{M}^0_{22}, \mathcal{M}_{22}) \geq 3$. If $\mathfrak{Pic}^{25}_{22}$ denotes the Picard stack of degree 25 over $\mathcal{M}^0_{22}$ (that is, the étale sheafification of the Picard functor), then we consider the substack $\mathfrak{G}^6_{25} \subset \mathfrak{Pic}^{25}_{22}$ parametrizing pairs $[C,L]$ where $[C] \in \mathcal{M}^0_{22}$ and $L \in W^6_{25}(C)$. We denote by $\sigma : \mathfrak{G}^6_{25} \to \mathcal{M}^0_{22}$ the forgetful morphism. Note that if $L \in W^6_{25}(C)$ then $K_C \otimes L^\vee \in W^2_{17}(C)$ and since the classical Severi variety of plane curves of given degree and genus is irreducible (cf. [H2]), it follows that $\mathfrak{G}^6_{25}$ is irreducible as well. For a general $[C] \in \mathcal{M}^0_{22}$, the fibre $\sigma^{-1}([C]) = W^6_{25}(C)$ is a smooth curve and $\mathfrak{G}^6_{25}$ is an irreducible stack of dimension dim$(\mathfrak{G}^6_{25}) =$ dim$(\mathcal{M}_{22}) + 1$. Moreover, $\mathfrak{G}^6_{25}$ is smooth at a point $[C,L]$ if an only if $H^1(C, N_f) = 0$, where N_f is normal line bundle of the map $f : \stackrel{|K_C \otimes L^\vee|}{\longrightarrow} \mathbf{P}^2$. As explained in [AC2], Proposition 2.9, the vanishing condition $H^1(C, N_f) = 0$ is satisfied outside a subset of $\mathfrak{G}^6_{25}$ of dimension $\leq g - 8 = 14$, hence for the purpose of codimension ≤ 2 calculations we carry out in this paper, we can work with $\mathfrak{G}^6_{25}$ as if it was a smooth stack.

We denote by $\pi : \mathcal{M}^0_{22,1} \to \mathcal{M}^0_{22}$ the universal curve over the moduli stack and by $p_2 : \mathcal{M}^0_{22,1} \times_{\mathcal{M}^0_{22}} \mathfrak{G}^6_{25} \to \mathfrak{G}^6_{25}$ the natural projection. If $\mathcal{L}$ is the Poincaré bundle over $\mathcal{M}^0_{22,1} \times_{\mathcal{M}^0_{22}} \mathfrak{G}^6_{25}$, then by Grauert's Theorem $\mathcal{E} := (p_2)_*(\mathcal{L})$ and $\mathcal{F} := (p_2)_*(\mathcal{L}^{\otimes 2})$ are vector bundles over $\mathfrak{G}^6_{25}$ with rank$(\mathcal{E}) = 7$ and rank$(\mathcal{F}) = 29$. There is a natural vector bundle map $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ and we denote by $\mathcal{U}_{22} \subset \mathfrak{G}^6_{25}$ its first degeneracy locus. We set $\mathfrak{D}_{22} := \sigma_*(\mathcal{U}_{22})$ and clearly $\mathcal{U}_{22}$ has expected codimension 2 inside $\mathfrak{G}^6_{25}$ hence $\mathfrak{D}_{22}$ is a virtual divisor on $\mathcal{M}_{22}$.

We shall extend the vector bundles $\mathcal{E}$ and $\mathcal{F}$ over a partial compactification of $\mathfrak{G}^6_{25}$. We denote by $\Delta^0_1 \subset \Delta_1 \subset \overline{\mathcal{M}}_{22}$ the locus of curves $[C \cup_y E]$, where E is an arbitrary elliptic curve, $[C] \in \mathcal{M}_{21}$ is a Brill-Noether general curve of genus 21 and $y \in C$ is an arbitrary point. We also denote by $\Delta^0_0 \subset \Delta_0 \subset \overline{\mathcal{M}}_{22}$ the locus consisting of curves $C/y \sim q$, where $[C, q] \in \mathcal{M}_{21,1}$ is Brill-Noether general and $y \in C$ is arbitary, as well as their degenerations $[C \cup_q E_\infty]$ where E_∞ is a rational nodal curve (that is, $j(E_\infty) = \infty$). Once we set $\widetilde{\mathcal{M}}_{22} := \mathcal{M}_{22}^0 \cup \Delta_0^0 \cup \Delta_1^0$, one can extend the map $\sigma : \mathfrak{G}_{25}^6 \to \mathcal{M}_{22}^0$ to a proper map $\sigma : \widetilde{\mathfrak{G}}_{25}^6 \to \widetilde{\mathcal{M}}_{22}$ from the variety $\widetilde{\mathfrak{G}}_{25}^6$ of (generalized) limit linear series $\mathfrak{g}_{25}^6$ over the tree-like curves from $\widetilde{\mathcal{M}}_{22}$.

Like in [F2], [F3], our technique for determining the class of the divisor $\overline{\mathfrak{D}}_{22}$ is to intersect it with two standard test curves sitting in the boundary of $\overline{\mathcal{M}}_{22}$: we fix a general pointed curve $[C, q] \in \mathcal{M}_{21,1}$ and a general elliptic curve $[E, y] \in \mathcal{M}_{1,1}$. Then we define the families

$$C^0 := \{C/y \sim q : y \in C\} \subset \Delta_0 \subset \overline{\mathcal{M}}_{22} \text{ and } C^1 := \{C \cup_y E : y \in C\} \subset \Delta_1 \subset \overline{\mathcal{M}}_{22}.$$

These curves intersect the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{22})$ as follows:

$$C^0 \cdot \lambda = 0,\ C^0 \cdot \delta_0 = -42,\ C^0 \cdot \delta_1 = 1 \text{ and } C^0 \cdot \delta_j = 0 \text{ for } 2 \le j \le 11, \text{ and}$$
$$C^1 \cdot \lambda = 0,\ C^1 \cdot \delta_0 = 0,\ C^1 \cdot \delta_1 = -40 \text{ and } C^1 \cdot \delta_j = 0 \text{ for } 2 \le j \le 11.$$

Before we state the next results, we recall that if X is a stable curve whose dual graph is a tree and l is a limit $\mathfrak{g}_d^r$ on X, for an irreducible component Y of X, we denote by $l_Y = (L_Y, V_Y \subset H^0(L_Y))$ the Y-aspect of l. For $y \in Y$ we denote by $\{a_s^{l_Y}(C)\}_{s=0\ldots r}$ the *vanishing sequence* of l at y and by $\rho(l_Y, y) := \rho(g, r, d) - \sum_{i=0}^r (a_i^{l_Y}(y) - i)$ the adjusted Brill-Noether number with respect to the point y.

Proposition 2.1. *Fix general curves $[C] \in \mathcal{M}_{21}$ and $[E, y] \in \mathcal{M}_{1,1}$ and consider the associated test curve $C^1 \subset \Delta_1 \subset \overline{\mathcal{M}}_{22}$. Then we have the following equality of 2-cycles in $\widetilde{\mathfrak{G}}_{25}^6$:*

$$\sigma^*(C^1) = X + X_1 \times X_2 + \Gamma_0 \times Z_0 + n_1 \cdot Z_1 + n_2 \cdot Z_2 + n_3 \cdot Z_3,$$

where

$$X := \{(y, L) \in C \times W_{25}^6(C) : h^0(L \otimes \mathcal{O}_C(-2y)) = 6\},$$
$$X_1 := \{(y, L) \in C \times W_{25}^6(C) : a^L(y) = (0, 2, 3, 4, 5, 6, 8)\},$$
$$X_2 := \{l_E \in G_8^6(E) : a_1^{l_E}(y) \ge 2, a_6^{l_E}(y) = 8\} \cong \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E(8y))}{H^0(\mathcal{O}_E(6y))}\Big)$$
$$\Gamma_0 := \{(y, A \otimes \mathcal{O}_C(y)) : y \in C, A \in W_{24}^6(C)\},\ Z_0 = G_7^6(E) \cong E,$$
$$Z_1 := \{l_E \in G_9^6(E) : a_1^{l_E}(y) \ge 3, a_6^{l_E}(y) = 9\} \cong \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E(9y))}{H^0(\mathcal{O}_E(6y))}\Big),$$
$$Z_2 := \{l_E \in G_8^6(E) : a_2^{l_E}(y) \ge 3, a_6^{l_E}(y) = 8\} \cong \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E(8y))}{H^0(\mathcal{O}_E(5y))}\Big),$$
$$Z_3 := \{l_E \in G_8^6(E) : a^{l_E}(y) \ge (0, 2, 3, 4, 5, 6, 7)\} \cong \bigcup_{z \in E} \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E(7y + z))}{H^0(\mathcal{O}_E(5y + z))}\Big),$$

where the constants n_1, n_2 and n_3 are explicitly known positive integers.

Remark 2.2. The constants $n_i, 1 \le i \le 3$ have the following enumerative interpretation. First n_1 is the number of linear series $L \in W_{25}^6(C)$ such that there exists an unspecified point $y \in C$ with $a^L(y) = (0, 2, 3, 4, 5, 6, 9)$. Similarly, n_2 is the number of those $L \in W_{25}^6(C)$ for which there exists $y \in C$ with $a^L(y) = (0, 2, 3, 4, 5, 7, 8)$. Finally n_3 is the number of points $y \in C$ such that there exists $L \in W_{24}^6(C)$ which is ramified at y. If n_0 is the number of $\mathfrak{g}_{24}^6$'s on C, then Γ_0 consists of n_0 disjoint copies of the curve C.

Proof. By the additivity of the Brill-Noether number, if $\{l_C, l_E\}$ is a limit $\mathfrak{g}^6_{25}$ on $C \cup_y E$, we have that $1 = \rho(22, 6, 25) \geq \rho(l_C, y) + \rho(l_E, y)$. Since $\rho(l_E, y) \geq 0$, we obtain that $\rho(l_C, y) \leq 1$. If $\rho(l_E, y) = 0$, then $l_E = 18y + |\mathcal{O}_E(7y)|$, that is, l_E is uniquely determined, while the C-aspect l_C is a complete $\mathfrak{g}^6_{25}$ with a cusp at the variable point $y \in C$. This gives rise to an element from X. In the case when $\rho(l_C, y) = 0$ and $\rho(l_E, y) = 1$, we obtain that the underlying line bundle L_C of l_C either has a base point at y and then $(y, L_C) \in \Gamma_0$, or else, L_C belongs to the curve X_1 and $l_E(-17y)$ is a $\mathfrak{g}^6_8$ on E having vanishing sequence $\geq (0, 2, 3, 4, 5, 6, 8)$, that is, $l_E(-17y)$ is an element of X_2. Finally, we have to consider the case when $\rho(l_C, y) = -1$ and $\rho(l_E, y) = 2$. There are a finite number of such points $y \in C$, and running through all the possibilities we obtain the components Z_i for $1 \leq i \leq 3$. □

Before stating our next result we introduce some notation. We fix a general pointed curve $[C, q] \in \mathcal{M}_{21,1}$ and denote by Y the following surface:

$$Y := \{(y, L) \in C \times W^6_{25}(C) : h^0(C, L \otimes \mathcal{O}_C(-y - q)) = 6\}$$

and by $\pi_1 : Y \to C$ the first projection. Inside Y we consider two curves corresponding to $\mathfrak{g}^6_{25}$'s with a base point at q:

$$\Gamma_1 := \{(y, A \otimes \mathcal{O}_C(y)) : y \in C, A \in W^6_{24}(C)\} \text{ and } \Gamma_2 := \{(y, A \otimes \mathcal{O}_C(q)) : y \in C, A \in W^6_{24}(C)\}$$

intersecting transversally in $n_0 = |W^6_{24}(C)|$ points (Note that sine $[C] \in \mathcal{M}_{21}$ is Brill-Noether general, $W^6_{24}(C)$ is a reduced 0-dimensional scheme consisting of n_0 very ample (in particular, base point free) $\mathfrak{g}^6_{24}$'s). We denote by Y' the blow-up of Y at these n_0 points and at the points $(q, B) \in Y$ where $B \in W^6_{25}(C)$ is a linear series with the property that $h^0(C, B \otimes \mathcal{O}_C(-8q)) \geq 1$. We denote by $E_A, E_B \subset Y'$ the exceptional divisors corresponding to $(q, A \otimes \mathcal{O}_C(q))$ and (q, B) respectively, by $\epsilon : Y' \to Y$ the projection and by $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2 \subset Y'$ the strict transforms of Γ_1 and Γ_2.

Proposition 2.3. *Fix a general curve $[C, q] \in \mathcal{M}_{21,1}$ and consider the associated test curve $C^0 \subset \Delta_0 \subset \overline{\mathcal{M}}_{22}$. Then we have the following equality of 2-cycles in $\widetilde{\mathfrak{G}}^6_{25}$:*

$$\sigma^*(C^0) = Y'/\widetilde{\Gamma}_1 \cong \widetilde{\Gamma}_2,$$

that is, $\sigma^*(C^0)$ can be naturally identified with the surface obtained from Y' by identifying the disjoint curves $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ over each pair $(y, A) \in C \times W^6_{24}(C)$.

Proof. We fix a point $y \in C - \{q\}$, denote by $[C^0_y := C/y \sim q] \in \overline{\mathcal{M}}_{22}$, $\nu : C \to C^y_0$ the normalization map, and we investigate the variety $\overline{W}^6_{25}(C^0_y) \subset \overline{\mathrm{Pic}}^{25}(C^0_y)$ of torsion-free sheaves L on C^0_y with $\deg(L) = 25$ and $h^0(C^0_y, L) \geq 7$. If $L \in W^6_{25}(C^0_y)$, that is, L is locally free, then L is determined by $\nu^*(L) \in W^6_{25}(C)$ which has the property that $h^0(C, \nu^* L \otimes \mathcal{O}_C(-y - q)) = 6$. However, the line bundles of type $A \otimes \mathcal{O}_C(y)$ or $A \otimes \mathcal{O}_C(q)$ with $A \in W^6_{24}(C)$, do not appear in this association even though they have this property. In fact they correspond to the situation when $L \in \overline{W}^6_{25}(C^y_0)$ is not locally free, in which case necessarily $L = \nu_*(A)$ for some $A \in W^6_{24}(C)$. Thus $Y \cap \pi_1^{-1}(y)$ is the partial normalization of $\overline{W}^6_{25}(C^0_y)$ at the n_0 points of the form $\nu_*(A)$ with $A \in W^6_{24}(C)$. A special analysis is required when $y = q$, that is, when C^0_y degenerates to $C \cup_q E_\infty$, where E_∞ is a rational nodal cubic. If $\{l_C, l_{E_\infty}\} \in \sigma^{-1}([C \cup_q E_\infty])$, then an analysis along the lines of Theorem 2.1 shows that $\rho(l_C, q) \geq 0$ and $\rho(l_{E_\infty}, q) \leq 1$. Then either l_C has a base point at q and then the underlying line bundle of l_C is of type $A \otimes \mathcal{O}_C(q)$ while $l_{E_\infty}(-18q) \in \overline{W}^6_7(E_\infty)$, or else, $a^{l_C}(q) = (0, 2, 3, 4, 5, 6, 8)$ and then $l_{E_\infty}(-17q) \in \mathbf{P}\big(H^0(E_\infty(8q))/H^0(E_\infty(6q))\big) \cong E_B$, where $B \in W^6_{25}(C)$ is the underlying line bundle of l_C. $\square$

Throughout the paper we use a few facts about intersection theory on Jacobians which we briefly recall (see [ACGH] for a general reference). If $[C] \in \mathcal{M}_g$ is Brill-Noether general, we denote by $\mathcal{P}$ the Poincaré bundle on $C \times \mathrm{Pic}^d(C)$ and by $\pi_1 : C \times \mathrm{Pic}^d(C) \to C$ and $\pi_2 : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ the projections. We define the cohomology class $\eta = \pi_1^*([point]) \in H^2(C \times \mathrm{Pic}^d(C))$, and if $\delta_1, \ldots, \delta_{2g} \in H^1(C, \mathbb{Z}) \cong H^1(\mathrm{Pic}^d(C), \mathbb{Z})$ is a symplectic basis, then we set

$$\gamma := -\sum_{\alpha=1}^{g} \Big(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Big).$$

We have the formula $c_1(\mathcal{P}) = d\eta + \gamma$, corresponding to the Hodge decomposition of $c_1(\mathcal{P})$. We also record that $\gamma^3 = \gamma\eta = 0$, $\eta^2 = 0$ and $\gamma^2 = -2\eta\pi_2^*(\theta)$. On $W^r_d(C)$ we have the tautological rank $r+1$ vector bundle $\mathcal{M} := (\pi_2)_*(\mathcal{P}_{|C \times W^r_d(C)})$. The Chern numbers of $\mathcal{M}$ can be computed using the Harris-Tu formula (cf. [HT]): if we write $\sum_{i=0}^r c_i(\mathcal{M}^\vee) = (1 + x_1) \cdots (1 + x_{r+1})$, then for every class $\zeta \in H^*(\mathrm{Pic}^d(C), \mathbb{Z})$ one has the following formula:

$$x_1^{i_1} \cdots x_{r+1}^{i_{r+1}}\, \zeta = \det\Big(\frac{\theta^{g+r-d+i_j-j+l}}{(g+r-d+i_j-j+l)!}\Big)_{1 \leq j,l \leq r+1}\, \zeta.$$

If we use the expression of the Vandermonde determinant, we get the identity

$$\det\Big(\frac{1}{(a_j+l-1)!}\Big)_{1 \leq j,l \leq r+1} = \frac{\Pi_{j>l}\,(a_l - a_j)}{\Pi_{j=1}^{r+1}\,(a_j+r)!}.$$

By repeatedly applying this we get all intersection numbers on $W^r_d(C)$ which we'll need:

Lemma 2.4. *If $[C] \in \mathcal{M}_{21}$ is Brill-Noether general and $c_i := c_i(\mathcal{M}^\vee)$ are the Chern classes of the dual of the tautological bundle on $W^2_{17}(C)$, we have the following identities in $H^*(W^2_{17}(C), \mathbb{Z})$:*

$$[W^2_{17}(C)] = \frac{\theta^{18}}{73156608000}.$$

$$x_1 \cdot \xi = \frac{\theta^{19} \cdot \xi}{219469824000},\ x_2 \cdot \xi = x_3 \cdot \xi = 0,\ \textit{for any } \xi \in H^4(\mathrm{Pic}^{21}(\mathrm{C})).$$

$$x_1x_2 \cdot \xi = \frac{\theta^{20}}{1755758592000} \cdot \xi,\ x_1x_3 \cdot \xi = x_2x_3 \cdot \xi = 0,\ \textit{for any } \xi \in H^2(\mathrm{Pic}^{21}(\mathrm{C})).$$

$$x_1^2 \cdot \xi = \frac{\theta^{20}}{1097349120000} \cdot \xi,\ x_2^2 \cdot \xi = -x_1x_2 \cdot \xi,\ x_3^2 \cdot \xi = 0,\ \textit{for any } \xi \in H^2(\mathrm{Pic}^{21}(\mathrm{C})).$$

$$x_1^3 = \frac{\theta^{21}}{7242504192000},\ x_2^3 = -\frac{t^{21}}{6584094720000},\ x_3^3 = x_1x_2x_3 = \frac{\theta^{21}}{36870930432000}.$$

$$x_1^2x_2 = -x_2^3,\ x_1x_2^2 = x_1^2x_3 = x_2x_3^2 = 0,\ x_1x_3^2 = x_2^2x_3 = -x_1x_2x_3.$$

We are going to extend the vector bundles $\mathcal{E}$ and $\mathcal{F}$ over the space $\widetilde{\mathfrak{G}}^6_{25}$ of limit linear series:

Proposition 2.5. *There exist two vector bundles $\mathcal{E}$ and $\mathcal{F}$ defined over $\widetilde{\mathfrak{G}}^6_{25}$ with $\mathrm{rank}(\mathcal{E}) = 7$ and $\mathrm{rank}(\mathcal{F}) = 29$ together with a vector bundle morphism $\mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$, such that the following statements hold:*

- *For $(C, L) \in \mathfrak{G}^6_{25}$ we have that $\mathcal{E}(L) = H^0(C, L)$ and $\mathcal{F}(L) = H^0(C, L^{\otimes 2})$.*
- *For $t = (C \cup_y E, l_C, l_E) \in \sigma^{-1}(\Delta_1^0)$, where $g(C) = 21, g(E) = 1$ and $l_C = |L_C|$ is such that $L_C \in W^6_{25}(C)$ has a cusp at $y \in C$, then $\mathcal{E}(t) = H^0(C, L_C)$ and*
$$\mathcal{F}(t) = H^0(C, L_C^{\otimes 2}(-2y)) \oplus \mathbb{C} \cdot u^2,$$
where $u \in H^0(C, L_C)$ is any section such that $\mathrm{ord}_y(u) = 0$. If L_C has a base point at y, then $\mathcal{E}(t) = H^0(C, L_C) = H^0(C, L_C \otimes \mathcal{O}_C(-y))$ and the image of a natural map $\mathcal{F}(t) \to H^0(C, L_C^{\otimes 2})$ is the subspace $H^0(C, L_C^{\otimes 2} \otimes \mathcal{O}_C(-2y))$.
- *Fix $t = (C^0_y := C/y \sim q, L) \in \sigma^{-1}(\Delta_0^0)$, with $q, y \in C$ and $L \in \overline{W}^6_{25}(C^0_y)$ such that $h^0(C, \nu^* L \otimes \mathcal{O}_C(-y-q)) = 6$, where $\nu : C \to C^0_y$ is the normalization map. In the case when L is locally free we have that*
$$\mathcal{E}(t) = H^0(C, \nu^* L) \ \text{ and } \ \mathcal{F}(t) = H^0(C, \nu^* L^{\otimes 2} \otimes \mathcal{O}_C(-y-q)) \oplus \mathbb{C} \cdot u^2,$$
where $u \in H^0(C, \nu^* L)$ is any section not vanishing at y and q. In the case when L is not locally free, that is, $L \in \overline{W}^6_{25}(C^y_0) - W^6_{25}(C^y_0)$, then $L = \nu_*(A)$, where $A \in W^6_{24}(C)$ and the image of the natural map $\mathcal{F}(t) \to H^0(C, \nu^* L^{\otimes 2})$ is the subspace $H^0(C, A^{\otimes 2})$.

We determine the cohomology classes of the surfaces X and Y introduced in Propositions 2.3 and 2.1.

Proposition 2.6. *Let $[C] \in \mathcal{M}_{21}$ be a Brill-Noether general curve and $q \in C$ a general point. If $\mathcal{M}$ denotes the tautological rank 3 vector bundle over $W^2_{17}(C)$ and $c_i := c_i(\mathcal{M}^\vee)$, then one has the following relations:*

(1) $[X] = \pi_2^*(c_2) - 6\eta\theta + (74\eta + 2\gamma)\pi_2^*(c_1) \in H^4(C \times W^2_{17}(C))$.
(2) $[Y] = \pi_2^*(c_2) - 2\eta\theta + (16\eta + \gamma)\pi_2^*(c_1) \in H^4(C \times W^2_{17}(C))$.

Proof. By Riemann-Roch, if $(y, L) \in X$, then the line bundle $M := K_C \otimes L^\vee \otimes \mathcal{O}_C(2y) \in W^2_{17}(C)$ has a cusp at y. We realize X as the degeneracy locus of a vector bundle map over $C \times W^2_{17}(C)$. For each pair $(y, M) \in C \times W^2_{17}(C)$, there is a natural map
$$H^0(C, M \otimes \mathcal{O}_{2y})^\vee \to H^0(C, M)^\vee$$
which globalizes to a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ over $C \times W^2_{17}(C)$ (Note that $W^2_{17}(C)$ is a smooth 3-fold). Then we have the identification $X = Z_1(\zeta)$ and the Thom-Porteous formula gives that $[X] = c_2\big(\pi_2^*(\mathcal{M}) - J_1(\mathcal{P}^\vee)\big)$. From the usual exact sequence over $C \times \mathrm{Pic}^{17}(C)$
$$0 \longrightarrow \pi_1^*(K_C) \otimes \mathcal{P} \longrightarrow J_1(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow 0,$$
we can compute the total Chern class of the jet bundle
$$c_t(J_1(\mathcal{P})^\vee) = \Big(\sum_{j \geq 0} (17\eta + \gamma)^j\Big) \cdot \Big(\sum_{j \geq 0} (57\eta + \gamma)^j\Big) = 1 - 6\eta\theta + 74\eta + 2\gamma,$$
which quickly leads to the formula for $[X]$. To compute $[Y]$ we proceed in a similar way. We denote by $p_1, p_2 : C \times C \times \mathrm{Pic}^{17}(C) \to C \times \mathrm{Pic}^{17}(C)$ the two projections, by

$\Delta \subset C \times C \times \mathrm{Pic}^{17}(C)$ the diagonal and we set $\Gamma_q := \{q\} \times \mathrm{Pic}^{17}(C)$. We introduce the rank 2 vector bundle $\mathcal{B} := (p_1)_*\big(p_2^*(\mathcal{P}) \otimes \mathcal{O}_{\Delta + p_2^*(\Gamma_q)}\big)$ defined over $C \times W^2_{17}(C)$ and we note that there is a bundle morphism $\chi : \mathcal{B}^\vee \to (\pi_2)^*(\mathcal{M})^\vee$ such that $Y = Z_1(\chi)$. Since we also have that

$$c_t(\mathcal{B}^\vee)^{-1} = \big(1 + (17\eta + \gamma) + (17\eta + \gamma)^2 + \cdots\big)(1 - \eta),$$

we immediately obtained the desired expression for $[Y]$. $\square$

For future reference we also record the following formulas:

(1) $$c_3(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) = \pi_2^*(c_3) - 6\eta\theta\pi_2^*(c_1) + (74\eta + 2\gamma)\pi_2^*(c_2) \text{ and}$$

(2) $$c_4(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) = \pi_2^*(c_3)(74\eta + 2\gamma) - 6\pi_2^*(c_2)\eta\theta.$$

Proposition 2.7. *Let $[C] \in \mathcal{M}_{21}$ be a Brill-Noether general curve and denote by $\mathcal{P}$ the Poincaré bundle on $C \times Pic^{17}(C)$. We have the following identities in $H^*(\mathrm{Pic}^{17}(C))$:*

$$c_1\big(R^1\pi_{2*}(\mathcal{P}_{|C \times W^2_{17}(C)})\big) = \theta - c_1 \ \textit{and} \ c_2\big(R^1\pi_{2*}(\mathcal{P}_{|C \times W^2_{17}(C)})\big) = \frac{\theta^2}{2} - \theta c_1 + c_2.$$

Proof. We recall that in order to obtain a determinantal structure on $W^2_{17}(C)$ one fixes a divisor $D \in C_e$ of degree $e >> 0$ and considers the morphism

$$(\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D)\big) \to (\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D_{|\pi_1^*D})\big).$$

Then $W^2_{17}(C)$ is the degeneration locus of rank $e - 6$ of this map and there is an exact sequence of vector bundles over $W^2_{17}(C)$:

$$0 \longrightarrow \mathcal{M} \longrightarrow (\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi^*D)\big) \longrightarrow (\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\big) \longrightarrow R^1\pi_{2*}\big(\mathcal{P}_{|C \times W^2_{17}(C)}\big) \longrightarrow 0,$$

from which our claim easily follow once we take into account that $(\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\big)$ is numerically trivial and $c_t\big((\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D))\big) = e^{-\theta}$. $\square$

Proposition 2.8. *Let $[C] \in \mathcal{M}_{21}$ and denote by $p_1, p_2 : C \times C \times \mathrm{Pic}^{17}(C) \to C \times \mathrm{Pic}^{17}(C)$ the natural projections. We denote by $\mathcal{A}_2$ the vector bundle on $C \times \mathrm{Pic}^{17}(C)$ with fibre at each point $\mathcal{A}_2(y, M) = H^0(C, K_C^{\otimes 2} \otimes M^{\otimes(-2)} \otimes \mathcal{O}_C(2y))$. We have the following formulas:*

$$c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 28\eta \ \textit{and} \ c_2(\mathcal{A}_2) = 8\theta^2 + 104\eta\theta + 16\gamma\theta.$$

Proof. Recall that if $M \in W^2_{17}(C)$ then by duality $L := K_C \otimes \mathcal{O}_C(2y) \otimes M^\vee \in W^6_{25}(C)$ is a linear series with a cusp at y. In this notation, $\mathcal{A}_2$ is the vector bundle with fibre $\mathcal{A}_2(y, M) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-2y))$. To compute $c_i(\mathcal{A}_2)$ we apply Grothendieck-Riemann-Roch to the map p_2. If $\nu_1 : C \times C \times \mathrm{Pic}^{17}(C) \to C$ denotes the projection onto the first coordinate, then one obtains that

$$ch\big(p_{2!}\big(\nu_1^*(K_C^{\otimes 2}) \otimes \mathcal{P}^{\otimes(-2)} \otimes \mathcal{O}(2\Delta)\big)\big) = ch(\mathcal{A}_2) =$$

$$= (p_2)_*\Big(ch\big(\nu_1^*(K_C^{\otimes 2}) \otimes p_1^*(\mathcal{P}^{\otimes(-2)}) \otimes \mathcal{O}(2\Delta)\big) \cdot (1 - \frac{1}{2}\nu_1^*(K_C))\Big)$$

and looking at terms of degree 2 and 3 one finds $c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 28\eta$ and $ch_2(\mathcal{A}_2) = -8\eta\theta$. $\square$

The next proposition is proved along the lines of Proposition 2.8:

Proposition 2.9. *Let $[C,q]\in\mathcal{M}_{21,1}$ be a general pointed curve an we denote by $\mathcal{B}_2$ the vector bundle on $C\times \mathrm{Pic}^{17}(C)$ having fibre $\mathcal{B}_2(y,M)=H^0\big(C,K_C^{\otimes 2}\otimes M^{\otimes(-2)}\otimes \mathcal{O}_C(y+q)\big)$ at each point $(y,M)\in C\times \mathrm{Pic}^{17}(C)$. Then we have that:*

$$c_1(\mathcal{B}_2)=-4\theta+7\eta-2\gamma \ \textit{and}\ c_2(\mathcal{B}_2)=8\theta^2-28\eta\theta+8\theta\gamma.$$

Now we prove that the virtual degeneracy locus $\overline{\mathfrak{D}}_{22}$ is an "honest" divisor on $\mathcal{M}_{22}$, that is, the vector bundle morphism $\mathrm{Sym}^2(\mathcal{E})\rightarrow\mathcal{F}$ is generically non-degenerate:

Theorem 2.10. *If $[C]\in\mathcal{M}_{22}$ is a sufficiently general smooth curve, then the multiplication map $\mathrm{Sym}^2H^0(L)\rightarrow H^0(L^{\otimes 2})$ is injective for every $L\in W^6_{25}(C)$. It follows that $\mathfrak{D}_{22}$ is a divisor on $\mathcal{M}_{22}$.*

Proof. Throughout this proof we use the set-up described in Section 4 of [FP] for understanding degenerations of multiplication maps on curves. We consider a degenerate curve $[S:=C\cup_q D]\in\Delta_8\subset\overline{\mathcal{M}}_{22}$ where (C,q) and (D,q) are suitable general pointed curves of genus 8 and 14 respectively. Suppose by contradiction that $[S]\in\overline{\mathfrak{D}}_{22}$. Then there exists a limit $\mathfrak{g}^6_{25}$ on S, say

$$l=\{l_C=(L_C,V_C\subset H^0(L_C)),\ l_D=(L_D,V_D\subset H^0(L_D)\}\in\sigma^{-1}([S]),$$

together with $\neq 0$ elements

$$\rho_C\in \mathrm{Ker}\{\mathrm{Sym}^2(V_C)\rightarrow H^0(L_C^{\otimes 2})\}\text{ and }\rho_D\in \mathrm{Ker}\{\mathrm{Sym}^2(V_D)\rightarrow H^0(L_D^{\otimes 2})\},$$

such that $\mathrm{ord}_q(\rho_C)+\mathrm{ord}_q(\rho_D)\geq 50=\deg(L_C)+\deg(L_D)$. Limit linear $\mathfrak{g}^6_{25}$'s on S are indexed by partitions $0\leq\beta_0\leq\beta_1\leq\ldots\leq\beta_6\leq 3$ such that $\sum_{i=0}^6\beta_i=g(C)=8$. If we pick such a partition, then $l_C(-11q)\in G^6_{14}(C)$ is a linear series with ramification sequence $(\beta_0,\ldots,\beta_6)$ at q, while $l_D(-5q)\in G^6_{20}(D)$ is a linear series with complementary ramification sequence $(3-\beta_6,\ldots,3-\beta_0)$ at q. We claim that by analyzing all the partitions $(\beta_i)_{0\leq i\leq 6}$ one can always choose (C,q) and (D,q) general enough such that either ρ_C or ρ_D must be 0. For simplicity, we carry this out only in a single case, the other being rather similar. Say we choose the partition $(\beta_i)_{0\leq i\leq 6}=(0,0,1,1,1,2,3)$. Then if $L_C\in\mathrm{Pic}^{14}(C)$ denotes the underlying line bundle of $l_C(-11q)$, we have that $a^{L_C}(q)=(0,1,3,4,5,7,9)$, whereas if $L_D\in\mathrm{Pic}^{20}(D)$ denotes the underlying line bundle of $l_D(-5q)$ then $a^{L_D}(q)=(0,2,4,5,6,8,9)$. A careful analysis similar to the one in the proof of Theorem 5.1 in [FP], shows that the only possible situation when ρ_C and ρ_D could satisfy the compatibility condition at q, is when $\mathrm{ord}_q(\rho_C)=9(=0+9=4+5)$. More precisely, if $W_C\subset\mathrm{Sym}^2H^0(L_C)$ denotes the 13-dimensional space of sections with order ≥ 13 at q, then

$$0\neq\rho_C\in\mathrm{Ker}\{W_C\rightarrow H^0(L_C^{\otimes 2}\otimes\mathcal{O}_C(-9q))\}$$

(Note that the kernel is indeed 1-dimensional if $[C,q]\in\mathcal{M}_{8,1}$ is sufficiently generic). By compatibility, then $\mathrm{ord}_q(\rho_D)\geq 9(=0+9=4+5)$. If $W_D\subset\mathrm{Sym}^2H^0(L_D)$ is the space of those ρ_D with $\mathrm{ord}_q(\rho_D)\geq 9$, then $\dim(W_D)=17$ and we have that

$$0\neq\rho_D\in\mathrm{Ker}\{W_D\rightarrow H^0(L_D^{\otimes 2}\otimes\mathcal{O}_D(-9q))\},$$

where the target is 18-dimensional. One has to show that this is indeed a divisorial condition on $\mathcal{M}_{14,1}$, which can be seen by further degenerating $[D,q]$ to reducible curves of smaller genus.

□

As a first step towards computing $[\overline{\mathfrak{D}}_{22}]$ we determine the δ_1 coefficient in its expression:

Theorem 2.11. *Let $[C] \in \mathcal{M}_{21}$ be Brill-Noether general and denote by $C^1 \subset \Delta_1$ the associated test curve. Then $\sigma^*(C^1) \cdot c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})) = 4847375988$. It follows that the coefficient of δ_1 in the expansion of $\overline{\mathfrak{D}}_{22}$ is equal to $b_1 = 731180268$.*

Proof. We intersect the degeneracy locus of the map $\mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the surface $\sigma^*(C^1)$ and use that the vector bundles $\mathcal{E}$ and $\mathcal{F}$ were defined by retaining the sections of the genus 21 aspect of each limit linear series and dropping the information coming from the elliptic curve. It follows that $Z_i \cdot c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})) = 0$ for $1 \le i \le 3$ (since $\mathcal{F}$ and $Sym^2(\mathcal{E})$) are both trivial along the surfaces Z_i), and $[X_1 \times X_2] \cdot c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})) = 0$ (because $c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))_{|X_1 \times X_2}$ is in fact the pull-back of a codimension 2 class from the 1-dimensional cycle X_1, therefore the intersection number is 0 for dimensional reasons). We are left with estimating the contribution coming from X and we write

$$\sigma^*(C^1) \cdot c_2(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})) = c_2(\mathcal{F}_{|X}) - c_1(\mathcal{F}_{|X}) c_1(\mathrm{Sym}^2 \mathcal{E}_{|X}) + c_1^2(\mathrm{Sym}^2 \mathcal{E}_{|X}) - c_2(\mathrm{Sym}^2 \mathcal{E}_{|X})$$

and we are going to compute each term in the right-hand-side of this expression.

Recall that we have constructed in Proposition 2.6 a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ and we denote by $U := \mathrm{Ker}(\zeta)$. In other words, U is a line bundle on X with fibre

$$U(y, M) = \frac{H^1(C, M \otimes \mathcal{O}_C(-2y))^\vee}{H^1(C, M)^\vee} = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-2y))}$$

over a point $(y, M) \in X$. The Chern class of U can be computed from the Harris-Tu formula and we find that (cf. (1)):

$$c_1(U) \cdot \xi_{|X} = -c_3(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) \cdot \xi_{|X} = -(\pi_2^*(c_3) - 6\eta\theta\pi_2^*(c_1) + (74\eta + 2\gamma)\pi_2^*(c_2)) \cdot \xi_{|X},$$

for any class $\xi \in H^2(C \times W^2_{17}(C))$, and

$$c_1^2(U) = c_4(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) = \pi_2^*(c_3)(74\eta + 2\gamma) - 6\pi_2^*(c_2)\eta\theta.$$

If $\mathcal{A}_3$ denotes the rank 30 vector bundle on X having fibres

$$\mathcal{A}_3(y, M) = H^0(C, L^{\otimes 2}) = H^0(C, K_C^{\otimes 2} \otimes M^{\otimes(-2)} \otimes \mathcal{O}_C(4y)),$$

then there is an injective bundle morphism $U^{\otimes 2} \hookrightarrow \mathcal{A}_3/\mathcal{A}_2$ and we consider the quotient sheaf

$$\mathcal{G} := \frac{\mathcal{A}_3/\mathcal{A}_2}{U^{\otimes 2}}$$

We note that since the morphism $U^{\otimes 2} \to \mathcal{A}_3/\mathcal{A}_2$ vanishes along the curve Γ_0 corresponding to pairs (y, M) where M has a base point, $\mathcal{G}$ has torsion along Γ_0. A straightforward local analysis now shows that $\mathcal{F}_{|X}$ can be identified as a subsheaf of $\mathcal{A}_3$ with the kernel of the map $\mathcal{A}_3 \to \mathcal{G}$. Therefore, there is an exact sequence of vector bundles on X

$$0 \longrightarrow \mathcal{A}_{2|X} \longrightarrow \mathcal{F}_{|X} \longrightarrow U^{\otimes 2} \longrightarrow 0,$$

which over a generic point of X corresponds to the decomposition

$$\mathcal{F}(y,M)=H^0(C,L^{\otimes 2}\otimes \mathcal{O}_C(-2y))\oplus \mathbb{C}\cdot u^2,$$

where $u\in H^0(C,L)$ is such that $\mathrm{ord}_y(u)=1$ (The analysis above, shows that the sequence stays exact over Γ_0 as well). Hence $c_1(\mathcal{F}_{|X})=c_1(\mathcal{A}_{2|X})+2c_1(U)$ and $c_2(\mathcal{F}_{|X})=c_2(\mathcal{A}_{2|X})$. Furthermore, we note that the vector bundle $\pi_2^*\big(R^1\pi_{2*}(\mathcal{P})\big)^{\vee}_{|X}$ is a subbundle of $\mathcal{E}_{|X}$ and we have an exact sequence

$$0\longrightarrow \pi_2^*\big(R^1\pi_{2*}(\mathcal{P})\big)^{\vee}_{|X}\longrightarrow \mathcal{E}_{|X}\longrightarrow U\longrightarrow 0$$

from which we find that $c_1(\mathcal{E}_{|X})=-\theta+\pi_2^*(c_1)+c_1(U)$. Similarly, we have that

$$c_2(\mathcal{E}_{|X})=\frac{\theta^2}{2}+\pi_2^*(c_2)-\theta\pi_2^*(c_1)-c_1(U)\pi_2^*(c_1)-\theta c_1(U). \tag{3}$$

It is elementary to check that $c_1(\mathrm{Sym}^2\mathcal{E}_{|X})=8\,c_1(\mathcal{E}_{|X})$ and that $c_2(\mathrm{Sym}^2\mathcal{E}_{|X})=27\,c_1^2(\mathcal{E}_{|X})+9\,c_2(\mathcal{E}_{|X})$, therefore we obtain that

$$\sigma^*(C^1)\cdot c_2(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))=c_2(\mathcal{A}_{2|X})+c_1(\mathcal{A}_{2|X})c_1(U^{\otimes 2})-$$

$$-8c_1(\mathcal{A}_{2|X})c_1(\mathcal{E}_{|X})-8c_1(\mathcal{E}_{|X})c_1(U^{\otimes 2})+37c_1^2(\mathcal{E}_{|X})-9c_2(\mathcal{E}_{|X})=$$

$$=\Big(-120\,\eta\theta+\frac{17}{2}\theta^2-16\,\theta\gamma-9\,\pi_2^*(c_2)+(224\,\eta+32\,\gamma-33\,\theta)\pi_2^*(c_1)+37\pi_2^*(c_1^2)\Big)\cdot[X]+$$

$$+(168\,\eta+24\,\gamma-25\,\theta+49\,\pi_2^*(c_1))\cdot c_1(U)+21c_1^2(U)=$$

$$=1754\,\eta\theta\pi_2^*(c_2)+1386\,\eta\pi_2^*(c_3)-2498\,\eta\theta\pi_2^*(c_1^2)+741\,\eta\theta^2\pi_2^*(c_1)-4068\,\eta\pi_2^*(c_1)\pi_2^*(c_2)-$$

$$-51\,\eta\theta^3+2738\,\eta\pi_2^*(c_1^3),$$

where the last expression lives inside $H^4(C\times W^2_{17}(C))$. Using Lemma 2.4 we can evaluate each term in this sum to find that $\sigma^*(C^1)\cdot c_2(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))=691\,\theta^{21}/1207084032000$, which implies the stated formula for b_1. □

Theorem 2.12. *Let $[C,q]\in\mathcal{M}_{21,1}$ be a suitably general pointed curve and $L\in W^6_{25}(C)$ a linear series with a cusp at q. Then the multiplication map $\mathrm{Sym}^2H^0(C,L)\to H^0(C,L^{\otimes 2})$ is injective. It follows that we have the relation $a-12b_0+b_1=0$.*

Proof. We consider the pencil $R\subset\overline{\mathcal{M}}_g$ obtained by attaching to C at the point q a pencil of plane cubics. It is well-known that $R\cdot\lambda=1, R\cdot\delta_0=12$ and $R\cdot\delta_1=-1$, thus the relation $a-12b_0+b_1=0$ would be immediate once we show that $R\cap\overline{\mathfrak{D}}_{22}=\emptyset$. Assume by contradiction that $R\cap\overline{\mathfrak{D}}_{22}\neq\emptyset$ and then according to Proposition 2.1 there exists $M_C\in W^6_{25}(C)$ with $h^0(M_C\otimes\mathcal{O}_C(-2q))=6$ such that the multiplication map $\mathrm{Sym}^2H^0(M_C)\to H^0(M_C^{\otimes 2})$ is not injective. There are two cases to consider. If $q\in\mathrm{Bs}|M_C|$, then $N:=M_C\otimes\mathcal{O}_C(-q)\in W^6_{24}(C)$ is such that the map $\mathrm{Sym}^2H^0(N)\to H^0(N^{\otimes 2})$ is not an isomorphism. This is a divisorial condition on $[C]\in\mathcal{M}_{21}$ (cf. [Kh] or [F3]) and therefore it does not occur if we choose $[C]\in\mathcal{M}_{21}$ sufficiently generically. We are left with the case when M_C has a cusp at q, hence $(q,M_C)\in X$. This case is covered by Theorem 2.10 which finishes the proof.

□

Theorem 2.13. *Let $[C,q] \in \mathcal{M}_{21,1}$ be a Brill-Noether general pointed curve and denote by $C^0 \subset \Delta_0$ the associated test curve. Then $\sigma^*(C^0)\cdot c_2(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})) = 42b_0 - b_1 = 4847375988$. It follows that $b_0 = 132822768$.*

Proof. This time we look at the virtual degeneracy locus of the morphism $\mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ along the surface $\sigma^*(C^0)$. The first thing to note is that the vector bundles $\mathcal{E}_{|\sigma^*(C^0)}$ and $\mathcal{F}_{|\sigma^*(C^0)}$ are both pull-backs of vector bundles on Y. For convenience we denote this vector bundles also by $\mathcal{E}$ and $\mathcal{F}$, hence to use the notation of Proposition 2.3, $\mathcal{E}_{|\sigma^*(C^0)}) = \epsilon^*(\mathcal{E}_{|Y})$ and $\mathcal{F}_{|\sigma^*(C^0)} = \epsilon^*(\mathcal{F}_{|Y})$. We find that

$$\sigma^*(C^0)\cdot c_2(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E})) = c_2(\mathcal{F}_{|Y}) - c_1(\mathcal{F}_{|Y})\cdot c_1(\mathcal{E}_{|Y}) + c_1^2(\mathcal{E}_{|Y}) - c_2(\mathcal{E}_{|Y})$$

and like in the proof of Theorem 2.11, we are going to compute each term in this expression. We denote by $V := \mathrm{Ker}(\chi)$, where $\chi : \mathcal{B}^\vee \to \pi_2^*(\mathcal{M})^\vee$ is the bundle morphism coming from Proposition 2.6. Thus V is a line bundle on Y with fibre

$$V(y,M) = \frac{H^0(C,L)}{H^0(C, L\otimes \mathcal{O}_C(-y-q))},$$

over each point $(y,M)\in Y$, and where $L := K_C\otimes M^\vee \otimes \mathcal{O}_C(y+q) \in W^6_{25}(C)$. By using again the Harris-Tu Theorem, we find the following formulas for the Chern numbers of V:

$$c_1(V)\cdot \xi_{|Y} = -(c_3(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee)\cdot \xi_{|Y}) = (\pi_2^*(c_3) + \pi_2^*(c_2)(16\eta+\gamma) - 2\pi_2^*(c_1)\eta\theta)\cdot \xi_{|Y},$$

for any class $\xi \in H^2(C\times W^2_{17}(C))$, and

$$c_1^2(V) = c_4(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee) = \pi_2^*(c_3)(16\eta+\gamma) - 2\pi_2^*(c_2)\eta\theta.$$

Recall that we introduced the rank 28 vector bundle $\mathcal{B}_2$ over $C\times W^2_{17}(C)$ with fibre $\mathcal{B}_2(y,M) = H^0(C, L^{\otimes 2}\otimes \mathcal{O}_C(-y-q))$. We claim that one has an exact sequence of bundles over Y

$$0 \longrightarrow \mathcal{B}_{2|Y} \longrightarrow \mathcal{F}_{|Y} \longrightarrow V^{\otimes 2} \longrightarrow 0. \tag{4}$$

If $\mathcal{B}_3$ is the rank 30 vector bundle on Y with fibres

$$\mathcal{B}_3(y,M) = H^0(C, L^{\otimes 2}) = H^0\big(C, K_C^{\otimes 2}\otimes M^{\otimes(-2)}\otimes \mathcal{O}_C(2y+2q)\big),$$

we have an injective morphism of sheaves $V^{\otimes 2} \hookrightarrow \mathcal{B}_3/\mathcal{B}_2$ locally given by

$$v^{\otimes 2} \mapsto v^2 \bmod H^0(C, L^{\otimes 2}\otimes \mathcal{O}_C(-y-q)),$$

where $v \in H^0(C,L)$ is any section not vanishing at q and y. Then $\mathcal{F}_{|Y}$ is canonically identified with the kernel of the projection morphism

$$\mathcal{B}_3 \to \frac{\mathcal{B}_3/\mathcal{B}_2}{V^{\otimes 2}}$$

and the exact sequence (4) now becomes clear. Therefore $c_1(\mathcal{F}_{|Y}) = c_1(\mathcal{B}_{2|Y}) + 2c_1(V)$ and $c_2(\mathcal{F}_{|Y}) = c_2(\mathcal{B}_{2|Y})$. Reasoning along the lines of Theorem 2.11, we also have an exact sequence

$$0 \longrightarrow \pi_2^*\big(R^1\pi_{2*}(\mathcal{P})\big)^\vee_{|Y} \longrightarrow \mathcal{E}_{|Y} \longrightarrow V \longrightarrow 0$$

and from this we obtain that

$$c_1(\mathcal{E}_{|Y}) = -\theta + \pi_2^*(c_1) + c_1(V) \text{ and } c_2(\mathcal{E}_{|Y}) = \frac{\theta^2}{2} + \pi_2^*(c_2) - \theta\pi_2^*(c_1) - \theta c_1(V) + c_1(V)\pi_2^*(c_1).$$

All in all, we can write the following expression for the total intersection number

$$\sigma^*(C^0)\cdot c_2(\mathcal{F}-\mathrm{Sym}^2(\mathcal{E}))=c_2(\mathcal{B}_{2|Y})+c_1(\mathcal{B}_{2|Y})c_1(V^{\otimes 2})-$$

$$-8c_1(\mathcal{B}_{2|Y})c_1(\mathcal{E}_{|Y})-8c_1(\mathcal{E}_{|Y})c_1(V^{\otimes 2})+37c_1^2(\mathcal{E}_{|Y})-9c_2(\mathcal{E}_{|Y})=$$

$$=\Big(\frac{17}{2}\theta^2+28\eta\theta-8\theta\gamma-9\pi_2^*(c_2)+(16\gamma-33\theta-56\eta)\pi_2^*(c_1)+37\pi_2^*(c_1^2)\Big)\cdot[Y]+$$

$$+(49\ \pi_2^*(c_1)-25\ \theta-42\ \eta+12\ \gamma)c_1(V)+21c_1^2(V)=$$

$$=428\ \eta\theta\pi_2^*(c_2)-536\ \eta\theta\pi_2^*(c_1^2)+168\ \eta\theta^2\pi_2^*(c_1)-984\ \eta\pi_2^*(c_1)\pi_2^*(c_2)+$$

$$+378\eta\pi_2^*(c_3)-17\ \eta\theta^3+592\eta\pi_2^*(c_1^3),$$

and using once more Lemma 2.4, we get that $42b_0-b_1=509\theta^{21}/5364817920000$. Since we already know the value of b_1 and $a-12b_0+b_1=0$, this allows us to calculate a and b_0. □

End of the proof of Theorem 1.1. We write $\overline{\mathfrak{D}}_{22}\equiv a\lambda-\sum_{j=0}^{11}b_j\delta_j$. Since $a/b_0=17121/2636\leq 71/10$, we are in a position to apply Corollary 1.2 from [FP] which gives the inequalities $b_j\geq b_0$ for $1\leq j\leq 11$, hence $s(\overline{\mathfrak{D}}_{22})=a/b_0<13/2$. □

References

[AC1] E. Arbarello and M. Cornalba, *Footnotes to a paper of Beniamino Segre*, Math. Ann. **256** (1981), 341-362.

[AC2] E. Arbarello and M. Cornalba, *A few remarks about the variety of irreducible plane curves of given degree and genus*, Ann. Scient. Éc. Norm. Sup. **16** (1983), 467-488.

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of algebraic curves*, Grundlehren der mathematischen Wissenschaften 267, Springer.

[CR1] M.C. Chang and Z. Ran, *Unirationality of the moduli space of curves of genus* $11, 13$ *(and* 12*)*, Invent. Math. **76** (1984), 41-54.

[CR2] M.C. Chang and Z. Ran, *The Kodaira dimension of the moduli space of curves of genus* 15, J. Diff. Geom. **24** (1986), 205-220.

[EH] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90** (1987), 359-387.

[H1] J. Harris, *On the Kodaira dimension of the moduli space of curves II: The even genus case*, Invent. Math. **75** (1984), 437-466.

[H2] J. Harris, *On the Severi problem*, Invent. Math. **84** (1984), 445-461.

[HM] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23-88.

[HMo] J. Harris and I. Morrison, *Slopes of effective divisors on the moduli space of stable curves*, Invent. Math. **99** (1990), 321-355.

[HT] J. Harris and L. Tu, *Chern numbers of kernel and cokernel bundles*, Invent. Math. **75** (1984), 467-475.

[F1] G. Farkas, *The geometry of the moduli space of curves of genus* 23, Math. Ann. **318** (2000), 43-65.

[F2] G. Farkas, *Syzygies of curves and the effective cone of* $\overline{\mathcal{M}}_g$, math.AG/0503498, to appear in Duke Math. J. (2006).

[F3] G. Farkas, *Koszul divisors on moduli spaces of curves*, math.AG/0607475.

[FP] G. Farkas and M. Popa, *Effective divisors on* $\overline{\mathcal{M}}_g$*, curves on* $K3$ *surfaces and the Slope Conjecture*, J. Algebraic Geometry, **14** (2005), 241-267.

[Kh] D. Khosla, *Moduli spaces of curves with linear series and the Slope Conjecture*, math.AG/0608024.

[Se] E. Sernesi, *L'unirazionalita della varieta dei moduli delle curve di genere dodici*, Ann. Scuola Norm. Sup. Pisa **8** (1981), 405-439.

[V] A. Verra, *The unirationality of the moduli space of curves of genus* 14 *or lower*, Compositio Math. **141** (2005), 1425-1444.

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