HIGHER RANK BRILL-NOETHER THEORY ON SECTIONS OF $K3$ SURFACES

GAVRIL FARKAS AND ANGELA ORTEGA

ABSTRACT. We discuss the role of K3 surfaces in the context of Mercat's conjecture in higher rank Brill-Noether theory. Using liftings of Koszul classes, we show that Mercat's conjecture in rank 2 fails for any number of sections and for any gonality stratum along a Noether-Lefschetz divisor inside the locus of curves lying on K3 surfaces. Then we show that Mercat's conjecture in rank 3 fails even for curves lying on K3 surfaces with Picard number 1. Finally, we provide a detailed proof of Mercat's conjecture in rank 2 for general curves of genus 11, and describe explicitly the action of the Fourier-Mukai involution on the moduli space of curves.

1. INTRODUCTION

The Clifford index $\mathrm{Cliff}(C)$ of an algebraic curve C is the second most important invariant of C after the genus, measuring the complexity of the curve in its moduli space. Its geometric significance is amply illustrated for instance in the statement

$$K_{p,2}(C, K_C) = 0 \Leftrightarrow p < \mathrm{Cliff}(C)$$

of Green's Conjecture [G] on syzygies of canonical curves. It has been a long-standing problem to find an adequate generalization of $\mathrm{Cliff}(C)$ for higher rank vector bundles. A definition in this sense has been proposed by Lange and Newstead [LN1]: If $E \in \mathcal{U}_C(n,d)$ denotes a semistable vector bundle of rank n and degree d on a curve C of genus g, one defines its Clifford index as

$$\gamma(E) := \mu(E) - \frac{2}{n} h^0(C, E) + 2 \geq 0,$$

and then the *higher Clifford indices* of C are defined as the quantities

$$\mathrm{Cliff}_n(C) := \min\big\{\gamma(E) : E \in \mathcal{U}_C(n,d),\ \ d \leq n(g-1),\ \ h^0(C,E) \geq 2n\big\}^1.$$

Note that $\mathrm{Cliff}_1(C) = \mathrm{Cliff}(C)$ is the classical Clifford index of C. By specializing to sums of line bundles, it is easy to check that $\mathrm{Cliff}_n(C) \leq \mathrm{Cliff}(C)$ for all $n \geq 1$. Mercat [Me] proposed the following interesting conjecture, which we state in the form of [LN1] Conjecture 9.3, linking the newly-defined invariants $\mathrm{Cliff}_n(C)$ to the classical geometry of C:

$$(M_n): \ \ \mathrm{Cliff}_n(C) = \mathrm{Cliff}(C).$$

Mercat's conjecture (M_2) holds for various classes of curves, in particular general k-gonal curves of genus $g > 4k-4$, or arbitrary smooth plane curves, see [LN1]. In [FO] Theorem 1.7, we have verified (M_2) for a general curve $[C] \in \mathcal{M}_g$ with $g \leq 16$. More generally, the statement (M_2) is a consequence of the *Maximal Rank Conjecture* (see [FO] Conjecture 2.2), therefore it is expected to be true for a general curve $[C] \in$

[1]The invariant $\mathrm{Cliff}_n(C)$ is denoted in the paper [LN1] by $\gamma'_n(C)$. Since the appearance of [LN1], it has become abundantly clear that $\mathrm{Cliff}_n(C)$, defined as above, is the most relevant Clifford type invariant for rank n vector bundles on C. Accordingly, the notation $\mathrm{Cliff}_n(C)$ seems appropriate.

$\mathcal{M}_g$. However, for every genus $g \geq 11$ there exist curves $[C] \in \mathcal{M}_g$ with maximal Clifford index $\mathrm{Cliff}(C) = [\frac{g-1}{2}]$ carrying stable rank 2 vector bundles E with $h^0(C, E) = 4$ and $\gamma(E) < \mathrm{Cliff}(C)$, see [FO] Theorems 3.6 and 3.7 and [LN2] Theorem 1.1 for an improvement. For these curves, the inequality $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ holds.

Obvious questions emerging from this discussion are whether such results are specific to (i) rank 2 bundles with 4 sections, or to (ii) curves with maximal Clifford index $[\frac{g-1}{2}]$. First we prove that under general circumstances, curves on $K3$ surfaces carry rank 2 vector bundles E with a prescribed (and exceptionally high) number of sections invalidating Mercat's inequality $\gamma(E) \geq \mathrm{Cliff}(C)$:

Theorem 1.1. *We fix integers $p \geq 1$ and $a \geq 2p+3$. There exists a smooth curve C of genus $2a+1$ and Clifford index $\mathrm{Cliff}(C) = a$, lying on a $K3$ surface $C \subset S \subset \mathbf{P}^{2p+2}$ with $\mathrm{Pic}(S) = \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H$, where $H^2 = 4p+2$, $H \cdot C = \deg(C) = 2a+2p+1$, as well as a stable rank 2 vector bundle $E \in \mathcal{SU}_C\big(2, \mathcal{O}_C(H)\big)$, such that $h^0(C, E) = p+3$. In particular $\gamma(E) = a - \frac{1}{2} < \mathrm{Cliff}(C)$ and Mercat's conjecture (M_2) fails for C.*

It is well-known cf. [M2], [V1], that a curve $[C] \in \mathcal{M}_{2a+1}$ lying on a $K3$ surface S possesses a rank 2 vector bundle $F \in \mathcal{SU}_C(2, K_C)$ with $h^0(C, F) = a+2$. In particular, $\gamma(F) = a \geq \mathrm{Cliff}(C)$ (with equality if $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$), hence such bundles satisfy condition (M_2). Let us consider the $K3$ locus in the moduli space of curves

$$\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}.$$

When $g = 11$ or $g \geq 13$, the variety $\mathcal{K}_g$ is irreducible and $\dim(\mathcal{K}_g) = 19 + g$, see [CLM] Theorem 5. For integers $r, d \geq 1$ such that $d^2 > 4(r-1)g$ and $2r-2 \nmid d$, we define the *Noether-Lefschetz* divisor inside the locus of sections of $K3$ surfaces

$$\mathfrak{NL}^r_{g,d} := \left\{ [C] \in \mathcal{K}_g \;\middle|\; \begin{array}{l} C \text{ lies on a } K3 \text{ surface } S, \ \mathrm{Pic}(S) \supset \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H, \\ H \in \mathrm{Pic}(S) \text{ is nef}, H^2 = 2r-2, \ \ C \cdot H = d, \ \ C^2 = 2g-2 \end{array} \right\}.$$

A consequence of Theorem 1.1 can be formulated as follows:

Corollary 1.2. *We fix integers $p \geq 1$ and $a \geq 2p+3$ and set $g := 2a+1$. Then Mercat's conjecture (M_2) fails generically along the Noether-Lefschetz locus $\mathfrak{NL}^{2p+2}_{g,2a+2p+1}$ inside $\mathcal{K}_g$, that is, $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$ for a general point $[C] \in \mathfrak{NL}^{2p+2}_{g,2a+2p+1}$.*

It is natural to wonder whether it is necessary to pass to a Noether-Lefschetz divisor in $\mathcal{K}_g$, or perhaps, all curves $[C] \in \mathcal{K}_g$ give counterexamples to conjecture (M_2). To see that this is not always the case and all conditions in Theorem 1.1 are necessary, we study in detail the case $g = 11$. Mukai [M3] proved that a general curve $[C] \in \mathcal{M}_{11}$ lies on a unique $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, thus, $\mathcal{M}_{11} = \mathcal{K}_{11}$.

Theorem 1.3. *For a general curve $[C] \in \mathcal{M}_{11}$ one has the equality $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$, that is, Mercat's conjecture holds generically on $\mathcal{M}_{11}$. Furthermore, the locus*

$$\{[C] \in \mathcal{M}_{11} : \mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)\}$$

can be identified with the Noether-Lefschetz divisor $\mathfrak{NL}^4_{11,13}$ on $\mathcal{M}_{11}$.

In Section 5, we describe in detail the divisor $\mathfrak{NL}^4_{11,13}$ and discuss, in connection with Mercat's conjecture, the action of the Fourier-Mukai involution $FM : \mathcal{F}_{11} \to \mathcal{F}_{11}$ on the moduli space of polarized $K3$ surfaces of genus 11. The automorphism FM acts on the set of Noether-Lefschetz divisors and in particular it (i) fixes the 6-gonal locus

$\mathcal{M}^1_{11,6}$ and it maps the divisor $\mathfrak{NL}^4_{11,13}$ which corresponds to certain elliptic $K3$ surfaces, to the Noether-Lefschetz divisor corresponding to $K3$ surfaces carrying a rational curve of degree 3.

Next we turn our attention to the conjecture (M_n) for $n \geq 3$. It was observed in [LMN] that Mukai's description [M4] of a general curve of genus 9 in terms of linear sections of a certain rational homogeneous variety, and especially the connection to rank 3 Brill-Noether theory, can be used to construct, on a general curve $[C] \in \mathcal{M}_9$, a stable vector bundle $E \in \mathcal{SU}_C(3, K_C)$ such that $h^0(C, E) = 6$. In particular $\gamma(E) = \frac{10}{3} < \mathrm{Cliff}(C)$, that is, Mercat's conjecture (M_3) fails for a general curve $[C] \in \mathcal{M}_9$. A similar construction is provided in [LMN] for a general curve of genus 11. In what follows we outline a construction illustrating that the results from [LMN] are part of a larger picture and curves on $K3$ surfaces carry vector bundles E of rank at least 3 with $\gamma(E) < \mathrm{Cliff}(C)$.

Let S be a $K3$ surface and $C \subset S$ a smooth curve of genus g. We choose a linear series $A \in W^r_d(C)$ of minimal degree such that the Brill-Noether number $\rho(g, r, d)$ is non-negative, that is, $d := r + [\frac{r(g+1)}{r+1}]$. The *Lazarsfeld bundle* M_A on C is defined as the kernel of the evaluation map, that is,

$$0 \longrightarrow M_A \longrightarrow H^0(C, A) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}_C} A \longrightarrow 0.$$

As usual, we set $Q_A := M_A^\vee$, hence $\mathrm{rank}(Q_A) = r$ and $\det(Q_A) = A$. Following a procedure that already appeared in [L], [M2], [V1], we note that C carries a vector bundle of rank $r+1$ with canonical determinant and unexpectedly many global sections:

Theorem 1.4. *For a curve $C \subset S$ and $A \in W^r_d(C)$ as above there exists a globally generated vector bundle E on C with $\mathrm{rank}(E) = r + 1$ and $\det(E) = K_C$, expressible as an extension*

$$0 \longrightarrow Q_A \longrightarrow E \longrightarrow K_C \otimes A^\vee \longrightarrow 0,$$

satisfying the condition $h^0(C, E) = h^0(C, A) + h^0(C, K_C \otimes A^\vee) = g - d + 2r + 1$. If moreover $r \leq 2$ and $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, then the above extension is non-trivial.

When $r = 1$ the rank 2 bundle E constructed in Theorem 1.4 is well-known and plays an essential role in [V1]. In this case $\gamma(E) \geq [\frac{g-1}{2}]$. For $r = 2$ and $g = 9$ (in which case $A \in W^2_8(C)$), or for $g = 11$ (and then $A \in W^2_{10}(C)$), Theorem 1.4 specializes to the construction in [LMN]. When $\mathrm{rank}(E) = 3$, we observe by direct calculation that $\gamma(E) < [\frac{g-1}{2}]$. In view of providing counterexamples to Mercat's conjecture (M_3), it is thus important to determine whether E is stable.

Theorem 1.5. *Fix $C \subset S$ as above with $g = 7, 9$ or $g \geq 11$ such that $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, as well as $A \in W^2_d(C)$, where $d := [\frac{2g+8}{3}]$. Then any globally generated rank 3 vector bundle E on C lying non-trivially in the extension*

$$0 \longrightarrow Q_A \longrightarrow E \longrightarrow K_C \otimes A^\vee \longrightarrow 0,$$

and with $h^0(C, E) = h^0(C, A) + h^0(C, K_C \otimes A^\vee) = g - d + 5$, is stable.

As a corollary, we note that for sufficiently high genus Mercat's statement (M_3) fails to hold for *any* smooth curve of maximal Clifford index lying on a $K3$ surface.

Corollary 1.6. *We fix an integer $g = 9$ or $g \geq 11$ and a curve $[C] \in \mathcal{K}_g$. Then the inequality $\mathrm{Cliff}_3(C) < [\frac{g-1}{2}]$ holds. In particular, Mercat's conjecture (M_3) fails generically along $\mathcal{K}_g$.*

We close the Introduction by thanking Herbert Lange and Peter Newstead for making a number of very pertinent comments on the first version of this paper.

2. Higher rank vector bundles with canonical determinant

In this section we treat Mercat's conjecture (M_3) and prove Theorems 1.4 and 1.5. We begin with a curve C of genus g lying on a smooth $K3$ surface S such that $\mathrm{Pic}(S)=\mathbb{Z}\cdot C$, and fix a linear series $A\in W^2_d(C)$ of minimal degree $d:=[\frac{2g+8}{3}]$. Under such assumptions both A and $K_C\otimes A^{\vee}$ are base point free. From the onset, we point out that the existence of vector bundles of higher rank on C having exceptional Brill-Noether behaviour has been repeatedly used in [L], [M2] and [V1]. Our aim is to study these bundles from the point of view of Mercat's conjecture and discuss their stability.

We define the *Lazarsfeld-Mukai* sheaf $\mathcal{F}_A$ via the following exact sequence on S:

$$0\longrightarrow \mathcal{F}_A\longrightarrow H^0(C,A)\otimes \mathcal{O}_S\stackrel{\mathrm{ev}_S}{\longrightarrow} A\longrightarrow 0.$$

Since A is base point free, $\mathcal{F}_A$ is locally free. We consider the vector bundle $\mathcal{E}_A:=\mathcal{F}_A^{\vee}$ on S, which by dualizing, sits in an exact sequence

$$0\longrightarrow H^0(C,A)^{\vee}\otimes\mathcal{O}_S\longrightarrow \mathcal{E}_A\longrightarrow K_C\otimes A^{\vee}\longrightarrow 0. \tag{1}$$

Since $K_C\otimes A^{\vee}$ is assumed to be base point free, the bundle $\mathcal{E}_A$ is globally generated. It is well-known (and follows from the sequence (1), that $c_1(\mathcal{E}_A)=\mathcal{O}_S(C)$ and $c_2(\mathcal{E}_A)=d$.

Proof of Theorem 1.4. We write down the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^0(C,A)\otimes\mathcal{O}_S(-C) & \stackrel{=}{\longrightarrow} & H^0(C,A)\otimes\mathcal{O}_S(-C) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{F}_A & \longrightarrow & H^0(C,A)\otimes\mathcal{O}_S & \longrightarrow & A & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & M_A & \longrightarrow & H^0(C,A)\otimes\mathcal{O}_C & \longrightarrow & A & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

from which, if we set $F_A:=\mathcal{F}_A\otimes\mathcal{O}_C$ and $E_A:=\mathcal{E}_A\otimes\mathcal{O}_C$, we obtain the exact sequence

$$0\longrightarrow M_A\otimes K_C^{\vee}\longrightarrow H^0(C,A)\otimes K_C^{\vee}\longrightarrow F_A\longrightarrow M_A\longrightarrow 0$$

(use that $\mathrm{Tor}^1_{\mathcal{O}_S}(M_A,\mathcal{O}_C)=M_A\otimes K_C^{\vee}$). Taking duals, we find the exact sequence

$$0\longrightarrow Q_A\longrightarrow E_A\longrightarrow K_C\otimes A^{\vee}\longrightarrow 0. \tag{2}$$

Since S is regular, from (1) we obtain that $h^0(S,\mathcal{E}_A)=h^0(C,A)+h^0(C,K_C\otimes A^{\vee})$ while $H^0(S,\mathcal{E}_A\otimes\mathcal{O}_S(-C))=0$, that is,

$$h^0(S,\mathcal{E}_A)\le h^0(C,E_A)\le h^0(C,A)+h^0(C,K_C\otimes A^{\vee}).$$

Thus the sequence (2) is exact on global sections.

We are left with proving that the extension (2) is non-trivial. We set $r=2$ and then $\mathrm{rank}(\mathcal{E}_A)=3$ and place ourselves in the situation when $\mathrm{Pic}(S)=\mathbb{Z}\cdot C$ (the case $r=1$ works similarly). By contradiction we assume that $E_A=Q_A\oplus(K_C\otimes A^{\vee})$ and denote by $s : E_A \to Q_A$ a retract and by $\tilde{s} : \mathcal{E}_A \to Q_A$ the induced map. We set $\mathcal{M} := \mathrm{Ker}\{\mathcal{E}_A \stackrel{\tilde{s}}{\longrightarrow} Q_A\}$, hence $\mathcal{M}$ can be regarded as an elementary transformation of the Lazarsfeld-Mukai bundle $\mathcal{E}_A$ along C. By direct calculation we find that

$$c_1(\mathcal{M}) = \mathcal{O}_S(-C) \text{ and } c_2(\mathcal{M}) = 2d - 2g + 2,$$

hence the discriminant of $\mathcal{M}$ equals $\Delta(\mathcal{M}) := 6c_2(\mathcal{M}) - 2c_1^2(\mathcal{M}) = 4(3d - 4g + 4) < 0$. Thus the sheaf $\mathcal{M}$ is $\mathcal{O}_S(C)$-unstable. Applying [HL] Theorems 7.3.3 and 7.3.4, there exists a subsheaf $\mathcal{M}' \subset \mathcal{M}$ such that if $\xi_{\mathcal{M},\mathcal{M}'} := \frac{c_1(\mathcal{M}')}{\mathrm{rank}(\mathcal{M}')} - \frac{c_1(\mathcal{M})}{\mathrm{rank}(\mathcal{M})} \in \mathrm{Pic}(S)_{\mathbb{R}}$, then

$$(i)\ \ \xi_{\mathcal{M},\mathcal{M}'} \cdot C > 0 \ \text{ and } \ (ii)\ \ \xi^2_{\mathcal{M},\mathcal{M}'} \geq -\frac{\Delta(\mathcal{M})}{18}.$$

Since $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, we may write $c_1(\mathcal{M}') = \mathcal{O}_S(aC)$ and also set $r' := \mathrm{rank}(\mathcal{M}')$. The Lazarsfeld-Mukai bundle $\mathcal{E}_A$ is $\mathcal{O}_S(C)$-stable, in particular $\mu_C(\mathcal{M}') \leq \mu_C(\mathcal{E}_A)$, which yields $a \leq 0$. Then from (i) we write that $0 \leq \frac{a}{r'} + \frac{1}{3} \leq \frac{1}{3}$, whereas from (ii) one finds

$$\frac{1}{9} \geq \frac{4(g-1) - 3d}{9(g-1)} \Leftrightarrow d \geq g - 1,$$

which is a contradiction. It follows that the extension (2) is non-trivial. $\square$

It is natural to ask when is the above constructed bundle E_A stable. We give an affirmative answer under certain generality assumptions, when $r < 3$.

We fix a $K3$ surface S such that $\mathrm{Pic}(S) = \mathbb{Z}{\cdot}C$ and as before, set $d := [\frac{2g+8}{3}]$. Under these assumptions, it follows from [L] that C satisfies the Brill-Noether theorem. We prove the stability of every globally generated non-split bundle E sitting in an extension of the form (2) and having a maximal number of sections.

Proof of Theorem 1.5. We first discuss the possibility of a destabilizing sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow B \longrightarrow 0,$$

where F is a vector bundle of rank 2 and $\deg(F) \geq \frac{4}{3}(g-1)$. Since E is globally generated, it follows that B is globally generated as well, hence $h^0(C, B) \geq 2$, in particular $\deg(B) \geq (g+2)/2$ and hence $\deg(F) \leq \frac{3}{2}g - 3$. Since $\deg(B) \leq \frac{2}{3}(g-1)$ and C is Brill-Noether general, it follows that $h^0(C, B) = 2$, therefore $h^0(C, F) \geq g - d + 3$. There are two cases to distinguish, depending on whether F possesses a subpencil or not.

Assume first that F has no subpencils. We apply [PR] Lemma 3.9 to find that $h^0(C, \det(F)) \geq 2h^0(C, F) - 3 \geq 2g - 2d + 3$. Writing down the inequality

$$\rho\big(g, 2g - 2d + 2, \deg(F)\big) \geq 0$$

and using that $\deg(F) < \frac{3}{2}g - 3$, we obtain a contradiction. If on the other hand, F has a subpencil, then as pointed out in [FO] Lemma 3.2, $\gamma(F) \geq \mathrm{Cliff}(C)$, but again this is a contradiction. This shows that E cannot have a rank 2 destabilizing subsheaf.

We are left with the possibility of a destabilizing short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where B is a line bundle with $\deg(B) \geq \frac{2}{3}(g-1)$ and F is a rank 2 bundle. The bundle Q_A is well-known to be stable and based on slope considerations, B cannot be a subbundle of Q_A, that is, necessarily $H^0(C, K_C \otimes A^{\vee} \otimes B^{\vee}) \neq 0$. Since the bundle E is not decomposable, it follows that $\deg(B) \leq \deg(K_C \otimes A^{\vee}) - 1 = 2g - 3 - d$. Furthermore $h^1(C, B) \geq 3$.

If F is not stable, we reason along the lines of [LMN] Proposition 3.5 and pull-back a destabilizing line subbundle of F to obtain a rank 2 subbundle $F' \subset E$ such that

$$\deg(F') \geq \deg(B) + \frac{1}{2}\Big(\deg(E) - \deg(B)\Big) \geq \frac{4}{3}(g-1),$$

which is the case which we have already ruled out. So we may assume that F is stable. We write $h^0(C,B) = a+1$, hence $h^0(C,F) \geq g-d-a+4$. Assume first that F admits no subpencils. Then from [PR] Lemma 3.9 we find the following estimate for the number of sections of the line bundle $\det(F) = K_C \otimes B^\vee$,

$$h^0(C, K_C \otimes B^\vee) \geq 2h^0(C,F) - 3 \geq 2g - 2d - 2a + 5,$$

which, after applying Riemann-Roch to B, leads to the inequality

$$3a \geq g - 2d + 5 + \deg(B).$$

Combining this estimate with the Brill-Noether inequality $\rho\big(g, a, \deg(B)\big) \geq 0$ and substituting the actual value of d, we find that $3a + 3 \geq g$. On the other hand $a \leq h^0(C, K_C \otimes A^\vee) - 2 = g - d < \frac{g-3}{3}$, and this is a contradiction.

Finally, if F admits a subpencil, then $\gamma(F) \geq \mathrm{Cliff}(C)$. Combining this with the classical Clifford inequality for B, we find that $\gamma(E) \geq \mathrm{Cliff}(C)$, which again is a contradiction. We conclude that the rank 3 bundle E must be stable. □

3. Rank 2 bundles and Koszul classes

The aim of this section is to prove Theorem 1.1. We shall construct rank 2 vector bundles on curves using a connection between vector bundles on curves and Koszul cohomology of line bundles, cf. [AN] and [V2]. Let us recall that for a smooth projective variety X, a sheaf $\mathcal{F}$ and a globally generated line bundle L on X, the Koszul cohomology group $K_{p,q}(X;\mathcal{F},L)$ is defined as the cohomology of the complex:

$$\bigwedge^{p+1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q-1}) \stackrel{d_{p+1,q-1}}{\longrightarrow} \bigwedge^{p} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q}) \stackrel{d_{p,q}}{\longrightarrow} \bigwedge^{p-1} H^0(L) \otimes H^0(\mathcal{F} \otimes L^{q+1}).$$

Most of the time $\mathcal{F} = \mathcal{O}_X$, and then one writes $K_{p,q}(X;\mathcal{O}_X,L) := K_{p,q}(X,L)$.

A Koszul class $[\zeta] \in K_{p,1}(X,L)$ is said to have rank $\leq n$, if there exists a subspace $W \subset H^0(X,L)$ with $\dim(W) = n$ and a representative $\zeta \in \wedge^p W \otimes H^0(X,L)$. The smallest number n with this property is the rank of the syzygy $[\zeta]$.

Next we discuss a connection due to Voisin [V2] and expanded in [AN], between rank 2 vector bundles on curves and syzygies. Let E be a rank 2 bundle on a smooth curve C with $h^0(C,E) \geq p+3 \geq 4$ and set $L := \det(E)$. Let

$$\lambda : \wedge^2 H^0(C,E) \to H^0(C,L)$$

be the determinant map, and we assume that there exists linearly independent sections $e_1 \in H^0(C,E)$ and $e_2, \ldots, e_{p+3} \in H^0(C,E)$, such that the map

$$\lambda\big(e_1 \wedge -\big) : \langle e_2, \ldots, e_{p+3} \rangle \to H^0(C,L)$$

in injective onto its image. Such an assumption is automatically satisfied for instance if E admits no subpencils. We introduce the subspace

$$W := \big\langle s_2 := \lambda(e_1 \wedge e_2), \ldots, s_{p+3} := \lambda(e_1 \wedge e_{p+3})\big\rangle \subset H^0(C,L).$$

By assumption, $\dim(W) = p+2$. Following [AN] and [V2], we define the tensor

$$\zeta(E) := \sum_{i<j} (-1)^{i+j}\, s_2 \wedge \ldots \wedge \hat{s}_i \wedge \ldots \wedge \hat{s}_j \wedge \ldots \wedge s_{p+3} \otimes \lambda(e_i \wedge e_j) \in \wedge^p W \otimes H^0(C, L).$$

One checks that $d_{p,1}(\zeta(E)) = 0$, hence $[\zeta(E)] \in K_{p,1}(C, L)$ is a non-trivial Koszul class of rank at most $p+2$. Conversely, starting with a non-trivial class $[\zeta] \in K_{p,1}(C, L)$ represented by an element ζ of $\wedge^p W \otimes H^0(C, L)$ where $\dim(W) = p+2$, Aprodu and Nagel [AN] Theorem 3.4 constructed a rank 2 vector bundle E on C with $\det(E) = L$, $h^0(C, E) \geq p+3$ and such that $[\zeta(E)] = [\zeta]$. This correspondence sets up a dictionary between the Brill-Noether loci in $\{E \in \mathcal{SU}_C(2, L) : h^0(C, E) \geq p+3\}$ and Koszul classes of rank at most $p+2$ in $K_{p,1}(C, L)$.

Let us now fix integers $p \geq 1$ and $a \geq 2p+3$. Using the surjectivity of the period mapping, see e.g. [K] Theorem 1.1, one can construct a smooth $K3$ surface $S \subset \mathbf{P}^{2p+2}$ of degree $4p+2$ containing a smooth curve $C \subset S$ of degree $d := 2a + 2p + 1$ and genus $g := 2a+1$. The surface S can be chosen with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$, where $H^2 = 4p+2$, $H \cdot C = d$ and $C^2 = 4a$. The smooth curve $H \subset C$ is the hyperplane section of S and has genus $g(H) = 2p+2$. The following observation is trivial:

Lemma 3.1. *Keeping the notation above, we have that $H^0(S, \mathcal{O}_S(H - C)) = 0$.*

Proof. It is enough to notice that H is nef and $(H - C) \cdot H = 2p - 2a + 1 < 0$. ☐

We consider the decomposable rank 2 bundle $K_H = A \oplus (K_H \otimes A^\vee)$ on H, where $A \in W^1_{p+2}(H)$. Via the Green-Lazarsfeld non-vanishing theorem [GL1] (or equivalently, applying [AN]), one obtains a non-zero Koszul class of rank $p+1$

$$\beta := \Big[\zeta\big(A \oplus (K_H \otimes A^\vee)\big)\Big] \in K_{p,1}(H, K_H).$$

Since S is a regular surface, there exist an exact sequence

$$0 \longrightarrow H^0(S, \mathcal{O}_S) \longrightarrow H^0(S, \mathcal{O}_S(H)) \longrightarrow H^0(H, K_H) \longrightarrow 0,$$

which induces an isomorphism [G] Theorem (3.b.7)

$$\mathrm{res}_H : K_{p,1}(S, \mathcal{O}_S(H)) \cong K_{p,1}(H, K_H).$$

By construction, the non-trivial class $\alpha := \mathrm{res}_H^{-1}(\beta) \in K_{p,1}(S, \mathcal{O}_S(H))$ has rank at most $\mathrm{rank}(\beta) + 1 = p+2$. Using [G] Theorem (3.b.1), we write the following exact sequence in Koszul cohomology:

$$\cdots \to K_{p,1}(S; -C, H) \to K_{p,1}(S, H) \to K_{p,1}(C, H \otimes \mathcal{O}_C) \to K_{p-1,2}(S; -C, H) \to \cdots.$$

Since $H^0(S, \mathcal{O}_S(H - C)) = 0$, it follows that $K_{p,1}(S; -C, H) = 0$, in particular the non-zero class $\alpha \in K_{p,1}(S, H)$ can be viewed as a Koszul class of rank at most $p+2$ inside the group $K_{p,1}(C, \mathcal{O}_C(H))$. This class corresponds to a *stable* rank 2 bundle on C:

Proposition 3.2. *Let $C \subset S \subset \mathbf{P}^{2p+2}$ as above and $L := \mathcal{O}_C(1) \in \mathrm{Pic}^{2a+2p+1}(C)$. Then there exists a stable vector bundle $E \in \mathcal{SU}_C(2, L)$ with $h^0(C, E) = p+3$.*

Proof. From [AN] we know that there exists a rank 2 vector bundle E on C with $\det(E) = L$ such that $[\zeta(E)] = \alpha \in K_{p,1}(C, L)$, in particular $h^0(C, E) \geq p+3$. Geometrically, E is the restriction to C of the Lazarsfeld-Mukai bundle $\mathcal{E}_A$ on S corresponding to a pencil $A \in W^1_{p+2}(H)$. In particular, E is globally generated, being the restriction of a globally generated bundle on S. We also know that $\mathrm{Cliff}(C) = a$ (to be proved in Proposition

3.3). Since $\gamma(E) \leq a - \frac{1}{2} < \mathrm{Cliff}(C)$, it follows that E admits no subpencils (If $B \subset E$ is a subpencil, then $h^0(C, L \otimes B^\vee) \geq 2$ because E is globally generated. It is easily verified that both B and $L \otimes B^\vee$ contribute to $\mathrm{Cliff}(C)$, which brings about a contradiction). Assume now that

$$0 \longrightarrow B \longrightarrow E \longrightarrow L \otimes B^\vee \longrightarrow 0$$

is a destabilizing sequence, where $B \in \mathrm{Pic}(C)$ has degree at least $a+p+1$. As already pointed out, $h^0(C, B) \leq 1$, hence $h^0(C, L \otimes B^\vee) \geq p+2$. If $h^1(C, L \otimes B^\vee) \leq 1$, then $p+2 \leq h^0(C, L\otimes B^\vee) \leq 1 + \deg(L \otimes B^\vee) - 2a$, which leads to a contradiction. If on the other hand $h^1(C, L \otimes B^\vee) \geq 2$, then $\mathrm{Cliff}(L \otimes B^\vee) \leq a - p - 2 < a$, which is impossible. Thus E is a stable vector bundle. $\square$

We are left with showing that the curve $C \subset S$ constructed above has maximal Clifford index a. Note that the corresponding statement when $p = 1$ has been proved in [FO] Theorem 3.6.

Proposition 3.3. *We fix integers $p \geq 1$, $a \geq 2p+3$ and a $K3$ surface S with Picard lattice $\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$ where $C^2 = 4a$, $H^2 = 4p+2$ and $C \cdot H = 2a + 2p + 1$. Then $\mathrm{Cliff}(C) = a$.*

Proof. First note that C has Clifford dimension 1, for curves $C \subset S$ of higher Clifford dimension have even genus. Observe also that $h^0(C, \mathcal{O}_C(1)) = 2p+3$ and $h^1(C, \mathcal{O}_C(1)) = 2$, hence $\mathcal{O}_C(1)$ contributes to the Clifford index of C and

$$\mathrm{Cliff}(C) \leq \mathrm{Cliff}(C, \mathcal{O}(1)) = C \cdot H - 2(2p+2) = 2a - 2p - 3 \ \big(\geq a\big).$$

Assume by contradiction that $\mathrm{Cliff}(C) < a$. According to [GL2], there exists an effective divisor $D \equiv mH + nC$ on S satisfying the conditions

$$h^0(S, \mathcal{O}_S(D)) \geq 2, \ \ h^0(S, \mathcal{O}_S(C-D)) \geq 2, \ C \cdot D \leq g-1, \tag{3}$$

and with $\mathrm{Cliff}(\mathcal{O}_C(D)) = \mathrm{Cliff}(C)$. By [Ma] Lemma 2.2, the dimension $h^0(C', \mathcal{O}_{C'}(D))$ stays constant for all smooth curves $C' \in |C|$ and its value equals $h^0(S, D)$. We conclude that $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{O}_C(D)) = D \cdot C - 2\dim|D|$. We summarize the numerical consequences of the inequalities (3):

$$\begin{array}{rl} (i) & md + 2n(g-1) \leq g-1 \\ (ii) & (2p+1)m^2 + mnd + n^2(g-1) \geq 0 \\ (iii) & (4p+2)m + dn > 2, \end{array}$$

We claim that for any divisor $D \subset S$ verifying (i)-(iii), the following inequality holds:

$$\mathrm{Cliff}(\mathcal{O}_C(D)) = D \cdot C - D^2 - 2 \geq H \cdot C - H^2 - 2 = 2a - 2p - 3 \geq a.$$

This will contradict the assumption $\mathrm{Cliff}(C) < a$. The proof proceeds along the lines of Theorem 3 in [F], with the difference that we must also consider curves with $D^2 = 0$, that is, elliptic pencils which we now characterize. By direct calculation, we note that there are no (-2)-curves in S. Equality holds in (ii) when $m = -n$ or $m = -un$ with $u := 2a/(2p+1)$.
First, we describe the effective divisors $D \subset S$ with self-intersection $D^2 = 0$. Consider the case $m = -un$. If $2p+1$ does not divide a, then $D \equiv 2aH - (2p+1)C$ and $D \cdot C = 2a(2a - 2p - 1) > g - 1$, that is, D does not verify condition (i). If $a = k(2p+1)$, for $k \geq 2$, then $D \equiv 2kH - C$. Notice that $D \cdot C = a(4k-4) + 2k(2p+1) > 2a$ for $k \geq 2$,

that is, D does not satisfies (i).
In the the case $m=-n$, the effective divisor $D \equiv C - H$, satisfies (i)-(iii) and

$$\mathrm{Cliff}(\mathcal{O}_C(C-H)) = 2a-2p-3 \geq a.$$

Case $n<0$. From (ii) we have either $m<-n$ or $m>-un$. In the first case, by using inequality (iii), we obtain $2<-(4p+2)n+dn = n(2a-2p-1)$, which is a contradiction since $n<0$ and $2a>2p+1$. Suppose $m>-un>0$. Inequality (i) implies that

$$(-n)\frac{2ad}{2p+1} < -(g-1)(2n-1) = -2a(2n-1),$$

then $(-n)(d-(4p+2)) < 2p+1$ and since $d>4p+2$, this yields $2a+2p+1 = d < 6p+3$ which contradicts the hypothesis $a \geq 2p+3$.

Case $n>0$. Again, by condition (ii), we have either that $m<-un$ or $m>-n$. In the first case, using (iii) we write that

$$0 < (4p+2)m + dn < n\left(d-(4p+2)\frac{2a}{2p+1}\right),$$

but one can get easily check that $d(2p+1) < 2a(4p+2)$, which yields a contradiction. Suppose now $-n<m<0$. By (i) we have $2a(2n-1) \leq -md < nd$, so $n < \frac{2a}{4a-d} = \frac{2a}{2a-2p-1} < 2$, since $a \geq 2p+1$. This implies $n=1$, therefore for $n>0$ there are no divisors $D \subset S$ with $D^2>0$ satisfying the inequalities (i)-(iii).

Case $n=0$. From (i), one writes $m \leq \frac{g-1}{d} = \frac{2a}{2a+2p+1} < 1$, but this yields to a contradiction since by (iii) it follows that $m>0$. The proof is thus finished.

□

4. Curves with prescribed gonality and small rank 2 Clifford index

The equality $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ is known to be valid for *arbitrary* k-gonal curves $[C] \in \mathcal{M}^1_{g,k}$ of genus $g > (k-1)(2k-4)$. It is thus of some interest to study Mercat's question for arbitrary curves in a given gonality stratum in $\mathcal{M}_g$ and decide how sharp is this quadratic bound. We shall construct curves C of unbounded genus and relatively small gonality, carrying a stable rank 2 vector bundle E with $h^0(C,E)=4$ such that $\gamma(E) < \mathrm{Cliff}(C)$. In order to be able to determine the gonality of C, we realize it as a section of a $K3$ surface S in $\mathbf{P}^4$ which is special in the sense of Noether-Lefschetz theory. The pencil computing the gonality is the restriction of an elliptic pencil on the surface. The constraint of having a Picard lattice of rank 2 containing, apart from the hyperplane class, both an elliptic pencil and a curve C of prescribed genus, implies that the discriminant of $\mathrm{Pic}(S)$ must be a perfect square. This imposes severe restrictions on the genera for which such a construction could work.

Theorem 4.1. *We fix integers $a \geq 3$ and $b=4,5,6$. There exists a smooth curve $C \subset \mathbf{P}^4$ with*

$$\deg(C) = 6a+b, \ \ g(C) = 3a^2+ab+1 \ \ \text{and gonality} \ \ \mathrm{gon}(C) = ab,$$

such that C lies on a $(2,3)$ complete intersection $K3$ surface. In particular $K_{1,1}(C, \mathcal{O}_C(1)) \neq 0$ and conjecture (M_2) fails for C.

Before presenting the proof, we discuss the connection between Theorem 4.1 and conjecture (M_2). For $C \subset S \subset \mathbf{P}^4$ as above, we construct a vector bundle E with $\det(E) = \mathcal{O}_C(1)$ and $h^0(C, E) = 4$, lying in an exact sequence

$$0 \longrightarrow E \longrightarrow W \otimes \mathcal{O}_C(1) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0,$$

where $W \in G(3, H^0(C, \mathcal{O}_C(1))$ has the property that the quadric $Q \in \mathrm{Sym}^2 H^0(C, \mathcal{O}_C(1))$ induced by S is representable by a tensor in $W \otimes H^0(C, L)$. This construction is a particular procedure of associating vector bundles to non-trivial syzygies, cf. [AN]. The proof that E is stable is standard and proceeds along the lines of e.g. [GMN] Theorem 3.2. Next we compute the Clifford invariant:

$$\gamma(E) = 3a + \frac{b}{2} < ab - 2 = \mathrm{Cliff}(C),$$

since $b \geq 4$, so not only $\mathrm{Cliff}_2(C) < \mathrm{Cliff}(C)$, but the difference $\mathrm{Cliff}(C) - \mathrm{Cliff}_2(C)$ becomes arbitrarily positive.

Proof. By means of [K] Theorem 6.1, there exist a smooth complete intersection surface $S \subset \mathbf{P}^4$ of type $(2, 3)$ such that $\mathrm{Pic}(S) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot C$, where $H^2 = 6,\ H \cdot C = d = 6a + b$ and $C^2 = 2(g-1)$ (Note that such a surface exists when $d^2 > 12g$, which is satisfied when $b \geq 4$). The divisor $E := C - aH$ verifies $E^2 = 0$, $E \cdot H = b$ and $E \cdot C = ab$. In particular E is effective. The class E is primitive, hence it follows that $h^0(S, E) = h^0(C, \mathcal{O}_C(E)) = 2$, where the last equality follows by noting that $H^1(S, \mathcal{O}_S(E - C)) = 0$ by Kodaira vanishing. Furthermore, $h^1(C, \mathcal{O}_C(E)) \geq 3a^2 + 2$, that is, $\mathcal{O}_C(E)$ contributes to Cliff(C) and then we write that

$$\mathrm{gon}(C) = \mathrm{Cliff}(C) + 2 \leq \mathrm{Cliff}(C, \mathcal{O}_C(E)) + 2 = ab.$$

We shall show that $\mathcal{O}_C(E)$ computes the Clifford index of C.
First, we classify the primitive effective divisors $F \equiv mH + nC \subset S$ having self-intersection zero. By solving the equation $(mH + nC)^2 = 0$, where $m, n \in \mathbb{Z}$, we find the following primitive solutions: $E_1 \equiv (3a + b)H - 3C$ for $b \neq 6$ (respectively $E_2 \equiv (a+2)H - C$ for $b = 6$), and $E_3 = E \equiv C - aH$. A simple computation shows that $E_i \cdot C > ab$ for $i = 1, 2$.

Since $\mathrm{Cliff}(C) \leq ab - 2 < [\frac{g-1}{2}]$, the Clifford index of C is computed by a bundle defined on S. Following [GL2], there exists an effective divisor $D \equiv mH + nC$ on S, satisfying the following numerical conditions:

(4) $$h^0(S, D) = h^0(C, \mathcal{O}_C(D)) \geq 2,\ h^0(S, C - D) \geq 2,\ \ D^2 \geq 0 \text{ and } \ D \cdot C \leq g - 1,$$

and such that

$$f(D) := \mathrm{Cliff}(\mathcal{O}_C(D)) + 2 = D \cdot C - D^2 = \mathrm{Cliff}(C) + 2.$$

Furthermore, D can be chosen such that $h^1(S, D) = 0$, cf. [Ma]. To bound $f(D)$ and show that $f(D) \geq ab$, we distinguish two cases depending on whether $D^2 > 0$ or $D^2 = 0$.

By a complete classification of curves with self-intersection zero, we have already seen that for any elliptic pencil $|D|$ satisfying (4), one has $f(D) \geq ab = f(E)$. We are left with the case $D^2 > 0$ and rewrite the inequalities (4):

$$\begin{array}{rl} (i) & (6a + b)m + (2n - 1)(3a^2 + ab) \leq 0 \\ (ii) & (m + an)(3an + 3m + bn) > 0 \\ (iii) & 6m + (6a + b)n > 2, \end{array}$$

where (ii) comes from the assumption $D^2 > 0$ and (iii) from the fact that $D \cdot H > 2$. Furthermore,

$$f(m,n) := D \cdot C - D^2 = -6m^2 + m(d - 2nd) + (n - n^2)(2g - 2). \tag{5}$$

We prove that for any divisor D satisfying (i) -(iii), the inequality $f(m,n) \geq ab$ holds, from which we conclude that $\mathrm{Cliff}(C) = ab - 2$.

Case $n < 0$. From (iii) we find that $m > 0$. Then $m < -an$ or $3m > -(3a + b)n$. When $m < -an$, from (iii) we have that $2 < 6m + dn < -6an + dn = nb < 0$, which is a contradiction. Suppose $(3a + b)n + 3m > 0$. For a fixed n the function $f(m,n)$ reaches its maximum at $m_0 := \frac{d(1-2n)}{12}$. So when $3m_0 + (3a + b)n \leq 0$, we have $f(m,n) \geq f\big(\frac{(1-2n)(g-1)}{d}, n\big)$, since by condition (i), $m \leq \frac{(1-2n)(g-1)}{d}$. A simple computation gives that whenever $n < 0$, one has the inequality:

$$\begin{aligned} f\left(\frac{(1-2n)(g-1)}{d}, n\right) &= (2n^2 - 2n)(g-1)\frac{b^2}{d^2} + (g-1)\left(1 - \frac{6(g-1)}{d^2}\right) \\ &\geq 4(g-1)\frac{b^2}{d^2} + \frac{g-1}{d^2}\big(18a^2 + b^2 + 6ab\big) \geq \frac{3a^2 + ab}{2} \geq ab. \end{aligned}$$

Assume now that $3m_0 + (3a+b)n > 0$. Since $m \in \big(-\frac{(3a+b)n}{3}, \frac{(1-2n)(g-1)}{d}\big]$, we have

$$f(m,n) \geq \min\Big\{f\Big(-\frac{(3a+b)n}{3}, n\Big), f\Big(\frac{(1-2n)(g-1)}{d}, n\Big)\Big\}.$$

A direct computation yields

$$f\Big(-\frac{(3a+b)n}{3}, n\Big) = -n\left(ab + \frac{b^2}{3}\right) \geq ab + \frac{b^2}{3} \geq ab.$$

Case $n > 0$. If $m \geq 0$ we get a contradiction to (i). Suppose $m < 0$, then we have either $3m + (3a+b)n < 0$, or else $m > -an$. The first case contradicts (iii), so it does not appear. Suppose $m > -an$. Reasoning as before, observe that $m_0 < (1-2n)(g-1)/d$, where m_0 is the maximum of $f(m,n)$ for a fixed n, and m takes values in the interval $\big(-an, \frac{(1-2n)(g-1)}{d}\big]$. If $-an \geq m_0$, then $f(m,n) \geq f\big(\frac{(1-2n)(g-1)}{d}, n\big)$. Since we are assuming $-an < \frac{(1-2n)(g-1)}{d}$, we have that $n < \frac{3a}{b} + 1$. We use this bound to directly show, like in the previous case, that $f\big(\frac{(1-2n)(g-1)}{d}, n\big) \geq ab$. When $-an < m_0$ we have that

$$f(m,n) \geq \min\Big\{f(-an,n), f\Big(\frac{(1-2n)(g-1)}{d}, n\Big)\Big\}.$$

In this case it is enough to note that $f(-an, n) = nab \geq ab$.

Case $n = 0$. From inequalities (i) and (iii) with $n = 0$, we have $1 \leq m \leq \frac{g-1}{d}$. Note that $f(m,0) = -6m^2 + md$ reaches its maximum at $\frac{d}{12}$. So, since $\frac{g-1}{d} \leq \frac{d}{12}$, we conclude that $f(m,0) \geq f(1,0) = 6a + b - 6$. Finally, we observe that $6a + b - 6 \geq ab$ if and only if $b \leq 6$. This finishes the proof. □

5. The Fourier-Mukai involution on $\mathcal{F}_{11}$

The aim of this section is to provide a detailed proof of Mercat's conjecture (M_2) in one non-trivial case, that of genus 11, and discuss the connection to Mukai's work [M1], [M3]. We denote as usual by $\mathcal{F}_g$ the moduli space parametrizing pairs $[S,\ell]$, where S is a smooth $K3$ surface and $\ell \in \mathrm{Pic}(S)$ is a primitive nef line bundle with $\ell^2 = 2g-2$. Furthermore, we introduce the parameter space

$$\mathcal{P}_g := \big\{[S,C] : S \text{ is a smooth } K3 \text{ surface}, \quad C \subset S \text{ is a smooth curve}, \ \ [S, \mathcal{O}_S(C)] \in \mathcal{F}_g\big\}$$

and denote by $\pi : \mathcal{P}_g \to \mathcal{F}_g$ the projection map $[S,C] \mapsto [S, \mathcal{O}_S(C)]$. If S is a $K3$ surface, following [M1], we set $\widetilde{H}(S,\mathbb{Z}) := H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$ and

$$\widetilde{NS}(S) := H^0(S,\mathbb{Z}) \oplus NS(S) \oplus H^4(S,\mathbb{Z}).$$

We recall the definition of the *Mukai pairing* on $\widetilde{H}(S,\mathbb{Z})$:

$$(\alpha_0,\alpha_2,\alpha_4)\cdot(\beta_0,\beta_2,\beta_4) := \alpha_2\cup\beta_2 - \alpha_4\cup\beta_0 - \alpha_0\cup\beta_4 \in H^4(S,\mathbb{Z}) = \mathbb{Z}.$$

Let now $r,s \geq 1$ be relatively prime integers such that $g = 1+rs$. For a polarized $K3$ surface $[S,\ell] \in \mathcal{F}_g$ one defines the *Fourier-Mukai dual* $\hat{S} := M_S(r,\ell,s)$, where

$$M_S(r,\ell,s) = \big\{E : E \text{ is an } \ell-\text{stable sheaf on } S, \ \mathrm{rk}(E) = r, \ c_1(E) = \ell, \chi(S,E) = r+s\big\}.$$

Setting $v := (r,\ell,s) \in \widetilde{H}(S,\mathbb{Z})$, there is a Hodge isometry, see [M1] Theorem 1.4:

$$\psi : H^2(M_S(r,\ell,s),\mathbb{Z}) \xrightarrow{\cong} v^{\perp}/\mathbb{Z}v.$$

We observe that $\hat{\ell} := \psi^{-1}((0,\ell,2s))$ is a nef primitive vector with $(\hat{\ell})^2 = 2g-2$, and in this way the pair $(\hat{S},\hat{\ell})$ becomes a polarized $K3$ surface of genus g. The *Fourier-Mukai involution* is the morphism $FM : \mathcal{F}_g \to \mathcal{F}_g$ defined by $FM([S,\ell]) := [\hat{S},\hat{\ell}]$.

We turn to the case $g = 11$, when we set $r = 2$ and $s = 5$. For a general curve $[C] \in \mathcal{M}_{11}$, the Lagrangian Brill-Noether locus

$$\mathcal{SU}_C(2,K_C,7) := \{E \in \mathcal{U}_C(2,20) : \det(E) = K_C, \ h^0(C,E) = 7\}$$

is a smooth $K3$ surface. The main result of [M3] can be summarized as saying a general $[C] \in \mathcal{M}_{11}$ lies on a unique $K3$ surface which moreover can be realized as $\widehat{\mathcal{SU}_C(2,K_C},7)$. Furthermore, there is a birational isomorphism

$$\phi_{11} : \mathcal{M}_{11} \dashrightarrow \mathcal{P}_{11}, \quad \phi_{11}([C]) := \big[\widehat{\mathcal{SU}_C(2,K_C},7), C\big]$$

and we set $q_{11} := \pi \circ \phi_{11} : \mathcal{M}_{11} \dashrightarrow \mathcal{F}_{11}$. On the moduli space $\mathcal{M}_{11}$ there exist two distinct irreducible Brill-Noether divisors

$$\mathcal{M}^1_{11,6} := \{[C] \in \mathcal{M}_{11} : W^1_6(C) \neq \emptyset\} \text{ and } \mathcal{M}^2_{11,9} := \{[C] \in \mathcal{M}_{11} : W^2_9(C) \neq \emptyset\}.$$

Via the residuation morphism $W^1_6(C) \ni L \mapsto K_C \otimes L^\vee \in W^5_{14}(C)$, the Hurwitz divisor is the pull-back of a Noether-Lefschetz divisor on $\mathcal{F}_{11}$, that is, $\mathcal{M}^1_{11,6} = q^*_{11}(D^1_6)$ where

$$D^1_6 := \{[S,\ell] \in \mathcal{F}_{11} : \exists H \in \mathrm{Pic}(S), \ H^2 = 8, \ H\cdot\ell = 14\}.$$

Similarly, via the residuation map $W^2_9(C) \ni L \mapsto K_C \otimes L^\vee \in W^3_{11}(C)$, one has the equality of divisors $\mathcal{M}^2_{11,9} = q^*_{11}(D^2_9)$, where

$$D^2_9 := \{[S,\ell] \in \mathcal{F}_{11} : \exists H \in \mathrm{Pic}(S), \ H^2 = 4, \ H\cdot\ell = 11\}.$$

Next we establish Mercat's conjecture for general curves of genus 11.

Theorem 5.1. *The equality* $\mathrm{Cliff}_2(C) = \mathrm{Cliff}(C)$ *holds for a general curve* $[C] \in \mathcal{M}_{11}$.

Proof. We fix a curve $[C] \in \mathcal{M}_{11}$ such that (i) $W^1_7(C)$ is a smooth curve, (ii) $W^2_9(C) = \emptyset$ (in particular, any Petri general curve will satisfy these conditions) and (iii) the rank 2 Brill-Noether locus $\mathcal{SU}_C(2, K_C, 7)$ is a smooth $K3$ surface of Picard number 1. As discussed in both [LMN] Proposition 4.5 and [FO] Question 3.5, in order to verify (M_2), it suffices to show that C possesses no bundles $E \in \mathcal{U}_C(2, 13)$ with $h^0(C, E) = 4$. Suppose E is such a vector bundle. Then $L := \det(E) \in W^4_{13}(C)$ is a linear series such that the multiplication map $\nu_2(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is not injective. For each extension class

$$e \in \mathbf{P}_L := \mathbf{P}(\mathrm{Coker}\ \nu_2(L))^\vee \subset \mathbf{P}(H^0(C, L^{\otimes 2}))^\vee = \mathbf{P}\mathrm{Ext}^1(L, K_C \otimes L^\vee),$$

one obtains a rank 2 vector bundle F on C sitting in an exact sequence

$$0 \longrightarrow K_C \otimes L^\vee \longrightarrow F \longrightarrow L \longrightarrow 0, \tag{6}$$

such that $h^0(C, F) = h^0(C, L) + h^0(C, K_C \otimes L^\vee) = 7$. We claim that any non-split vector bundle F with $h^0(C, F) = 7$ and which sits in an exact sequence (6), is semistable. Indeed, let us assume by contradiction that $M \subset F$ is a destabilizing line subbundle with $\deg(M) \geq 11$. Since $\deg(M) > \deg(K_C \otimes L^\vee)$, the composite morphism $M \to L$ is non-zero, hence we can write that $M = L(-D)$, where D is an effective divisor of degree 1 or 2. Because $W^2_9(C) = \emptyset$, one finds that $h^0(C, K_C \otimes L^\vee(D)) = 2$ and L must be very ample, that is, $h^0(C, L(-D)) = h^0(C, L) - \deg(D)$. We obtain that

$$h^0(L)+h^0(K_C\otimes L^\vee) = h^0(F) \leq h^0(M)+h^0(K_C\otimes M^\vee) = h^0(L)-\deg(D)+h^0(K_C\otimes L^\vee),$$

a contradiction. Thus one obtains an induced morphism $u : \mathbf{P}_L \to \mathcal{SU}_C(2, K_C, 7)$. Since $\mathcal{SU}_C(2, K_C, 7)$ is a $K3$ surface, this also implies that $\mathrm{Coker}\ \nu_2(L)$ is 2-dimensional, hence $\mathbf{P}_L = \mathbf{P}^1$.

We claim that u is an embedding. Setting $A := K_C \otimes L^\vee \in W^1_7(C)$, we write the exact sequence $0 \to H^0(C, \mathcal{O}_C) \to H^0(C, F^\vee \otimes L) \to H^0(C, K_C \otimes A^{\otimes(-2)})$, and note that the last vector space is the kernel of the Petri map $H^0(C, A) \otimes H^0(C, L) \to H^0(C, K_C)$, which is injective, hence $h^0(C, F^\vee \otimes L) = 1$. This implies that u is an embedding. But this contradicts the fact that $\mathrm{Pic}\ \mathcal{SU}_C(2, K_C, 7) = \mathbb{Z}$, in particular $\mathcal{SU}_C(2, K_C, 7)$ contains no (-2)-curves. We conclude that $\nu_2(L)$ is injective for *every* $L \in W^4_{13}(C)$. $\square$

This proof also shows that the failure locus of statement (M_2) on $\mathcal{M}_{11}$ is equal to the Koszul divisor

$$\mathfrak{Syz}^4_{11,13} := \{[C] \in \mathcal{M}_{11} : \exists L \in W^4_{13}(C) \text{ such that } K_{1,1}(C, L) \neq 0\}.$$

Suppose now that $[C] \in \mathfrak{Syz}^4_{11,13}$ is a general point corresponding to an embedding $C \overset{|L|}{\hookrightarrow} \mathbf{P}^4$ such that C lies on a $(2, 3)$ complete intersection $K3$ surface $S \subset \mathbf{P}^4$. Then $S = \widehat{\mathcal{SU}_C(2, K_C, 7)}$ and $\rho(S) = 2$ and furthermore $\mathrm{Pic}(S) = \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H$, where $H^2 = 6, C \cdot H = 13$ and $C^2 = 20$. In particular we note that S contains no (-2)-curves, hence S and $\hat{S}$ are not isomorphic.

Let us define the Noether-Lefschetz divisor

$$D^4_{13} := \{[S, \ell] \in \mathcal{F}_{11} : \exists H \in \mathrm{Pic}(S),\ H^2 = 6, H \cdot \ell = 13\},$$

therefore $\mathfrak{Syz}^4_{11,13} = q^*_{11}(D^4_{13})$.

Proposition 5.2. *The action of the Fourier-Mukai involution $FM : \mathcal{F}_{11} \to \mathcal{F}_{11}$ on the three distinguished Noether-Lefschetz divisors is described as follows:*

(i) $FM(D_6^1) = D_6^1$.
(ii) $FM(D_9^2) = \{[S, \ell] \in \mathcal{F}_{11} : \exists R \in \mathrm{Pic}(S)$ *such that* $R^2 = -2,\ R \cdot \ell = 1\}$.
(iii) $FM(D_{13}^4) = \{[S, \ell] \in \mathcal{F}_{11} : \exists R \in \mathrm{Pic}(S)$ *such that* $R^2 = -2,\ R \cdot \ell = 3\}$.

Proof. For $[S, \ell] \in \mathcal{F}_{11}$, we set $v := (2, \ell, 5) \in \widetilde{H}(S, \mathbb{Z})$ and $\hat{\ell} := (0, \ell, 10) \in \widetilde{H}(S, \mathbb{Z})$ for the class giving the genus 11 polarization. We describe the lattice $\psi\big(NS(\hat{S})\big) \subset \widetilde{NS}(S)$.

In the case of a general point of D_6^1 with lattice $NS(S) = \mathbb{Z} \cdot \ell \oplus \mathbb{Z} \cdot H$, by direct calculation we find that $\psi\big(NS(\hat{S})\big)$ is generated by the vectors $\hat{\ell}$ and $(2, \ell + H, 12)$. Furthermore, $(2, \ell + H, 12)^2 = 8$ and $(2, H + \ell, 12) \cdot \hat{\ell} = 14$, that is, $\mathrm{Pic}(\hat{S}) \cong \mathrm{Pic}(S)$, hence D_6^1 is a fixed divisor for the automorphism FM.

A similar reasoning for a general point of the divisor D_9^2 shows that the Neron-Severi groups $\psi\big(NS(\hat{S})\big)$ is generated by $\hat{\ell}$ and $(-1, H - \ell, -2)$, where $(-1, H - \ell, -2)^2 = -2$ and $(-1, H - \ell, -2) \cdot \hat{\ell} = 1$. In other words, the class $(-1, H - \ell, -2)$ corresponds to a line in the embedding $\hat{S} \stackrel{|\hat{\ell}|}{\hookrightarrow} \mathbf{P}^{11}$. Finally, for a general point of D_{13}^4 corresponding to a lattice $\mathbb{Z} \cdot \ell \oplus \mathbb{Z} \cdot H$, the Picard lattice of the Fourier-Mukai partner is spanned by the vectors $\hat{\ell}$ and $(-1, H - \ell, -1)$, where $(-1, H - \ell, -1)^2 = -2$ and $(-1, H - \ell, -1) \cdot \hat{\ell} = 3$. $\square$

Remark 5.3. The fact that the divisor D_6^1 is fixed by the automorphism FM is already observed and proved with geometric methods in [M3] Theorem 3.

Remark 5.4. It is instructive to point out the difference between a general element of D_{13}^4 and its Fourier-Mukai partner. As a polarized $K3$ surface, $\mathcal{SU}_C(2, K_C, 7)$ is characterized by the existence of a degree 3 rational curve $u(\mathbf{P}_L) \subset \mathcal{SU}_C(2, K_C, 7)$. On the other hand, the complete intersection surface $S \subset \mathbf{P}^4$ containing $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^4$, where $L \in W_{13}^4(C)$, carries no smooth rational curves. It contains however elliptic curves in the linear system $|\mathcal{O}_S(C - H)|$. Thus the involution FM assigns to a $K3$ surface with a degree 7 elliptic pencil, a $K3$ surface containing a (-2)-curve. Since $S = \widehat{\mathcal{SU}_C(2, K_C, 7)}$, it also follows that the complete intersection S is a smooth $K3$ surface, which a priori is not at all obvious.

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HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, UNTER DEN LINDEN 6
10099 BERLIN, GERMANY
E-mail address: farkas@math.hu-berlin.de

HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, UNTER DEN LINDEN 6
10099 BERLIN, GERMANY
E-mail address: ortega@math.hu-berlin.de