MODULI OF THETA-CHARACTERISTICS VIA NIKULIN SURFACES

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The importance of the locus $\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C$ lies on a $K3$ surface$\}$ has been recognized for some time. Fundamental results in the theory of algebraic curves like the Brill-Noether Theorem [Laz], or Green's Conjecture for generic curves [Vo] have been proved by specialization to a general point $[C] \in \mathcal{K}_g$. The variety $\mathcal{K}_g$ viewed as a subvariety of $\mathcal{M}_g$ serves as an obstruction for effective divisors on $\overline{\mathcal{M}}_g$ to having small slope [FP] and thus plays a significant role in determining the cone of effective divisors on $\overline{\mathcal{M}}_g$.

The first aim of this paper is to show that at the level of the the *Prym moduli space* $\mathcal{R}_g$ classifying étale double covers of curves of genus g, the locus of curves lying on a *Nikulin $K3$ surfaces* plays a similar role. The analogy is far-reaching: Nikulin surfaces furnish an explicit unirational parametrization of $\mathcal{R}_g$ in small genus, see Theorem 0.2, just like ordinary $K3$ surfaces do the same for $\mathcal{M}_g$; numerous results involving curves on $K3$ surfaces have a Prym-Nikulin analogue, see Theorem 0.4, and even exceptions to uniform statements concerning curves on $K3$ surfaces carry over in this analogy!

Our other aim is to complete the birational classification of the moduli space $\overline{\mathcal{S}}_g^+$ of even spin curves of genus g. It is known [F] that $\overline{\mathcal{S}}_g^+$ is of general type when $g \geq 9$. Using Nikulin surfaces we show that $\overline{\mathcal{S}}_g^+$ is uniruled for $g \leq 7$, see Theorem 0.7, which leaves $\overline{\mathcal{S}}_8^+$ as the only case missing from the classification. We prove the following:

Theorem 0.1. *The Kodaira dimension of $\overline{\mathcal{S}}_8^+$ is equal to zero.*

Theorems 0.1 and 0.7 highlight the fact that the birational type of $\overline{\mathcal{S}}_g^+$ is entirely governed by the world of $K3$ surfaces, in the sense that $\overline{\mathcal{S}}_g^+$ is uniruled precisely when a general even spin curve of genus g moves on a special $K3$ surface. This is in contrast to $\overline{\mathcal{M}}_g$ which is known to be uniruled at least for $g \leq 16$, whereas the general curve of genus $g \geq 12$ does not lie on a $K3$ surface.

A *Nikulin surface* [Ni] is a $K3$ surface S endowed with a non-trivial double cover

$$f : \tilde{S} \to S$$

with a branch divisor $N := N_1 + \cdots + N_8$ consisting of 8 disjoint smooth rational curves $N_i \subset S$. Blowing down the (-1)-curves $E_i := f^{-1}(N_i) \subset \tilde{S}$, one obtains a minimal $K3$ surface $\sigma : \tilde{S} \to Y$, together with an involution $\iota \in \mathrm{Aut}(Y)$ having 8 fixed points corresponding to the images $\sigma(E_i)$ of the exceptional divisors. The class $\mathcal{O}_S(N)$ is divisible by 2 in $\mathrm{Pic}(S)$ and we set $e := \frac{1}{2}\mathcal{O}_S(N_1 + \cdots + N_8) \in \mathrm{Pic}(S)$. Assume that $C \subset S$ is a smooth curve of genus g such that $C \cdot N_i = 0$ for $i = 1, \ldots, 8$. We say that the triple $(S, e, \mathcal{O}_S(C))$ is a *polarized Nikulin surface of genus g* and denote by $\mathcal{F}_g^{\mathfrak{N}}$ the 11-dimensional moduli space of such objects. Over $\mathcal{F}_g^{\mathfrak{N}}$ we consider the $\mathbf{P}^g$-bundle

$$\mathcal{P}_g^{\mathfrak{N}} := \Big\{(S, e, C) : C \subset S \text{ is a smooth curve such that } [S, e, \mathcal{O}_S(C)] \in \mathcal{F}_g^{\mathfrak{N}}\Big\},$$

which comes equipped with two maps

$$\begin{array}{ccc} & \mathcal{P}_g^{\mathfrak{N}} & \\ {\scriptstyle p_g}\swarrow & & \searrow{\scriptstyle \chi_g} \\ \mathcal{F}_g^{\mathfrak{N}} & & \mathcal{R}_g \end{array}$$

where $p_g([S,e,C]) := [S,e,\mathcal{O}_S(C)]$ and $\chi_g([S,e,C]) := [C, e_C := e \otimes \mathcal{O}_C]$. Since $C \cdot N = 0$, it follows that $e_C^{\otimes 2} = \mathcal{O}_C$. The étale double cover induced by e_C is precisely the restriction $f_C := f_{|\tilde{C}} : \tilde{C} \to C$, where $\tilde{C} := f^{-1}(C)$. Note that $\dim(\mathcal{P}_g^{\mathfrak{N}}) = 11 + g$ and it is natural to ask when is χ_g dominant and induces a uniruled parametrization of $\mathcal{R}_g$.

Theorem 0.2. *The general Prym curve $[C, e_C] \in \mathcal{R}_g$ lies on a Nikulin surface if and only if $g \leq 7$ and $g \neq 6$, that is, the morphism $\chi_g : \mathcal{P}_g^{\mathfrak{N}} \to \mathcal{R}_g$ is dominant precisely in this range.*

In contrast, the general Prym curve $[C, e_C] \in \mathcal{R}_6$ lies on an Enriques surface [V1] but not on a Nikulin surface. Since $\mathcal{P}_g^{\mathfrak{N}}$ is a uniruled variety being a $\mathbf{P}^g$-bundle over $\mathcal{F}_g^{\mathfrak{N}}$, we derive from Theorem 0.2 the following immediate consequence:

Corollary 0.3. *The Prym moduli space $\mathcal{R}_g$ is uniruled for $g \leq 7$.*

The discussion in Sections 2 and 3 implies the stronger result that $\mathcal{F}_g^{\mathfrak{N}}$ (and thus $\mathcal{N}_g := \mathrm{Im}(\chi_g)$) is unirational for $g \leq 6$. It was known that $\mathcal{R}_g$ is rational for $g \leq 4$, see [Do2], [Ca], and unirational for $g = 5, 6$, see [Do], [ILS], [V1], [V2]. Apart from the result in genus 7 which is new, the significance of Corollary 0.3 is that Nikulin surfaces provide an *explicit uniform parametrization* of $\mathcal{R}_g$ that works for all genera $g \leq 7$.

Before going into a more detailed explanation of our results on $\mathcal{F}_g^{\mathfrak{N}}$, it is instructive to recall Mukai's work on the moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces of genus g:

Mukai's results [M1], [M2], [M3]:

(1) A general curve $[C] \in \mathcal{M}_g$ lies on a $K3$ surface if and only if $g \leq 11$ and $g \neq 10$, that is, the equality $\mathcal{K}_g = \mathcal{M}_g$ holds precisely in this range.

(2) $\mathcal{M}_{11}$ is birationally isomorphic to the tautological $\mathbf{P}^{11}$-bundle $\mathcal{P}_{11}$ over the moduli space $\mathcal{F}_{11}$ of polarized $K3$ surfaces of genus 11. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{11} & \overset{\cong}{\underset{q_{11}}{\dashleftarrow}} & \mathcal{P}_{11} \\ & \searrow \quad \swarrow{\scriptstyle p_{11}} & \\ & \mathcal{F}_{11} & \end{array}$$

with $q_{11}^{-1}([C]) = [S, C]$, where S is the unique $K3$ surface containing a general $[C] \in \mathcal{M}_{11}$.

(3) The locus $\mathcal{K}_{10}$ is a divisor on $\mathcal{M}_{10}$ which has the following set-theoretic incarnation:

$$\mathcal{K}_{10} = \bigl\{[C] \in \mathcal{M}_{10} : \exists L \in W^4_{12}(C) \text{ such that } \mu_0(L) : \mathrm{Sym}^2 H^0(C, L) \xrightarrow{\ncong} H^0(C, L^{\otimes 2})\bigr\}.$$

(4) There exists a rational variety $X \subset \mathbf{P}^{13}$ with $K_X = \mathcal{O}_X(-3)$ and $\dim(X) = 5$, such that the general $K3$ surface of genus 10 appears as a 2-dimensional linear section of X. Such a realization is unique up to the action of $\mathrm{Aut}(X)$ and one has birational isomorphisms:

$$\mathcal{F}_{10} \overset{\cong}{\dashrightarrow} G\bigl(\mathbf{P}^{10}, \mathbf{P}^{13}\bigr)^{\mathrm{ss}} /\!/ \mathrm{Aut}(X) \quad \text{and} \quad \mathcal{K}_{10} \overset{\cong}{\dashrightarrow} G\bigl(\mathbf{P}^{9}, \mathbf{P}^{13}\bigr)^{\mathrm{ss}} /\!/ \mathrm{Aut}(X).$$

To this list of well-known results, one could add the following statement from [FP]: (5) The closure $\overline{\mathcal{K}}_{10}$ of $\mathcal{K}_{10}$ inside $\overline{\mathcal{M}}_{10}$ is an extremal point in the effective cone $\mathrm{Eff}(\overline{\mathcal{M}}_{10})$; its class $\overline{\mathcal{K}}_{10} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - \cdots \in \mathrm{Pic}(\overline{\mathcal{M}}_{10})$ has minimal slope among all effective divisors on $\overline{\mathcal{M}}_{10}$ and provides a counterexample to the Slope Conjecture [HMo].

Quite remarkably, each of the statements (1)-(5) has a precise Prym-Nikulin analogue. Theorem 0.2 is the analogue of (1). For the highest genus when the Prym-Nikulin condition is generic, the moduli space acquires a surprising Mori fibre space structure:

Theorem 0.4. *The moduli space $\mathcal{R}_7$ is birationally isomorphic to the tautological $\mathbf{P}^7$-bundle $\mathcal{P}_7^{\mathfrak{N}}$ and there is a commutative diagram:*

$$\begin{array}{ccc} \mathcal{R}_7 & \overset{\cong}{\underset{\chi_7}{\dashleftarrow}} & \mathcal{P}_7^{\mathfrak{N}} \\ & \searrow \quad \swarrow {\scriptstyle p_7} & \\ & \mathcal{F}_7^{\mathfrak{N}} & \end{array}$$

Furthermore, $\chi_7^{-1}([C,\eta]) = [S, C]$, where the unique Nikulin surface S containing C is given by the base locus of the net of quadrics containing the Prym-canonical embedding $\phi_{K_C \otimes \eta} : C \to \mathbf{P}^5$.

Just like in Mukai's work, the genus next to maximal from the point of view of Prym-Nikulin theory, behaves exotically.

Theorem 0.5. *The Prym-Nikulin locus $\mathcal{N}_6 := \mathrm{Im}(\chi_6)$ is a divisor on $\mathcal{R}_6$ which can be identified with the ramification locus of the Prym map $\mathrm{Pr}_6 : \mathcal{R}_6 \to \mathcal{A}_5$:*

$$\mathcal{N}_6 = \big\{[C,\eta] \in \mathcal{R}_6 : \mu_0(K_C \otimes \eta) : \mathrm{Sym}^2 H^0(C, K_C \otimes \eta) \stackrel{\ncong}{\longrightarrow} H^0(C, K_C^{\otimes 2})\big\}.$$

Observe that both divisors $\mathcal{K}_{10}$ and $\mathcal{N}_6$ share the same Koszul-theoretic description. Furthermore, they are both extremal points in their respective effective cones, cf. Proposition 3.6. Is there a Prym analogue of the genus 10 *Mukai G_2-variety* $X := G_2/P \subset \mathbf{P}^{13}$? The answer to this question is in the affirmative and we outline the construction of a Grassmannian model for $\mathcal{F}_6^{\mathfrak{N}}$ while referring to Section 3 for details.

Set $V := \mathbb{C}^5$ and $U := \mathbb{C}^4$ and view $\mathbf{P}^3 = \mathbf{P}(U)$ as the space of planes inside $\mathbf{P}(U^\vee)$. Let us choose a smooth quadric $Q \subset \mathbf{P}(V)$. The quadratic line complex $W_Q \subset G(2, V) \subset \mathbf{P}(\wedge^2 V)$ consisting of tangent lines to Q is singular along the codimension 2 subvariety V_Q of lines contained in Q. One can identify V_Q with the Veronese 3-fold

$$\nu_2(\mathbf{P}^3) \subset \mathbf{P}\big(\mathrm{Sym}^2(U)\big) = \mathbf{P}(\wedge^2 V) = \mathbf{P}^9.$$

The projective tangent bundle $\mathbf{P}_Q$ of Q, viewed as the blow-up of W_Q along V_Q, is endowed with a double cover branched along V_Q and induced by the map

$$\mathbf{P}^3 \times \mathbf{P}^3 \stackrel{2:1}{\longrightarrow} \mathbf{P}\big(\mathrm{Sym}^2(U)\big), \quad (H_1, H_2) \mapsto H_1 + H_2.$$

We show in Theorem 3.4 that codimension 3 linear sections of W_Q are Nikulin surfaces of genus 6 with general moduli. Moreover there is a birational isomorphism

$$\mathcal{F}_6^{\mathfrak{N}} \stackrel{\cong}{\dashrightarrow} G(7, \wedge^2 V)^{\mathrm{ss}} /\!/ \mathrm{Aut}(Q).$$

Taking codimension 4 linear sections of W_Q one obtains a similar realization of $\mathcal{N}_6$, which should be viewed as the Prym counterpart of Mukai's construction of $\mathcal{K}_{10}$.

The subvariety $\mathcal{K}_g \subset \mathcal{M}_g$ is *intrinsic in moduli*, that is, its generic point $[C]$ admits characterizations that involve C alone and the $K3$ surface containing C is a result of some peculiarity of the canonical curve. For instance [BM], if $[C] \in \mathcal{K}_g$ then the *Wahl map*

$$\psi_{K_C} : \wedge^2 H^0(C, K_C) \to H^0(C, K_C^{\otimes 3}),$$

is not surjective. It is natural to ask for similar intrinsic characterizations of the Prym-Nikulin locus $\mathcal{N}_g \subset \mathcal{R}_g$ in terms of Prym curves alone, without making reference to Nikulin surfaces. In this direction, we prove in Section 1 the following result:

Theorem 0.6. *Set $g := 2i + 6$. Then $K_{i,2}(C, K_C \otimes \eta) \neq 0$ for any $[C, \eta] \in \mathcal{N}_g$, that is, the Prym-canonical curve $C \stackrel{|K_C \otimes \eta|}{\longrightarrow} \mathbf{P}^{g-2}$ of a Prym-Nikulin section fails to satisfy property (N_i).*

It is the content of the *Prym-Green Conjecture* [FL] that $K_{i,2}(C, K_C \otimes \eta) = 0$ for a general Prym curve $[C, \eta] \in \mathcal{R}_{2i+6}$. This suggests that curves on Nikulin surfaces can be recognized by extra syzygies of their Prym-canonical embedding.

Our initial motivation for considering Nikulin surfaces was to use them for the birational classification of moduli spaces of even theta-characteristics and we propose to turn our attention to the moduli space $\mathcal{S}_g^+$ of even spin curves classifying pairs $[C, \eta]$, where $[C] \in \mathcal{M}_g$ is a smooth curve of genus g and $\eta \in \mathrm{Pic}^{g-1}(C)$ is an even theta-characteristic. Let $\overline{\mathcal{S}}_g^+$ be the coarse moduli space associated to the Deligne-Mumford stack of even stable spin curves of genus g, cf. [Cor]. The projection $\pi : \mathcal{S}_g^+ \to \mathcal{M}_g$ extends to a finite covering $\pi : \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ branched along the boundary divisor Δ_0 of $\overline{\mathcal{M}}_g$. It is shown in [F] that $\overline{\mathcal{S}}_g^+$ is a variety of general type as soon as $g \geq 9$.

The existence of the dominant morphism $\chi_g : \mathcal{P}_g^{\mathfrak{N}} \to \mathcal{R}_g$ when $g \leq 7$ and $g \neq 6$, leads to a straightforward uniruled parametrization of $\overline{\mathcal{S}}_g^+$, which we briefly describe. Let us start with a general even spin curve $[C, \eta] \in \mathcal{S}_g^+$ and a non-trivial point of order two $e_C \in \mathrm{Pic}^0(C)$ in the Jacobian, such that $h^0(C, e_C \otimes \eta) \geq 1$. Since the curve $[C] \in \mathcal{M}_g$ is general, it follows that $h^0(C, e_C \otimes \eta) = 1$ and $Z := \mathrm{supp}(e_C \otimes \eta)$ consists of $g-1$ distinct points. Applying Theorem 0.2, if $g \neq 6$ there exists a Nikulin $K3$ surface (S, e) containing C such that $e_C = e \otimes \mathcal{O}_C$. When $g = 6$, there exists an Enriques surface (S, e) satisfying the same property, see [V1], and the construction described below goes through in that case as well. In the embedding $\phi_{|\mathcal{O}_S(C)|} : S \to \mathbf{P}^g$, the span $\langle Z \rangle \subset \mathbf{P}^g$ is a codimension 2 linear subspace and $h^0(S, \mathcal{I}_{Z/S}(1)) = 2$. Let

$$P := \mathbf{P}H^0\big(S, \mathcal{I}_{Z/S}(1)\big) \subset |\mathcal{O}_S(C)|$$

be the corresponding pencil of curves on S. Each curve $D \in P$ is endowed with the odd theta-characteristic $\mathcal{O}_D(Z)$. Twisting this line bundle with $e \otimes \mathcal{O}_D \in \mathrm{Pic}^0(D)$, we obtain an even theta-characteristic on D. This procedure induces a rational curve in moduli

$$m : P \to \overline{\mathcal{S}}_g^+, \quad P \ni D \mapsto [D, e \otimes \mathcal{O}_D(Z)],$$

which passes through the general point $[C, \eta] \in \overline{\mathcal{S}}_g^+$. This proves the following result:

Theorem 0.7. *The moduli space $\overline{\mathcal{S}}_g^+$ is uniruled for $g \leq 7$.*

It is known [F] that $\overline{\mathcal{S}}_g^+$ is of general type when $g \geq 9$. We complete the birational classification of $\overline{\mathcal{S}}_g^+$ and wish to highlight the following result, see Theorem 0.1:

$\boxed{\overline{\mathcal{S}}_8^+ \text{ is a variety of Calabi-Yau type.}}$

We observe the curious fact that $\overline{\mathcal{S}}_8^-$ is unirational [FV] whereas $\overline{\mathcal{S}}_8^+$ is not even uniruled. In contrast to the case of $\overline{\mathcal{S}}_g^{\mp}$, the birational classification of other important classes of moduli spaces is not complete. The Kodaira dimension of $\overline{\mathcal{M}}_g$ is unknown for $17 \leq g \leq 21$, see [HM], [EH1], the birational type of $\overline{\mathcal{R}}_g$ is not understood in the range $8 \leq g \leq 13$, see [FL], whereas finding the Kodaira dimension of $\mathcal{A}_6$ is a notorious open problem. Settling these outstanding cases is expected to require genuinely new ideas.

The proof of Theorem 0.1 relies on two main ideas: Following [F], one finds an *explicit* effective representative for the canonical divisor $K_{\overline{\mathcal{S}}_8^+}$ as a $\mathbb{Q}$-combination of the divisor $\overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_8^+$ of vanishing theta-nulls, the pull-back $\pi^*(\overline{\mathcal{M}}^2_{8,7})$ of the Brill-Noether divisor $\overline{\mathcal{M}}^2_{8,7}$ on $\overline{\mathcal{M}}_8$ of curves with a $\mathfrak{g}^2_7$, and boundary divisor classes corresponding to spin curves whose underlying stable model is of compact type. This already implies the inequality $\kappa(\overline{\mathcal{S}}_8^+) \geq 0$. Each irreducible component of this particular representative of $K_{\overline{\mathcal{S}}_8^+}$ is rigid (see Section 3), and the goal is to show that $K_{\overline{\mathcal{S}}_8^+}$ is rigid as well. To that end, we use the existence of a birational model $\mathfrak{M}_8$ of $\overline{\mathcal{M}}_8$ inspired by Mukai's work [M2]. The space $\mathfrak{M}_8$ is realized as the following GIT quotient

$$\mathfrak{M}_8 := G(8, \wedge^2 V)^{\mathrm{ss}} /\!/ SL(V),$$

where $V = \mathbb{C}^6$. We note that $\rho(\mathfrak{M}_8) = 1$ and there exists a birational morphism

$$f : \overline{\mathcal{M}}_8 \dashrightarrow \mathfrak{M}_8,$$

which contracts all the boundary divisors $\Delta_1, \ldots, \Delta_4$ as well as $\overline{\mathcal{M}}^2_{8,7}$. Using the geometric description of f, we establish a geometric characterization of points inside $\overline{\Theta}_{\mathrm{null}}$:

Proposition 0.8. *Let C be a smooth curve of genus 8 without a $\mathfrak{g}^2_7$. The following are equivalent:*

- *There exists a vanishing theta-null L on C, that is, $[C, L] \in \overline{\Theta}_{\mathrm{null}}$.*
- *There exists a smooth $K3$ surface S together with elliptic pencils $|F_1|$ and $|F_2|$ on S, such that $C \in |F_1 + F_2|$ and $L = \mathcal{O}_C(F_1) = \mathcal{O}_C(F_2)$.*

The existence of such a doubly elliptic $K3$ surface S is equivalent to stating that there exists a smooth $K3$ extension $S \subset \mathbf{P}^8$ of the canonical curve $C \subset \mathbf{P}^7$, such that the rank three quadric $C \subset Q \subset \mathbf{P}^7$ which induces the theta-null L, lifts to a *rank 4* quadric $S \subset Q_S \subset \mathbf{P}^8$. Having produced S, the pencils $|F_1|$ and $|F_2|$ define a product map

$$\phi : S \to \mathbf{P}^1 \times \mathbf{P}^1,$$

such that each smooth member $D \in I := |\phi^*\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1)|$ is a canonical curve contained in a rank 3 quadric. A general pencil in I passing through C induces a rational curve $R \subset \overline{\mathcal{S}}_8^+$, and after intersection theoretic calculations on the stack $\overline{\mathcal{S}}_8^+$, we prove the following:

Proposition 0.9. *The theta-null divisor $\overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_8^+$ is uniruled and swept by rational curves $R \subset \overline{\mathcal{S}}_8^+$ such that $R \cdot \overline{\Theta}_{\mathrm{null}} < 0$ and $R \cdot \pi^*(\overline{\mathcal{M}}^2_{8,7}) = 0$. Furthermore R is disjoint from all boundary divisors $\pi^*(\Delta_i)$ for $i = 1, \ldots, 4$.*

Proposition 0.9 implies that $K_{\overline{\mathcal{S}}_8^+}$, expressed as a weighted sum of $\overline{\Theta}_{\mathrm{null}}$, the pull-back $\pi^*(\overline{\mathcal{M}}_{8,7}^2)$ and boundary divisors $\pi^*(\Delta_i)$ for $i = 1, \ldots, 4$, is rigid as well. Equivalently, $\kappa(\overline{\mathcal{S}}_8^+) = 0$. Note that since $K_{\overline{\mathcal{S}}_8^+}$ consists of 10 uniruled base components which can be blown-down, the variety $\overline{\mathcal{S}}_8^+$ is not minimal and there exists a birational model $\mathcal{S}$ of $\overline{\mathcal{S}}_8^+$ which is a genuine Calabi-Yau variety in the sense that $K_{\mathcal{S}} = 0$. Finding an explicit modular interpretation of this Calabi-Yau 21-fold (or perhaps even its equations!) is a very interesting question.

1. Prym-canonical curves on Nikulin surfaces

Let us start with a smooth $K3$ surface Y. A *Nikulin involution* on Y is an automorphism $\iota \in \mathrm{Aut}(Y)$ of order 2 which is symplectic, that is, $\iota^*(\omega) = \omega$, for all $\omega \in H^{2,0}(Y)$. A Nikulin involution has 8 fixed points, see [Ni] Lemma 3, and the quotient $\bar{Y} := Y/\langle \iota \rangle$ has 8 ordinary double point singularities. Let $\sigma : \tilde{S} \to Y$ be the blow-up of the 8 fixed points and denote by $E_1, \ldots, E_8 \subset \tilde{S}$ the exceptional divisors and by $\tilde{\iota} \in \mathrm{Aut}(\tilde{S})$ the automorphism induced by ι. Then $S := \tilde{S}/\langle \tilde{\iota} \rangle$ is a smooth $K3$ surface and if $f : \tilde{S} \to S$ is the projection, then $N_i := f(E_i)$ are (-2)-curves on S. The branch divisor of f is equal to $N := \sum_{i=1}^8 N_i$. We summarize the situation in the following diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\sigma} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \longrightarrow & \bar{Y} \end{array} \tag{1}$$

Sometimes we shall refer to the pair (Y, ι) as a Nikulin surface, while keeping the previous diagram in mind. We refer to [Mo], [vGS] for a lattice-theoretic study on the action of the Nikulin involution on the cohomology $H^2(Y, \mathbb{Z}) = U^3 \oplus E_8(-1) \oplus E_8(-1)$, where U is the standard rank 2 hyperbolic lattice and E_8 is the unique even, negative-definite unimodular lattice of rank 8. It follows from [Mo] Theorem 5.7 that the orthogonal complement $E_8(-2) \cong \big(H^2(Y, \mathbb{Z})^{\iota}\big)^{\perp}$ is contained in $\mathrm{Pic}(Y)$, hence Y has Picard number at least 9. The class $\mathcal{O}_S(N_1 + \cdots + N_8)$ is divisible by 2, and we denote by $e \in \mathrm{Pic}(S)$ the class such that $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)$.

Definition 1.1. The *Nikulin lattice* is an even lattice $\mathfrak{N}$ of rank 8 generated by elements $\{\mathfrak{n}_i\}_{i=1}^8$ and $\mathfrak{e} := \frac{1}{2}\sum_{i=1}^8 \mathfrak{n}_i$, with the bilinear form induced by $\mathfrak{n}_i^2 = -2$ for $i = 1, \ldots, 8$ and $\mathfrak{n}_i \cdot \mathfrak{n}_j = 0$ for $i \neq j$.

Note that $\mathfrak{N}$ is the minimal primitive sublattice of $H^2(S, \mathbb{Z})$ containing the classes $N_1, \ldots, N_8$ and e. For any Nikulin surface one has an embedding $\mathfrak{N} \subset \mathrm{Pic}(S)$. Assuming that (Y, ι) defines a general point in an irreducible component of the moduli space of Nikulin involutions, both Y and S have Picard number 9 and there is a decomposition $\mathrm{Pic}(S) = \mathbb{Z} \cdot [C] \oplus \mathfrak{N}$, where C is an integral curve of genus $g \geq 2$. According to [vGS] Proposition 2.2, only two cases are possible: either $C \cdot e = 0$ so that the previous decomposition is an orthogonal sum, or else, $C \cdot e \neq 0$, this second case being possible only when g is odd. In this paper we consider only Nikulin surfaces of the first kind.

We fix an integer $g \geq 2$ and consider the lattice $\Lambda_g := \mathbb{Z} \cdot \mathfrak{c} \oplus \mathfrak{N}$, where $\mathfrak{c} \cdot \mathfrak{c} = 2g - 2$.

Definition 1.2. A *Nikulin surface of genus* g is a $K3$ surface S together with a primitive embedding of lattices $j : \Lambda_g \hookrightarrow \mathrm{Pic}(S)$ such that $C := j(\mathfrak{c})$ is a nef class.

The coarse moduli space $\mathcal{F}_g^{\mathfrak{N}}$ of Nikulin surfaces of genus g is the quotient of the 11-dimensional domain

$$\mathcal{D}_{\Lambda_g} := \{\omega \in \mathbf{P}(\Lambda_g \otimes_{\mathbb{Z}} \mathbb{C}) : \omega^2 = 0, \ \omega \cdot \bar{\omega} > 0\}$$

by an arithmetic subgroup of $\mathbf{O}(\Lambda_g)$. Its existence follows e.g. from [Do1] Section 3.

We now consider a Nikulin surface $f : \tilde{S} \to S$, together with a smooth curve $C \subset S$ of genus g such that $C \cdot N = 0$. If $\tilde{C} := f^{-1}(C)$, then $f_C := f_{|\tilde{C}} : \tilde{C} \to C$ is an étale double covering. By the Hodge index theorem, $\tilde{C}$ cannot split in two disjoint connected components, hence f_C is non-trivial and $e_C := \mathcal{O}_C(e) \in \mathrm{Pic}^0(C)$ is the non trivial 2-torsion element defining the covering f_C. We set $H \equiv C - e \in NS(S)$, hence $H^2 = 2g - 6$ and $H \cdot C = 2g - 2$. For further reference we collect a few easy facts:

Lemma 1.3. *Let $[S, e, \mathcal{O}_S(C)] \in \mathcal{F}_g^{\mathfrak{N}}$ be a Nikulin surface such that $\mathrm{Pic}(S) = \Lambda_g$. The following statements hold:*

(i) $H^i(S, e) = 0$ *for all* $i \geq 0$.
(ii) $\mathrm{Cliff}(C) = [\frac{g-1}{2}]$.
(iii) *The line bundle $\mathcal{O}_S(H)$ is ample for $g \geq 4$ and very ample for $g \geq 6$. In this range, it defines an embedding $\phi_H : S \to \mathbf{P}^{g-2}$ such that the images $\phi_H(N_i)$ are lines for all $i = 1, \ldots, 8$.*
(iv) *If $g \geq 7$, the ideal of the surface $\Phi_H(S) \subset \mathbf{P}^{g-2}$ is cut out by quadrics.*

Proof. Recalling that $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)$ and that the curves $\{N_i\}_{i=1}^8$ are pairwise disjoint, it follows that $H^0(S, e) = 0$ and clearly $H^2(S, e) = 0$. Since $e^2 = -4$, by Riemann-Roch one finds that $H^1(S, e) = 0$ as well.

In order to prove (ii) we assume that $\mathrm{Cliff}(C) < [\frac{g-1}{2}]$. From [GL2] it follows that there exists a divisor $D \in \mathrm{Pic}(S)$ such that $h^i(S, \mathcal{O}_C(D)) \geq 2$ for $i = 0, 1$ and $C \cdot D \leq g - 1$, such that $\mathcal{O}_C(D)$ computes the Clifford index of C, that is, $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{O}_C(D))$. But $C \cdot \ell \equiv 0 \bmod 2g - 2$ for every class $\ell \in \mathrm{Pic}(S)$, hence no such divisor D can exist.

Moving to (iii), the ampleness (respectively very ampleness) of $\mathcal{O}_S(H)$ is proved in [GS] Proposition 3.2 (respectively Lemma 3.1). From the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-H) \longrightarrow \mathcal{O}_S(e) \longrightarrow O_C(e) \longrightarrow 0,$$

one finds that $h^1(S, \mathcal{O}_S(H)) = 0$ and then $\dim |H| = g - 2$. Furthermore $H \cdot N_i = 1$ for $i = 1, \ldots, 8$ and the claim follows.

To prove (iv), following [SD] Theorem 7.2, it suffices to show that there exists no irreducible curve $\Gamma \subset S$ with $\Gamma^2 = 0$ and $H \cdot \Gamma = 3$. Assume by contradiction that $\Gamma \equiv aC - b_1 N_1 - \cdots - b_8 N_8$ is such a curve, where necessarily $a, b_i \in \mathbb{Z}_{\leq 0}$. Then $\sum_{i=1}^8 b_i = 2ag - 2a - 3$ and $\sum_{i=1}^8 b_i^2 = a^2(g-1)$. Applying the Cauchy-Schwarz inequality $\left(\sum_{i=1}^8 b_i\right)^2 \leq 8\left(\sum_{i=1}^8 b_i^2\right)$, we obtain an immediate contradiction. □

We consider the $\mathbf{P}^g$-bundle $p_g : \mathcal{P}_g^{\mathfrak{N}} \to \mathcal{F}_g^{\mathfrak{N}}$, as well as the map

$$\chi_g : \mathcal{P}_g^{\mathfrak{N}} \to \mathcal{R}_g, \quad \chi_g([S, e, C]) := [C, e_C := e \otimes \mathcal{O}_C]$$

defined in the introduction. We fix a Nikulin surface $[S, e, \mathcal{O}_S(C)] \in \mathcal{P}_g^{\mathfrak{N}}$. A Lefschetz pencil of curves $\{C_\lambda\}_{\lambda \in \mathbf{P}^1}$ inside $|\mathcal{O}_S(C)|$ induces a rational curve

$$\Xi_g := \{[C_\lambda, \ e_{C_\lambda} := e \otimes \mathcal{O}_{C_\lambda}] : \lambda \in \mathbf{P}^1\} \subset \overline{\mathcal{R}}_g.$$

In the range where χ_g is a dominant map, Ξ_g is a rational curve passing through a general point of $\overline{\mathcal{R}}_g$, and it is of some interest to compute its numerical characters. If $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ denotes the projection map, we recall the formula [FL] Example 1.4

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}}, \tag{2}$$

where $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$ and $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$ are boundary divisor classes on $\overline{\mathcal{R}}_g$ whose meaning we recall. Let us fix a general point $[C_{xy}] \in \Delta_0$ induced by a 2-pointed curve $[C, x, y] \in \mathcal{M}_{g-1,2}$ and the normalization map $\nu : C \to C_{xy}$, where $\nu(x) = \nu(y)$. A general point of $\Delta_0^{'}$ (respectively of $\Delta_0^{''}$) corresponds to a stable Prym curve $[C_{xy}, \eta]$, where $\eta \in \mathrm{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \mathrm{Pic}^0(C)$ is non-trivial (respectively, $\nu^*(\eta) = \mathcal{O}_C$). A general point of Δ_0^{ram} is of the form $[X, \eta]$, where $X := C \cup_{\{x,y\}} \mathbf{P}^1$ is a quasi-stable curve, whereas $\eta \in \mathrm{Pic}^0(X)$ is characterized by $\eta_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^{\otimes 2} = \mathcal{O}_C(-x-y)$.

Proposition 1.4. *If $\Xi_g \subset \overline{\mathcal{R}}_g$ is the curve induced by a pencil on a Nikulin surface, then*

$$\Xi_g \cdot \lambda = g + 1,\ \Xi_g \cdot \delta_0^{'} = 6g + 2,\ \ \Xi_g \cdot \delta_0^{''} = 0 \ \textit{and}\ \Xi_g \cdot \delta_0^{\mathrm{ram}} = 8.$$

It follows that $\Xi_g \cdot K_{\overline{\mathcal{R}}_g} = g - 15$.

Proof. We use [FP] Lemma 2.4 to find that $\Xi_g \cdot \lambda = \pi_*(\Xi_g) \cdot \lambda = g + 1$ and $\Xi_g \cdot \pi^*(\delta_0) = \pi_*(\Xi_g) \cdot \delta_0 = 6g + 18$, as well as $\Xi_g \cdot \pi^*(\delta_i) = 0$ for $1 \le i \le [g/2]$. For each $1 \le i \le 8$, the sublinear system $\mathbf{P}\ H^0(\mathcal{O}_S(C - N_i)) \subset \mathbf{P}\ H^0(\mathcal{O}_S(C))$ intersects Ξ_g transversally in one point which corresponds to a curve $N_i + C_i \in |\mathcal{O}_S(C)|$, where $N_i \cdot C_i = -N_i^2 = 2$ and $C_i \equiv C - N_i$. Furthermore $e \otimes \mathcal{O}_{N_i} = \mathcal{O}_{N_i}(1)$ and $e_{C_i}^{\otimes 2} = \mathcal{O}_{C_i}(-N_i \cdot C_i)$. Each of these points lie in the intersection $\Xi_g \cap \Delta_0^{\mathrm{ram}}$. All remaining curves in Ξ_g are irreducible, hence $\Xi_g \cdot \delta_0^{\mathrm{ram}} = 8$. Since $\Xi_g \cdot \delta_0^{''} = 0$, from (2) we find that $\Xi_g \cdot \delta_0^{'} = 6g + 2$. Finally, according to [FL] Theorem 1.5 the formula $K_{\overline{\mathcal{R}}_g} \equiv 13\lambda - 2(\delta_0^{'} + \delta_0^{''}) - 3\delta_0^{\mathrm{ram}} - \cdots \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ holds, therefore putting everything together, $\Xi_g \cdot K_{\overline{\mathcal{R}}_g} = g - 15$. □

The calculations in Proposition 1.4 are applied now to show that syzygies of Prym-canonical curves on Nikulin surfaces are exceptional when compared to those of general Prym-canonical curves. To make this statement precise, let us recall the *Prym-Green Conjecture*, see [FL] Conjecture 0.7: If $g := 2i + 6$ with $i \ge 0$, then the locus

$$\mathcal{U}_{g,i} := \{[C, \eta] \in \mathcal{R}_{2i+6} : K_{i,2}(C, K_C \otimes \eta) \neq 0\}$$

is a *virtual divisor*, that is, the degeneracy locus of two vector bundles of the same rank defined over $\mathcal{R}_{2i+6}$. The statement of the Prym-Green Conjecture is that this vector bundle morphism is generically non-degenerate:

Prym-Green Conjecture: $K_{i,2}(C, K_C \otimes \eta) = 0$ for a general Prym curve $[C, \eta] \in \mathcal{R}_{2i+6}$.

The conjecture is known to hold in bounded genus and has been used in [FL] to show that $\overline{\mathcal{R}}_g$ is of general type when $g \ge 14$ is even.

Theorem 1.5. *For each $[S, e, C] \in \mathcal{P}_{2i+6}^{\mathfrak{N}}$ one has $K_{i,2}(C, K_C \otimes e_C) \neq 0$. In particular, the Prym-Green Conjecture fails along the locus $\mathcal{N}_{2i+6}$.*

Proof. If the non-vanishing $K_{i,2}(C, K_C \otimes \eta) \neq 0$ holds for a general point $[C, \eta] \in \mathcal{R}_g$, then there is nothing to prove, hence we may assume that $\mathcal{U}_{g,i}$ is a genuine divisor on $\mathcal{R}_g$. The class of its closure inside $\overline{\mathcal{R}}_g$ has been calculated [FL] Theorem 0.6:

$$\overline{\mathcal{U}}_{g,i} \equiv \binom{2i+2}{i} \Big(\frac{3(2i+7)}{i+3}\lambda - \frac{3}{2}\delta_0^{\mathrm{ram}} - \delta_0^{'} - \alpha\, \delta_0^{''} - \cdots\Big) \in \mathrm{Pic}(\overline{\mathcal{R}}_{2i+6}).$$

From Proposition 1.4, by direct calculation one finds that $\Xi_g \cdot \overline{\mathcal{U}}_{g,i} = -\binom{2i+3}{i} < 0$, thus $\Xi_g \subset \overline{\mathcal{U}}_{g,i}$. By varying Ξ_g inside $\overline{\mathcal{R}}_g$, we obtain that $\mathcal{N}_g \subset \overline{\mathcal{U}}_{g,i}$, which ends the proof. □

Remark 1.6. A geometric proof of Theorem 1.5 using the Lefschetz hyperplane principle for Koszul cohomology is given in [AF] Theorem 3.5. The indirect proof presented here is however shorter and illustrates how cohomology calculations on $\overline{\mathcal{R}}_g$ can be used to derive geometric consequences for individual Prym curves.

Remark 1.7. One might ask whether similar applications to $\mathcal{R}_g$ can be obtained using Enriques surfaces. There is a major difference between Prym curves $[C, \eta] \in \mathcal{R}_g$ lying on a Nikulin surface and those lying on an Enriques surface. For instance, if $C \subset S$ is a curve of genus g lying on an Enriques surface S, then from [CD] Corollary 2.7.1

$$\mathrm{gon}(C) \leq 2\inf\big\{F \cdot C : F \in \mathrm{Pic}(S),\ F^2 = 0,\ F \not\equiv 0\big\} \leq\ 2\sqrt{2g-2}.$$

In particular, for g sufficiently high, C is far from being Brill-Noether general. On the other hand, we have seen that for $[S, e, C] \in \mathcal{P}_g^{\mathfrak{N}}$ such that $\mathrm{Pic}(S) = \Lambda_g$, one has that $\mathrm{gon}(C) = [\frac{g+3}{2}]$. For this reason, the *Prym-Nikulin* locus $\mathcal{N}_g := \mathrm{Im}(\chi_g) \subset \mathcal{R}_g$ appears as a more promising and less constrained locus than the *Prym-Enriques* locus in $\mathcal{R}_g$, being transversal to stratifications of $\mathcal{R}_g$ coming from Brill-Noether theory.

2. The Prym-Nikulin locus in $\mathcal{R}_g$ for $g \leq 7$

In this section we give constructive proofs of Theorems 0.2 and 0.4. Comparing the dimensions $\dim(\mathcal{P}_g^{\mathfrak{N}}) = 11 + g$ and $\dim(\mathcal{R}_g) = 3g - 3$, one may inquire whether the morphism $\chi_g : \mathcal{P}_g^{\mathfrak{N}} \to \mathcal{R}_g$ is dominant when $g \leq 7$. The similar question for ordinary $K3$ surfaces has been answered by Mukai [M1]. Let $\mathcal{F}_g$ denote the 19-dimensional moduli space of polarized $K3$ surfaces of genus g and consider the associated $\mathbf{P}^g$-bundle

$$\mathcal{P}_g := \big\{[S, C] : C \subset S \text{ is a smooth curve such that } [S, \mathcal{O}_S(C)] \in \mathcal{F}_g\big\}.$$

The map $q_g : \mathcal{P}_g \to \mathcal{M}_g$ forgetting the $K3$ surface is dominant if and only if $g \leq 11$ and $g \neq 10$. The result for $g = 10$ is contrary to untutored expectation since the general fibre of q_{10} is 3-dimensional, hence $\dim(\mathrm{Im}(q_{10})) = \dim(\mathcal{P}_{10}) - 3 = 26$. A strikingly similar picture emerges for Nikulin surfaces and Prym curves. The morphism $\chi_g : \mathcal{P}_g^{\mathfrak{N}} \to \mathcal{R}_g$ is dominant when $g \leq 7$ and $g \neq 6$. For each genus we describe a geometric construction that furnishes a Nikulin surface in the fibre $\chi_g^{-1}\big([C, \eta]\big)$ over a general point $[C, \eta] \in \mathcal{R}_g$.

2.1. **Nikulin surfaces of genus 7.** We start with a general element $[C, \eta] \in \mathcal{R}_7$ and construct a Nikulin surface containing C. One may assume that $\mathrm{gon}(C) = 5$ and that the line bundle η does not lie in the difference variety $C_2 - C_2 \subset \mathrm{Pic}^0(C)$, or equivalently, the linear series $L := K_C \otimes \eta \in W^5_{12}(C)$ is very ample. It is a consequence of [GL1] Theorem 2.1 that the Prym-canonical image $C \xrightarrow{|L|} \mathbf{P}^5$ is quadratically normal, that is, $h^0(\mathbf{P}^5, \mathcal{I}_{C/\mathbf{P}^5}(2)) = 3$.

Lemma 2.1. *For a general $[C, \eta] \in \mathcal{R}_7$, the base locus of $|\mathcal{I}_{C/\mathbf{P}^5}(2)|$ is a smooth K3 surface.*

Proof. The property that the base locus of $|\mathcal{I}_{C/\mathbf{P}^5}(2)|$ is smooth, is open in $\mathcal{R}_7$ and it suffices to exhibit a single Prym-canonical curve $[C,\eta] \in \mathcal{R}_7$ satisfying it. Let us fix an element $(S,e,C) \in \mathcal{P}_7^{\mathfrak{N}}$ such that $\mathrm{Pic}(S) = \Lambda_7$ and set $H \equiv C - e$. Then according to Lemma 1.3, $\phi_H : S \to \mathbf{P}^5$ is an embedding whose image $\phi_H(S)$ is ideal-theoretically cut out by quadrics. Moreover gon$(C) = 5$, hence $K_C \otimes e_C \in W^5_{12}(C)$ is quadratically normal. This implies that $H^0(S, \mathcal{O}_S(2H - C)) = H^1(S, \mathcal{O}_S(2H - C)) = 0$, and then $H^0(\mathbf{P}^5, \mathcal{I}_{S/\mathbf{P}^5}(2)) \cong H^0(\mathbf{P}^5, \mathcal{I}_{C/\mathbf{P}^5}(2))$, therefore the quadrics in $|\mathcal{I}_{C/\mathbf{P}^5}(2)|$ cut out precisely the surface S. □

Remark 2.2. This proof shows that if $[S,e,C] \in \mathcal{P}_7^{\mathfrak{N}}$ is general then $\chi_7^{-1}\big([C,e_C]\big) = [S,e,C]$ and in particular the fibre $\chi_7^{-1}\big([C,e_C]\big)$ is reduced. Indeed, let $[S',e',C] \in \mathcal{P}_7^{\mathfrak{N}}$ be an arbitrary Nikulin surface containing C. Set $H' \equiv C - e' \in NS(S')$. We may assume that $\mathrm{Pic}(S') = \Lambda_7$, therefore the map $\phi_{H'} : S' \to \mathbf{P}^5$ is an embedding whose image is cut out by quadrics. Since Cliff$(C) = 3$, from Lemma 1.3 we find that $K_C \otimes e_C$ is quadratically normal and then S' is cut out by the quadrics contained in Prym-canonical embedding of $C \subset \mathbf{P}^5$.

Since both $\mathcal{P}_7^{\mathfrak{N}}$ and $\mathcal{R}_7$ are irreducible varieties of dimension 18, Remark 2.2 shows that $\chi_7 : \mathcal{P}_7^{\mathfrak{N}} \to \mathcal{R}_7$ is a birational morphism and we now describe χ_7^{-1}.

Proposition 2.3. *For a general $[C,\eta] \in \mathcal{R}_7$, the surface $S := \mathrm{bs}\,|\mathcal{I}_{C/\mathbf{P}^5}(2)|$ is a polarized Nikulin surface of genus 7.*

Proof. We show that $\mathrm{Pic}(S) \supset \mathbb{Z}\cdot C \oplus \mathfrak{N}$. Denote by $H \subset S$ the hyperplane class and let $N :\equiv 2(C - H)$, thus $N^2 = -16$, $N \cdot H = 8$ and $N \cdot C = 0$. We aim to prove that N is linearly equivalent to a sum of 8 pairwise disjoint integral (-2) curves on S. We consider the following exact sequence

$$0 \longrightarrow \mathcal{O}_S(N - C) \longrightarrow \mathcal{O}_S(N) \longrightarrow \mathcal{O}_C(N) \longrightarrow 0.$$

Note that $\mathcal{O}_C(N)$ is trivial because $e_C = \mathcal{O}_C(C - H)$ and that $h^1(S, \mathcal{O}_S(N - C)) = h^1(S, \mathcal{O}_S(C - 2H)) = 0$, because $C \subset \mathbf{P}^5$ is quadratically normal. Passing to the long exact sequence, it follows that $h^0(S, \mathcal{O}_S(N)) = 1$. Using Remark 2.2 it follows that $N \equiv N_1 + \cdots + N_8$, where $N_i \cdot N_j = -2\delta_{ij}$. Finally, to conclude that $[S, \mathbb{Z}\cdot C \oplus \mathfrak{N}] \in \mathcal{F}_7^{\mathfrak{N}}$ we must show that there is a primitive embedding $\mathbb{Z}\cdot C \oplus \mathfrak{N} \hookrightarrow \mathrm{Pic}(S)$. We apply [vGS] Proposition 2.7. Since $H^0(\tilde{S}, \mathcal{O}_S(\tilde{C})) = H^0(S, \mathcal{O}_S(C)) \oplus H^0(S, \mathcal{O}_S(C) \otimes e^{\vee})$ and sections in the second summand vanish on the exceptional divisor of the morphism $\sigma : \tilde{S} \to Y$, it follows that this is precisely the decomposition of $H^0(Y, \mathcal{O}_Y(\tilde{C}))$ into ι_Y^*-eigenspaces. Invoking *loc. cit.*, we finish the proof. □

2.2. **The symmetric determinantal cubic hypersurface and Prym curves.** We provide a general set-up that allows us to reconstruct a Nikulin surface from a Prym curve of genus $g \leq 5$. Let us start with a curve $[C,\eta] \in \mathcal{R}_g$ inducing an étale double cover $f : \tilde{C} \to C$ together with an involution $\iota : \tilde{C} \to \tilde{C}$ such that $f \circ \iota = f$. For each integer $r \geq -1$, the *Prym-Brill-Noether* locus is defined as the locus

$$V^r(C,\eta) := \{L \in \mathrm{Pic}^{2g-2}(\tilde{C}) : \mathrm{Nm}_f(L) = K_C,\ h^0(L) \geq r+1 \text{ and } h^0(L) \equiv r+1 \bmod 2\}.$$

Note that $V^{-1}(C,\eta) = \mathrm{Pr}(C,\eta)$. For each line bundle $L \in V^r(C,\eta)$, the Petri map

$$\mu_0(L) : H^0(\tilde{C}, L) \otimes H^0(\tilde{C}, K_{\tilde{C}} \otimes L^{\vee}) \to H^0(\tilde{C}, K_{\tilde{C}})$$

splits into an ι-anti-invariant part

$$\mu_0^-(L) : \Lambda^2 H^0(\tilde{C}, L) \to H^0(C, K_C \otimes \eta), \quad s \wedge t \mapsto s \cdot \iota^*(t) - t \cdot \iota^*(s),$$

and an ι-invariant part respectively

$$\mu_0^+(L) : \mathrm{Sym}^2 H^0(\tilde{C}, L) \to H^0(C, K_C), \quad s \otimes t + t \otimes s \mapsto s \cdot \iota^*(t) + t \cdot \iota^*(s).$$

For a general $[C, \eta] \in \mathcal{R}_g$, the Prym-Petri map $\mu_0^-(L)$ is injective for every $L \in V^r(C, \eta)$ and $V^r(C, \eta)$ is equidimensional of dimension $g - 1 - \binom{r+1}{2}$, see [We]. We introduce the *universal Prym-Brill-Noether variety*

$$\mathcal{R}_g^r := \Big\{ \big([C, \eta], L\big) : [C, \eta] \in \mathcal{R}_g, \ \ L \in V^r(C, \eta) \Big\}.$$

When $g - 1 - \binom{r+1}{2} \geq 0$, the variety $\mathcal{R}_g^r$ is irreducible of dimension $4g - 4 - \binom{r+1}{2}$. We propose to focus on the case $r = 2$ and $g \geq 4$ and choose a general triple $(f : \tilde{C} \to C, L) \in \mathcal{R}_g^2$, such that L is base point free and $h^0(\tilde{C}, L) = 3$.

Setting $\mathbf{P}^2 := \mathbf{P}\big(H^0(L)^\vee\big)$, we consider the quasi-étale double cover $q : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^5$ obtained by projecting via the Segre embedding to the space of symmetric tensors. Note that q is ramified along the diagonal $\Delta \subset \mathbf{P}^2 \times \mathbf{P}^2$ and $V_4 := q(\Delta) \subset \mathbf{P}^5$ is the Veronese surface. Moreover $\Sigma := \mathrm{Im}(q)$ is the determinantal symmetric cubic hypersurface isomorphic to the secant variety of V_4. We have the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{C} & \xrightarrow{(L, \iota^* L)} & \mathbf{P}^2 \times \mathbf{P}^2 & \searrow & \\
\Big\downarrow f & & \Big\downarrow q & & \mathbf{P}^8 = \mathbf{P}\big(H^0(L)^\vee \otimes H^0(L)^\vee\big) \\
C & \xrightarrow{\mu_0^+(L)} & \mathbf{P}^5 = \mathbf{P}(\mathrm{Sym}^2 H^0(L)^\vee) & \dashleftarrow &
\end{array}$$

Observe that the involution $\iota : \mathbf{P}^8 \to \mathbf{P}^8$ given by $\iota[v \otimes w] := [w \otimes v]$ where $v, w \in H^0(L)$, is compatible with $\iota : \tilde{C} \to \tilde{C}$. To summarize, giving a point $(\tilde{C} \to C, L) \in \mathcal{R}_g^2$ is equivalent to specifying a symmetric determinantal cubic hypersurface $\Sigma \in H^0(\mathbf{P}^{g-1}, \mathcal{I}_{C/\mathbf{P}^{g-1}}(3))$ containing the canonical curve.

2.3. **A birational model of $\mathcal{F}_4^{\mathfrak{N}}$.** As a warm-up, we indicate how the set-up described above is a generalization of the construction that Catanese [Ca] used to prove that $\mathcal{R}_4$ is rational. For a general point $[C, \eta] \in \mathcal{R}_4$ we find that $V^2(C, \eta) = \{L, \iota^* L\}$, that is, the pair $(L, \iota^* L)$ is uniquely determined. The map $\mu_0(L)$ has corank 2 and $\mathbf{P}^6_{\tilde{C}} := \mathbf{P}\big(\mathrm{Im}\ \mu_0(L)\big) \subset \mathbf{P}^8$ has codimension 2. The intersection $\tilde{T} := (\mathbf{P}^2 \times \mathbf{P}^2) \cap \mathbf{P}^6_{\tilde{C}}$ is a del Pezzo surface of degree 6, whereas $T := \Sigma \cap \mathbf{P}^3_+$ is a 4-nodal Cayley cubic. Here we set $\mathbf{P}^3_+ := \mathbf{P}\big(H^0(K_C)^\vee\big)$. The double cover $q : \tilde{T} \to T$ is ramified at the singular points of T.

To obtain a Nikulin surface containing $[C, \eta]$, we reverse this construction and start with a quartic rational normal curve $R \subset \mathbf{P}^4$ and denote by $\bar{\mathcal{Y}} := \mathrm{Sec}(R) \subset \mathbf{P}^4$ its secant variety, which we view as a hyperplane section of $\Sigma \subset \mathbf{P}^5$. Retaining the notation of diagram (1), for a general quadric $Q \in |\mathcal{O}_{\mathbf{P}^4}(2)|$, the intersection $\bar{Y} := \bar{\mathcal{Y}} \cap Q$ is a $K3$ surface with 8 rational double points at $R \cap Q$. There exists a cover $q : Y \overset{2:1}{\to} \bar{Y}$ ramified at the singular points of Y, induced by restriction from the map $q : \mathbf{P}^2 \times \mathbf{P}^2 \to \Sigma$. Clearly

$q : Y \to \bar{Y}$ is a Nikulin covering, and a hyperplane section in $|\mathcal{O}_{\bar{Y}}(1)|$ induces a Prym curve $[C, \eta] \in \mathcal{R}_4$ having general moduli. Moreover we have a birational isomorphism

$$\mathcal{F}_4^{\mathfrak{N}} \overset{\cong}{\dashrightarrow} \mathbf{P}\Big(H^0(\mathcal{O}_{\mathbf{P}^4}(2))\Big)^{\mathrm{ss}}//SL_2,$$

where $PGL_2 = \mathrm{Aut}(R) \subset PGL_5$. An immediate consequence is that $\mathcal{F}_4^{\mathfrak{N}}$ is unirational.

2.4. **Nikulin surfaces of genus** 3. We prove that $\chi_3 : \mathcal{P}_3^{\mathfrak{N}} \to \mathcal{R}_3$ is dominant and fix a complete intersection of 3 quadrics $Y \subset \mathbf{P}^5$ invariant with respect to an involution fixing a line $L \subset \mathbf{P}^5$ and a 3-dimensional linear subspace $\Lambda \subset \mathbf{P}^5$. The projection $\pi_L : \mathbf{P}^5 \dashrightarrow \Lambda$ induces a quartic $\bar{Y} := \pi_L(Y)$ with 8 nodes, which is a Nikulin surface. We check that a general Prym curve $[C, \eta] \in \mathcal{R}_3$ corresponding to an étale cover $f : \tilde{C} \to C$ embeds in such a surface.

Indeed, the canonical model $\tilde{C} \subset \mathbf{P}^4$ is a complete intersection of 3 quadrics. Fixing projective coordinates on $\mathbf{P}^4$, we can assume that the involution $\iota : \tilde{C} \to \tilde{C}$ is induced by the projective involution $[x : y : u : v : t] \leftrightarrow [-x : -y : u : v : t]$. Note that the ι^*anti-invariant quadratic forms are vectors $q = ax + by$, where a, b are linear forms in u, v, t. Since $\tilde{C}$ is complete intersection of 3 quadrics, no non-zero quadric $q = ax + by$ vanishes on $\tilde{C}$, for not, $\tilde{C}$ would intersect the plane $\{x = y = 0\}$ and then ι would have fixed points. Thus ι acts as the identity on the space $H^0(\mathbf{P}^4, \mathcal{I}_{\tilde{C}/\mathbf{P}^4}(2))$. Hence it follows $\tilde{C} = \{a_1 + b_1 = a_2 + b_2 = a_3 + b_3 = 0\}$, where a_i, b_i are quadratic forms in x, y and u, v, t. Passing to $\mathbf{P}^5$ by adding one coordinate h, we can choose quadratic forms $a_i + b_i + hl_i$, where l_i is a general linear form in h, u, v, t. Consider the surface $Y \subset \mathbf{P}^5$ defined by the latter 3 equations. Then $[x : y : h : u : v : t] \leftrightarrow [-x : -y : h : u : v : t]$ induces a Nikulin involution on Y. Let $\pi_L : Y \to \mathbf{P}^3$ be the projection of center $L = \{h = u = v = t = 0\}$. Then $\overline{Y} := \pi_L(Y)$ is a quartic Nikulin surface and $C = \pi_L(\tilde{C})$ is a plane section of it.

2.5. **Nikulin surfaces of genus** 5. To describe the morphism $\chi_5 : \mathcal{P}_5^{\mathfrak{N}} \to \mathcal{R}_5$ more geometrically, we use the set-up introduced in Subsection 2.2. If $[C, \eta] \in \mathcal{R}_5$ is general, then $\dim V^2(C, \eta) = 1$, the ι-invariant Petri map $\mu_0^-(L)$ is injective, $\mu_0^+(L)$ surjective, thus $\dim\big(\mathrm{Coker}\ \mu_0(L)\big) = 1$. We consider the hyperplane

$$\mathbf{P}^7_{\tilde{C}} := \mathbf{P}\big(\mathrm{Im}(\mu_0(L)\big) \subset \mathbf{P}\big(H^0(L)^\vee \otimes H^0(L)^\vee\big)$$

and also set $\mathbf{P}^4_+ := \mathbf{P}\big(H^0(K_C)^\vee\big) \subset \mathbf{P}^5$. Then we further denote

$$\tilde{T} := (\mathbf{P}^2 \times \mathbf{P}^2) \cap \mathbf{P}^7_{\tilde{C}} \text{ and } T := \Sigma \cap \mathbf{P}^4_+.$$

Note that $\tilde{T}$ is a degree 6 threefold in $\mathbf{P}^7_{\tilde{C}}$. Since the hyperplane $\mathbf{P}^7_{\tilde{C}}$ is ι-invariant, it follows $\tilde{T}$ is also endowed with the involution $\iota_{\tilde{T}} \in \mathrm{Aut}(\tilde{T})$ such that $\mathrm{Fix}(\iota_{\tilde{T}}) = \Delta \cap \tilde{T}$ is a rational quartic curve in $\mathbf{P}^4_+$. Furthermore $T \subset \mathbf{P}^4_+$ is the secant variety of R.

Proposition 2.4. *For a general point $[C, \eta, L] \in \mathcal{R}_5^2$ the following statements hold:*

(i) *The threefold $\tilde{T} \subset \mathbf{P}^2 \times \mathbf{P}^2$ is smooth, while $T \subset \mathbf{P}^4_+$ is singular precisely along R.*
(ii) $h^0(\tilde{T}, \mathcal{I}_{\tilde{C}/\tilde{T}}(2)) = 3$. *Moreover $H^i(\tilde{T}, \mathcal{I}_{\tilde{C}/\tilde{T}}(2)) = 0$ for $i = 1, 2$.*
(iii) *Every quadratic section in the linear system $|\mathcal{I}_{\tilde{C}/\tilde{T}}(2)|$ is ι-invariant, that is,*

$$H^0(\tilde{T}, \mathcal{I}_{\tilde{C}/\tilde{T}}(2)) = q^* H^0(T, \mathcal{I}_{C/T}(2)).$$

(iv) *A general quadratic section $Y \in |\mathcal{I}_{\tilde{C}/\tilde{T}}(2)|$ is a smooth $K3$ surface endowed with an involution ι_Y with fixed points precisely at the 8 points in the intersection $R \cap Y$.*

Proof. We take cohomology in the following exact sequence

$$0 \longrightarrow \mathcal{I}_{\tilde{C}/\mathbf{P}^2\times\mathbf{P}^2}(2) \longrightarrow \mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(2) \longrightarrow K_{\tilde{C}}^{\otimes 2} \longrightarrow 0,$$

to note that $h^0(\mathcal{I}_{\tilde{C}/\tilde{T}}(2)) = 3 (\Leftrightarrow H^1(\mathcal{I}_{\tilde{C}/\mathbf{P}^2\times\mathbf{P}^2}(2)) = 0)$, if and only if the composed map

$$\mathrm{Sym}^2 H^0(\tilde{C}, L) \otimes \mathrm{Sym}^2 H^0(\tilde{C}, \iota^* L) \to H^0(\tilde{C}, L^{\otimes 2}) \otimes H^0(\tilde{C}, \iota^*(L^{\otimes 2})) \to H^0(\tilde{C}, K_{\tilde{C}}^{\otimes 2})$$

is surjective. This is an open condition and a triple $(\tilde{C} \xrightarrow{f} C, L) \in \mathcal{R}_g^2$ satisfying it, and for which moreover $\tilde{T} \subset \mathbf{P}^2 \times \mathbf{P}^2$ is smooth, has been constructed in [V2] Section 4. Finally, from the exact sequence

$$0 \longrightarrow \mathcal{I}_{T/\mathbf{P}^4_+}(2) \longrightarrow \mathcal{I}_{C/\mathbf{P}^4_+}(2) \longrightarrow \mathcal{I}_{C/T}(2) \to 0,$$

we compute that $h^0(T, \mathcal{I}_{C,T}(2)) = 3$, therefore $q^* : H^0(T, \mathcal{I}_{C/T}(2)) \to H^0(\tilde{T}, \mathcal{I}_{\tilde{C}/\tilde{T}}(2))$ is an isomorphism, based on dimension count. Part (iv) is a consequence of (i)-(iii). Assume that $\bar{Y} = T \cap Q$, where $Q \in H^0(\mathcal{I}_{C/\mathbf{P}^4_+}(2))$. Then $Y = \tilde{T} \cap q^*(Q)$ and $\bar{Y}$ is the quotient of Y by the involution ι_Y obtained by restriction from $\iota \in \mathrm{Aut}(\mathbf{P}^2 \times \mathbf{P}^2)$. It follows that the covering $q : Y \to \bar{Y}$ is a Nikulin surface such that $C \subset \bar{Y} \subset \mathbf{P}^4_+$. To conclude, we must check that for a general choice of $Y \in |\mathcal{I}_{\tilde{C}/\tilde{T}}(2)|$, the point $[Y, \iota_Y]$ gives rise to an element of $\mathcal{F}_5^{\mathfrak{N}}$, that is, using the notation of diagram (1), that $\mathrm{Pic}(S) = \Lambda_5$. Proposition 2.7 from [vGS] picks out two possibilities for $\mathrm{Pic}(Y)$ (or equivalently for $\mathrm{Pic}(S)$), and we must check that $\mathbb{Z} \cdot \mathcal{O}_Y(\tilde{C}) \oplus E_8(-2)$ has index 2 in $\mathrm{Pic}(Y)$, see also [GS] Corollary 2.2. [1]

This is achieved by finding the decomposition of $H^0(\mathcal{O}_Y(\tilde{C}))$ into ι_Y^*-eigenspaces. In the course of the proof of [V2] Proposition 5.2 an example of a smooth quadratic section $Y \in |\mathcal{I}_{\tilde{C}/\tilde{T}}(2)|$ is constructed such that

$$H^0(Y, \mathcal{O}_Y(\tilde{C}))^+ = q^* H^0(\bar{Y}, \mathcal{O}_{\bar{Y}}(C)).$$

In particular the $(+1)$-eigenspace of $H^0(Y, \mathcal{O}_Y(\tilde{C}))$ is 6-dimensional and invoking once more [vGS] Proposition 2.7, we conclude that $[Y, \iota_Y] \in \mathcal{F}_5^{\mathfrak{N}}$. □

We close this subsection with an amusing result on a geometric divisor on $\mathcal{R}_5$. For a Prym curve $[C, \eta] \in \mathcal{R}_5$ and $L := K_C \otimes \eta \in W^3_8(C)$, we observe that the vector spaces entering the multiplication map $\nu_3(L) : \mathrm{Sym}^3 H^0(C, L) \to H^0(C, L^{\otimes 3})$ have the same dimension. The condition that $\nu_3(L)$ be not an isomorphism is divisorial in $\mathcal{R}_5$. We have not been able to find a direct proof of the following equality of cycles on $\mathcal{R}_5$, even though one inclusion is straightforward:

Theorem 2.5. *Let $[C, \eta] \in \mathcal{R}_5$ be a Prym curve such that the Prym-canonical line bundle $K_C \otimes \eta$ is very ample. Then $\phi_{K_C \otimes \eta} : C \to \mathbf{P}^3$ lies on a cubic surface if and only if C is trigonal.*

Proof. Let $\mathfrak{D}_1$ be the locus of Prym curves whose Prym-canonical model lies on a cubic

$$\mathfrak{D}_1 := \{[C, \eta] \in \overline{\mathcal{R}}_5 : \nu_3(\omega_C \otimes \eta) : \mathrm{Sym}^3 H^0(C, \omega_C \otimes \eta) \xrightarrow{\ncong} H^0(C, \omega_C^{\otimes 3} \otimes \eta^{\otimes 3})\},$$

[1]We are grateful to the referee for raising this point that he have initially overlooked.

and $\mathfrak{D}_2$ the closure inside $\overline{\mathcal{R}}_5$ of the divisor $\{[C,\eta]\in\mathcal{R}_5 : \eta\in C_2-C_2\}$ of smooth Prym curves for which $L := K_C\otimes\eta\in W^3_8(C)$ is not very ample. Obviously, $\mathfrak{D}_1-\mathfrak{D}_2\geq 0$, for if L is not very ample, then the multiplication map $\nu_3(L):\mathrm{Sym}^3H^0(C,L)\to H^0(C,L^{\otimes 3})$ cannot be an isomorphism. The class of $\mathfrak{D}_2$ can be read off [FL] Theorem 5.2:

$$\mathfrak{D}_2\equiv 14\lambda-2(\delta_0^{'}+\delta_0^{''})-\frac{5}{2}\delta_0^{\mathrm{ram}}-\cdots\in\mathrm{Pic}(\overline{\mathcal{R}}_5).$$

For $i\geq 1$, let $\mathbb{E}_i$ be the vector bundle over $\overline{\mathcal{R}}_5$ with fibre $\mathbb{E}_i[C,\eta]=H^0(C,\omega_C^{\otimes i}\otimes\eta^{\otimes i})$ for every $[C,\eta]\in\overline{\mathcal{R}}_g$. One has the following formulas from [FL] Proposition 1.7:

$$c_1(\mathbb{E}_i)=\binom{i}{2}(12\lambda-\delta_0^{'}-\delta_0^{''}-2\delta_0^{\mathrm{ram}})+\lambda-\frac{i^2}{4}\delta_0^{\mathrm{ram}}\in\mathrm{Pic}(\overline{\mathcal{R}}_g).$$

As a consequence, $\mathfrak{D}_1\equiv c_1(\mathbb{E}_3)-c_1(\mathrm{Sym}^3\mathbb{E}_1)\equiv 37\lambda-3(\delta_0+\delta_0^{''})-\frac{33}{4}\delta_0^{\mathrm{ram}}-\cdots\in\mathrm{Pic}(\overline{\mathcal{R}}_5)$, therefore $\mathfrak{D}_1-\mathfrak{D}_2\equiv 8\lambda-(\delta_0^{'}+\delta_0^{''})-2\delta_0^{\mathrm{ram}}-\cdots=\pi^*(8\lambda-\delta_0-\cdots)\geq 0$, where the terms left out are combinations of boundary divisors $\pi^*(\delta_i)$ with $i\geq 1$, corresponding to reducible curves. The only effective divisors $D\equiv a\lambda-b_0\delta_0-b_1\delta_1-b_2\delta_2$ on $\overline{\mathcal{M}}_5$ such that $\frac{a}{b_0}\leq 8$ and satisfying $\Delta_i\nsubseteq\mathrm{supp}(D)$ for $i=1,2$, are multiples of the trigonal locus $\overline{\mathcal{M}}^1_{5,3}$ (the proof is identical to that of Proposition 5.1). This proves that if $[C,\eta]\in\mathfrak{D}_1-\mathfrak{D}_2$, with C being a smooth curve, then necessarily $[C]\in\mathcal{M}^1_{5,3}$, which finishes the proof. □

3. A SINGULAR QUADRATIC COMPLEX AND A BIRATIONAL MODEL FOR $\mathcal{F}_6^{\mathfrak{N}}$

Let us set $V:=\mathbb{C}^{n+1}$ and denote by $\mathbf{G}:=G(2,V)\subset\mathbf{P}(\wedge^2V)$ the Grassmannian of lines in $\mathbf{P}(V)$. We fix once and for all a smooth quadric $Q\subset\mathbf{P}(V)$. The projective tangent bundle $\mathbf{P}_Q:=\mathbf{P}(T_Q)$ can be realized as the incidence correspondence

$$\mathbf{P}_Q=\big\{(x,\ell)\in Q\times\mathbf{G}: x\in\ell\subset\mathbf{P}(T_xQ)\big\}.$$

For each point $x\in Q$, the fibre $\mathbf{P}_Q(x)$ is the space of lines tangent to Q at x. We introduce the projections $p:\mathbf{P}_Q\to\mathbf{G}$ and $q:\mathbf{P}_Q\to Q$, then set

$$W_Q:=p(\mathbf{P}_Q)=\{\ell\in\mathbf{G}:\ell\ \text{ is tangent to the quadric } Q\}.$$

Note that W_Q contains the Hilbert scheme of lines in Q, which we denote by $V_Q\subset W_Q$. It is well-known that V_Q is smooth, irreducible and $\dim(V_Q)=2n-5$. The restriction $p_{|p^{-1}(W_Q-V_Q)}$ is an isomorphism and $E_Q:=p^{-1}(V_Q)\subset\mathbf{P}_Q$ is the exceptional divisor of p.

Proposition 3.1. *The variety W_Q is a quadratic complex of lines in $\mathbf{G}$. Its singular locus is equal to V_Q and each point of V_Q is an ordinary double point of W_Q.*

Proof. Let $Q:V\to\mathbb{C}$ be the quadratic form whose zero locus is the quadric hypersurface also denoted by $Q\subset\mathbf{P}(V)$, and $\tilde{Q}:V\times V\to\mathbb{C}$ the associated bilinear map. We define the bilinear map $\nu_2(\tilde{Q}):\wedge^2V\times\wedge^2V\to\mathbb{C}$ by the formula

$$\nu_2(\tilde{Q})(u\wedge v,s\wedge t):=\tilde{Q}(u,s)\tilde{Q}(v,t)-\tilde{Q}(v,s)\tilde{Q}(u,t)$$

for $u,v,s,t\in V$, and denote by $\nu_2(Q):\wedge^2V\to\mathbb{C}$ the induced quadratic form.

For fixed points $x=[u]\in Q$ and $y=[v]\in\mathbf{P}(V)$, we observe that the line $\ell=\langle x,y\rangle$ is tangent to Q if and only if $\tilde{Q}(u,v)=0\Leftrightarrow\nu_2(Q)(u\wedge v)=0$. Therefore $W_Q=\mathbf{G}\cap\nu_2(Q)$ is a quadratic line complex in $\mathbf{G}$, being the vanishing locus of $\nu_2(Q)$.

Keeping the same notation, a point $\ell=[u\wedge v]\in W_Q$ is a singular point, if and only if the linear form $\nu_2(\tilde{Q})(u\wedge v,\ -)$ vanishes along $\mathbf{P}(T_\ell\,\mathbf{G})$. Since $\mathbf{P}(T_\ell\,\mathbf{G})$ is spanned by the Schubert cycle $\{m \in \mathbf{G} : m \cap \ell \neq \emptyset\}$, any tangent vector in $T_\ell(\mathbf{G})$ has a representative of the form $u \wedge a - v \wedge b$, where $a, b \in V$. We obtain that $[u \wedge v] \in \mathrm{Sing}(W_Q)$ if and only if $Q(v, v) = 0$, that is, $\ell = [u \wedge v] \in V_Q$. Since W_Q is a quadratic complex, each point $\ell \in V_Q$ has multiplicity 2. $\Box$

The map $p : \mathbf{P}_Q \to W_Q$ appears as a desingularization of the quadratic complex W_Q. We shall compute the class of the exceptional divisor E_Q of $\mathbf{P}_Q$. Let $H := p^*(\mathcal{O}_{\mathbf{G}}(1))$ be the class of the family of tangent lines to Q intersecting a fixed $(n-2)$-plane in $\mathbf{P}(V)$ and $B := q^*(\mathcal{O}_Q(1)) \in \mathrm{Pic}(\mathbf{P}_Q)$. Furthermore, we consider the class $h \in NS_1(\mathbf{P}_Q)$ of the pencil of tangent lines to Q with center a given point $x \in Q$. It is clear that

$$h \cdot H = 1 \quad \text{and} \quad h \cdot B = 0.$$

If $\ell \in V_Q$ is a fixed line, let $s \in NS_1(\mathbf{P}_Q)$ be the class of the family $\{(x, \ell) : x \in \ell\}$. Then

$$s \cdot H = 0 \ \text{ and } \ s \cdot B = 1.$$

Lemma 3.2. *The linear equivalence $E_Q \equiv 2H - 2B$ in $\mathrm{Pic}(\mathbf{P}_Q)$ holds. In particular, the class E_Q is divisible by 2 and it is the branch divisor of a double cover*

$$f : \tilde{\mathbf{P}}_Q \to \mathbf{P}_Q.$$

Proof. To compute the class of E_Q it suffices to compute $h \cdot E_Q$ and $s \cdot E_Q$. First we note that $h \cdot E_Q = 2$. Indeed a pencil of tangent lines to Q through a fixed point $x \in Q$ has two elements which are in Q. Finally, recalling that $V_Q = \mathrm{Sing}(W_Q)$ consists of ordinary double points, we obtain that $s \cdot E_Q = -2$, since $p^{-1}(\ell)$ is a conic inside $\mathbf{P}\big(N_{V_Q/\mathbf{G}}(\ell)\big)$. $\Box$

3.1. **A birational model for $\mathcal{F}_6^{\mathfrak{N}}$.** Let us now specialize to the case $n = 4$, that is,

$$Q \subset \mathbf{P}^4, \ \ \mathbf{G} = G(2,5) \subset \mathbf{P}^9 \text{ and } \dim(W_Q) = 5.$$

The class of V_Q equals $4\sigma_{2,1} \in H^6(\mathbf{G}, \mathbb{Z})$, therefore $\deg(V_Q) = 4\sigma_{2,1} \cdot \sigma_1^3 = 8$. This can also be seen by recalling that V_Q is isomorphic to the Veronese 3-fold $\nu_2(\mathbf{P}^3) \subset \mathbf{P}^9$.

The double covering $f : \tilde{\mathbf{P}}_Q \to \mathbf{P}_Q$ constructed above has a transparent projective interpretation. For $(x, \ell) \in \mathbf{P}_Q$, we denote by $\Pi_\ell \in G(3, V)$ the *polar space* of ℓ defined as the base locus of the pencil of polar hyperplanes $\{z \in \mathbf{P}(V) : \tilde{Q}(y, z) = 0\}_{y \in \ell}$. Clearly $x \in \Pi_\ell \subset \mathbf{P}(T_x Q)$ and $Q \cap \Pi_\ell$ is a conic of rank at most 2 in Π_ℓ. When $\ell \in W_Q - V_Q$, the quadric has rank exactly 2 which corresponds to a pair of lines $\ell_1 + \ell_2$ with $\ell_1, \ell_2 \in V_Q$. The double cover is induced by the map from the parameter space of the lines themselves.

In the next statement we shall keep in mind the notation of diagram (1):

Proposition 3.3. *A general codimension 3 linear section $\bar{Y} := \Lambda \cap W_Q$ of the quadratic complex W_Q where $\Lambda \in G(7, \wedge^2 V)$, is a 8-nodal $K3$ surface with desingularization*

$$p : S := p^{-1}(\bar{Y}) \to \bar{Y}.$$

The triple $[S, \mathcal{O}_S(H - B), \mathcal{O}_S(H)] \in \mathcal{F}_6^{\mathfrak{N}}$ is a Nikulin surface of genus 6 and the induced double cover is the restriction $f : \tilde{S} := f^{-1}(S) \to S$.

Proof. We fix a general 6-plane $\Lambda \in G(7, \wedge^2 V)$. Since $K_{W_Q} = \mathcal{O}_{W_Q}(-3H)$, by adjunction we obtain that $\bar{Y} := \Lambda \cap W_Q$ is a $K3$ surface. From Bertini's theorem, $\bar{Y}$ has ordinary double points at the 8 points of intersection $\Lambda \cap V_Q$. General hyperplane sections of $C \in |\mathcal{O}_{\bar{Y}}(H)|$, viewed as codimension 4 linear sections of W_Q, are canonical curves of genus 6, endowed with a line bundle of order 2 given by $\mathcal{O}_C(H-B)$. The remaining statements are immediate. □

It turns out that the general Nikulin surface of genus 6 arises in this way:

Theorem 3.4. *Let $V := \mathbb{C}^5$ and $Q \subset \textbf{P}(V)$ be a smooth quadric. One has a dominant map*

$$\varphi : G(7, \wedge^2 V)^{\mathrm{ss}} // \mathrm{Aut}(Q) \dashrightarrow \mathcal{F}_6^{\mathfrak{N}},$$

given by $\varphi(\Lambda) := \bigl[S := p^{-1}(\Lambda \cap W_Q),\ \mathcal{O}_S(H-B),\ \mathcal{O}_S(H)\bigr]$.

Proof. Via the embedding $\mathrm{Aut}(Q) \subset PGL(V) \hookrightarrow PGL(\wedge^2 V)$, we observe that every automorphism of Q induces an automorphism of $\mathbf{P}(\wedge^2 V)$ that fixes both W_Q and V_Q. Since (i) the moduli space $\mathcal{F}_6^{\mathfrak{N}}$ is irreducible and (ii) polarized Nikulin surfaces have finite automorphism groups, it suffices to observe that dim $G(7, \wedge^2 V)//\mathrm{Aut}(Q) = 21 - 10 = 11$ and $\dim(\mathcal{F}_6^{\mathfrak{N}}) = 11$ as well. □

Corollary 3.5. *The Prym-Nikulin locus $\mathcal{N}_6 \subset \mathcal{R}_6$ is an irreducible unirational divisor, which is set-theoretically equal to the ramification locus of the Prym map* $\mathrm{Pr} : \mathcal{R}_6 \to \mathcal{A}_5$

$$\mathcal{U}_{6,0} = \{[C, \eta] \in \mathcal{R}_6 : K_{0,2}(C, K_C \otimes \eta) \neq 0\}.$$

Furthermore, the exists a dominant rational map $G(6, \wedge^2 V)^{\mathrm{ss}}//\mathrm{Aut}(Q) \dashrightarrow \mathcal{N}_6$.

Proof. Just observe that $\langle C \rangle = \mathbf{P}^5$ and that this has codimension 4 in $\mathbf{P}(\wedge^2 V)$, hence there is a $\mathbf{P}^3$ of Nikulin sections of W_Q containing C. □

The divisor $\overline{\mathcal{K}}_{10} \subset \overline{\mathcal{M}}_{10}$ of sections of $K3$ surfaces is known to be an extremal point of the effective cone Eff$(\overline{\mathcal{M}}_{10})$. An analogous result holds for the closure of $\mathcal{N}_6$:

Proposition 3.6. *The Prym-Nikulin divisor $\overline{\mathcal{N}}_6$ is extremal in the effective cone* $\mathrm{Eff}(\overline{\mathcal{R}}_6)$*:*

Proof. It follows from [FL] Theorem 0.6 that $\overline{\mathcal{N}}_6 \equiv 7\lambda - \frac{3}{2}\delta_0^{\mathrm{ram}} - (\delta_0' + \delta_0'') - \cdots \in \mathrm{Pic}(\overline{\mathcal{R}}_6)$. The divisor $\overline{\mathcal{KN}}_6$ is filled-up by the rational curves $\Xi_6 \subset \overline{\mathcal{R}}_6$ constructed in the course of proving Theorem 1.4. We compute that $\Xi_6 \cdot \overline{\mathcal{N}}_6 = -1$, which completes the proof. □

4. Spin curves and the divisor $\overline{\Theta}_{\mathrm{null}}$

We turn our attention to the moduli space of spin curves and begin by setting notation and terminology. If $\mathbf{M}$ is a Deligne-Mumford stack, we denote by $\mathcal{M}$ its associated coarse moduli space. A $\mathbb{Q}$-Weil divisor D on a normal $\mathbb{Q}$-factorial projective variety X is said to be *movable* if $\mathrm{codim}\bigl(\bigcap_m \mathrm{Bs}|mD|, X\bigr) \geq 2$, where the intersection is taken over all m which are sufficiently large and divisible. We say that D is *rigid* if $|mD| = \{mD\}$, for all $m \geq 1$ such that mD is an integral Cartier divisor. The *Kodaira-Iitaka dimension* of a divisor D on X is denoted by $\kappa(X, D)$.

If $D = m_1 D_1 + \cdots + m_s D_s$ is an effective $\mathbb{Q}$-divisor on X, with irreducible components $D_i \subset X$ and $m_i > 0$ for $i = 1, \ldots, s$, a (trivial) way of showing that $\kappa(X, D) = 0$ is by exhibiting for each $1 \leq i \leq s$, an irreducible curve $\Gamma_i \subset X$ passing through a general point of D_i, such that $\Gamma_i \cdot D_i < 0$ and $\Gamma_i \cdot D_j = 0$ for $i \neq j$.

We recall basic facts about the moduli space $\overline{\mathcal{S}}_g^+$ and refer to [Cor], [F] for details.

Definition 4.1. An *even spin curve* of genus g consists of a triple (X, η, β), where X is a genus g quasi-stable curve, $\eta \in \mathrm{Pic}^{g-1}(X)$ is a line bundle of degree $g-1$ such that $\eta_E = \mathcal{O}_E(1)$ for every rational component $E \subset X$ with $|E \cap (\overline{X-E})| = 2$ (such a component is called *exceptional*), $h^0(X, \eta) \equiv 0 \mbox{ mod } 2$, and $\beta : \eta^{\otimes 2} \rightarrow \omega_X$ is a morphism of sheaves which is generically non-zero along each non-exceptional component of X.

Even spin curves of genus g form a smooth Deligne-Mumford stack $\pi : \overline{\mathbf{S}}_g^+ \rightarrow \overline{\mathbf{M}}_g$. At the level of coarse moduli schemes, the morphism $\pi : \overline{\mathcal{S}}_g^+ \rightarrow \overline{\mathcal{M}}_g$ is the stabilization map $\pi([X, \eta, \beta]) := [\mathrm{st}(X)]$, which associates to a quasi-stable curve its stable model.

We explain the boundary structure of $\overline{\mathcal{S}}_g^+$: If $[X, \eta, \beta] \in \pi^{-1}([C \cup_y D])$, where $[C, y] \in \mathcal{M}_{i,1}, [D, y] \in \mathcal{M}_{g-i,1}$ and $1 \leq i \leq [g/2]$, then necessarily $X = C \cup_{y_1} E \cup_{y_2} D$, where E is an exceptional component such that $C \cap E = \{y_1\}$ and $D \cap E = \{y_2\}$. Moreover $\eta = \bigl(\eta_C, \eta_D, \eta_E = \mathcal{O}_E(1)\bigr) \in \mathrm{Pic}^{g-1}(X)$, where $\eta_C^{\otimes 2} = K_C, \eta_D^{\otimes 2} = K_D$. The condition $h^0(X, \eta) \equiv 0 \mbox{ mod } 2$, implies that the theta-characteristics η_C and η_D have the same parity. We denote by $A_i \subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs

$$([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^+ \times \mathcal{S}_{g-i,1}^+$$

and by $B_i \subset \overline{\mathcal{S}}_g^+$ the closure of the locus corresponding to pairs

$$([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^- \times \mathcal{S}_{g-i,1}^-.$$

We set $\alpha_i := [A_i] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+), \beta_i := [B_i] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, and then one has the relation

$$\pi^*(\delta_i) = \alpha_i + \beta_i. \tag{3}$$

We recall the description of the ramification divisor of the covering $\pi : \overline{\mathcal{S}}_g^+ \rightarrow \overline{\mathcal{M}}_g$. For a point $[X, \eta, \beta] \in \overline{\mathcal{S}}_g^+$ corresponding to a stable model $\mathrm{st}(X) = C_{yq} := C/y \sim q$, with $[C, y, q] \in \mathcal{M}_{g-1,2}$, there are two possibilities depending on whether X possesses an exceptional component or not. If $X = C_{yq}$ (i.e. X has no exceptional component) and $\eta_C := \nu^*(\eta)$ where $\nu : C \rightarrow X$ denotes the normalization map, then $\eta_C^{\otimes 2} = K_C(y+q)$. For each choice of $\eta_C \in \mathrm{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X, \eta) \equiv 0 \mbox{ mod } 2$. We denote by A_0 the closure in $\overline{\mathcal{S}}_g^+$ of the locus of spin curves $[C_{yq}, \eta_C \in \sqrt{K_C(y+q)}]$ as above.

If $X = C \cup_{\{y,q\}} E$, where E is an exceptional component, then $\eta_C := \eta \otimes \mathcal{O}_C$ is a theta-characteristic on C. Since $H^0(X, \omega) \cong H^0(C, \omega_C)$, it follows that $[C, \eta_C] \in \mathcal{S}_{g-1}^+$. We denote by $B_0 \subset \overline{\mathcal{S}}_g^+$ the closure of the locus of spin curves

$$\bigl[C \cup_{\{y,q\}} E,\ E \cong \mathbf{P}^1,\ \eta_C \in \sqrt{K_C},\ \eta_E = \mathcal{O}_E(1)\bigr] \in \mathcal{S}_g^+.$$

If $\alpha_0 := [A_0], \beta_0 := [B_0] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, we have the relation, see [Cor]:

$$\pi^*(\delta_0) = \alpha_0 + 2\beta_0. \tag{4}$$

In particular, B_0 is the ramification divisor of π. An important effective divisor on $\overline{\mathcal{S}}_g^+$ is the locus of vanishing theta-nulls

$$\Theta_{\mathrm{null}} := \{[C, \eta] \in \mathcal{S}_g^+ : H^0(C, \eta) \neq 0\}.$$

The class of its compactification inside $\overline{\mathcal{S}}_g^+$ is given by the formula, cf. [F]:

$$\overline{\Theta}_{\mathrm{null}} \equiv \frac{1}{4}\lambda - \frac{1}{16}\alpha_0 - \frac{1}{2}\sum_{i=1}^{[g/2]}\beta_i \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+). \tag{5}$$

It is also useful to recall the formula for the canonical class of $\overline{\mathcal{S}}_g^+$:

$$K_{\overline{\mathcal{S}}_g^+} \equiv \pi^*(K_{\overline{\mathcal{M}}_g}) + \beta_0 \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - 2\sum_{i=1}^{[g/2]}(\alpha_i+\beta_i) - (\alpha_1+\beta_1).$$

An argument involving spin curves on certain singular canonical surfaces in $\mathbf{P}^6$, implies that for $g \leq 9$, the divisor $\overline{\Theta}_{\mathrm{null}}$ is uniruled and a rigid point in the cone of effective divisors $\mathrm{Eff}(\overline{\mathcal{S}}_g^+)$:

Theorem 4.2. *For $g \leq 9$ the divisor $\overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_g^+$ is uniruled and rigid. Precisely, through a general point of $\overline{\Theta}_{\mathrm{null}}$ there passes a rational curve $\Gamma \subset \overline{\mathcal{S}}_g^+$ such that $\Gamma \cdot \overline{\Theta}_{\mathrm{null}} < 0$. In particular, if D is an effective divisor on $\overline{\mathcal{S}}_g^+$ with $D \equiv n\overline{\Theta}_{\mathrm{null}}$ for some $n \geq 1$, then $D = n\overline{\Theta}_{\mathrm{null}}$.*

Proof. A general point $[C, \eta_C] \in \Theta_{\mathrm{null}}$ corresponds to a canonical curve $C \stackrel{|K_C|}{\hookrightarrow} \mathbf{P}^{g-1}$ lying on a rank 3 quadric $Q \subset \mathbf{P}^{g-1}$ such that $C \cap \mathrm{Sing}(Q) = \emptyset$. The pencil η_C is recovered from the ruling of Q. We construct the pencil $\Gamma \subset \overline{\mathcal{S}}_g^+$ by representing C as a section of a nodal canonical surface $S \subset Q$ and noting that dim $|\mathcal{O}_S(C)| = 1$. The construction of S depends on the genus and we describe the various cases separately.

(i) $7 \leq g \leq 9$. We choose $V \in G\big(7, H^0(C, K_C)\big)$ such that if $\pi_V : \mathbf{P}^{g-1} \dashrightarrow \mathbf{P}(V^\vee)$ denotes the projection, then $\tilde{Q} := \pi_V(Q)$ is a quadric of rank 3. Let $C' := \pi_V(C) \subset \mathbf{P}(V^\vee)$ be the projection of the canonical curve C. By counting dimensions we find that

$$\dim\Big\{I_{C'/\mathbf{P}(V^\vee)}(2) := \mathrm{Ker}\big\{\mathrm{Sym}^2(V) \to H^0(C, K_C^{\otimes 2})\big\}\Big\} \geq 31 - 3g \geq 4,$$

that is, the embedded curve $C' \subset \mathbf{P}^6$ lies on at least 4 independent quadrics, namely the rank 3 quadric $\tilde{Q}$ and $Q_1, Q_2, Q_3 \in |I_{C'/\mathbf{P}(V^\vee)}(2)|$. By choosing V sufficiently general we make sure that $S := \tilde{Q} \cap Q_1 \cap Q_2 \cap Q_3$ is a canonical surface in $\mathbf{P}(V^\vee)$ with 8 nodes corresponding to the intersection $\bigcap_{i=1}^3 Q_i \cap \mathrm{Sing}(\tilde{Q})$ (This transversality statement can also be checked with Macaulay by representing C as a section of the corresponding Mukai variety). From the exact sequence on S,

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0,$$

coupled with the adjunction formula $\mathcal{O}_C(C) = K_C \otimes K_{S|C}^\vee = \mathcal{O}_C$, as well as the fact $H^1(S, \mathcal{O}_S) = 0$, it follows that dim $|C| = 1$, that is, $C \subset S$ moves in its linear system. In particular, $\overline{\Theta}_{\mathrm{null}}$ is a uniruled divisor for $g \leq 9$.

We determine the numerical parameters of the family $\Gamma \subset \overline{\mathcal{S}}_g^+$ induced by varying $C \subset S$. Since $C^2 = 0$, the pencil $|C|$ is base point free and gives rise to a fibration $f : \tilde{S} \to \mathbf{P}^1$, where $\tilde{S} := \mathrm{Bl}_8(S)$ is the blow-up of the nodes of S. This in turn induces a moduli map $m : \mathbf{P}^1 \to \overline{\mathcal{S}}_g^+$ and $\Gamma =: m(\mathbf{P}^1)$. We have the formulas

$$\Gamma \cdot \lambda = m^*(\lambda) = \chi(S, \mathcal{O}_S) + g - 1 = 8 + g - 1 = g + 7,$$

and

$$\Gamma \cdot \alpha_0 + 2\Gamma \cdot \beta_0 = m^*(\pi^*(\delta_0)) = m^*(\alpha_0) + 2m^*(\beta_0) = c_2(\tilde{S}) + 4(g-1).$$

Noether's formula gives that $c_2(\tilde{S}) = 12\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) - K_{\tilde{S}}^2 = 12\chi(S, \mathcal{O}_S) - K_S^2 = 80$, hence $m^*(\alpha_0) + 2m^*(\beta_0) = 4g + 76$. The singular fibres corresponding to spin curves lying in B_0 are those in the fibres over the blown-up nodes and all contribute with multiplicity 1, that is, $\Gamma \cdot \beta_0 = 8$ and then $\Gamma \cdot \alpha_0 = 4g + 60$. It follows that $\Gamma \cdot \overline{\Theta}_{\mathrm{null}} = -2 < 0$ (independent of g!), which finishes the proof.

(ii) $g = 5$. In the case $C \subset Q \subset \mathbf{P}^4$ and we choose a general quartic $X \in H^0(\mathbf{P}^4, \mathcal{I}_{C/\mathbf{P}^4}(4))$ and set $S := Q \cap X$. Then S is a canonical surface with nodes at the 4 points $X \cap \mathrm{Sing}(Q)$. As in the previous case $\dim |C| = 1$, and the numerical characters of the induced family $\Gamma \subset \overline{\mathcal{S}}_5^+$ can be readily computed:

$$\Gamma \cdot \lambda = g + 5 = 10, \ \Gamma \cdot \beta_0 = |\mathrm{Sing}(S)| = 4, \text{ and } \Gamma \cdot \alpha_0 = 4g + 52,$$

where the last equality is a consequence of Noether's formula $\Gamma \cdot (\alpha_0 + 2\beta_0) = 12\chi(S, \mathcal{O}_S) - K_S^2 + 4(g-1) = 4g + 60$. By direct calculation, we obtain once more that $\Gamma \cdot \overline{\Theta}_{\mathrm{null}} = -2$. The case $g = 6$ is similar, except that the canonical surface S is a $(2, 2, 3)$ complete intersection in $\mathbf{P}^5$, where one of the quadrics is the rank 3 quadric Q.

(iii) $g = 4$. In this last case we proceed slightly differently and denote by $S = \mathbb{F}_2$ the blow-up of the vertex of a cone $Q \subset \mathbf{P}^3$ over a conic in $\mathbf{P}^3$ and write $\mathrm{Pic}(S) = \mathbb{Z} \cdot F + \mathbb{Z} \cdot C_0$, where $F^2 = 0$, $C_0^2 = -2$ and $C_0 \cdot F = 1$. We choose a Lefschetz pencil of genus 4 curves in the linear system $|3(C_0 + 2F)|$. By blowing-up the $18 = 9(C_0 + 2F)^2$ base points, we obtain a fibration $f : \tilde{S} := \mathrm{Bl}_{18}(S) \to \mathbf{P}^1$ which induces a family of spin curves $m : \mathbf{P}^1 \to \overline{\mathcal{S}}_4^+$ given by $m(t) := [f^{-1}(t), \mathcal{O}_{f^{-1}(t)}(F)]$. We have the formulas

$$m^*(\lambda) = \chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) + g - 1 = 4, \quad \text{and}$$

$$m^*(\pi^*(\delta_0)) = m^*(\alpha_0) + 2m^*(\beta_0) = c_2(\tilde{S}) + 4(g-1) = 34.$$

The singular fibres lying in B_0 correspond to curves in the Lefschetz pencil on Q passing through the vertex of the cone, that is, when $f^{-1}(t_0)$ splits as $C_0 + D$, where $D \subset \tilde{S}$ is the residual curve. Since $C_0 \cdot D = 2$ and $\mathcal{O}_{C_0}(F) = \mathcal{O}_{C_0}(1)$, it follows that $m(t_0) \in B_0$. One finds that $m^*(\beta_0) = 1$, hence $m^*(\alpha_0) = 32$ and $m^*(\overline{\Theta}_{\mathrm{null}}) = -1$. Since $\Gamma := m(\mathbf{P}^1)$ fills-up the divisor $\overline{\Theta}_{\mathrm{null}}$, we obtain that $[\overline{\Theta}_{\mathrm{null}}] \in \mathrm{Eff}(\overline{\mathcal{S}}_4^+)$ is rigid. □

5. SPIN CURVES OF GENUS 8

The moduli space $\mathcal{M}_8$ carries one Brill-Noether divisor, the locus of plane septics

$$\mathcal{M}_{8,7}^2 := \{[C] \in \mathcal{M}_8 : G_7^2(C) \neq \emptyset\}.$$

The locus $\overline{\mathcal{M}}_{8,7}^2$ is irreducible and for a known constant $c_{8,7}^2 \in \mathbb{Z}_{>0}$, one has, cf. [EH1],

$$\mathfrak{bn}_8 := \frac{1}{c_{8,7}^2} \overline{\mathcal{M}}_{8,7}^2 \equiv 22\lambda - 3\delta_0 - 14\delta_1 - 24\delta_2 - 30\delta_3 - 32\delta_4 \in \mathrm{Pic}(\overline{\mathcal{M}}_8).$$

In particular, $s(\overline{\mathcal{M}}_{8,7}^2) = 6 + 12/(g+1)$ and this is the minimal slope of an effective divisor on $\overline{\mathcal{M}}_8$. The following fact is probably well-known:

Proposition 5.1. *Through a general point of $\overline{\mathcal{M}}_{8,7}^2$ there passes a rational curve $R \subset \overline{\mathcal{M}}_8$ such that $R \cdot \overline{\mathcal{M}}_{8,7}^2 < 0$. In particular, the class $[\overline{\mathcal{M}}_{8,7}^2] \in \mathrm{Eff}(\overline{\mathcal{M}}_8)$ is rigid.*

Proof. One takes a Lefschetz pencil of nodal plane septic curves with 7 assigned nodes in general position (and 21 unassigned base points). After blowing up the 21 unassigned base points as well as the 7 nodes, we obtain a fibration $f : S := \mathrm{Bl}_{28}(\mathbf{P}^2) \to \mathbf{P}^1$, and the corresponding moduli map $m : \mathbf{P}^1 \to \overline{\mathcal{M}}_8$ is a covering curve for the irreducible divisor $\overline{\mathcal{M}}^2_{8,7}$. The numerical invariants of this pencil are

$$m^*(\lambda) = \chi(S, \mathcal{O}_S) + g - 1 = 8 \text{ and } m^*(\delta_0) = c_2(S) + 4(g-1) = 59,$$

while $m^*(\delta_i) = 0$ for $i = 1, \ldots, 4$. We find $m^*([\overline{\mathcal{M}}^2_{8,7}]) = c^2_{8,7}(8\cdot 22 - 3\cdot 59) = -c^2_{8,7} < 0$. □

Using (5) we find the following explicit representative for the canonical class $K_{\overline{\mathcal{S}}^+_8}$:

$$K_{\overline{\mathcal{S}}^+_8} \equiv \frac{1}{2}\pi^*(\mathfrak{bn}_8) + 8\overline{\Theta}_{\mathrm{null}} + \sum_{i=1}^{4}(a_i\, \alpha_i + b_i\, \beta_i), \tag{6}$$

where $a_i, b_i > 0$ for $i = 1, \ldots, 4$. The multiples of each irreducible component appearing in (6) are rigid divisors on $\overline{\mathcal{S}}^+_8$, but in principle, their sum could still be a movable class. Assuming for a moment Proposition 0.9, we explain how this implies Theorem 0.1:

Proof of Theorem 0.1. The covering curve $R \subset \overline{\Theta}_{\mathrm{null}}$ constructed in Proposition 0.9, satisfies $R \cdot \overline{\Theta}_{\mathrm{null}} < 0$ as well as $R \cdot \pi^*(\overline{\mathcal{M}}^2_{8,7}) = 0$ and $R \cdot \alpha_i = R \cdot \beta_i = 0$ for $i = 1, \ldots, 4$. It follows from (6) that for each $n \geq 1$, one has an equality of linear series on $\overline{\mathcal{S}}^+_8$

$$|nK_{\overline{\mathcal{S}}^+_8}| = 8n\overline{\Theta}_{\mathrm{null}} + |n(K_{\overline{\mathcal{S}}^+_8} - 8\overline{\Theta}_{\mathrm{null}})|.$$

Furthermore, from (6) one finds constants $a'_i > 0$ for $i = 1, \ldots, 4$, such that if

$$D \equiv 22\lambda - 3\delta_0 - \sum_{i=1}^{4} a'_i\, \delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_8),$$

then the difference $\frac{1}{2}\pi^*(D) - (K_{\overline{\mathcal{S}}^+_8} - 8\overline{\Theta}_{\mathrm{null}})$ is still effective on $\overline{\mathcal{S}}^+_8$. We can thus write

$$0 \leq \kappa(\overline{\mathcal{S}}^+_8) = \kappa\big(\overline{\mathcal{S}}^+_8, K_{\overline{\mathcal{S}}^+_8} - 8\overline{\Theta}_{\mathrm{null}}\big) \leq \kappa\big(\overline{\mathcal{S}}^+_8, \frac{1}{2}\pi^*(D)\big) = \kappa\big(\overline{\mathcal{S}}^+_8, \pi^*(D)\big).$$

We claim that $\kappa\big(\overline{\mathcal{S}}^+_8, \pi^*(D)\big) = 0$. Indeed, in the course of the proof of Proposition 5.1 we have constructed a covering family $B \subset \overline{\mathcal{M}}_8$ for the divisor $\overline{\mathcal{M}}^2_{8,7}$ such that $B \cdot \overline{\mathcal{M}}^2_{8,7} < 0$ and $B \cdot \delta_i = 0$ for $i = 1, \ldots, 4$. We lift B to a family $R \subset \overline{\mathcal{S}}^+_8$ of spin curves by taking

$$\tilde{B} := B \times_{\overline{\mathcal{M}}_8} \overline{\mathcal{S}}^+_8 = \{[C_t,\ \eta_{C_t}] \in \overline{\mathcal{S}}^+_8 : [C_t] \in B, \eta_{C_t} \in \overline{\mathrm{Pic}}^7(C_t), t \in \mathbf{P}^1\} \subset \overline{\mathcal{S}}^+_8.$$

One notes that $\tilde{B}$ is disjoint from the boundary divisors $A_i, B_i \subset \overline{\mathcal{S}}^+_8$ for $i = 1, \ldots, 4$, hence $\tilde{B} \cdot \pi^*(D) = 2^{g-1}(2^g + 1)(B \cdot \overline{\mathcal{M}}^2_{8,7})_{\overline{\mathcal{M}}_8} < 0$. Thus we write that

$$\kappa\big(\overline{\mathcal{S}}^+_8, \pi^*(D)\big) = \kappa\big(\overline{\mathcal{S}}^+_8, \pi^*(D - (22\lambda - 3\delta_0)\big) = \kappa\big(\overline{\mathcal{S}}^+_8, \sum_{i=1}^{4} a'_i(\alpha_i + \beta_i)\big) = 0.$$

□

6. A FAMILY OF SPIN CURVES $R \subset \overline{\mathcal{S}}_8^+$ WITH $R \cdot \pi^*(\overline{\mathcal{M}}^2_{8,7}) = 0$ AND $R \cdot \overline{\Theta}_{\mathrm{null}} < 0$

The aim of this section is to prove Proposition 0.9, which is the key ingredient in the proof of Theorem 0.1. We begin by reviewing facts about the geometry of $\overline{\mathcal{M}}_8$, in particular the construction of general curves of genus 8 as complete intersections in a rational homogeneous variety, see [M2].

We fix $V := \mathbb{C}^6$ and denote by $\mathbf{G} := G(2, V) \subset \mathbf{P}(\wedge^2 V)$ the Grassmannian of lines. Noting that smooth codimension 7 linear sections of $\mathbf{G}$ are canonical curves of genus 8, one is led to consider the *Mukai model* of the moduli space of curves of genus 8

$$\mathfrak{M}_8 := G(8, \wedge^2 V)^{\mathrm{st}} // SL(V).$$

There is a birational map $f : \overline{\mathcal{M}}_8 \dashrightarrow \mathfrak{M}_8$, whose inverse is given by $f^{-1}(H) := \mathbf{G} \cap H$, for a general $H \in G(8, \wedge^2 V)$. The map f is constructed as follows: Starting with a curve $[C] \in \mathcal{M}_8 - \mathcal{M}^2_{8,7}$, one notes that C has a finite number of pencils $\mathfrak{g}^1_5$. We choose $A \in W^1_5(C)$ and set $L := K_C \otimes A^\vee \in W^3_9(C)$. There exists a unique rank 2 vector bundle $E \in SU_C(2, K_C)$ (independent of A!), sitting in an extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow L \longrightarrow 0,$$

such that $h^0(E) = h^0(A) + h^0(L) = 6$. Since E is globally generated, we define the map

$$\phi_E : C \to G\big(2, H^0(E)^\vee\big), \quad \phi_E(p) := E(p)^\vee \ \big(\hookrightarrow H^0(E)^\vee\big),$$

and let $\wp : G(2, H^0(E)^\vee) \to \mathbf{P}(\wedge^2 H^0(E)^\vee)$ be the Plücker embedding. The determinant map $u : \wedge^2 H^0(E) \to H^0(K_C)$ is surjective and we can view $H^0(K_C)^\vee \in G(8, \wedge^2 H^0(E)^\vee)$, see [M2] Theorem C. We set

$$f([C]) := H^0(K_C)^\vee \bmod SL(H^0(E)^\vee) \in \mathfrak{M}_8,$$

that is, we assign to C its linear span $\langle C \rangle$ under the Plücker map $\wp \circ \phi_E : C \to \mathbf{P}\big(\wedge^2 H^0(E)^\vee\big)$.

Even though this is not strictly needed for our proof, it follows from [M2] that the exceptional divisors of f are the Brill-Noether locus $\overline{\mathcal{M}}^2_{8,7}$ and the boundary divisors $\Delta_1, \ldots, \Delta_4$. The map f^{-1} does not contract any divisors.

Inside the moduli space $\mathcal{F}_8$ of polarized $K3$ surfaces $[S, h]$ of degree $h^2 = 14$, we consider the following *Noether-Lefschetz* divisor

$$\mathfrak{NL} := \{[S, \mathcal{O}_S(C_1 + C_2)] \in \mathcal{F}_8 : \mathrm{Pic}(S) \supset \mathbb{Z} \cdot C_1 \oplus \mathbb{Z} \cdot C_2, \ \ C_1^2 = C_2^2 = 0, \ C_1 \cdot C_2 = 7\},$$

of doubly-elliptic $K3$ surfaces. For a general element $[S, \mathcal{O}_S(C)] \in \mathfrak{NL}$, the embedded surface $\phi_{\mathcal{O}_S(C)} : S \hookrightarrow \mathbf{P}^8$ lies on a rank 4 quadric whose rulings induce the elliptic pencils $|C_1|$ and $|C_2|$ on S.

Let $\mathcal{U} \to \mathfrak{NL}$ be the space classifying pairs $\big([S, \mathcal{O}_S(C_1 + C_2)], C \subset S\big)$, where

$$C \in |H^0(S, \mathcal{O}_S(C_1)) \otimes H^0(S, \mathcal{O}_S(C_2))| \subset |H^0(S, \mathcal{O}_S(C_1 + C_2))|.$$

An element of $\mathcal{U}$ corresponds to a hyperplane section $C \subset S \subset \mathbf{P}^8$ of a doubly-elliptic $K3$ surface, such that the intersection of $\langle C \rangle$ with the rank 4 quadric induced by the elliptic pencils, has rank at most 3. There exists a rational map

$$q : \mathcal{U} \dashrightarrow \overline{\Theta}_{\mathrm{null}}, \quad q\big([S, \mathcal{O}_S(C_1 + C_2)], C\big) := [C, \mathcal{O}_C(C_1) = \mathcal{O}_C(C_2)].$$

Since $\mathcal{U}$ is birational to a $\mathbf{P}^3$-bundle over an open subvariety of $\mathfrak{NL}$, we obtain that $\mathcal{U}$ is irreducible and $\dim(\mathcal{U}) = 21\big(= 3 + \dim(\mathfrak{NL})\big)$. We shall show that the morphism q is dominant (see Corollary 6.3) and begin with some preparations.

We fix a general point $[C,\eta]\in\overline{\Theta}_{\mathrm{null}}\subset\overline{\mathcal{S}}_8^+$, with η a vanishing theta-null. Then

$$C\subset Q\subset \mathbf{P}^7:=\mathbf{P}\big(H^0(C,K_C)^\vee\big),$$

where $Q\in H^0(\mathbf{P}^7,\mathcal{I}_{C/\mathbf{P}^7}(2))$ is the rank 3 quadric such that the ruling of Q cuts out on C precisely η. As explained, there exists a linear embedding $\mathbf{P}^7\subset\mathbf{P}^{14}:=\mathbf{P}\big(\wedge^2 H^0(E)^\vee\big)$ such that $\mathbf{P}^7\cap\mathbf{G}=C$. The restriction map yields an isomorphism between spaces of quadrics, cf. [M2],

$$\mathrm{res}_C: H^0(\mathbf{G},\mathcal{I}_{\mathbf{G}/\mathbf{P}^{14}}(2))\xrightarrow{\cong} H^0(\mathbf{P}^7,\mathcal{I}_{C/\mathbf{P}^7}(2)).$$

In particular there is a unique quadric $\mathbf{G}\subset\tilde{Q}\subset\mathbf{P}^{14}$ such that $\tilde{Q}\cap\mathbf{P}^7=Q$.

There are three possibilities for the rank of any quadric $\tilde{Q}\in H^0(\mathbf{P}^{14},\mathcal{I}_{\mathbf{G}/\mathbf{P}^{14}}(2))$: (a) $\mathrm{rk}(\tilde{Q})=15$, (b) $\mathrm{rk}(\tilde{Q})=6$ and then $\tilde{Q}$ is a *Plücker quadric*, or (c) $\mathrm{rk}(\tilde{Q})=10$, in which case $\tilde{Q}$ is a sum of two Plücker quadrics, see [M2] Proposition 1.4.

Proposition 6.1. *For a general $[C,\eta]\in\overline{\Theta}_{\mathrm{null}}$, the quadric $\tilde{Q}$ is smooth, that is, $\mathrm{rk}(\tilde{Q})=15$.*

Proof. We may assume that $\dim G^1_5(C)=0$ (in particular C has no $\mathfrak{g}^1_4$'s), and $G^2_7(C)=\emptyset$. The space $\mathbf{P}(\mathrm{Ker}(u))\subset\mathbf{P}\big(\wedge^2H^0(E)\big)$ is identified with the space of hyperplanes $H\in(\mathbf{P}^{14})^\vee$ containing the canonical space $\mathbf{P}^7$.

Claim: If $\mathrm{rk}(\tilde{Q})<15$, there exists a pencil of 8-dimensional planes $\mathbf{P}^7\subset\Xi\subset\mathbf{P}^{14}$, such that $S:=\mathbf{G}\cap\Xi$ is a $K3$ surface containing C as a hyperplane section, and

$$\mathrm{rk}\big\{Q_\Xi:=\tilde{Q}\cap\Xi\in H^0(\Xi,\mathcal{I}_{S/\Xi}(2))\big\}=3.$$

The conclusion of the claim contradicts the assumption that $[C,\eta]\in\overline{\Theta}_{\mathrm{null}}$ is general. Indeed, we pick such an 8-plane Ξ and corresponding $K3$ surface S. Since $\mathrm{Sing}(Q)\cap C=\emptyset$, where $Q_\Xi\cap\mathbf{P}^7=Q$, it follows that $S\cap\mathrm{Sing}(Q_\Xi)$ is finite. The ruling of Q_Ξ cuts out an elliptic pencil $|E|$ on S. Furthermore, S has nodes at the points $S\cap\mathrm{Sing}(Q_\Xi)$. For numerical reasons, $|\mathrm{Sing}(S)|=7$, and then on the surface $\tilde{S}$ obtained from S by resolving the 7 nodes, one has the linear equivalence

$$C\equiv 2E+\Gamma_1+\cdots+\Gamma_7,$$

where $\Gamma_i^2=-2$, $\Gamma_i\cdot E=1$ for $i=1,\ldots,7$ and $\Gamma_i\cdot\Gamma_j=0$ for $i\neq j$. In particular $\mathrm{rk}(\mathrm{Pic}(\tilde{S}))\geq 8$. A standard parameter count, see e.g. [Do1], shows that

$$\dim\big\{(S,C):C\in|\mathcal{O}_S(2E+\Gamma_1+\cdots+\Gamma_7)|\big\}\leq 19-7+\dim|\mathcal{O}_{\tilde{S}}(C)|=20.$$

Since $\dim(\overline{\Theta}_{\mathrm{null}})=20$ and a general curve $[C]\in\overline{\Theta}_{\mathrm{null}}$ lies on infinitely many such $K3$ surfaces S, one obtains a contradiction.

We are left with proving the claim made in the course of the proof. The key point is to describe the intersection $\mathbf{P}(\mathrm{Ker}(u))\cap\tilde{Q}^\vee$, where we recall that the linear span $\langle\tilde{Q}^\vee\rangle$ classifies hyperplanes $H\in(\mathbf{P}^{14})^\vee$ such that $\mathrm{rk}(\tilde{Q}\cap H)\leq\mathrm{rk}(\tilde{Q})-1$. Note also that $\dim\langle\tilde{Q}\rangle=\mathrm{rk}(\tilde{Q})-2$.

If $\mathrm{rk}(\tilde{Q})=6$, then $\tilde{Q}^\vee$ is contained in the dual Grassmannian $\mathbf{G}^\vee:=G(2,H^0(E))$, cf. [M2] Proposition 1.8. Points in the intersection $\mathbf{P}(\mathrm{Ker}(u))\cap\mathbf{G}^\vee$ correspond to decomposable tensors $s_1\wedge s_2$, with $s_1,s_2\in H^0(C,E)$, such that $u(s_1\wedge s_2)=0$. The image of the morphism $\mathcal{O}_C^{\oplus 2}\stackrel{(s_1,s_2)}{\longrightarrow}E$ is thus a subbundle $\mathfrak{g}^1_5$ of E and there is a bijection

$$\mathbf{P}(\mathrm{Ker}(u))\cap\mathbf{G}\big(2,H^0(E)\big)\cong W^1_5(C).$$

It follows, there are at most finitely many tangent hyperplanes to $\tilde{Q}$ containing the space $\mathbf{P}^7 = \langle C\rangle$, and consequently, $\dim\big(\mathbf{P}(\mathrm{Ker}(u))\cap\langle\tilde{Q}^\vee\rangle\big) \leq 1$. Then there exists a codimension 2 linear space $W^{12} \subset \mathbf{P}^{14}$ such that $\mathrm{rk}(\tilde{Q}\cap W) = 3$, which proves the claim (and much more), in the case $\mathrm{rk}(\tilde{Q}) = 6$.

When $\mathrm{rk}(\tilde{Q}) = 10$, using the explicit description of the dual quadric $\tilde{Q}^\vee$ provided in [M2] Proposition 1.8, one finds that $\dim\big(\mathbf{P}(\mathrm{Ker}(u))\cap\langle\tilde{Q}^\vee\rangle\big) \leq 4$. Thus there exists a codimension 5 linear section $W^9 \subset \mathbf{P}^{14}$ such that $\mathrm{rk}(\tilde{Q}\cap W) = 3$, which implies the claim when $\mathrm{rk}(\tilde{Q}) = 10$ as well.

□

We consider an 8-dimensional linear extension $\mathbf{P}^7 \subset \Lambda^8 \subset \mathbf{P}^{14}$ of the canonical space $\mathbf{P}^7 = \langle C\rangle$, such that $S_\Lambda := \Lambda \cap \mathbf{G}$ is a smooth K3 surface. The restriction map

$$\mathrm{res}_{C/S_\Lambda} : H^0(\Lambda, \mathcal{I}_{S_\Lambda/\Lambda}(2)) \to H^0(\mathbf{P}^7, \mathcal{I}_{C/\mathbf{P}^7}(2))$$

is an isomorphism, see [SD]. Thus there exists a *unique* quadric $S_\Lambda \subset Q_\Lambda \subset \Lambda$ with $Q_\Lambda \cap \mathbf{P}^7 = Q$. Since $\mathrm{rk}(Q) = 3$, it follows that $3 \leq \mathrm{rk}(Q_\Lambda) \leq 5$ and it is easy to see that for a general Λ, the corresponding quadric $Q_\Lambda \subset \Lambda$ is of rank 5. We show however, that one can find $K3$-extensions of the canonical curve C, which lie on quadrics of rank 4:

Proposition 6.2. *For a general $[C, \eta] \in \overline{\Theta}_{\mathrm{null}}$, there exists a pencil of 8-dimensional extensions*

$$\boldsymbol{P}(H^0(C, K_C)^\vee) \subset \Lambda \subset \boldsymbol{P}^{14}$$

such that $\mathrm{rk}(Q_\Lambda) = 4$. It follows that there exists a smooth $K3$ surface $S_\Lambda \subset \Lambda$ containing C as a transversal hyperplane section, such that $\mathrm{rk}(Q_\Lambda) = 4$.

Proof. We pass from projective to vector spaces and view the rank 15 quadric

$$\tilde{Q} : \wedge^2 H^0(C, E)^\vee \xrightarrow{\sim} \wedge^2 H^0(C, E)$$

as an isomorphism, which by restriction to $H^0(C, K_C)^\vee \subset \wedge^2 H^0(C, E)^\vee$, induces the rank 3 quadric $Q : H^0(C, K_C)^\vee \to H^0(C, K_C)$. The map $u \circ \tilde{Q} : \wedge^2 H^0(E)^\vee \to H^0(K_C)$ being surjective, its kernel $\mathrm{Ker}(u \circ \tilde{Q})$ is a 7-dimensional vector space containing the 5-dimensional subspace $\mathrm{Ker}(Q)$. We choose an arbitrary element

$$[\bar{v} := v + \mathrm{Ker}(Q)] \in \mathbf{P}\Big(\frac{\mathrm{Ker}(u \circ \tilde{Q})}{\mathrm{Ker}(Q)}\Big) = \mathbf{P}^1,$$

inducing a subspace $H^0(C, K_C)^\vee \subset \Lambda := H^0(C, K_C)^\vee + \mathbb{C}v \subset \wedge^2 H^0(C, E)^\vee$, with the property that $\mathrm{Ker}(Q_\Lambda) = \mathrm{Ker}(Q)$, where $Q_\Lambda : \Lambda \to \Lambda^\vee$ is induced from $\tilde{Q}$ by restriction and projection. It follows that $\mathrm{rk}(Q_\Lambda) = 4$ and there is a pencil of 8-planes $\Lambda \supset \mathbf{P}^7$ with this property. □

Let $C \subset Q \subset \mathbf{P}^7$ be a general canonical curve endowed with a vanishing theta-null, where $Q \in H^0\big(\mathbf{P}^7, I_{C/\mathbf{P}^7}(2)\big)$ is the corresponding rank 3 quadric. We choose a general 8-plane $\mathbf{P}^7 \subset \Lambda \subset \mathbf{P}^{14}$ such that $S := \Lambda \cap \mathbf{G}$ is a smooth K3 surface, and the lift of Q to Λ

$$Q_\Lambda \in H^0\big(\Lambda, \mathcal{I}_{S/\Lambda}(2)\big)$$

has rank 4 (cf. Proposition 6.2). Moreover, we can assume that $S \cap \mathrm{Sing}(Q_\Lambda) = \emptyset$. The linear projection $f_\Lambda : \Lambda \dashrightarrow \mathbf{P}^3$ with center $\mathrm{Sing}(Q_\Lambda)$, induces a regular map $f : S \to \mathbf{P}^3$ with image the smooth quadric $Q_0 \subset \mathbf{P}^3$. Then S is endowed with two elliptic pencils

$|C_1|$ and $|C_2|$ corresponding to the projections of $Q_0 \cong \mathbf{P}^1 \times \mathbf{P}^1$ onto the two factors. Since $C \in |\mathcal{O}_S(1)|$, one has a linear equivalence $C \equiv C_1 + C_2$, on S. As already pointed out, $\deg(f) = C_1 \cdot C_2 = C^2/2 = 7$. The condition $\mathrm{rk}(Q_\Lambda \cap \mathbf{P}^7) = \mathrm{rk}(Q_\Lambda) - 1$, implies that the hyperplane $\mathbf{P}^7 \in (\Lambda)^\vee$ is the pull-back of a hyperplane from $\mathbf{P}^3$, that is, $\mathbf{P}^7 = f_\Lambda^{-1}(\Pi_0)$, where $\Pi_0 \in (\mathbf{P}^3)^\vee$. This proves the following:

Corollary 6.3. *The rational morphism $q : \mathcal{U} \dashrightarrow \overline{\Theta}_{\mathrm{null}}$ is dominant.*

Proof. Keeping the notation from above, if $[C] \in \overline{\Theta}_{\mathrm{null}}$ is a general point corresponding to the rank 3 quadric $Q \in H^0(\mathbf{P}^7, \mathcal{I}_{C/\mathbf{P}^7}(2))$, then $[S, \mathcal{O}_S(C_1 + C_2), C] \in q^{-1}([C])$. □

We begin the proof of Proposition 0.9 while retaining the set-up described above. Let us choose a general line $l_0 \subset \Pi_0$ and denote by $\{q_1, q_2\} := l_0 \cap Q_0$. We consider the pencil $\{\Pi_t\}_{t \in \mathbf{P}^1} \subset (\mathbf{P}^3)^\vee$ of planes through l_0 as well as the induced pencil of curves of genus 8

$$\{C_t := f^{-1}(\Pi_t) \subset S\}_{t \in \mathbf{P}^1},$$

each endowed with a vanishing theta-null induced by the pencil $f_t : C_t \to Q_0 \cap \Pi_t$.

This pencil contains precisely two *reducible* curves, corresponding to the planes Π_1, Π_2 in $\mathbf{P}^3$ spanned by the rulings of Q_0 passing through q_1 and q_2 respectively. Precisely, if $l_i, m_i \subset Q_0$ are the rulings passing through q_i such that $l_1 \cdot l_2 = m_1 \cdot m_2 = 0$, then it follows that for $\Pi_1 = \langle l_1, m_2 \rangle, \Pi_2 = \langle l_2, m_1 \rangle$, the fibres $f^{-1}(\Pi_1)$ and $f^{-1}(\Pi_2)$ split into two elliptic curves $f^{-1}(l_i)$ and $f^{-1}(m_j)$ meeting transversally in 7 points. The half-canonical $\mathfrak{g}^1_7$ specializes to a degree 7 admissible covering

$$f^{-1}(l_i) \cup f^{-1}(m_j) \xrightarrow{f} l_i \cup m_j, \ \ i \neq j,$$

such that the 7 points in $f^{-1}(l_i) \cap f^{-1}(m_j)$ map to $l_i \cap m_j$. To determine the point in $\overline{\mathcal{S}}_8^+$ corresponding to the admissible covering $\big(f^{-1}(l_i) \cup f^{-1}(m_j),\ f_{|f^{-1}(l_i) \cup f^{-1}(m_j)}\big)$, one must insert 7 exceptional components at all the points of intersection of the two components. We denote by $R \subset \overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_8^+$ the pencil of spin curves obtained via this construction.

Lemma 6.4. *Each member $C_t \subset S$ in the above constructed pencil is nodal. Moreover, each curve C_t different from $f^{-1}(l_1) \cup f^{-1}(m_2)$ and $f^{-1}(l_2) \cup f^{-1}(m_1)$ is irreducible. It follows that $R \cdot \alpha_i = R \cdot \beta_i = 0$ for $i = 1, \ldots, 4$.*

Proof. This follows since $f : S \to Q_0$ is a regular morphism and the base line $l_0 \subset H_0$ of the pencil $\{\Pi_t\}_{t \in \mathbf{P}^1}$ is chosen to be general. □

Lemma 6.5. $R \cdot \pi^*(\overline{\mathcal{M}}^2_{7,8}) = 0$.

Proof. We show instead that $\pi_*(R) \cdot \overline{\mathcal{M}}^2_{8,7} = 0$. From Lemma 6.4, the curve R is disjoint from the divisors A_i, B_i for $i = 1, \ldots, 4$, hence $\pi_*(R)$ has the numerical characteristics of a Lefschetz pencil of curves of genus 8 on a fixed $K3$ surface.

In particular, $\pi_*(R) \cdot \delta / \pi_*(R) \cdot \lambda = 6 + 12/(g+1) = s(\overline{\mathcal{M}}^2_{8,7})$ and $\pi_*(R) \cdot \delta_i = 0$ for $i = 1, \ldots, 4$. This implies the statement. □

Lemma 6.6. $R \cdot \overline{\Theta}_{\mathrm{null}} = -1$.

Proof. We have already determined that $R\cdot\lambda=\pi_*(R)\cdot\lambda=\chi(\tilde{S},\mathcal{O}_{\tilde{S}})+g-1=9$, where $\tilde{S}:=\mathrm{Bl}_{2g-2}(S)$ is the blow-up of S at the points $f^{-1}(q_1)\cup f^{-1}(q_2)$. Moreover,

$$R\cdot\alpha_0+2R\cdot\beta_0=\pi_*(R)\cdot\delta_0=c_2(\tilde{X})+4(g-1)=38+28=66. \tag{7}$$

To determine $R\cdot\beta_0$ we study the local structure of $\overline{\mathcal{S}}_8^+$ in a neighbourhood of one of the two points, say $t^*\in R$ corresponding to a reducible curve, say $f^{-1}(l_1)\cup f^{-1}(m_2)$, the situation for $f^{-1}(l_2)\cup f^{-1}(m_1)$ being of course identical. We set $\{p\}:=l_1\cap m_2\in Q_0$ and $\{x_1,\ldots,x_7\}:=f^{-1}(p)\subset S$. We insert exceptional components $E_1,\ldots,E_7$ at the nodes $x_1,\ldots,x_7$ of $f^{-1}(l_1)\cup f^{-1}(m_2)$ and denote by X the resulting quasi-stable curve. If

$$\mu: f^{-1}(l_1)\cup f^{-1}(m_2)\cup E_1\cup\ldots\cup E_7\to f^{-1}(l_1)\cup f^{-1}(m_2)$$

is the stabilization morphism, we set $\{y_i,z_i\}:=\mu^{-1}(x_i)$, where $y_i\in E_i\cap f^{-1}(l_1)$ and $z_i\in E_i\cap f^{-1}(m_2)$ for $i=1,\ldots,7$. If $t^*=[X,\eta,\beta]$, then $\eta_{f^{-1}(l_1)}=\mathcal{O}_{f^{-1}(l_1)}$, $\eta_{f^{-1}(m_2)}=\mathcal{O}_{f^{-1}(m_2)}$, and of course $\eta_{E_i}=\mathcal{O}_{E_i}(1)$. Moreover, one computes that $\mathrm{Aut}(X,\eta,\beta)=\mathbb{Z}_2$, see [Cor] Lemma 2.2, while clearly $\mathrm{Aut}(f^{-1}(l_1)\cup f^{-1}(m_2))=\{\mathrm{Id}\}$.

If $\mathbb{C}_\tau^{3g-3}$ denotes the versal deformation space of $[X,\eta,\beta]\in\overline{\mathcal{S}}_g^+$, then there are local parameters $(\tau_1,\ldots,\tau_{3g-3})$, such that for $i=1,\ldots,7$, the locus $\big(\tau_i=0\big)\subset\mathbb{C}_\tau^{3g-3}$ parameterizes spin curves for which the exceptional component E_i persists. It particular, the pull-back $\mathbb{C}_\tau^{3g-3}\times_{\overline{\mathcal{S}}_g^+}B_0$ of the boundary divisor $B_0\subset\overline{\mathcal{S}}_g^+$ is given by the equation $\big(\tau_1\cdots\tau_7=0\big)\subset\mathbb{C}_\tau^{3g-3}$. The group $\mathrm{Aut}(X,\eta,\beta)$ acts on $\mathbb{C}_\tau^{3g-3}$ by

$$(\tau_1,\ldots,\tau_7,\tau_8,\ldots,\tau_{3g-3})\mapsto(-\tau_1,\ldots,-\tau_7,\tau_8,\ldots,\tau_{3g-3}),$$

and since an étale neighbourhood of $t^*\in\overline{\mathcal{S}}_g^+$ is isomorphic to $\mathbb{C}_\tau^{3g-3}/\mathrm{Aut}(X,\eta,\beta)$, we find that the Weil divisor B_0 is not Cartier around t^* (though $2B_0$ is Cartier). It follows that the intersection multiplicity of $R\times_{\overline{\mathcal{S}}_g^+}\mathbb{C}_\tau^{3g-3}$ with the locus $(\tau_1\cdots\tau_7)=0$ equals 7, that is, the intersection multiplicity of $R\cap\beta_0$ at the point t^* equals $7/2$, hence

$$R\cdot\beta_0=\big(R\cdot\beta_0\big)_{f^{-1}(l_1)\cup f^{-1}(m_2)}+\big(R\cdot\beta_0\big)_{f^{-1}(l_2)\cap f^{-1}(m_1)}=\frac{7}{2}+\frac{7}{2}=7.$$

Then using (7) we find that $R\cdot\alpha_0=66-14=52$, and finally

$$R\cdot\overline{\Theta}_{\mathrm{null}}=\frac{1}{4}R\cdot\lambda-\frac{1}{16}R\cdot\alpha_0=\frac{9}{4}-\frac{52}{16}=-1.$$

□

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