$\overline{\mathcal{M}}_{16}$ IS NOT OF GENERAL TYPE

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ABSTRACT. We prove that the moduli space of curves of genus 16 is not of general type.

The problem of determining the nature of the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g has long been one of the key questions in the field, motivating important developments in moduli theory. Severi [Sev] observed that $\overline{\mathcal{M}}_g$ is unirational for $g \leq 10$, see [AC] for a modern presentation. Much later, in the celebrated series of papers [HM], [H], [EH], Harris, Mumford and Eisenbud showed that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$. Very recently, it has been showed in [FJP] that both $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type. On the other hand, due to work of Sernesi [Ser], Chang-Ran [CR1], [CR2] and Verra [Ve] it is known that $\overline{\mathcal{M}}_g$ is unirational also for $11 \leq g \leq 14$. Finally, Bruno and Verra [BV] proved that $\overline{\mathcal{M}}_{15}$ is rationally connected. Our result is the following:

Theorem 1. *The moduli space $\overline{\mathcal{M}}_{16}$ of stable curves of genus 16 is not of general type.*

Note that 16 is the highest genus for which it is known that $\overline{\mathcal{M}}_{16}$ is *not* of general type. We further refer to Tseng's recent paper [Ts] for further details on the convoluted history of determining the Kodaira dimension of $\overline{\mathcal{M}}_{16}$.

Before explaining our strategy of proving Theorem 1, recall the standard notation $\Delta_0, \ldots, \Delta_{\lfloor \frac{g}{2} \rfloor}$ for the irreducible boundary divisors on $\overline{\mathcal{M}}_g$, see [HM]. Here Δ_0 denotes the closure in $\overline{\mathcal{M}}_g$ of the locus of irreducible 1-nodal curves of arithmetic genus g. Our approach relies on the explicit *uniruled parametrization* of $\overline{\mathcal{M}}_{15}$ found by Bruno and Verra [BV]. Their work establishes that through a general point of $\overline{\mathcal{M}}_{15}$ there passes not only a rational curve, but in fact a *rational surface*. This extra degree of freedom, yields a uniruled parametrization of $\overline{\mathcal{M}}_{15,2}$, therefore also a parametrization the boundary divisor Δ_0 inside $\overline{\mathcal{M}}_{16}$. We show the following:

Theorem 2. *The boundary divisor Δ_0 of $\overline{\mathcal{M}}_{16}$ is uniruled and swept by a family of rational curves, whose general member $\Gamma \subseteq \Delta_0$ satisfies $\Gamma \cdot K_{\overline{\mathcal{M}}_{16}} = 0$ and $\Gamma \cdot \Delta_0 > 0$.*

Assuming Theorem 2, we conclude that $\overline{\mathcal{M}}_{16}$ cannot be of general type, thus establishing Theorem 1. To that end, note first that in any effective representation of the canonical divisor

$$K_{\overline{\mathcal{M}}_{16}} \equiv \alpha \cdot \Delta_0 + D,$$

where $\alpha \in \mathbb{Q}_{>0}$ and D is an effective $\mathbb{Q}$-divisor on $\overline{\mathcal{M}}_{16}$ not containing Δ_0 in its support, we must have $\alpha = 0$. Indeed, we can choose the curve Γ such that $\Gamma \nsubseteq D$, then we write

$$0 = \Gamma \cdot K_{\overline{\mathcal{M}}_{16}} = \alpha \Gamma \cdot \Delta_0 + \Gamma \cdot D \geq \alpha \Gamma \cdot \Delta_0 \geq 0,$$

hence $\alpha = 0$. Furthermore, since the singularities of $\overline{\mathcal{M}}_g$ do not impose adjunction conditions [HM, Theorem 1], $\overline{\mathcal{M}}_g$ is a variety of general type for a given $g \geq 4$ if and only if the canonical class $K_{\overline{\mathcal{M}}_g}$ is a big divisor class, that is, it can be written as

$$K_{\overline{\mathcal{M}}_g} \equiv A + E, \tag{1}$$

where A is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor respectively. Assume that $K_{\overline{\mathcal{M}}_{16}}$ can be written like in (1). It has already been observed that $\Delta_0 \nsubseteq \mathrm{supp}(E)$, in particular $\Gamma \cdot E \geq 0$. Using Kleiman's ampleness criterion, $\Gamma \cdot A > 0$, which yields the immediate contradiction $0 = \Gamma \cdot K_{\overline{\mathcal{M}}_{16}} = \Gamma \cdot A + \Gamma \cdot E \geq \Gamma \cdot A > 0$.

We are left therefore with proving Theorem 2, which is what we do in the rest of the paper. The rational curve Γ constructed in Theorem 2 is the moduli curve corresponding to an appropriate pencil of curves of genus 15 on a certain canonical surface $S \subseteq \mathbf{P}^6$. Establishing that this pencil can be chosen in such a way to contain only stable curves will take up most of Section 2.

1. The Bruno-Verra parametrization of $\overline{\mathcal{M}}_{15}$

The parametrization of the boundary divisor Δ_0 of $\overline{\mathcal{M}}_{16}$ and the proof of Theorem 2 uses several results from [BV], which we now recall. We denote by $\mathcal{H}_{15,9}$ the Hurwitz space parametrizing degree 9 covers $C \to \mathbf{P}^1$ having simple ramification, where C is a smooth curve of genus 15. Then $\mathcal{H}_{15,9}$ is birational to the parameter space $\mathcal{G}^1_{15,9}$ classifying pairs (C, A), where $[C] \in \mathcal{M}_{15}$ and $A \in W^1_9(C)$ is a pencil. By residuation, $\mathcal{G}^1_{15,9}$ is isomorphic to the parameter space $\mathcal{G}^6_{15,19}$ of pairs $[C, L]$, where C is a smooth curve of genus 15 and $L \in W^6_{19}(C)$. Note that the general fibre of the forgetful map

$$\pi \colon \mathcal{H}_{15,9} \to \mathcal{M}_{15}, \quad [C, A] \mapsto [C] \tag{2}$$

is 1-dimensional. Clearly, $\mathcal{H}_{15,9}$ and thus $\mathcal{G}^6_{15,19}$ is irreducible.

We pick a general element $[C, L] \in \mathcal{G}^6_{15,19}$, in particular L is very ample and $h^0(C, L) = 7$. We set $A := \omega_C \otimes L^\vee \in W^1_9(C)$. We may assume that A is base point free and the pencil $|A|$ has simple ramification. We consider the multiplication map

$$\phi_L \colon \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^2).$$

Since C is Petri general, $h^1(C, L^2) = 0$, therefore $h^0(C, L^2) = 2 \cdot 19 + 1 - 15 = 24$. Furthermore, via a degeneration argument it is shown in [BV, Theorem 3.11], that for a general choice of (C, L), the map ϕ_L is surjective, hence $h^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(2)) = \dim\big(\mathrm{Ker}(\phi_L)\big) = 4$, that is, the degree 19 curve $C \subseteq \mathbf{P}^6$ lies on precisely 4 independent quadrics. We let

$$S := \mathrm{Bs}\big|\mathcal{I}_{C/\mathbf{P}^6}(2)\big| \tag{3}$$

be the base locus of the system of quadrics containing C. It is further established in [BV, Theorem 3.11] that under our generality assumptions, S is a smooth surface. From the adjunction formula it follows that $\omega_S = \mathcal{O}_S(1)$, that is, S is a canonical surface. We write down the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0. \tag{4}$$

From the adjunction formula $\mathcal{O}_C(C) \cong \omega_C \otimes \omega_{S|C}^\vee = \omega_C \otimes L^\vee = A \in W^1_9(C)$. Since S is a regular surface, by taking cohomology in (4), we obtain

$$h^0(S, \mathcal{O}_S(C)) = h^0(S, \mathcal{O}_S) + h^0(C, A) = 3.$$

Observe also from the sequence (4) that the linear system $|\mathcal{O}_S(C)|$ is base point free, for $|\mathcal{O}_C(C)| = |A|$ is so. This brings to an end our summary of the results from [BV].

In what follows, we denote by

$$f\colon S \to \mathbf{P}^2 = |\mathcal{O}_S(C)|^\vee \tag{5}$$

the induced map. For what we intend to do, it is important to show that f is a finite map, or equivalently, that $\mathcal{O}_S(C)$ is ample.

Theorem 3. *For a general pair $(C, A) \in \mathcal{H}_{15,9}$, the line bundle $\mathcal{O}_S(C)$ is ample.*

In order to prove Theorem 3 it suffices to exhibit a single pair $(C, A) \in \mathcal{H}_{15,9}$ for which the corresponding map $f\colon S \to \mathbf{P}^2$ given by (5) is finite. We shall realize the canonical surface $S \subseteq \mathbf{P}^6$ as the double cover of a suitable $K3$ surface $Y \subseteq \mathbf{P}^5$ of genus 5 (that is, of degree 8). It will prove advantageous to consider $K3$ surfaces having a certain Picard lattice of rank 3. We first discuss the geometry of such $K3$ surfaces.

Definition 4. We denote by Λ the even lattice of signature $(1, 2)$ generated by elements $\bar{H}, \bar{F}$ and $\bar{R}$ having the following intersection matrix:

$$\begin{pmatrix} \bar{H}^2 & \bar{H}\cdot\bar{F} & \bar{H}\cdot\bar{R} \\ \bar{F}\cdot\bar{H} & \bar{F}^2 & \bar{F}\cdot\bar{R} \\ \bar{R}\cdot\bar{H} & \bar{R}\cdot\bar{F} & \bar{R}^2 \end{pmatrix} = \begin{pmatrix} 8 & 9 & 1 \\ 9 & 4 & 2 \\ 1 & 2 & -2 \end{pmatrix}.$$

We denote by $\mathcal{F}_5^\Lambda$ the moduli space of polarized $K3$ surfaces $[Y, \bar{H}]$, where $\bar{H}^2 = 8$, admitting a primitive embedding $\Lambda \hookrightarrow \mathrm{Pic}(Y)$, such that the classes $\bar{H}, \bar{F}, \bar{R}$ correspond to curve classes on Y which we denote by the same symbol. Furthermore, $\bar{H} \in \mathrm{Pic}(Y)$ is assumed to be ample.

For details on the construction of the moduli space $\mathcal{F}_5^\Lambda$ we refer to [Do, Section 3]. It follows from *loc.cit.* that $\mathcal{F}_5^\Lambda$ is an irreducible variety of dimension $17 = 20 - \mathrm{rk}(\Lambda)$. Let us now fix a general element $[Y, \bar{H}]$, where $\mathrm{Pic}(Y) \cong \mathbb{Z}\langle \bar{H}, \bar{F}, \bar{R}\rangle$ as in Definition 4. Then $\mathcal{O}_Y(\bar{H})$ is very ample and we denote by

$$\varphi_{\bar{H}}\colon Y \hookrightarrow \mathbf{P}^5 \tag{6}$$

the embedding induced by this linear system. Observe that $h^0\big(\mathbf{P}^5, \mathcal{I}_{Y/\mathbf{P}^5}(2)\big) = 3$ and that $Y = \mathrm{Bs}|\mathcal{I}_{Y/\mathbf{P}^5}(2)|$ is in fact a complete intersection of three quadrics. Note that $\bar{F} \subseteq Y$ is a curve of genus 3, whereas $\bar{R} \subseteq Y$ is a smooth rational curve embedded as a line under the map $\varphi_{\bar{H}}$. The class $\bar{E} := 2\bar{H} - \bar{F}$ satisfies $\bar{E}^2 = 0$. Since $\bar{E}\cdot\bar{H} = 7 > 0$, it follows that $|\bar{E}|$ is an elliptic pencil and furthermore $\bar{E}\cdot\bar{R} = 0$. Setting also

$$\bar{D} := 2\bar{H} + \bar{R} - \bar{E} \in \mathrm{Pic}(Y),$$

we compute $\bar{D}^2 = 6$, $\bar{D}\cdot\bar{E} = 14$ and $\bar{D}\cdot\bar{R} = 0$. In the basis $\big(\bar{D}, \bar{E}, \bar{R}\big)$ of $\mathrm{Pic}(Y)$, the intersection form on Y is described by the following simpler matrix:

$$\begin{pmatrix} \bar{D}^2 & \bar{D}\cdot\bar{E} & \bar{D}\cdot\bar{R} \\ \bar{E}\cdot\bar{D} & \bar{E}^2 & \bar{E}\cdot\bar{R} \\ \bar{R}\cdot\bar{D} & \bar{R}\cdot\bar{E} & \bar{R}^2 \end{pmatrix} = \begin{pmatrix} 6 & 14 & 0 \\ 14 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{7}$$

On our way to proving Theorem 3, we establish the following result:

Proposition 5. *The line bundle $\mathcal{O}_Y(\bar{F})$ is very ample.*

Proof. We first claim that $\bar{F}$ is nef. Since $\bar{F}^2 = 4 > 0$, it suffices to check that for any smooth rational curve $\bar{\Gamma} \subseteq Y$, one has $\bar{\Gamma} \cdot \bar{F} \geq 0$. We write $\bar{\Gamma} \equiv a\bar{D} + b\bar{E} + c\bar{R}$, where a, b and c are integers. We may assume $\bar{\Gamma} \neq \bar{R}$, thus $\bar{\Gamma} \cdot \bar{R} \geq 0$, implying $c \leq 0$. Furthermore, $\bar{\Gamma} \cdot \bar{E} \geq 0$, hence $a \geq 0$. Using (7), one has $\bar{\Gamma}^2 = 6a^2 - 2c^2 + 28ab = -2$. Assume by contradiction $\bar{\Gamma} \cdot \bar{F} = \bar{\Gamma} \cdot (\bar{D} - \bar{R}) = 6a + 14b + 2c \leq -2$. Multiplying this inequality with $2a \geq 0$ and substituting in the equality $\bar{\Gamma}^2 = -2$ we obtain that $(a+c)^2 + 2a^2 + 2a - 1 \geq 0$, implying $a = 0$ and $c \in \{-1, 1\}$. If, say $c = 1$, then $\bar{\Gamma} \equiv \bar{R} + b\bar{E}$. From the assumption $\bar{\Gamma} \cdot \bar{F} \leq -2$, we obtain that $b \leq -1$, hence $\bar{\Gamma} \cdot \bar{H} < 0$, thus $\bar{\Gamma}$ cannot be effective, a contradiction. The case $c = -1$, implying $b \leq 0$ is ruled out similarly

Thus $\bar{F}$ is a nef curve. To conclude that $\bar{F}$ is very ample, we invoke [SD]. It suffices to rule out the existence of a divisor class $\bar{M} \in \mathrm{Pic}(Y)$ such that (i) $\bar{M}^2 = 0$ and $\bar{M} \cdot \bar{F} \in \{1, 2\}$, or satisfying (ii) $\bar{M}^2 = -2$ and $\bar{M} \cdot \bar{F} = 0$. We discuss only (i), the remaining case being similar. Write $\bar{M} = a\bar{D} + b\bar{E} + c\bar{R}$. Since $\bar{M}^2 = 0$, from (7) we obtain $3a^2 - c^2 + 14ab = 0$, whereas from $\bar{M} \cdot \bar{F} = 2$, we obtain that $3a + 7b + c = 1$. Eliminating c, we find $6a^2 + a(28b - 6) + 49b^2 - 14b + 1 = 0$. Since the discriminant of this equation is negative, this case is excluded. We conclude that $\bar{F}$ is very ample. □

We fix a general polarized $K3$ surface $[Y, \bar{H}] \in \mathcal{F}_5^{\Lambda}$, while keeping the notation from above. Choose a smooth divisor $\bar{Q} \in \left|\mathcal{O}_Y(2\bar{H})\right|$ and consider the double cover

$$\sigma \colon S \to Y \tag{8}$$

branched along $\bar{Q}$. We denote by $Q \subseteq S$ the ramification divisor of σ, hence $\sigma^*(\bar{Q}) = 2Q$. We set $H := \sigma^*(\bar{H})$, where $\bar{H} \in \left|\mathcal{O}_Y(1)\right|$ is a linear section of Y. Note that $Q \in \left|\mathcal{O}_S(H)\right|$.

Proposition 6. *The induced morphism $\varphi_H \colon S \to \mathbf{P}^6$ embeds S as a canonical surface which is the complete intersection of 4 quadrics in $\mathbf{P}^6$. More precisely, S is a quadratic section of the cone $\mathcal{C}_Y \subseteq \mathbf{P}^6$ over the $K3$ surface $Y \subseteq \mathbf{P}^5$.*

Proof. From the adjunction formula we find $\omega_S = \mathcal{O}_S(Q) = \mathcal{O}_S(H)$. Furthermore, we have $\sigma_*(\mathcal{O}_S) = \mathcal{O}_Y \oplus \mathcal{O}_Y(-H)$, hence from the projection formula we can write

$$H^0\big(S, \mathcal{O}_S(H)\big) \cong H^0\big(Y, \mathcal{O}_Y(\bar{H})\big) \oplus H^0\big(Y, \mathcal{O}_Y\big) \cong H^0\big(Y, \mathcal{O}_Y(\bar{H})\big) \oplus \mathbb{C}\langle Q \rangle,$$

where recall that $Q \in \left|\mathcal{O}_S(H)\right|$, as well as

$$H^0\big(S, \mathcal{O}_S(2H)\big) \cong H^0\big(Y, \mathcal{O}_Y(2\bar{H})\big) \oplus H^0\big(Y, \mathcal{O}_Y(\bar{H})\big) \cdot Q.$$

Thus $h^0\big(S, \mathcal{O}_S(H)\big) = 6$ and $h^0(S, \mathcal{O}_S(2H)) = h^0(Y, \mathcal{O}_Y(2)) + h^0(Y, \mathcal{O}_Y(1)) = 2 + 2\bar{H}^2 + 6 = 24$. Furthermore, $S \subseteq \mathbf{P}^6$ is projectively normal, so $h^0\big(\mathbf{P}^6, \mathcal{I}_{S/\mathbf{P}^6}(2)\big) = 4$. Since clearly $S \subseteq \mathcal{C}_Y$, it follows that S can be viewed as a quadratic section of the cone $\mathcal{C}_Y$, precisely the intersection of $\mathcal{C}_Y$ with one of the quadrics containing S not lying in the subsystem $\left|\sigma^* H^0\big(\mathbf{P}^5, \mathcal{I}_{Y/\mathbf{P}^5}(2)\big)\right|$. □

We are now in a position to prove Theorem 3. We denote by $\mathrm{Hilb}_{15,19}$ the unique component of the Hilbert scheme of curves $C \subseteq \mathbf{P}^6$ of genus 15 and degree 19 dominating $\mathcal{M}_{15}$. A general point of $\mathrm{Hilb}_{15,19}$ corresponds to a smooth projectively normal curve $C \subseteq \mathbf{P}^6$ such that the canonical surface S defined by (3) is smooth.

Proof of Theorem 3. We choose a $K3$ surface $[Y, \mathcal{O}_Y(\bar{H})] \in \mathcal{F}_5^{\Lambda}$ with $\mathrm{Pic}(Y) = \mathbb{Z}\langle \bar{H}, \bar{F}, \bar{R} \rangle$, where the intersection matrix is given as in Definition 4. The restriction map

$$H^0\big(Y, \mathcal{O}_Y(2\bar{H})\big) \to H^0\big(\bar{R}, \mathcal{O}_{\bar{R}}(2\bar{H})\big)$$

being surjective, we can choose a smooth curve $\bar{Q} \in \left|\mathcal{O}_Y(2\bar{H})\right|$ which is *tangent* to $\bar{R}$, that is, $\bar{Q} \cdot \bar{R} = 2y$, for a point $y \in Y$. Construct the double cover $\sigma\colon S \to Y$ defined in (8). The pull-back $\sigma^*(\bar{R})$ is then a double cover of $\bar{R}$ branched over the single point y, hence necessarily

$$\sigma^*(\bar{R}) = R + R' \subseteq S,$$

where R and R' are lines on $S \subseteq \mathbf{P}^6$ meeting at a single point. Next, we choose a smooth genus 3 curve $\bar{F} \subseteq Y$ general in its linear system and set

$$C' := \sigma^*(\bar{F}) \subseteq S.$$

Since $\bar{F} \cdot \bar{Q} = 2\bar{F} \cdot \bar{H} = 18$, we obtain that C' is a smooth curve of genus 14 and degree 18 endowed with the double cover $C' \to \bar{F}$. Note that the linear system

$$\left|\mathcal{O}_S(C')\right| = \pi^*\left|\mathcal{O}_Y(\bar{F})\right|$$

is 3-dimensional. Applying Theorem 5, since $\mathcal{O}_Y(\bar{F})$ is ample and σ is finite, we obtain that $\mathcal{O}_S(C')$ is ample as well. Observe that $C' \cdot R = \bar{F} \cdot \bar{R} = 2$. Choosing $\bar{F}$ general in its linear system, we can arrange the intersection of R and C' to be transverse, therefore

$$C := C' + R \subseteq S \subseteq \mathbf{P}^6 \tag{9}$$

is a nodal curve of genus 15 and degree 19. Note that the linear system $\left|\mathcal{O}_S(C)\right|$ has R as a fixed component, and $\left|\mathcal{O}_S(C)\right| = R + \pi^*\left|\mathcal{O}_Y(\bar{F})\right|$.

Despite the fact that $\left|\mathcal{O}_S(C)\right|$ is not ample, we can complete the proof of Theorem 3. Indeed, let us pick a general family $\left\{[C_t \hookrightarrow \mathbf{P}^6]\right\}_{t\in T} \subseteq \mathrm{Hilb}_{15,19}$ over a pointed base (T, o), whose fibre over $o \in T$ is the curve C described in (9). If $S_t = \mathrm{Bs}\left|\mathcal{I}_{C_t/\mathbf{P}^6}(2)\right|$, assume the line bundle $\mathcal{O}_{S_t}(C_t)$ is not ample for each $t \in T$. As we have already observed, we may assume that $\mathcal{O}_{S_t}(C_t)$ is nef for all $t \in T$ and we denote by $f_t\colon S_t \to \mathbf{P}^2$ the map induced by the linear system $\left|\mathcal{O}_{S_t}(C_t)\right|$ for $t \in T \setminus \{o\}$. The limiting map of this family

$$f_o\colon S \to \mathbf{P}^2,$$

satisfies then $f_o^*\big(\mathcal{O}_{\mathbf{P}^2}(1)\big) = \mathcal{O}_S(R + C')$ and is induced by a subspace of sections $\sigma^*(V)$, where $V \subseteq H^0\big(Y, \mathcal{O}_Y(\bar{F})\big)$ is 3-dimensional. By assumption, there exists a family of curves $\Gamma_t \subseteq S_t$ such that $\Gamma_t \cdot C_t = 0$. We denote by $\Gamma_o \subseteq S$ the limiting curve of Γ_t, therefore $\Gamma_o \cdot (C' + R) = 0$. We write $\Gamma_o = G + mR$, where $m \geq 0$ and $G \subseteq S$ is a curve not having R in its support. From the adjunction formula, we find $R^2 = -3$. Since $R \cdot C' = 2$, it follows that $R \cdot (C' + R) = 1$, thus $G \neq 0$. Furthermore, the morphism f_o contracts G, which we argue, leads to a contradiction. Indeed, f_o admits a factorization

$$S \xrightarrow{\ \sigma\ } Y \xrightarrow{\ |\bar{F}|\ } \mathbf{P}^3 \xrightarrow{\ p\ } \mathbf{P}^2, \qquad f_o = p \circ |\bar{F}| \circ \sigma$$

where $p\colon \mathbf{P}^3 \to \mathbf{P}^2$ is the linear projection corresponding to $V \subseteq H^0\big(Y, \mathcal{O}_Y(\bar{F})\big)$. Since σ is finite and $|\bar{F}|$ is very ample, it follows that $\sigma(G)$ must be contracted by the projection p, that is, $\sigma(C)$ is a line in $\mathbf{P}^3$. By inspecting the intersection matrix (7) of $\mathrm{Pic}(Y)$ we immediately see that no such line can exist on Y, which finishes the proof. □

2. THE UNIRULEDNESS OF THE BOUNDARY DIVISOR Δ_0 IN $\overline{\mathcal{M}}_{16}$

We now lift the construction discussed above from $\overline{\mathcal{M}}_{15}$ to the moduli space $\overline{\mathcal{M}}_{15,2}$ of 2-pointed stable curves of genus 15 and eventually to $\overline{\mathcal{M}}_{16}$. Recall that $\mathrm{Hilb}_{15,19}$ is the component of the Hilbert scheme of curves $C \subseteq \mathbf{P}^6$ of genus 15 and degree 19 dominating $\mathcal{M}_{15}$. We denote by $\mathrm{Hilb}_{2,2,2,2}$ the Hilbert scheme of complete intersections of 4 quadrics in $\mathbf{P}^6$. Since $\mathrm{Hilb}_{15,19}//PGL(7)$ is birational to the Hurwitz space $\mathcal{H}_{15,9}$, we have a rational map

(10) $$\chi\colon \mathcal{H}_{15,9} \dashrightarrow \mathrm{Hilb}_{2,2,2,2}//PGL(7), \quad [C,A] \mapsto S := \mathrm{Bs}\big|\mathcal{I}_{C/\mathbf{P}^6}(2)\big| \mod PGL(7),$$

where the canonical surface $S \subseteq \mathbf{P}^6$ is defined by (3). We set

(11) $$\mathcal{S} := \chi(\mathcal{H}_{15,9}).$$

The general fibre of the morphism $\chi\colon \mathcal{H}_{15,9} \to \mathcal{S}$ consists of finitely many linear nonempty open subsets of linear systems $\big|\mathcal{O}_S(C)\big|$, where $C \subseteq S \subseteq \mathbf{P}^6$ is a smooth curve of genus 15 and degree 19. In particular, $\mathcal{S}$ is an irreducible variety of dimension $41 = \dim(\mathcal{H}_{15,9}) - 2$. Recall that $\pi\colon \mathcal{H}_{15,9} \to \mathcal{M}_{15}$ denotes the forgetful map. The next observation will prove to be useful in several moduli counts.

Proposition 7. *If $\mathcal{S}'$ is an irreducible subvariety of $\mathcal{S}$ of dimension $\dim(\mathcal{S}') \leq 39$, then $\pi\big(\chi^{-1}(\mathcal{S}')\big)$ is a proper subvariety of $\mathcal{M}_{15}$.*

Proof. Since $\dim\big(\chi^{-1}(\mathcal{S}')\big) \leq \dim(\mathcal{S}') + 2 \leq 41 = \dim(\mathcal{M}_{15}) - 1$, the claim follows. □

Let us now take a general curve C of genus 15 and consider the correspondence

$$\Sigma := \Big\{(A, x+y) \in W^1_9(C) \times C_2 : H^0(C, A(-x-y)) \neq 0\Big\},$$

endowed with the projections $\pi_1\colon \Sigma \to W^1_9(C)$ and $\pi_2\colon \Sigma \to C_2$ respectively. Here C_2 is the second symmetric product of C. It follows that Σ is an irreducible surface and that π_2 is generically finite. Indeed, for a general point $2x \in C_2$, we can invoke for instance [EH, Theorem 1.1] to conclude that $\pi_2^{-1}(2x)$ is finite. The fibre $\pi_1^{-1}(A)$ is irreducible whenever A has simple ramification.

We now fix a general element $[C, x, y] \in \overline{\mathcal{M}}_{15,2}$. Then there exist finitely many pencils $A \in W^1_9(C)$ containing both points x and y in the same fibre. Each of these pencils A may be assumed to be base point free with simple ramification and general enough such that $L := \omega_C \otimes A^{\vee} \in W^6_{19}(C)$ is very ample and in the embedding

$$\varphi_L\colon C \hookrightarrow \mathbf{P}^6$$

the curve C lies on precisely 4 independent quadrics intersecting in a smooth canonical surface S defined by (3).

Proposition 8. *With the notation above, if $h^0\big(C, A(-x-y)\big) = 1$, then $\dim\big|\mathcal{I}_{\{x,y\}}(C)\big| = 1$.*

Proof. It follows from the commutativity of the following diagram, keeping in mind that $h^0(S, \mathcal{O}_S(C)) = 3$ and that the first column is injective.

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0\big(S, \mathcal{I}_{\{x,y\}}(C)\big) & \longrightarrow & H^0\big(S, \mathcal{O}_S(C)\big) & \longrightarrow & H^0\big(\mathcal{O}_{\{x,y\}}(C)\big) \\
& \downarrow & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle =} \\
0 \longrightarrow & H^0\big(C, A(-x-y)\big) & \longrightarrow & H^0(C, A) & \longrightarrow & H^0\big(\mathcal{O}_{\{x,y\}}(C)\big)
\end{array}$$

□

We now introduce the moduli map of the pencil introduced in Proposition 8

$$m\colon \mathbf{P} = \left|\mathcal{I}_{\{x,y\}}(C)\right| \to \overline{\mathcal{M}}_{15,2}, \tag{12}$$

where the marked points of the pencil are the base points x and y respectively. Composing m with the clutching map $\overline{\mathcal{M}}_{15,2} \twoheadrightarrow \Delta_0 \subseteq \overline{\mathcal{M}}_{16}$, we obtain a pencil $\xi\colon \mathbf{P} \to \Delta_0$. We set

$$R := m_*(\mathbf{P}) \subseteq \overline{\mathcal{M}}_{15,2} \quad \text{and} \quad \Gamma := \xi_*(\mathbf{P}) \subseteq \overline{\mathcal{M}}_{16}. \tag{13}$$

Proposition 9. *Every curve inside the pencil $\Gamma \subseteq \overline{\mathcal{M}}_{16}$ corresponds to a nodal curve which does not belong to any of the boundary divisors $\Delta_1, \ldots, \Delta_8$.*

Proof. Keeping the notation above, for a generic choice of $(A, x+y) \in \Sigma$, the pencil

$$\mathbf{P} := \left|\mathcal{I}_{\{x,y\}}(C)\right|$$

corresponds to a generic line inside $\left|\mathcal{O}_S(C))\right|$. As pointed out in Theorem 3, $\left|\mathcal{O}_S(C)\right|$ is base point free and ample on the surface S defined by (3), giving rise to the finite map

$$f\colon S \to \mathbf{P}^2 = \left|\mathcal{O}_S(C)\right|^\vee$$

considered in (5). We show that the inverse image $\mathbf{P}$ under f of a general pencil of lines in $\left|\mathcal{O}_S(C)\right|^\vee$ consists only of integral curves with at most one node. This is achieved in several steps.

(i) Since $\mathcal{O}_S(C)$ is ample, we can apply [BL, Theorem A] and conclude that each curve $C' \in \left|\mathcal{O}_S(C)\right|$ is 2-connected, that is, it cannot be written as a sum of effective divisors $C' = F + M$, where $F \cdot M \leq 1$. This implies that $\left|\mathcal{O}_S(C)\right|$ does not contain any *tree-like* curves, that is, curves for which its irreducible components meet at a single point, which furthermore is a node.

(ii) The essential step in our argument involves proving that $\mathbf{P}$ contains no curves with singularities worse than nodes. Precisely, we show that $\left|\mathcal{O}_S(C)\right|$ contains only *finitely many* non-nodal curves. Note first that the *branch curve* $B \subseteq \mathbf{P}^2$ of f is reduced, else we contradict the assumption that the pencil $A \in W^1_9(C)$ on C has simple ramification. We introduce the discriminant curve

$$J := \Big\{C' \in \left|\mathcal{O}_S(C)\right| : C' \text{ is singular}\Big\}.$$

The dual curve $B^\vee$ is contained in J. Since B is reduced, the general tangent line to B is tangent at exactly one point $p \in B$ and with multiplicity 2. A standard local calculation shows that $f^*\big(\mathbb{T}_p(B)\big) \in |\mathcal{O}_S(C)|$ is a one-nodal curve, singular at exactly one point $z \in f^{-1}(p)$ such that the differential $df_z\colon T_z(S) \to T_p(\mathbf{P}^2)$ is not an isomorphism.

The complement $J \setminus B^\vee$ is the (possibly empty) union of (some of) the pencils $\mathbf{P}_b$, where $b \in B_{\text{sing}}$ and $\mathbf{P}_b$ is defined as the pull-back by f of the pencil of lines in $\mathbf{P}^2$ through b. In view of the numerical situation at hand (that is, $C^2 = 9$), the geometric possibilities for such a pencil

$$\mathbf{P}_b \subseteq J$$

are quite constrained. Since f is finite, the pencil $\mathbf{P}_b$ has no fixed components. Let $Z := \mathrm{Bs}(\mathbf{P}_b)$. Then a general $C' \in \mathbf{P}_b$ is integral and smooth along $C' \setminus Z$. Moreover, each

$C' \in \mathbf{P}_b$ is singular at a given point $z \in Z$ and a general such C' has multiplicity $m \geq 2$ at z. Necessarily, the differential $df_z \colon T_z(S) \to T_p(\mathbf{P}^2)$ is zero. Since $m^2 \leq C^2 = (C')^2 = 9$, we find $m \in \{2, 3\}$. Let

$$\sigma \colon S' \to S$$

be the blow-up of S at z and denote by $E \subseteq S'$ the exceptional divisor. The pencil $\bigl|\mathcal{O}_{S'}(\sigma^* C - mE)\bigr|$ is the strict transform of $\mathbf{P}_b$. Observe that the restriction map

$$r \colon H^0\bigl(S', \mathcal{O}_{S'}(\sigma^* C - mE)\bigr) \to H^0\bigl(E, \mathcal{O}_E(m)\bigr)$$

is not zero, hence $\mathrm{Im}(r)$ defines a linear series $\mathfrak{p}_b$ on $E \cong \mathbf{P}^1$. Either $\mathfrak{p}_b$ is a pencil or a constant divisor of degree $m \in \{2, 3\}$. We now list the possibilities for the pencil $\mathbf{P}_b$.

(P1) If $m = 3$, then $\mathrm{supp}(Z) = \{z\}$. Every curve $C' \in \mathbf{P}_b$ has a triple point at z.

(P2) If $m = 2$, then either each $C' \in \mathbf{P}_b$ has a node, or else, each $C' \in \mathbf{P}_b$ has a cusp at z. Indeed, if $\mathfrak{p}_b$ is a pencil on E, then each $C' \in \mathbf{P}_b$ is nodal at z. If $\mathfrak{p}_b = \{u_1 + u_2\}$ consists of a fixed divisor, then $\mathbf{P}_b$ contains a unique curve C_z having multiplicity at least 3 at z. If $u_1 \neq u_2$, all other curves $C' \in \mathbf{P}_b \setminus \{C_z\}$ are nodal at z, whereas if $u_1 = u_2$, then all such C' are cuspidal at z.

Both possibilities **(P1)** and **(P2)** can be ruled out by a parameter count that contradicts the generality of the pair $(C, A) \in \mathcal{H}_{15,9}$ we started with. We first rule out **(P1)**. Assume $C' \in \mathbf{P}_b$ has a triple point at z and no further singularities and denote by $\nu \colon \bar{C} \to C'$ the normalization. Set $\{z_1, z_2, z_3\} = \nu^{-1}(z)$ and $\bar{A} := \nu^*\bigl(\mathcal{O}_{C'}(C')\bigr) \in W^1_9(\bar{C})$. Since $\bar{A}$ is induced from a pencil of curves with a triple point at z, it follows that $\bigl|\bar{A}(-3z_1 - 3z_2 - 3z_3)\bigr| \neq \emptyset$, therefore for degree reason $\bar{A} = \mathcal{O}_{\bar{C}}(3z_1 + 3z_2 + 3z_3)$. We denote by $\mathcal{H}^{\mathrm{triple}}_{12,9}$ the Hurwitz space classifying degree 9 covers $\bar{C} \to \mathbf{P}^1$ having a divisor of the form $3(z_1 + z_2 + z_3)$ in a fibre, where $\bar{C}$ is of genus 12. Then $\mathcal{H}^{\mathrm{triple}}_{12,9}$ is pure of dimension $\dim(\mathcal{M}_{12}) - 1 = 32$. Let $\mathcal{Y}_1$ be the parameter space of pairs (S, C'), where $S \subseteq \mathbf{P}^6$ is a smooth complete intersection of 4 quadrics and $C' \subseteq S$ is an integral curve of arithmetic genus 15 with a triple point as described by **(P1)**. Let

$$\mathcal{S} \xleftarrow{\ \pi_1\ } \mathcal{Y}_1 \xrightarrow{\ \pi_2\ } \mathcal{H}^{\mathrm{triple}}_{12,9}$$

be the projections given by $\pi_1\bigl([S, C']\bigr) := [S]$ and $\pi_2\bigl([S, C']\bigr) := [\bar{C}, \bar{A}]$ respectively. With the notation above, from the adjunction formula $\nu^*(\mathcal{O}_{C'}(1)) = \omega_{\bar{C}}(-z_1 - z_2 - z_3)$. The fibre $\pi_2^{-1}\bigl(\pi_2([S, C'])\bigr)$ corresponds then to the choice of a 7-dimensional space of sections $V \subseteq H^0\bigl(\bar{C}, \omega_{\bar{C}}(-z_1 - z_2 - z_3)\bigr)$ satisfying $\dim\Bigl(V \cap H^0\bigl(\omega_{\bar{C}}(-2z_1 - 2z_2 - 2z_3)\bigr)\Bigr) \geq 6$. Since $h^0\bigl(\omega_{\bar{C}}(-2z_1 - 2z_2 - 2z_3)\bigr) = 6$, it follows that

$$\frac{V}{H^0\bigl(\omega_{\bar{C}}(-2z_1 - 2z_2 - 2z_3)\bigr)} \in \mathbf{P}\Bigl(\frac{H^0\bigl(\omega_{\bar{C}}(-z_1 - z_2 - z_3)\bigr)}{H^0\bigl(\omega_{\bar{C}}(-2z_1 - 2z_2 - 2z_3)\bigr)}\Bigr) \cong \mathbf{P}^2.$$

Therefore $\dim(\mathcal{Y}_1) = \dim(\mathcal{H}^{\mathrm{triple}}_{12,9}) + 2 = 34 \leq 39$, so we can invoke Proposition 7 to conclude that $\overline{\pi_1(\mathcal{Y}_1)} \neq \mathcal{S}$ and rule out possibility **(P1)**.

Next we rule out possibility **(P2)**, focusing on the case when each $C' \in \mathbf{P}_b$ is cuspidal at z. Passing to the normalization $\nu \colon \bar{C} \to C'$, setting $\bar{z} := \nu^{-1}(z)$ we obtain that $\bar{A} := \nu^*(\mathcal{O}_{C'}(C') \in W^1_9(\bar{C})$ verifies $h^0\bigl(\bar{C}, \bar{A}(-4\bar{z})\bigr) \geq 1$. Let $\mathcal{H}^{\mathrm{four}}_{14,9}$ be the Hurwitz space classifying degree 9 covers $\bar{C} \to \mathbf{P}^1$ containing a divisor of type $4\bar{z}$ in one of its fibres and where $\bar{C}$ has genus 14. Then $\mathcal{H}_{14,9}^{\mathrm{four}}$ is irreducible of dimension $39 = \dim(\mathcal{M}_{14})$. Let $\mathcal{Y}_2$ be the parameter space of pairs (S, C'), where $S \subseteq \mathbf{P}^6$ is a smooth complete intersection of 4 quadrics and $C' \subseteq S$ is an integral curve of arithmetic genus 15 with a cusp at z as described by **(P2)**. We consider the projections

$$\mathcal{S} \xleftarrow{\ \pi_1\ } \mathcal{Y}_2 \xrightarrow{\ \pi_2\ } \mathcal{H}_{14,9}^{\mathrm{four}}$$

given by $\pi_1\big([S, C']\big) := [S]$ and $\pi_2\big([S, C']\big) := [\bar{C}, \bar{A}]$ respectively. Observe that π_2 is birational onto its image. Indeed, given $[\bar{C}, \bar{A}] \in \pi_2(\mathcal{Y}_2)$, then we denote by C' the image of the map $\varphi_{\omega_{\bar{C}}(2y)\otimes \bar{A}^\vee} \colon \bar{C} \to \mathbf{P}^6$, in which case the canonical surface S is recovered by (3). We conclude by Proposition 7 again that $\pi_1(\mathcal{Y}_2)$ is not dense in $\mathcal{S}$. The final case when all curves $C' \in \mathbf{P}_b$ are (at least) nodal at z is ruled out analogously. □

Before stating our next result, recall that one sets $\delta_i := [\Delta_i] \in CH^1(\overline{\mathcal{M}}_g)$ for $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$. We denote as usual by $\lambda \in CH^1(\overline{\mathcal{M}}_g)$ the Hodge class. Recall also the formula [HM] for the canonical class of $\overline{\mathcal{M}}_g$:

$$K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor \frac{g}{2} \rfloor} \in CH^1(\overline{\mathcal{M}}_g). \tag{14}$$

Proposition 10. *The rational curve* Γ *is a sweeping pencil for the boundary divisor* Δ_0. *Its intersection numbers with the standard generators of* $CH^1(\overline{\mathcal{M}}_{16})$ *are as follows:*

$$\Gamma \cdot \lambda = 22, \quad \Gamma \cdot \delta_0 = 143, \quad \Gamma \cdot \delta_j = 0 \ \textit{ for } j = 2, \ldots, 8.$$

Proof. First we construct a fibration whose moduli map is precisely the rational curve $m \colon \mathbf{P}^1 \to \overline{\mathcal{M}}_{15,2}$ considered in (12). We consider the curve $C \subseteq S$ and observe that since $\mathcal{O}_C(C) \cong A \in W^1_9(C)$, we have that $C^2 = 9$, that is, the pencil $\big|\mathcal{I}_{\{x,y\}}(C)\big|$ has precisely 9 base points, namely x, y, as well as the 7 further points lying in the same fibre of the pencil $|A|$ as x and y. We consider the blow-up surface $\epsilon \colon \widetilde{S} = \mathrm{Bl}_9(S) \to S$ at these 9 points. It comes equipped with a fibration

$$\pi \colon \widetilde{S} \to \mathbf{P}^1,$$

as well as with two sections $E_x, E_y \subseteq \widetilde{S}$ corresponding to the exceptional divisors at x and y respectively.

In order to compute the intersection numbers of $R = m(\mathbf{P})$ with the tautological classes on $\overline{\mathcal{M}}_{15,2}$, we use for instance [Tan]. The subscript indicates the moduli space on which the intersection number is computed.

$$\big(R \cdot \lambda\big)_{\overline{\mathcal{M}}_{15,2}} = \chi(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) + g - 1 = h^2(S, \mathcal{O}_S) + g = h^0(S, \mathcal{O}_S(1)) + 15 = 22. \tag{15}$$

Here we have used $H^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = H^1(S, \mathcal{O}_S) = 0$, as well as the fact that S is a canonical surface, hence $\omega_S = \mathcal{O}_S(1)$, therefore $h^2(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = h^2(S, \mathcal{O}_S) = 7$. Furthermore, recalling that all curves in the fibres of m are irreducible, we find via [Tan] that

$$\big(R \cdot \delta_0\big)_{\overline{\mathcal{M}}_{15,2}} = c_2(\widetilde{S}) + 4(g-1) = c_2(\widetilde{S}) + 56.$$

From the Euler formula, $c_2(\widetilde{S}) = 12\chi(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) - K^2_{\widetilde{S}}$. We have already computed that $\chi(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = 8$, whereas $K^2_{\widetilde{S}} = K^2_S - 9 = \deg(S) - 9 = 7$, for S is an intersection of 4 quadrics. Thus $c_2(\widetilde{S}) = 12 \cdot 8 - 7 = 89$, leading to $\big(R \cdot \delta_0\big)_{\overline{\mathcal{M}}_{15,2}} = 89 + 4 \cdot 14 = 145$.

If we denote by $\psi_x, \psi_y \in CH^1(\overline{\mathcal{M}}_{15,2})$ the cotangent classes corresponding to the marked points labelled by x and y respectively, we compute furthermore

$$R \cdot \psi_x = -E_x^2 = 1 \quad \text{and} \quad R \cdot \psi_y = -E_y^2 = 1.$$

We now pass to the pencil $\xi\colon \mathbf{P}^1 \to \overline{\mathcal{M}}_{16}$ obtained from m by identifying pointwise the disjoint sections E_x and E_y on the surface $\widetilde{S}$. First, using (15) we observe that

$$\Gamma \cdot \lambda = \xi(\mathbf{P}) \cdot \lambda = \big(R \cdot \lambda\big)_{\overline{\mathcal{M}}_{15,2}} = 22.$$

Furthermore, using Proposition 9 we conclude that $\Gamma \cdot \delta_i = 0$ for $i = 1, \ldots, 8$. Finally, invoking for instance [CR3, page 271], we find that

$$\Gamma \cdot \delta_0 = \big(R \cdot \delta_0\big)_{\overline{\mathcal{M}}_{15,2}} - \big(R \cdot \psi_x\big)_{\overline{\mathcal{M}}_{15,2}} - \big(R \cdot \psi_y\big)_{\overline{\mathcal{M}}_{15,2}} = 145 - 2 = 143.$$

□

Proof of Theorem 2. Since the image of m passes through a general point of $\overline{\mathcal{M}}_{15,2}$, the rational curve $\Gamma \subseteq \overline{\mathcal{M}}_{16}$ constructed in Proposition 10 is a sweeping curve for the boundary divisor Δ_0. Using the expression (14) for the canonical divisor of $\overline{\mathcal{M}}_{16}$, we compute $\Gamma \cdot K_{\overline{\mathcal{M}}_{16}} = 13\,\Gamma \cdot \lambda - 2\,\Gamma \cdot \delta_0 = 13 \cdot 22 - 2 \cdot 143 = 0$. Also $\Gamma \cdot \Delta_0 = 143 > 0$. □

3. THE SLOPE OF $\overline{\mathcal{M}}_{16}$.

The *slope* of an effective divisor D on the moduli space $\overline{\mathcal{M}}_g$ not containing any boundary divisor Δ_i in its support is defined as the quantity $s(D) := \frac{a}{\min_{i\geq 0} b_i}$, where $[D] = a\lambda - \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor} b_i\delta_i \in CH^1(\overline{\mathcal{M}}_g)$, with $a, b_i \geq 0$. Then the slope $s(\overline{\mathcal{M}}_g)$ of the moduli space $\overline{\mathcal{M}}_g$ is defined as the infimum of the slopes $s(D)$ over such effective divisors D.

Corollary 11. *We have that* $s(\overline{\mathcal{M}}_{16}) \geq \frac{13}{2}$.

Proof. For any effective divisor D on $\overline{\mathcal{M}}_{16}$ containing no boundary divisor in its support, we may assume that the curve Γ constructed in Proposition 10 does not lie inside D, hence $\Gamma \cdot D \geq 0$. Writing $[D] = a\lambda - \sum_{i=0}^{8} b_i\delta_i$, using Theorem 2 we obtain $\frac{a}{b_0} \geq \frac{\Gamma\cdot\delta_0}{\Gamma\cdot\lambda} = \frac{13}{2}$. Furthermore, using [FP, Theorem 1.4], we conclude that for this divisor D we have $b_i \geq b_0$ for $i = 1, \ldots, 8$, that is, $s(D) = \frac{a}{b_0}$. □

Final remarks: Our results establish that $\overline{\mathcal{M}}_{16}$ is not of general type. Showing that the Kodaira dimension of $\overline{\mathcal{M}}_{16}$ is non-negative amounts to constructing an effective divisor D on $\overline{\mathcal{M}}_{16}$ havind slope $s(D) \leq s(K_{\overline{\mathcal{M}}_{16}}) = \frac{13}{2}$. Currently the known effective divisor on $\overline{\mathcal{M}}_{16}$ of smallest slope is the closure in $\overline{\mathcal{M}}_{16}$ of the *Koszul divisor* $\mathcal{Z}_{16}$ consisting of curves C having a linear system $L \in W^7_{21}(C)$ such that the image curve $\varphi_L\colon C \hookrightarrow \mathbf{P}^6$ is ideal-theoretically not cut out by quadrics. It is shown in [F1, Theorem 1.1] that $\overline{\mathcal{Z}}_{16}$ is an effective divisor on $\overline{\mathcal{M}}_{16}$ and $s(\overline{\mathcal{Z}}_{16}) = \frac{407}{61} = 6.705....$ In a related direction, it is shown in [F2] that the canonical class of the space of admissible covers $\overline{\mathcal{H}}_{16,9}$ is effective. Note that one has a generically finite cover $\overline{\mathcal{H}}_{16,9} \to \overline{\mathcal{M}}_{16}$.

Soon after the appearance of the first version of this paper, it has been pointed out by Agostini and Barros [AB] that our proof of Theorem 2 yields in fact the bound $\kappa(\overline{\mathcal{M}}_{16}) \leq \dim(\overline{\mathcal{M}}_{16}) - 2$. Indeed, consider the parameter space $\mathcal{Z}$ of elements $[C, A, x, y]$, where C is a genus 15 irreducible nodal curve, $A \in W^1_9(C)$ and $x, y \in C$ are points such that $\left|A(-x-y)\right| \neq 0$. As we explain in this paper, $\mathcal{Z}$ has the structure of a $\mathbf{P}^1$-bundle and one has a dominant morphism $v\colon \mathcal{Z} \to \Delta_0$ given by $[C, A, x, y] \mapsto [C/x \sim y]$. In Proposition 10 we establish that the restriction of $v^*(K_{\overline{\mathcal{M}}_{16}})$ to the general fibre of this fibration is trivial. Accordingly, $\kappa(\overline{\mathcal{M}}_{16}) \leq \dim(\mathcal{Z}) - 1 = \dim(\overline{\mathcal{M}}_{16}) - 2$.

REFERENCES

[AB] D. Agostini and I. Barros, *Pencils on surfaces with normal crossings and the Kodaira dimension of* $\overline{\mathcal{M}}_{g,n}$, arXiv:2008.12826.

[AC] E. Arbarello and M. Cornalba, *Footnotes to a paper of Beniamino Segre*, Mathematische Annalen **256** (1981), 341–362.

[BL] M. Beltrametti and A. Lanteri, *On the* 2 *and* 3*-connectedness of ample divisors on a surface*, Manuscripta Mathematca **58** (1987), 109–128.

[BV] A. Bruno and A. Verra, $\mathcal{M}_{15}$ *is rationally connected*, in: Projective varieties with unexpected properties, 51–65, Walter de Gruyter 2005.

[CR1] M. C. Chang and Z. Ran, *Unirationality of the moduli space of curves of genus* 11, 13 *(and* 12*)*, Inventiones Math. **76** (1984), 41–54.

[CR2] M. C. Chang and Z. Ran, *The Kodaira dimension of the moduli space of curves of genus* 15, Journal of Differential Geometry **24** (1986), 205–220.

[CR3] M. C. Chang and Z. Ran, *On the slope and Kodaira dimension of* $\overline{\mathcal{M}}_g$ *for small* g, Journal of Differential Geometry **34** (1991), 267–274.

[Do] I. Dolgachev, *Mirror symmetry for lattice polarized* $K3$ *surfaces*, J. Math. Sciences **81** (1996), 2599–2630.

[EH] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Inventiones Math. **90** (1987), 359–387.

[F1] G. Farkas, *Syzygies of curves and the effective cone of* $\overline{\mathcal{M}}_g$, Duke Mathematical Journal **135** (2006), 53–98.

[F2] G. Farkas, *Effective divisors on Hurwitz spaces*, arXiv:1804.01898v3, to appear in *Facets in Algebraic Geometry*, a volume in honor of Fulton's 80th birthday.

[FJP] G. Farkas, D. Jensen and S. Payne, *The Kodaira dimension of* $\overline{\mathcal{M}}_{22}$ *and* $\overline{\mathcal{M}}_{23}$, arXiv:2005.00622.

[FP] G. Farkas and M. Popa, *Effective divisors on* $\overline{\mathcal{M}}_g$*, curves on* $K3$ *surfaces and the Slope Conjecture*, Journal of Algebraic Geometry **14** (2005), 151–174.

[H] J. Harris, *On the Kodaira dimension of the moduli space of curves II: The even genus case*, Inventiones Math. **75** (1984), 437–466.

[HM] J. Harris and D. Mumford, *On the Kodaira dimension of* $\overline{\mathcal{M}}_g$, Inventiones Math. **67** (1982), 23–88.

[SD] B. Saint-Donat, *Projective models of* $K3$ *surfaces*, American Journal of Mathematics **96** (1974), 602–639.

[Ser] E. Sernesi, *L'unirazionalitá della varietá dei moduli delle curve di genere* 12, Annali della Scuola Normale Superiore di Pisa **8** (1981), 405–439.

[Sev] F. Severi, *Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann*, Rendiconti della Reale Accademia Naz. Lincei **24** (1915), 877–888.

[Tan] S.-L. Tan, *On the slopes of the moduli space of curves*, International Journal of Mathematics **9** (1998), 119–127.

[Ts] D. Tseng, *On the slope of the moduli space of genus 15 and 16 curves*, arXiv:1905.00449.

[Ve] A. Verra, *The unirationality of the moduli space of curves of genus* ≤ 14, Compositio Mathematica **141** (2005), 1425–1444.

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