\textbf{Theorem 1.} The moduli space $\overline{M}_{16}$ of stable curves of genus 16 is not of general type.

Note that 16 is the highest genus for which it is known that $\overline{M}_{16}$ is not of general type. We further refer to Tseng’s recent paper [Ts] for further details on the convoluted history of determining the Kodaira dimension of $\overline{M}_{16}$.

Before explaining our strategy of proving Theorem 1, recall the standard notation $\Delta_0, \ldots, \Delta_{\lfloor g/2 \rfloor}$ for the irreducible boundary divisors on $\overline{M}_g$, see [HM]. Here $\Delta_0$ denotes the closure in $\overline{M}_g$ of the locus of irreducible 1-nodal curves of arithmetic genus $g$. Our approach relies on the explicit uniruled parametrization of $\overline{M}_{15}$ found by Bruno and Verra [BV]. Their work establishes that through a general point of $\overline{M}_{15}$ there passes not only a rational curve, but in fact a rational surface. This extra degree of freedom, yields a uniruled parametrization of $\overline{M}_{15,2}$, therefore also a parametrization the boundary divisor $\Delta_0$ inside $\overline{M}_{16}$. We show the following:

\textbf{Theorem 2.} The boundary divisor $\Delta_0$ of $\overline{M}_{16}$ is uniruled and swept by a family of rational curves, whose general member $\Gamma \subseteq \Delta_0$ satisfies $\Gamma \cdot K_{\overline{M}_{16}} = 0$ and $\Gamma \cdot \Delta_0 > 0$.

Assuming Theorem 2, we conclude that $\overline{M}_{16}$ cannot be of general type, thus establishing Theorem 1. To that end, note first that in any effective representation of the canonical divisor

$$K_{\overline{M}_{16}} \equiv \alpha \cdot \Delta_0 + D,$$

where $\alpha \in \mathbb{Q}_{>0}$ and $D$ is an effective $\mathbb{Q}$-divisor on $\overline{M}_{16}$ not containing $\Delta_0$ in its support, we must have $\alpha = 0$. Indeed, we can choose the curve $\Gamma$ such that $\Gamma \not\subseteq D$, then we write

$$0 = \Gamma \cdot K_{\overline{M}_{16}} = \alpha \Gamma \cdot \Delta_0 + \Gamma \cdot D \geq \alpha \Gamma \cdot \Delta_0 \geq 0,$$

hence $\alpha = 0$. Furthermore, since the singularities of $\overline{M}_g$ do not impose adjunction conditions [HM, Theorem 1], $\overline{M}_g$ is a variety of general type for a given $g \geq 4$ if and
only if the canonical class \( K_{\overline{M}_g} \) is a big divisor class, that is, it can be written as

\[
K_{\overline{M}_g} \equiv A + E,
\]

where \( A \) is an ample \( \mathbb{Q} \)-divisor and \( E \) is an effective \( \mathbb{Q} \)-divisor respectively. Assume that \( K_{\overline{M}_{16}} \) can be written like in (1). It has already been observed that \( \Delta_0 \not\subseteq \text{supp}(E) \), in particular \( \Gamma \cdot E \geq 0 \). Using Kleiman’s ampleness criterion, \( \Gamma \cdot A > 0 \), which yields the immediate contradiction \( 0 = \Gamma \cdot K_{\overline{M}_{16}} = \Gamma \cdot A + \Gamma \cdot E \geq \Gamma \cdot A > 0 \).

We are left therefore with proving Theorem 2, which is what we do in the rest of the paper. The rational curve \( \Gamma \) constructed in Theorem 2 is the moduli curve corresponding to an appropriate pencil of curves of genus 15 on a certain canonical surface \( S \subseteq \mathbb{P}^6 \). Establishing that this pencil can be chosen in such a way to contain only stable curves will take up most of Section 2.

1. The Bruno-Verra Parametrization of \( \overline{M}_{15} \)

The parametrization of the boundary divisor \( \Delta_0 \) of \( \overline{M}_{15} \) and the proof of Theorem 2 uses several results from [BV], which we now recall. We denote by \( \mathcal{H}_{15,9} \) the Hurwitz space parametrizing degree 9 covers \( C \to \mathbb{P}^1 \) having simple ramification, where \( C \) is a smooth curve of genus 15. Then \( \mathcal{H}_{15,9} \) is birational to the parameter space \( \mathcal{G}_{15,9}^1 \) classifying pairs \((C, A)\), where \([C] \in \mathcal{M}_{15} \) and \( A \in W_9^1(C) \) is a pencil. By residduation, \( \mathcal{G}_{15,9}^1 \) is isomorphic to the parameter space \( \mathcal{G}_{15,19}^0 \) of pairs \([C, L]\), where \( C \) is a smooth curve of genus 15 and \( L \in W_{19}^1(C) \). Note that the general fibre of the forgetful map

\[
\pi: \mathcal{H}_{15,9} \to \mathcal{M}_{15}, \quad [C, A] \mapsto [C]
\]

is 1-dimensional. Clearly, \( \mathcal{H}_{15,9} \) and thus \( \mathcal{G}_{15,19}^0 \) is irreducible.

We pick a general element \([C, L] \in \mathcal{G}_{15,19}^0 \) in particular \( L \) is very ample and \( h^0(C, L) = 7 \). We set \( A := \omega_C \otimes L^\vee \in W_9^1(C) \). We may assume that \( A \) is base point free and the pencil \([A]\) has simple ramification. We consider the multiplication map

\[
\phi_L:\ Sym^2 H^0(C, L) \to H^0(C, L^2).
\]

Since \( C \) is Petri general, \( h^1(C, L^2) = 0 \), therefore \( h^0(C, L^2) = 2 \cdot 19 + 1 - 15 = 24 \). Furthermore, via a degeneration argument it is shown in [BV, Theorem 3.11], that for a general choice of \((C, L)\), the map \( \phi_L \) is surjective, hence \( h^0(\mathbb{P}^6, \mathcal{I}_C/\mathbb{P}^6(2)) = \dim(\ker(\phi_L)) = 4 \), that is, the degree 19 curve \( C \subseteq \mathbb{P}^6 \) lies on precisely 4 independent quadrics. We let

\[
S := \text{Bs}[\mathcal{I}_{C/\mathbb{P}^6}(2)]
\]

be the base locus of the system of quadrics containing \( C \). It is further established in [BV, Theorem 3.11] that under our generality assumptions, \( S \) is a smooth surface. From the adjunction formula it follows that \( \omega_S = \mathcal{O}_S(1) \), that is, \( S \) is a canonical surface. We write down the exact sequence

\[
0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.
\]

From the adjunction formula \( \mathcal{O}_C(C) \cong \omega_C \otimes \omega_S^\vee |_C = \omega_C \otimes L^\vee = A \in W_9^1(C) \). Since \( S \) is a regular surface, by taking cohomology in (4), we obtain

\[
h^0(S, \mathcal{O}_S(C)) = h^0(S, \mathcal{O}_S) + h^0(C, A) = 3.
\]
Observe also from the sequence (4) that the linear system $|O_S(C)|$ is base point free, for $|O_C(C)| = |A|$ is so. This brings to an end our summary of the results from [BV].

In what follows, we denote by

$$f : S \to \mathbb{P}^2 = |O_S(C)|^\vee$$

the induced map. For what we intend to do, it is important to show that $f$ is a finite map, or equivalently, that $O_S(C)$ is ample.

**Theorem 3.** For a general pair $(C, A) \in \mathcal{H}_{15,9}$, the line bundle $O_S(C)$ is ample.

In order to prove Theorem 3 it suffices to exhibit a single pair $(C, A) \in \mathcal{H}_{15,9}$ for which the corresponding map $f : S \to \mathbb{P}^2$ given by (5) is finite. We shall realize the canonical surface $S \subseteq \mathbb{P}^6$ as the double cover of a suitable $K3$ surface $Y \subseteq \mathbb{P}^5$ of genus 5 (that is, of degree 8). It will prove advantageous to consider $K3$ surfaces having a certain Picard lattice of rank 3. We first discuss the geometry of such $K3$ surfaces.

**Definition 4.** We denote by $\Lambda$ the even lattice of signature $(1, 2)$ generated by elements $H, F$ and $R$ having the following intersection matrix:

$$
\begin{pmatrix}
H^2 & H \cdot F & H \cdot R \\
F \cdot H & F^2 & F \cdot R \\
R \cdot H & R \cdot F & R^2
\end{pmatrix}
= 
\begin{pmatrix}
8 & 9 & 1 \\
9 & 4 & 2 \\
1 & 2 & -2
\end{pmatrix}.
$$

We denote by $\mathcal{F}_5^\Lambda$ the moduli space of polarized $K3$ surfaces $[Y, H]$, where $H^2 = 8$, admitting a primitive embedding $\Lambda \hookrightarrow \text{Pic}(Y)$, such that the classes $H, F, R$ correspond to curve classes on $Y$ which we denote by the same symbol. Furthermore, $H \in \text{Pic}(Y)$ is assumed to be ample.

For details on the construction of the moduli space $\mathcal{F}_5^\Lambda$ we refer to [Do, Section 3]. It follows from loc.cit. that $\mathcal{F}_5^\Lambda$ is an irreducible variety of dimension $17 = 20 - \text{rk}(\Lambda)$. Let us now fix a general element $[Y, H]$, where $\text{Pic}(Y) \cong \mathbb{Z}\langle H, F, R \rangle$ as in Definition 4. Then $O_Y(H)$ is very ample and we denote by

$$\varphi_H : Y \hookrightarrow \mathbb{P}^5$$

the embedding induced by this linear system. Observe that $h^0(\mathbb{P}^5, I_{Y/\mathbb{P}^5}(2)) = 3$ and that $Y = \text{Bs}[I_{Y/\mathbb{P}^5}(2)]$ is in fact a complete intersection of three quadrics. Note that $F \subseteq Y$ is a curve of genus 3, whereas $R \subseteq Y$ is a smooth rational curve embedded as a line under the map $\varphi_H$. The class $E := 2H - F$ satisfies $E^2 = 0$. Since $E \cdot H > 0$, it follows that $|E|$ is an elliptic pencil and furthermore $E \cdot R = 0$. Setting also

$$D := 2H + R - E \in \text{Pic}(Y),$$

we compute $D^2 = 6, D \cdot E = 14$ and $D \cdot R = 0$. In the basis $(D, E, R)$ of $\text{Pic}(Y)$, the intersection form on $Y$ is described by the following simpler matrix:

$$
\begin{pmatrix}
D^2 & D \cdot E & D \cdot R \\
E \cdot D & E^2 & E \cdot R \\
R \cdot D & R \cdot E & R^2
\end{pmatrix}
= 
\begin{pmatrix}
6 & 14 & 0 \\
14 & 0 & 0 \\
0 & 0 & -2
\end{pmatrix}.
$$

On our way to proving Theorem 3, we establish the following result:

**Proposition 5.** The line bundle $O_Y(F)$ is very ample.
Proof. We first claim that $\bar{F}$ is nef. Since $\bar{F}^2 = 4 > 0$, it suffices to check that for any smooth rational curve $\tilde{\Gamma} \subseteq Y$, one has $\tilde{\Gamma} \cdot \bar{F} \geq 0$. We write $\tilde{\Gamma} \equiv aD + bE + cR$, where $a, b$ and $c$ are integers. We may assume $\tilde{\Gamma} \neq R$, thus $\tilde{\Gamma} \cdot \bar{R} \geq 0$, implying $c \leq 0$. Furthermore, $\tilde{\Gamma} \cdot E \geq 0$, hence $a \geq 0$. Using (7), one has $\tilde{\Gamma}^2 = 6a^2 - 2c^2 + 28ab = -2$. Assume by contradiction $\tilde{\Gamma} \cdot \bar{F} = \tilde{\Gamma} \cdot (D - \bar{R}) = 6a + 14b + 2c \leq -2$. Multiplying this inequality with $2a \geq 0$ and substituting in the equality $\tilde{\Gamma}^2 = -2$ we obtain that $(a + c)^2 + 2a^2 + 2a - 1 \geq 0$, implying $a = 0$ and $c \in \{-1, 1\}$. If, say $c = 1$, then $\tilde{\Gamma} \equiv \bar{R} + bE$. From the assumption $\tilde{\Gamma} \cdot \bar{F} \leq -2$, we obtain that $b \leq -1$, hence $\tilde{\Gamma} \cdot \bar{H} < 0$, thus $\tilde{\Gamma}$ cannot be effective, a contradiction. The case $c = -1$, implying $b \leq 0$ is ruled out similarly.

Thus $\bar{F}$ is a nef curve. To conclude that $\bar{F}$ is very ample, we invoke [SD]. It suffices to rule out the existence of a divisor class $M \in \text{Pic}(Y)$ such that (i) $M^2 = 0$ and $M \cdot \bar{F} \in \{1, 2\}$, or satisfying (ii) $M^2 = -2$ and $M \cdot \bar{F} = 0$. We discuss only (i), the remaining case being similar. Write $M = aD + bE + c\bar{R}$. Since $M^2 = 0$, from (7) we obtain $3a^2 - c^2 + 14ab = 0$, whereas from $M \cdot \bar{F} = 2$, we obtain that $3a + 7b + c = 1$. Eliminating $c$, we find $6a^2 + a(28b - 6) + 49b^2 - 14b + 1 = 0$. Since the discriminant of this equation is negative, this case is excluded. We conclude that $\bar{F}$ is very ample.

We fix a general polarized $K3$ surface $[Y, \bar{H}] \in \mathcal{F}_5^A$, while keeping the notation from above. Choose a smooth divisor $\bar{Q} \in |\mathcal{O}_Y(2\bar{H})|$ and consider the double cover

$$\sigma : S \to Y$$

branched along $\bar{Q}$. We denote by $Q \subseteq S$ the ramification divisor of $\sigma$, hence $\sigma^*(\bar{Q}) = 2Q$. We set $H := \sigma^*(\bar{H})$, where $\bar{H} \in |\mathcal{O}_Y(1)|$ is a linear section of $Y$. Note that $Q \in |\mathcal{O}_S(H)|$.

**Proposition 6.** The induced morphism $\varphi_H : S \to P^6$ embeds $S$ as a canonical surface which is the complete intersection of 4 quadrics in $P^6$. More precisely, $S$ is a quadratic section of the cone $\mathcal{C}_Y \subseteq P^6$ over the $K3$ surface $Y \subseteq P^5$.

**Proof.** From the adjunction formula we find $\omega_S = \mathcal{O}_S(Q) = \mathcal{O}_S(H)$. Furthermore, we have $\sigma^*(\mathcal{O}_S) = \mathcal{O}_Y \oplus \mathcal{O}_Y(-H)$, hence from the projection formula we can write

$$H^0(S, \mathcal{O}_S(H)) \cong H^0(Y, \mathcal{O}_Y(\bar{H})) \oplus H^0(Y, \mathcal{O}_Y) \cong H^0(Y, \mathcal{O}_Y(\bar{H})) \oplus \mathbb{C}(Q),$$

where recall that $Q \in |\mathcal{O}_S(H)|$, as well as

$$H^0(S, \mathcal{O}_S(2H)) \cong H^0(Y, \mathcal{O}_Y(2\bar{H})) \oplus H^0(Y, \mathcal{O}_Y(\bar{H})) \cdot Q.$$

Thus $h^0(S, \mathcal{O}_S(H)) = 6$ and $h^0(S, \mathcal{O}_S(2H)) = h^0(Y, \mathcal{O}_Y(2)) = h^0(Y, \mathcal{O}_Y(1)) = 2 + 2\bar{H}^2 + 6 = 24$. Furthermore, $S \subseteq P^6$ is projectively normal, so $h^0(P^6, \mathcal{I}_{S/P^6}(2)) = 4$. Since clearly $S \subseteq \mathcal{C}_Y$, it follows that $S$ can be viewed as a quadratic section of the cone $\mathcal{C}_Y$, precisely the intersection of $\mathcal{C}_Y$ with one of the quadrics containing $S$ not lying in the subsystem $|\sigma^*H^0(P^6, \mathcal{I}_{S/P^6}(2))|$. □

We are now in a position to prove Theorem 3. We denote by $\text{Hilb}_{15,19}$ the unique component of the Hilbert scheme of curves $C \subseteq P^6$ of genus 15 and degree 19 dominating $\mathcal{M}_{15}$. A general point of $\text{Hilb}_{15,19}$ corresponds to a smooth projectively normal curve $C \subseteq P^6$ such that the canonical surface $S$ defined by (3) is smooth.

**Proof of Theorem 3.** We choose a $K3$ surface $[Y, \mathcal{O}_Y(\bar{H})] \in \mathcal{F}_5^A$ with $\text{Pic}(Y) = \mathbb{Z}(\bar{H}, \bar{F}, \bar{R})$, where the intersection matrix is given as in Definition 4. The restriction map

$$H^0(Y, \mathcal{O}_Y(2\bar{H})) \to H^0(\bar{R}, \mathcal{O}_R(2\bar{H}))$$
being surjective, we can choose a smooth curve $Q \subset \mathcal{O}_Y(2H)$ which is tangent to $R$, that is, $\bar{Q} \cdot \bar{R} = 2y$, for a point $y \in Y$. Construct the double cover $\sigma : S \to Y$ defined in (8). The pull-back $\sigma^*(\bar{R})$ is then a double cover of $\bar{R}$ branched over the single point $y$, hence necessarily
\[
\sigma^*(\bar{R}) = R + R' \subseteq S,
\]
where $R$ and $R'$ are lines on $S \subseteq \mathbb{P}^6$ meeting at a single point. Next, we choose a smooth genus 3 curve $\bar{F} \subseteq Y$ general in its linear system and set
\[
C' := \sigma^*(\bar{F}) \subseteq S.
\]
Since $\bar{F} \cdot \bar{Q} = 2\bar{F} \cdot \bar{H} = 18$, we obtain that $C'$ is a smooth curve of genus 14 and degree 18 endowed with the double cover $\bar{C}' \to \bar{F}$. Note that the linear system
\[
|\mathcal{O}_S(C')| = \pi^*|\mathcal{O}_Y(\bar{F})|
\]
is 3-dimensional. Applying Theorem 5, since $\mathcal{O}_Y(\bar{F})$ is ample and $\sigma$ is finite, we obtain that $\mathcal{O}_S(C')$ is ample as well. Observe that $C' \cdot R = \bar{F} \cdot \bar{R} = 2$. Choosing $\bar{F}$ general in its linear system, we can arrange the intersection of $R$ and $C'$ to be transverse, therefore
\[
C := C' + R \subseteq S \subseteq \mathbb{P}^6
\]
is a nodal curve of genus 15 and degree 19. Note that the linear system $|\mathcal{O}_S(C)|$ has $R$ as a fixed component, and $|\mathcal{O}_S(C)| = R + \pi^*|\mathcal{O}_Y(\bar{F})|$.

Despite the fact that $|\mathcal{O}_S(C)|$ is not ample, we can complete the proof of Theorem 3. Indeed, let us pick a general family $\{[C_t \to \mathbb{P}^6]\}_{t \in T}$ over a pointed base $(T, o)$, whose fibre over $o \in T$ is the curve $C$ described in (9). If $S_t = Bs|\mathcal{I}_{C_t/\mathbb{P}^6}(2)|$, assume the line bundle $\mathcal{O}_{S_t}(C_t)$ is not ample for each $t \in T$. As we have already observed, we may assume that $\mathcal{O}_{S_t}(C_t)$ is nef for all $t \in T$ and we denote by $f_t : S_t \to \mathbb{P}^2$ the map induced by the linear system $|\mathcal{O}_{S_t}(C_t)|$ for $t \in T \setminus \{o\}$. The limiting map of this family
\[
f_o : S \to \mathbb{P}^2,
\]
satisfies then $f_o^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_S(R + C')$ and is induced by a subspace of sections $\sigma^*(V)$, where $V \subseteq H^0(Y, \mathcal{O}_Y(\bar{F}))$ is 3-dimensional. By assumption, there exists a family of curves $\Gamma_t \subseteq S_t$ such that $\Gamma_t \cdot C_t = 0$. We denote by $\Gamma_o \subseteq S$ the limiting curve of $\Gamma_t$, therefore $\Gamma_o \cdot (C' + R) = 0$. We write $\Gamma_o = G + mR$, where $m \geq 0$ and $G \subseteq S$ is a curve not having $R$ in its support. From the adjunction formula, we find $R^2 = -3$. Since $R \cdot C' = 2$, it follows that $R \cdot (C' + R) = 1$, thus $G \neq 0$. Furthermore, the morphism $f_o$ contracts $G$, which we argue, leads to a contradiction. Indeed, $f_o$ admits a factorization
\[
\xymatrix{ S \ar[r]^\sigma & Y \ar[r]^{\bar{F}} & \mathbb{P}^3 \ar[r]^p & \mathbb{P}^2
}
\]
where $p : \mathbb{P}^3 \to \mathbb{P}^2$ is the linear projection corresponding to $V \subseteq H^0(Y, \mathcal{O}_Y(\bar{F}))$. Since $\sigma$ is finite and $|\bar{F}|$ is very ample, it follows that $\sigma(G)$ must be contracted by the projection $p$, that is, $\sigma(C)$ is a line in $\mathbb{P}^3$. By inspecting the intersection matrix (7) of Pic$(Y)$ we immediately see that no such line can exist on $Y$, which finishes the proof. \qed
2. The Uniruledness of the Boundary Divisor $\Delta_0$ in $\overline{M}_{16}$

We now lift the construction discussed above from $\overline{M}_{15}$ to the moduli space $\overline{M}_{15,2}$ of 2-pointed stable curves of genus 15 and eventually to $\overline{M}_{16}$. Recall that $\text{Hilb}_{15,19}$ is the component of the Hilbert scheme of curves $C \subseteq \mathbb{P}^6$ of genus 15 and degree 19 dominating $\mathcal{M}_{15}$. We denote by $\text{Hilb}_{2,2,2}$ the Hilbert scheme of complete intersections of 4 quadrics in $\mathbb{P}^6$. Since $\text{Hilb}_{15,19}/\text{PGL}(7)$ is birational to the Hurwitz space $\mathcal{H}_{15,9}$, we have a rational map

$$\chi: \mathcal{H}_{15,9} \dasharrow \text{Hilb}_{2,2,2}/\text{PGL}(7), \quad [C, A] \mapsto S := Bs|\mathcal{I}_{C, \mathbb{P}^6}(2)| \mod \text{PGL}(7),$$

where the canonical surface $S \subseteq \mathbb{P}^6$ is defined by (3). We set

$$S := \chi(\mathcal{H}_{15,9}).$$

The general fibre of the morphism $\chi: \mathcal{H}_{15,9} \to S$ consists of finitely many linear nonempty open subsets of linear systems $|O_S(C)|$, where $C \subseteq S \subseteq \mathbb{P}^6$ is a smooth curve of genus 15 and degree 19. In particular, $S$ is an irreducible variety of dimension $41 = \dim(\mathcal{H}_{15,9}) - 2$. Recall that $\pi: \mathcal{H}_{15,9} \to \mathcal{M}_{15}$ denotes the forgetful map. The next observation will prove to be useful in several moduli counts.

**Proposition 7.** If $S'$ is an irreducible subvariety of $S$ of dimension $\dim(S') \leq 39$, then $\pi(\chi^{-1}(S'))$ is a proper subvariety of $\mathcal{M}_{15}$.

**Proof.** Since $\dim(\chi^{-1}(S')) \leq \dim(S') + 2 \leq 41 = \dim(\mathcal{M}_{15}) - 1$, the claim follows. \qed

Let us now take a general curve $C$ of genus 15 and consider the correspondence

$$\Sigma := \{(A, x, y) \in W^1_0(C) \times C_2 : H^0(C, A(-x - y)) \neq 0\},$$

endowed with the projections $\pi_1: \Sigma \to W^1_0(C)$ and $\pi_2: \Sigma \to C_2$ respectively. Here $C_2$ is the second symmetric product of $C$. It follows that $\Sigma$ is an irreducible surface and that $\pi_2$ is generically finite. Indeed, for a general point $2x \in C_2$, we can invoke for instance [EH, Theorem 1.1] to conclude that $\pi_2^{-1}(2x)$ is finite. The fibre $\pi_1^{-1}(A)$ is irreducible whenever $A$ has simple ramification.

We now fix a general element $[C, x, y] \in \overline{M}_{15,2}$. Then there exist finitely many pencils $A \in W^1_0(C)$ containing both points $x$ and $y$ in the same fibre. Each of these pencils $A$ may be assumed to be base point free with simple ramification and general enough such that $L := \omega_C \otimes A^\vee \in W^1_0(C)$ is very ample and in the embedding

$$\varphi_L: C \hookrightarrow \mathbb{P}^6$$

the curve $C$ lies on precisely 4 independent quadrics intersecting in a smooth canonical surface $S$ defined by (3).

**Proposition 8.** With the notation above, if $h^0(C, A(-x - y)) = 1$, then $\dim|\mathcal{I}_{\{x, y\}}(C)| = 1$.

**Proof.** It follows from the commutativity of the following diagram, keeping in mind that $h^0(S, \mathcal{O}_S(C)) = 3$ and that the first column is injective.

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, \mathcal{I}_{\{x, y\}}(C)) \\
\downarrow & & \downarrow \text{res} \\
0 & \longrightarrow & H^0(C, A(-x - y))
\end{array}
\begin{array}{ccc}
& & \longrightarrow \ H^0(S, \mathcal{O}_S(C)) \\
\downarrow & & \downarrow \text{res} \\
& & \longrightarrow \ H^0(C, A)
\end{array}
\begin{array}{c}
\longrightarrow \ H^0(\mathcal{O}_{\{x, y\}}(C))
\end{array}
$$

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, \mathcal{I}_{\{x, y\}}(C)) \\
\downarrow & & \downarrow \text{res} \\
0 & \longrightarrow & H^0(C, A(-x - y))
\end{array}
\begin{array}{ccc}
& & \longrightarrow \ H^0(S, \mathcal{O}_S(C)) \\
\downarrow & & \downarrow \text{res} \\
& & \longrightarrow \ H^0(C, A)
\end{array}
\begin{array}{c}
\longrightarrow \ H^0(\mathcal{O}_{\{x, y\}}(C))
\end{array}
$$
We now introduce the moduli map of the pencil introduced in Proposition 8
(12) \[ m : P = |I_{(x,y)}(C)| \to \overline{M}_{15,2}, \]
where the marked points of the pencil are the base points \( x \) and \( y \) respectively. Composing \( m \) with the clutching map \( \overline{M}_{15,2} \to \Delta_0 \subseteq \overline{M}_{16} \), we obtain a pencil \( \xi : P \to \Delta_0 \).

We set
(13) \[ R := m_*(P) \subseteq \overline{M}_{15,2} \text{ and } \Gamma := \xi_*(P) \subseteq \overline{M}_{16}. \]

**Proposition 9.** Every curve inside the pencil \( \Gamma \subseteq \overline{M}_{16} \) corresponds to a nodal curve which does not belong to any of the boundary divisors \( \Delta_1, \ldots, \Delta_8 \).

**Proof.** Keeping the notation above, for a generic choice of \((A, x + y) \in \Sigma\), the pencil
\[ P := |I_{(x,y)}(C)| \]
corresponds to a generic line inside \( |O_S(C)| \). As pointed out in Theorem 3, \( |O_S(C)| \) is base point free and ample on the surface \( S \) defined by (3), giving rise to the finite map
\[ f : S \to P^2 = |O_S(C)|^\vee \]
considered in (5). We show that the inverse image \( P \) under \( f \) of a general pencil of lines in \( |O_S(C)|^\vee \) consists only of integral curves with at most one node. This is achieved in several steps.

(i) Since \( O_S(C) \) is ample, we can apply [BL, Theorem A] and conclude that each curve \( C' \in |O_S(C)| \) is 2-connected, that is, it cannot be written as a sum of effective divisors \( C' = F + M \), where \( F \cdot M \leq 1 \). This implies that \( |O_S(C)| \) does not contain any tree-like curves, that is, curves for which its irreducible components meet at a single point, which furthermore is a node.

(ii) The essential step in our argument involves proving that \( P \) contains no curves with singularities worse than nodes. Precisely, we show that \( |O_S(C)| \) contains only finitely many non-nodal curves. Note first that the branch curve \( B \subseteq P^2 \) of \( f \) is reduced, else we contradict the assumption that the pencil \( A \in W^1_0(C) \) on \( C \) has simple ramification. We introduce the discriminant curve
\[ J := \left\{ C' \in |O_S(C)| : C' \text{ is singular} \right\}. \]

The dual curve \( B^\vee \) is contained in \( J \). Since \( B \) is reduced, the general tangent line to \( B \) is tangent at exactly one point \( p \in B \) and with multiplicity 2. A standard local calculation shows that \( f^*(T_p(B)) \in |O_S(C)| \) is a one-nodal curve, singular at exactly one point \( z \in f^{-1}(p) \) such that the differential \( df_z : T_z(S) \to T_p(P^2) \) is not an isomorphism.

The complement \( J \setminus B^\vee \) is the (possibly empty) union of (some of) the pencils \( P_b \), where \( b \in B_{\text{sing}} \) and \( P_b \) is defined as the pull-back by \( f \) of the pencil of lines in \( P^2 \) through \( b \). In view of the numerical situation at hand (that is, \( C^2 = 9 \)), the geometric possibilities for such a pencil
\[ P_b \subseteq J \]
are quite constrained. Since \( f \) is finite, the pencil \( P_b \) has no fixed components. Let \( Z := B_\text{s}\{P_b\} \). Then a general \( C' \in P_b \) is integral and smooth along \( C' \setminus Z \). Moreover, each
$C' \in \mathbb{P}_b$ is singular at a given point $z \in Z$ and a general such $C'$ has multiplicity $m \geq 2$ at $z$. Necessarily, the differential $df_z : T_z(S) \to T_p(\mathbb{P}^2)$ is zero. Since $m^2 = C^2 = (C')^2 = 9$, we find $m \in \{2, 3\}$. Let

$$\sigma : S' \to S$$

be the blow-up of $S$ at $z$ and denote by $E \subset S'$ the exceptional divisor. The pencil $|O_{S'}(\sigma^*C - mE)|$ is the strict transform of $P_b$. Observe that the restriction map

$$r : H^0(S', O_{S'}(\sigma^*C - mE)) \to H^0(E, O_E(m))$$

is not zero, hence $\text{Im}(r)$ defines a linear series $p_b$ on $E \cong \mathbb{P}^1$. Either $p_b$ is a pencil or a constant divisor of degree $m \in \{2, 3\}$. We now list the possibilities for the pencil $p_b$.

(P1) If $m = 3$, then $\text{supp}(Z) = \{z\}$. Every curve $C' \in \mathbb{P}_b$ has a triple point at $z$.

(P2) If $m = 2$, then either each $C' \in \mathbb{P}_b$ has a node, or else, each $C' \in \mathbb{P}_b$ has a cusp at $z$. Indeed, if $p_b$ is a pencil on $E$, then each $C' \in \mathbb{P}_b$ is nodal at $z$. If $p_b = \{u_1 + u_2\}$ consists of a fixed divisor, then $P_b$ contains a unique curve $C_z$ having multiplicity at least 3 at $z$. If $u_1 \neq u_2$, all other curves $C' \in \mathbb{P}_b \setminus \{C_z\}$ are nodal at $z$, whereas if $u_1 = u_2$, then all such $C'$ are cuspidal at $z$.

Both possibilities (P1) and (P2) can be ruled out by a parameter count that contradicts the generality of the pair $(C, A) \in H_{15,9}$ we started with. We first rule out (P1). Assume $C' \in \mathbb{P}_b$ has a triple point at $z$ and no further singularities and denote by

$$\nu : C \to C'$$

the normalization. Set $\{z_1, z_2, z_3\} = \nu^{-1}(z)$ and $\tilde{A} := \nu^*(O_{C'}(C')) \in W^3_4(C)$.

Since $\tilde{A}$ is induced from a pencil of curves with a triple point at $z$, it follows that $|\tilde{A}(−3z_1 − 3z_2 − 3z_3)| \neq \emptyset$, therefore for degree reason $\tilde{A} = O_C(z_1 + z_2 + z_3)$. We denote by $H^\text{triple}_{12,9}$ the Hurwitz space classifying degree 9 covers $\tilde{C} \to \mathbb{P}^1$ having a divisor of the form $3(z_1 + z_2 + z_3)$ in a fibre, where $\tilde{C}$ is of genus 12. Then $H^\text{triple}_{12,9}$ is pure of dimension $\dim(\mathcal{M}_{12}) = 32$. Let $Y_1$ be the parameter space of pairs $(C', C')$, where $S \subset \mathbb{P}^6$ is a smooth complete intersection of 4 quadrics and $C' \subset S$ is an integral curve of arithmetic genus 15 with a triple point as described by (P1). Let

$$S \xleftarrow{\pi_1} Y_1 \xrightarrow{\pi_2} H^\text{triple}_{12,9}$$

be the projections given by $\pi_1([S, C']) := [S]$ and $\pi_2([S, C']) := [C, \tilde{A}]$ respectively. With the notation above, from the adjunction formula $\nu^*(O_{C'}(1)) = \omega_{\tilde{C}}(−z_1 − z_2 − z_3)$. The fibre $\pi_2^{-1}(\pi_2([S, C']))$ corresponds then to the choice of a 7-dimensional space of sections $V \subset H^0(\tilde{C}, \omega_{\tilde{C}}(−z_1 − z_2 − z_3))$ satisfying

$$\dim \left( V \cap H^0(\omega_{\tilde{C}}(−2z_1 − 2z_2 − 2z_3)) \right) \geq 6.$$ 

Since $h^0(\omega_{\tilde{C}}(−2z_1 − 2z_2 − 2z_3)) = 6$, it follows that

$$\frac{V}{H^0(\omega_{\tilde{C}}(−2z_1 − 2z_2 − 2z_3))} \in \mathbb{P} \left( \frac{H^0(\omega_{\tilde{C}}(−z_1 − z_2 − z_3))}{H^0(\omega_{\tilde{C}}(−2z_1 − 2z_2 − 2z_3))} \right) \cong \mathbb{P}^2.$$ 

Therefore $\dim(Y_1) = \dim(H^\text{triple}_{12,9}) + 2 = 34 \leq 39$, so we can invoke Proposition 7 to conclude that $\overline{\pi_1(Y_1)} \neq S$ and rule out possibility (P1).

Next we rule out possibility (P2), focusing on the case when each $C' \in \mathbb{P}_b$ is cuspidal at $z$. Passing to the normalization $\nu : \tilde{C} \to C'$, setting $\tilde{z} := \nu^{-1}(z)$ we obtain that $\tilde{A} := \nu^*(O_{C'}(C')) \in W^3_4(C)$ verifies $h^0(\tilde{C}, \tilde{A}(−4\tilde{z})) = 1$. Let $H^\text{four}_{14,9}$ be the Hurwitz space classifying degree 9 covers $\tilde{C} \to \mathbb{P}^1$ containing a divisor of type $4\tilde{z}$ in one of its fibres.
and where \( \overline{C} \) has genus 14. Then \( \mathcal{H}_{14,9}^{\text{four}} \) is irreducible of dimension 39 = \( \dim(\mathcal{M}_{14}) \). Let \( \mathcal{V}_2 \) be the parameter space of pairs \( (S, C') \), where \( S \subseteq \mathbb{P}^6 \) is a smooth complete intersection of 4 quadrics and \( C' \subseteq S \) is an integral curve of arithmetic genus 15 with a cusp at \( z \) as described by \((\text{P}2)\). We consider the projections

\[
S \leftarrow_{\pi_1} \mathcal{V}_2 \rightarrow_{\pi_2} \mathcal{H}_{14,9}^{\text{four}}
\]

given by \( \pi_1([S, C']) := [S] \) and \( \pi_2([S, C']) := [\overline{C}, \overline{A}] \) respectively. Observe that \( \pi_2 \) is birational onto its image. Indeed, given \( [\overline{C}, \overline{A}] \in \pi_2(\mathcal{V}_2) \), then we denote by \( C' \) the image of the map \( \overline{\omega}_{\overline{O}}(2y) \otimes \overline{\omega}_V \colon \overline{C} \to \mathbb{P}^6 \), in which case the canonical surface \( S \) is recovered by \((3)\). We conclude by Proposition 7 again that \( \pi_1(\mathcal{V}_2) \) is not dense in \( S \). The final case when all curves \( C' \in \mathbb{P}_b \) are (at least) nodal at \( z \) is ruled out analogously.

Before stating our next result, recall that one sets \( \delta_i := |\Delta_i| \in CH^1(\mathcal{M}_g) \) for \( 0 \leq i \leq \left\lfloor \frac{g}{2} \right\rfloor \). We denote as usual by \( \lambda \in CH^1(\mathcal{M}_g) \) the Hodge class. Recall also the formula [HM] for the canonical class of \( \mathcal{M}_g \):

\[
K_{\mathcal{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\left\lfloor \frac{g}{2} \right\rfloor} \in CH^1(\mathcal{M}_g).
\]

**Proposition 10.** The rational curve \( \Gamma \) is a sweeping pencil for the boundary divisor \( \Delta_0 \). Its intersection numbers with the standard generators of \( CH^1(\mathcal{M}_{16}) \) are as follows:

\[
\Gamma \cdot \lambda = 22, \quad \Gamma \cdot \delta_0 = 143, \quad \Gamma \cdot \delta_j = 0 \quad \text{for} \quad j = 2, \ldots, 8.
\]

**Proof.** First we construct a fibration whose moduli map is precisely the rational curve \( m \colon \mathbb{P}^1 \to \mathcal{M}_{15,2} \) considered in \((12)\). We consider the curve \( C \subseteq S \) and observe that since \( \mathcal{O}_C(C) \cong A \in W^1_9(C) \), we have that \( C^2 = 9 \), that is, the pencil \( |\mathcal{I}_{x,y}(C)| \) has precisely 9 base points, namely \( x, y \), as well as the 7 further points lying in the same fibre of the pencil \( |A| \) as \( x \) and \( y \). We consider the blow-up surface \( \pi \colon \overline{S} = Bl_y(S) \to S \) at these 9 points. It comes equipped with a fibration

\[
\pi \colon \overline{S} \to \mathbb{P}^1,
\]

as well as with two sections \( E_x, E_y \subseteq \overline{S} \) corresponding to the exceptional divisors at \( x \) and \( y \) respectively.

In order to compute the intersection numbers of \( R = m(\mathbb{P}) \) with the tautological classes on \( \mathcal{M}_{15,2} \), we use for instance [Tan]. The subscript indicates the moduli space on which the intersection number is computed.

\[
(R \cdot \lambda)_{\mathcal{M}_{15,2}} = \chi(\overline{S}, \mathcal{O}_S) + g - 1 = h^2(S, \mathcal{O}_S) + g = h_0(S, \mathcal{O}_S(1)) + 15 = 22.
\]

Here we have used \( H^1(\overline{S}, \mathcal{O}_S) = H^1(S, \mathcal{O}_S) = 0 \), as well as the fact that \( S \) is a canonical surface, hence \( \omega_S = \mathcal{O}_S(1) \), therefore \( h^2(\overline{S}, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 7 \). Furthermore, recalling that all curves in the fibres of \( m \) are irreducible, we find via [Tan] that

\[
(R \cdot \delta_0)_{\mathcal{M}_{15,2}} = c_2(\overline{S}) + 4(g - 1) = c_2(\overline{S}) + 56.
\]

From the Euler formula, \( c_2(\overline{S}) = 12\chi(\overline{S}, \mathcal{O}_S) - K_S^2 \). We have already computed that \( \chi(\overline{S}, \mathcal{O}_S) = 8 \), whereas \( K_S^2 = K_{\overline{S}}^2 - 9 = \deg(S) - 9 = 7 \), for \( S \) is an intersection of 4 quadrics. Thus \( c_2(\overline{S}) = 12 \cdot 8 - 7 = 89 \), leading to \( (R \cdot \delta_0)_{\mathcal{M}_{15,2}} = 89 + 4 \cdot 14 = 145 \).
If we denote by $\psi_x, \psi_y \in CH^1(\overline{M}_{15,2})$ the cotangent classes corresponding to the marked points labelled by $x$ and $y$ respectively, we compute furthermore
\[
R \cdot \psi_x = -E_x^2 = 1 \text{ and } R \cdot \psi_y = -E_y^2 = 1.
\]
We now pass to the pencil $\xi : \mathbb{P}^1 \to \overline{M}_{16}$ obtained from $m$ by identifying pointwise the disjoint sections $E_x$ and $E_y$ on the surface $\tilde{S}$. First, using (15) we observe that
\[
\Gamma \cdot \lambda = \xi(P) \cdot \lambda = (R \cdot \lambda)_{\overline{M}_{15,2}} = 22.
\]
Furthermore, using Proposition 9 we conclude that $\Gamma \cdot \delta_i = 0$ for $i = 1, \ldots, 8$. Finally, invoking for instance [CR3, page 271], we find that
\[
\Gamma \cdot \delta_0 = (R \cdot \delta_0)_{\overline{M}_{15,2}} - (R \cdot \psi_x)_{\overline{M}_{15,2}} - (R \cdot \psi_y)_{\overline{M}_{15,2}} = 145 - 2 = 143.
\]

\[\square\]

**Proof of Theorem 2.** Since the image of $m$ passes through a general point of $\overline{M}_{15,2}$, the rational curve $\Gamma \subseteq \overline{M}_{16}$ constructed in Proposition 10 is a sweeping curve for the boundary divisor $\Delta_0$. Using the expression (14) for the canonical divisor of $\overline{M}_{16}$, we compute
\[
\Gamma \cdot K_{\overline{M}_{16}} = 13 \Gamma \cdot \lambda - 2 \Gamma \cdot \delta_0 = 13 \cdot 22 - 2 \cdot 143 = 0. \text{ Also } \Gamma \cdot \Delta_0 = 143 > 0. \quad \square
\]

3. The slope of $\overline{M}_{16}$.

The slope of an effective divisor $D$ on the moduli space $\overline{M}_g$ not containing any boundary divisor $\Delta_i$ in its support is defined as the quantity $s(D) := \frac{a}{\min_i b_i}$, where $[D] = a\lambda - \sum_{i=0}^{15} b_i\delta_i \in CH^1(\overline{M}_g)$, with $a, b_i \geq 0$. Then the slope $s(\overline{M}_g)$ of the moduli space $\overline{M}_g$ is defined as the infimum of the slopes $s(D)$ over such effective divisors $D$.

**Corollary 11.** We have that $s(\overline{M}_{16}) \geq \frac{13}{2}$.

**Proof.** For any effective divisor $D$ on $\overline{M}_{16}$ containing no boundary divisor in its support, we may assume that the curve $\Gamma$ constructed in Proposition 10 does not lie inside $D$, hence $\Gamma \cdot D \geq 0$. Writing $[D] = a\lambda - \sum_{i=0}^{15} b_i\delta_i$, using Theorem 2 we obtain $\frac{a}{\min_i b_i} \geq \frac{13\delta_0}{\overline{M}_{16}} = \frac{13}{2}$. Furthermore, using [FP, Theorem 1.4], we conclude that for this divisor $D$ we have $b_i \geq b_0$ for $i = 1, \ldots, 8$, that is, $s(D) = \frac{a}{b_0}$. \[\square\]

**Final remarks:** Our results establish that $\overline{M}_{16}$ is not of general type. Showing that the Kodaira dimension of $\overline{M}_{16}$ is non-negative amounts to constructing an effective divisor $D$ on $\overline{M}_{16}$ havind slope $s(D) \leq s(K_{\overline{M}_{16}}) = \frac{13}{2}$. Currently the known effective divisor on $\overline{M}_{16}$ of smallest slope is the closure in $\overline{M}_{16}$ of the *Koszul divisor* $\mathcal{Z}_{16}$ consisting of curves $C$ having a linear system $L \in W^2_2(C)$ such that the image curve $\varphi_L : C \to \mathbb{P}^6$ is ideal-theoretically not cut out by quadrics. It is shown in [F1, Theorem 1.1] that $\mathcal{Z}_{16}$ is an effective divisor on $\overline{M}_{16}$ and $s(\mathcal{Z}_{16}) = \frac{407}{18} = 6.70516$. In a related direction, it is shown in [F2] that the canonical class of the space of admissible covers $\overline{H}_{16,9}$ is effective. Note that one has a generically finite cover $\overline{H}_{16,9} \to \overline{M}_{16}$.

Soon after the appearance of the first version of this paper, it has been pointed out by Agostini and Barros [AB] that our proof of Theorem 2 yields in fact the bound
\[
\kappa(\overline{M}_{16}) \leq \dim(\overline{M}_{16}) - 2.
\]
Indeed, consider the parameter space $\mathcal{Z}$ of elements $[C, A, x, y]$, where $C$ is a genus 15 irreducible nodal curve, $A \in W^1_C$ and $x, y \in C$ are points such
that $|A(-x-y)| \neq 0$. As we explain in this paper, $Z$ has the structure of a $\mathbb{P}^1$-bundle and one has a dominant morphism $v: Z \to \Delta_0$ given by $[C, A, x, y] \mapsto [C/x \sim y]$. In Proposition 10 we establish that the restriction of $v^*(K_{\overline{M}_{16}})$ to the general fibre of this fibration is trivial. Accordingly, $\kappa(\overline{M}_{16}) \leq \dim(Z) - 1 = \dim(\overline{M}_{16}) - 2$.

References


[FP] G. Farkas and M. Popa, Effective divisors on $\overline{M}_g$, curves on $K3$ surfaces and the Slope Conjecture, Journal of Algebraic Geometry 14 (2005), 151–174.


