\overline{M}_{16} IS UNIRULED

GAVRIL FARKAS AND ALESSANDRO VERRA

Abstract. We prove that the moduli space of curves of genus 16 is uniruled.

The problem of determining the nature of the moduli space $\overline{M}_g$ of stable curves of genus $g$ has long been one of the key questions in the field, motivating important developments in moduli theory. Severi [Sev] observed that $\overline{M}_g$ is unirational for $g \leq 10$, see [AC] for a modern presentation. Much later, in the celebrated series of papers [HM], [H], [EH], Harris, Mumford and Eisenbud showed that $\overline{M}_g$ is of general type for $g \geq 24$. Very recently, it has been showed in [FJP] that both $\overline{M}_{22}$ and $\overline{M}_{23}$ are of general type. On the other hand, due to work of Sernesi [Ser], Chang-Ran [CR1], [CR2] and Verra [Ve] it is known that $\overline{M}_g$ is unirational also for $11 \leq g \leq 14$. Finally, Bruno and Verra [BV] proved that $\overline{M}_{15}$ is rationally connected. Our result is the following:

**Theorem 1.** The moduli space $\overline{M}_{16}$ of stable curves of genus 16 is uniruled.

Note that 16 is the highest genus for which it is known that $\overline{M}_{16}$ is not of general type. We further refer to Tseng’s recent paper [Ts] for further details on the convoluted history of determining the Kodaira dimension of $\overline{M}_{16}$.

Before explaining our strategy of proving Theorem 1, recall the standard notation $\Delta_0, \ldots, \Delta_{\lfloor \frac{2g}{3} \rfloor}$ for the irreducible boundary divisors on $\overline{M}_g$, see [HM]. Here $\Delta_0$ denotes the closure in $\overline{M}_g$ of the locus of irreducible 1-nodal curves of arithmetic genus $g$. Our approach relies on the explicit uniruled parametrization of $\overline{M}_{15}$ found by Bruno and Verra [BV]. Their work establishes that through a general point of $\overline{M}_{15}$ there passes not only a rational curve, but in fact a rational surface. This extra degree of freedom, yields a uniruled parametrization of $\overline{M}_{15,2}$, therefore also a parametrization the boundary divisor $\Delta_0$ inside $\overline{M}_{16}$. We show the following:

**Theorem 2.** The boundary divisor $\Delta_0$ of $\overline{M}_{16}$ is uniruled and swept by a family of rational curves, whose general member $\Gamma \subseteq \Delta_0$ satisfies $\Gamma \cdot K_{\overline{M}_{16}} = 0$ and $\Gamma \cdot \Delta_0 > 0$.

Assuming Theorem 2, we conclude that $\overline{M}_{16}$ cannot be of general type, thus establishing Theorem 1. To that end, note first that in any effective representation of the canonical divisor

$$K_{\overline{M}_{16}} \equiv \alpha \cdot \Delta_0 + D,$$

where $\alpha \in \mathbb{Q}_{>0}$ and $D$ is an effective $\mathbb{Q}$-divisor on $\overline{M}_{16}$ not containing $\Delta_0$ in its support, we must have $\alpha = 0$. Indeed, we can choose the curve $\Gamma$ such that $\Gamma \not\subseteq D$, then we write

$$0 = \Gamma \cdot K_{\overline{M}_{16}} = \alpha \Gamma \cdot \Delta_0 + \Gamma \cdot D \geq \alpha \Gamma \cdot \Delta_0 \geq 0,$$

hence $\alpha = 0$. Furthermore, since the singularities of $\overline{M}_g$ do not impose adjunction conditions [HM, Theorem 1], $\overline{M}_g$ is a variety of general type for a given $g \geq 4$ if and
only if the canonical class \( K_{\overline{M}_g} \) is a big divisor class, that is, it can be written as
\[
K_{\overline{M}_g} \equiv A + E,
\]
where \( A \) is an ample \( \mathbb{Q} \)-divisor and \( E \) is an effective \( \mathbb{Q} \)-divisor respectively. Assume that \( K_{\overline{M}_{16}} \) can be written like in (1). It has already been observed that \( \Delta_0 \not\in \text{supp}(E) \), in particular \( \Gamma \cdot E \geq 0 \). Using Kleiman’s ampleness criterion, \( \Gamma \cdot A > 0 \), which yields the immediate contradiction \( 0 = \Gamma \cdot K_{\overline{M}_{16}} = \Gamma \cdot A + \Gamma \cdot E \geq \Gamma \cdot A > 0 \).

We are left therefore with proving Theorem 2, which is what we do in the rest of the paper. The rational curve \( \Gamma \) constructed in Theorem 2 is the moduli curve corresponding to an appropriate pencil of curves of genus 15 on a certain canonical surface \( S \subseteq \mathbb{P}^6 \). Establishing that this pencil can be chosen in such a way to contain only stable curves will take up most of Section 2.

1. THE BRUNO-VERRA PARAMETRIZATION OF \( \overline{M}_{15} \)

The parametrization of the boundary divisor \( \Delta_0 \) of \( \overline{M}_{16} \) and the proof of Theorem 2 uses several results from [BV], which we now recall. We denote by \( \mathcal{H}_{15,9} \) the Hurwitz space parametrizing degree 9 covers \( C \to \mathbb{P}^1 \) having simple ramification, where \( C \) is a smooth curve of genus 15. Then \( \mathcal{H}_{15,9} \) is birational to the parameter space \( \mathcal{G}_{15,9}^1 \) classifying pairs \((C, A)\), where \( [C] \in \overline{M}_{15} \) and \( A \in W_9^1(C) \) is a pencil. By residduation, \( \mathcal{G}_{15,9}^1 \) is isomorphic to the parameter space \( \mathcal{G}_{15,19}^6 \) of pairs \([C, L]\), where \( C \) is a smooth curve of genus 15 and \( L \in W_{19}^6(C) \). Note that the general fibre of the forgetful map
\[
\pi : \mathcal{H}_{15,9} \to \overline{M}_{15}, \quad [C, A] \mapsto [C]
\]
is 1-dimensional. Clearly, \( \mathcal{H}_{15,9} \) and thus \( \mathcal{G}_{15,19}^6 \) is irreducible.

We pick a general element \([C, L] \in \mathcal{G}_{15,19}^6\), in particular \( L \) is very ample and \( h^0(C, L) = 7 \). We set \( A := \omega_C \otimes L^\vee \in W_9^1(C) \). We may assume that \( A \) is base point free and the pencil \([A]\) has simple ramification. We consider the multiplication map
\[
\phi_L : \text{Sym}^2 H^0(C, L) \to H^0(C, L^2).
\]
Since \( C \) is Petri general, \( h^1(C, L^2) = 0 \), therefore \( h^0(C, L^2) = 2 \cdot 19 + 1 - 15 = 24 \). Furthermore, via a degeneration argument it is shown in [BV, Theorem 3.11], that for a general choice of \((C, L)\), the map \( \phi_L \) is surjective, hence \( h^0(\mathbb{P}^6, \mathcal{I}_C/\mathbb{P}^6(2)) = \dim(\ker(\phi_L)) = 4 \), that is, the degree 19 curve \( C \subseteq \mathbb{P}^6 \) lies on precisely 4 independent quadrics. We let
\[
S := \text{Bs}[\mathcal{I}_C/\mathbb{P}^6(2)]
\]
be the base locus of the system of quadrics containing \( C \). It is further established in [BV, Theorem 3.11] that under our generality assumptions, \( S \) is a smooth surface. From the adjunction formula it follows that \( \omega_S = \mathcal{O}_S(1) \), that is, \( S \) is a canonical surface. We write down the exact sequence
\[
0 \to \mathcal{O}_S \to \mathcal{O}_{S}(C) \to \mathcal{O}_{C}(C) \to 0.
\]
From the adjunction formula \( \mathcal{O}_C(C) \cong \omega_C \otimes \omega_S^{\vee} \mid_C = \omega_C \otimes L^\vee = A \in W_9^1(C) \). Since \( S \) is a regular surface, by taking cohomology in (4), we obtain
\[
h^0(S, \mathcal{O}_{S}(C)) = h^0(S, \mathcal{O}_{S}) + h^0(C, A) = 3.
\]
Observe also from the sequence (4) that the linear system \( |O_S(C)| \) is base point free, for \( |O_C(C)| = |A| \) is so. This brings to an end our summary of the results from [BV].

In what follows, we denote by
\[
(5) \\
f: S \to \mathbb{P}^2 = |O_S(C)|^Y
\]
the induced map. For what we intend to do, it is important to show that \( f \) is a finite map, or equivalently, that \( O_S(C) \) is ample.

**Theorem 3.** For a general pair \((C, A) \in \mathcal{H}_{15,9}\), the line bundle \( O_S(C) \) is ample.

In order to prove Theorem 3 it suffices to exhibit a single pair \((C, A) \in \mathcal{H}_{15,9}\) for which the corresponding map \( f: S \to \mathbb{P}^2 \) given by (5) is finite. We shall realize the canonical surface \( S \subseteq \mathbb{P}^6 \) as the double cover of a suitable \( K3 \) surface \( Y \subseteq \mathbb{P}^5 \) of genus 5 (that is, of degree 8). It will prove advantageous to consider \( K3 \) surfaces having a certain Picard lattice of rank 3. We first discuss the geometry of such \( K3 \) surfaces.

**Definition 4.** We denote by \( \Lambda \) the even lattice of signature \((1, 2)\) generated by elements \( H, F \) and \( R \) having the following intersection matrix:
\[
\begin{pmatrix}
H^2 & H \cdot F & H \cdot R \\
F \cdot H & F^2 & F \cdot R \\
R \cdot H & R \cdot F & R^2
\end{pmatrix} = \begin{pmatrix}
8 & 9 & 1 \\
9 & 4 & 2 \\
1 & 2 & -2
\end{pmatrix}.
\]

We denote by \( \mathcal{F}_5^\Lambda \) the moduli space of polarized \( K3 \) surfaces \([Y, H]\), where \( H^2 = 8 \), admitting a primitive embedding \( \Lambda \hookrightarrow \text{Pic}(Y) \), such that the classes \( H, F, R \) correspond to curve classes on \( Y \) which we denote by the same symbol. Furthermore, \( H \in \text{Pic}(Y) \) is assumed to be ample.

For details on the construction of the moduli space \( \mathcal{F}_5^\Lambda \) we refer to [Do, Section 3]. It follows from loc.cit. that \( \mathcal{F}_5^\Lambda \) is an irreducible variety of dimension \( 17 = 20 - \text{rk}(\Lambda) \). Let us now fix a general element \([Y, H]\), where \( \text{Pic}(Y) \cong \mathbb{Z}(H, F, R) \) as in Definition 4. Then \( O_Y(\bar{H}) \) is very ample and we denote by
\[
(6) \\
\varphi_H: Y \hookrightarrow \mathbb{P}^5
\]
the embedding induced by this linear system. Observe that \( h^0(\mathbb{P}^5, \mathcal{I}_Y(\mathbb{P}^5(2))) = 3 \) and that \( Y = Bs[\mathcal{I}_Y(\mathbb{P}^5(2))] \) is in fact a complete intersection of three quadrics. Note that \( F \subseteq Y \) is a curve of genus 3, whereas \( R \subseteq Y \) is a smooth rational curve embedded as a line under the map \( \varphi_H \). The class \( E := 2\bar{H} - F \) satisfies \( E^2 = 0 \). Since \( E \cdot H = 7 > 0 \), it follows that \( |E| \) is an elliptic pencil and furthermore \( E \cdot R = 0 \). Setting also
\[
\bar{D} := 2\bar{H} + R - E \in \text{Pic}(Y),
\]
we compute \( \bar{D}^2 = 6, \bar{D} \cdot E = 14 \) and \( \bar{D} \cdot R = 0 \). In the basis \((\bar{D}, E, R)\) of \( \text{Pic}(Y) \), the intersection form on \( Y \) is described by the following simpler matrix:
\[
(7) \\
\begin{pmatrix}
\bar{D}^2 & \bar{D} \cdot E & \bar{D} \cdot R \\
E \cdot \bar{D} & E^2 & E \cdot R \\
R \cdot \bar{D} & R \cdot E & R^2
\end{pmatrix} = \begin{pmatrix}
6 & 14 & 0 \\
14 & 0 & 0 \\
0 & 0 & -2
\end{pmatrix}.
\]

On our way to proving Theorem 3, we establish the following result:

**Proposition 5.** The line bundle \( O_Y(\bar{F}) \) is very ample.
Proof. We first claim that $F$ is nef. Since $F^2 = 4 > 0$, it suffices to check that for any smooth rational curve $\Gamma \subseteq Y$, one has $\Gamma \cdot F \geq 0$. We write $\Gamma = aD + bE + cR$, where $a, b$ and $c$ are integers. We may assume $\Gamma \neq R$, thus $\Gamma \cdot R \geq 0$, implying $c \leq 0$. Furthermore, $\Gamma \cdot E \geq 0$, hence $a \geq 0$. Using (7), one has $\Gamma^2 = 6a^2 - 2c^2 + 28ab = -2$. Assume by contradiction $\Gamma \cdot F = \Gamma \cdot (D - R) = 6a + 14b + 2c \leq -2$. Multiplying this inequality with $2a \geq 0$ and substituting in the equality $\Gamma^2 = -2$ we obtain that $(a + c)^2 + 2a^2 + 2a - 1 \geq 0$, implying $a = 0$ and $c \in \{-1, 1\}$. If, say $c = 1$, then $\Gamma \equiv R + bE$. From the assumption $\Gamma \cdot F \leq -2$, we obtain that $b \leq -1$, hence $\Gamma \cdot H < 0$, thus $\Gamma$ cannot be effective, a contradiction. The case $c = -1$, implying $b \leq 0$ is ruled out similarly.

Thus $F$ is a nef curve. To conclude that $F$ is very ample, we invoke [SD]. It suffices to rule out the existence of a divisor class $M \in \text{Pic}(Y)$ such that (i) $M^2 = 0$ and $M \cdot F \in \{1, 2\}$, or satisfying (ii) $M^2 = -2$ and $M \cdot F = 0$. We discuss only (i), the remaining case being similar. Write $M = aD + bE + cR$. Since $M^2 = 0$, from (7) we obtain $3a^2 - c^2 + 14ab = 0$, whereas from $M \cdot F = 2$, we obtain that $3a + 7b + c = 1$. Eliminating $c$, we find $6a^2 + a(28b - 6) + 49b^2 - 14b + 1 = 0$. Since the discriminant of this equation is negative, this case is excluded. We conclude that $F$ is very ample. □

We fix a general polarized $K3$ surface $[Y, \tilde{H}] \in \mathcal{F}_5^A$, while keeping the notation from above. Choose a smooth divisor $Q \in |O_Y(2\tilde{H})|$ and consider the double cover 

$\sigma : S \to Y$

branched along $Q$. We denote by $Q \subseteq S$ the ramification divisor of $\sigma$, hence $\sigma^*(Q) = 2Q$. We set $H := \sigma^*(\tilde{H})$, where $H \in |O_Y(1)|$ is a linear section of $Y$. Note that $Q \in |O_S(H)|$.

Proposition 6. The induced morphism $\varphi_H : S \to \mathbb{P}^6$ embeds $S$ as a canonical surface which is the complete intersection of 4 quadrics in $\mathbb{P}^6$. More precisely, $S$ is a quadratic section of the cone $C_Y \subseteq \mathbb{P}^6$ over the $K3$ surface $Y \subseteq \mathbb{P}^5$.

Proof. From the adjunction formula we find $\omega_S = O_S(Q) = O_S(H)$. Furthermore, we have $\sigma_*(\omega_S) = \omega_Y \oplus \omega_Y(-H)$, hence from the projection formula we can write

$$H^0(S, O_S(H)) \cong H^0(Y, O_Y(\tilde{H})) \oplus H^0(Y, \omega_Y) \cong H^0(Y, O_Y(\tilde{H})) \oplus \mathbb{C}(Q),$$

where recall that $Q \in |O_S(H)|$, as well as

$$H^0(S, O_S(2H)) \cong H^0(Y, O_Y(2\tilde{H})) \oplus H^0(Y, O_Y(\tilde{H})) \cdot Q.$$

Thus $h^0(S, O_S(H)) = 6$ and $h^0(S, O_S(2H)) = h^0(Y, O_Y(2)) + h^0(Y, O_Y(1)) = 2 + 2\tilde{H}^2 + 6 = 24$. Furthermore, $S \subseteq \mathbb{P}^6$ is projectively normal, so $h^0(\mathbb{P}^6, \mathcal{I}_{S/\mathbb{P}^6}(2)) = 4$. Since clearly $S \subseteq C_Y$, it follows that $S$ can be viewed as a quadratic section of the cone $C_Y$, precisely the intersection of $C_Y$ with one of the quadrics containing $S$ not lying in the subsystem $|\sigma^*H^0(\mathbb{P}^5, \mathcal{I}_{Y/\mathbb{P}^5}(2))|$. □

We are now in a position to prove Theorem 3. We denote by $\text{Hilb}_{15,19}$ the unique component of the Hilbert scheme of curves $C \subseteq \mathbb{P}^6$ of genus 15 and degree 19 dominating $\mathcal{M}_{15}$. A general point of $\text{Hilb}_{15,19}$ corresponds to a smooth projectively normal curve $C \subseteq \mathbb{P}^6$ such that the canonical surface $S$ defined by (3) is smooth.

Proof of Theorem 3. We choose a $K3$ surface $[Y, O_Y(\tilde{H})] \in \mathcal{F}_5^A$ with $\text{Pic}(Y) = \mathbb{Z}\langle \tilde{H}, \tilde{F}, \tilde{R} \rangle$, where the intersection matrix is given as in Definition 4. The restriction map

$$H^0(Y, O_Y(2\tilde{H})) \to H^0(\tilde{R}, O_{\tilde{R}}(2\tilde{H}))$$
being surjective, we can choose a smooth curve $Q \in |\mathcal{O}_Y(2H)|$ which is tangent to $\bar{R}$, that is, $Q \cdot \bar{R} = 2y$, for a point $y \in Y$. Construct the double cover $\sigma : S \to Y$ defined in (8). The pull-back $\sigma^*(\bar{R})$ is then a double cover of $\bar{R}$ branched over the single point $y$, hence necessarily

$$\sigma^*(\bar{R}) = R + R' \subseteq S,$$

where $R$ and $R'$ are lines on $S \subseteq \mathbb{P}^6$. Next, we choose a smooth genus 3 curve $\bar{F} \subseteq Y$ general in its linear system and set

$$C' := \sigma^*(\bar{F}) \subseteq S.$$

Since $\bar{F} \cdot \bar{Q} = 2\bar{F} \cdot \bar{H} = 18$, we obtain that $C'$ is a smooth curve of genus 14 and degree 18 endowed with the double cover $C' \to \bar{F}$. Note that the linear system

$$|\mathcal{O}_S(C')| = \pi^*|\mathcal{O}_Y(\bar{F})|$$

is 3-dimensional. Applying Theorem 5, since $\mathcal{O}_Y(\bar{F})$ is ample and $\sigma$ is finite, we obtain that $\mathcal{O}_S(C')$ is ample as well. Observe that $C' \cdot R = \bar{F} \cdot \bar{R} = 2$. Choosing $\bar{F}$ general in its linear system, we can arrange the intersection of $R$ and $C'$ to be transverse, therefore

$$C := C' + R \subseteq S \subseteq \mathbb{P}^6$$

is a nodal curve of genus 15 and degree 19. Note that the linear system $|\mathcal{O}_S(C)|$ has $R$ as a fixed component, and $|\mathcal{O}_S(C)| = R + \pi^*|\mathcal{O}_Y(\bar{F})|$.

Despite the fact that $|\mathcal{O}_S(C)|$ is not ample, we can complete the proof of Theorem 3. Indeed, let us pick a general family $\{[C_t \to \mathbb{P}^6]\}_{t \in T} \subseteq \text{Hilb}_{15,19}$ over a pointed base $(T, o)$, whose fibre over $o \in T$ is the curve $C$ described in (9). If $S_t = Bs|I_{C_t}/\mathbb{P}^6(2)|$, assume the line bundle $\mathcal{O}_{S_t}(C_t)$ is not ample for each $t \in T$. As we have already observed, we may assume that $\mathcal{O}_{S_t}(C_t)$ is nef for all $t \in T$ and we denote by $f_t : S_t \to \mathbb{P}^2$ the map induced by the linear system $|\mathcal{O}_{S_t}(C_t)|$ for $t \in T \setminus \{o\}$. The limiting map of this family

$$f_o : S \to \mathbb{P}^2,$$

satisfies then $f_o^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_S(R + C')$ and is induced by a subspace of sections $\sigma^*(V)$, where $V \subseteq H^0(Y, \mathcal{O}_Y(\bar{F}))$ is 3-dimensional. By assumption, there exists a family of curves $\Gamma_t \subseteq S$ such that $\Gamma_t \cdot C_t = 0$. We denote by $\Gamma_o \subseteq S$ the limiting curve of $\Gamma_t$, therefore $\Gamma_o \cdot (C' + R) = 0$. We write $\Gamma_o = G + mR$, where $m \geq 0$ and $G \subseteq S$ is a curve not having $R$ in its support. From the adjunction formula, we find $R^2 = -3$. Since $R \cdot C' = 2$, it follows that $R \cdot (C' + R) = 1$, thus $G \neq 0$. Furthermore, the morphism $f_o$ contracts $G$, which we argue, leads to a contradiction. Indeed, $f_o$ admits a factorization

$$\xymatrix{ S \ar[r]_-{\sigma} & Y \ar[r]^-{\bar{F}} & \mathbb{P}^3 \ar[r]_-p & \mathbb{P}^2 }$$

where $p : \mathbb{P}^3 \to \mathbb{P}^2$ is the linear projection corresponding to $V \subseteq H^0(Y, \mathcal{O}_Y(\bar{F}))$. Since $\sigma$ is finite and $|\bar{F}|$ is very ample, it follows that $\sigma(G)$ must be contracted by the projection $p$, that is, $\sigma(C)$ is a line in $\mathbb{P}^3$. By inspecting the intersection matrix (7) of Pic($Y$) we immediately see that no such line can exist on $Y$, which finishes the proof. \qed
2. The Uniruledness of the Boundary Divisor $\Delta_0$ in $\overline{M}_{16}$

We now lift the construction discussed above from $\overline{M}_{15}$ to the moduli space $\overline{M}_{15,2}$ of 2-pointed stable curves of genus 15 and eventually to $\overline{M}_{16}$. Recall that $\text{Hilb}_{15,19}$ is the component of the Hilbert scheme of curves $C \subseteq \mathbb{P}^6$ of genus 15 and degree 19 dominating $\mathcal{M}_{15}$. We denote by $\text{Hilb}_{2,2,2,2}$ the Hilbert scheme of complete intersections of 4 quadrics in $\mathbb{P}^6$. Since $\text{Hilb}_{15,19}/\mathbb{PGL}(7)$ is birational to the Hurwitz space $\mathcal{H}_{15,9}$, we have a rational map

$$\chi : \mathcal{H}_{15,9} \dashrightarrow \text{Hilb}_{2,2,2,2}/\mathbb{PGL}(7), \quad [C, A] \mapsto S := \text{Bs}[\mathcal{I}_{C/\mathbb{P}^6}(2)] \mod \mathbb{PGL}(7),$$

where the canonical surface $S \subseteq \mathbb{P}^6$ is defined by (3). We set

$$\mathcal{S} \coloneqq \chi(\mathcal{H}_{15,9}).$$

The general fibre of the morphism $\chi : \mathcal{H}_{15,9} \to S$ consists of finitely many linear nonempty open subsets of linear systems $|\mathcal{O}_S(C)|$, where $C \subseteq S \subseteq \mathbb{P}^6$ is a smooth curve of genus 15 and degree 19. In particular, $S$ is an irreducible variety of dimension $41 = \dim(\mathcal{H}_{15,9}) - 2$. Recall that $\pi : \mathcal{H}_{15,9} \to \mathcal{M}_{15}$ denotes the forgetful map. The next observation will prove to be useful in several moduli counts.

**Proposition 7.** If $S'$ is an irreducible subvariety of $S$ of dimension $\dim(S') \leq 39$, then $\pi(\chi^{-1}(S'))$ is a proper subvariety of $\mathcal{M}_{15}$.

**Proof.** Since $\dim(\chi^{-1}(S')) \leq \dim(S') + 2 \leq 41 = \dim(\mathcal{M}_{15}) - 1$, the claim follows. \qed

Let us now take a general curve $C$ of genus 15 and consider the correspondence

$$\Sigma := \left\{ (A, x + y) \in W^1_9(C) \times C_2 : H^0(C, A(-x - y)) \neq 0 \right\},$$

dowered with the projections $\pi_1 : \Sigma \to W^1_9(C)$ and $\pi_2 : \Sigma \to C_2$ respectively. Here $C_2$ is the second symmetric product of $C$. It follows that $\Sigma$ is an irreducible surface and that $\pi_2$ is generically finite. Indeed, for a general point $2x \in C_2$, we can invoke for instance [EH, Theorem 1.1] to conclude that $\pi_2^{-1}(2x)$ is finite. The fibre $\pi_1^{-1}(A)$ is irreducible whenever $A$ has simple ramification.

We now fix a general element $[C, x, y] \in \overline{M}_{15,2}$. Then there exist finitely many pencils $A \in W^1_9(C)$ containing both points $x$ and $y$ in the same fibre. Each of these pencils $A$ may be assumed to be base point free with simple ramification and general enough such that $L := \omega_C \otimes A^\vee \in W^1_{19}(C)$ is very ample and in the embedding

$$\varphi_L : C \hookrightarrow \mathbb{P}^6$$

the curve $C$ lies on precisely 4 independent quadrics intersecting in a smooth canonical surface $S$ defined by (3).

**Proposition 8.** With the notation above, if $h^0(C, A(-x - y)) = 1$, then $\dim \left| \mathcal{I}_{x,y}(C) \right| = 1$

**Proof.** It follows from the commutativity of the following diagram, keeping in mind that $h^0(S, \mathcal{O}_S(C)) = 3$ and that the first column is injective.

$$\begin{array}{ccc}
0 & \longrightarrow & H^0(S, \mathcal{I}_{x,y}(C)) \\
\downarrow & & \downarrow \text{res} \\
0 & \longrightarrow & H^0(C, A(-x - y)) \\
\end{array}$$
We now introduce the moduli map of the pencil introduced in Proposition 8

\[ m : \mathcal{P} = \mathcal{I}_{\{x,y\}}(C) \to \overline{\mathcal{M}}_{15,2}, \]

where the marked points of the pencil are the base points \( x \) and \( y \) respectively. Composing \( m \) with the clutching map \( \overline{\mathcal{M}}_{15,2} \to \Delta_0 \subseteq \overline{\mathcal{M}}_{16} \), we obtain a pencil \( \xi : \mathcal{P} \to \Delta_0 \).

We set

\[ R := m_*(\mathcal{P}) \subseteq \overline{\mathcal{M}}_{15,2} \quad \text{and} \quad \Gamma := \xi_*(\mathcal{P}) \subseteq \overline{\mathcal{M}}_{16}. \]

**Proposition 9.** Every curve inside the pencil \( \Gamma \subseteq \overline{\mathcal{M}}_{16} \) corresponds to a nodal curve which does not belong to any of the boundary divisors \( \Delta_1, \ldots, \Delta_8 \).

**Proof.** Keeping the notation above, for a generic choice of \( (A, x + y) \in \Sigma \), the pencil

\[ \mathcal{P} := \mathcal{I}_{\{x,y\}}(C) \]

corresponds to a generic line inside \( |\mathcal{O}_S(C)| \). As pointed out in Theorem 3, \( |\mathcal{O}_S(C)| \) is base point free and ample on the surface \( S \) defined by (3), giving rise to the finite map

\[ f : S \to \mathbb{P}^2 = |\mathcal{O}_S(C)|^\vee \]

considered in (5). We show that the inverse image \( \mathcal{P} \) under \( f \) of a general pencil of lines in \( |\mathcal{O}_S(C)|^\vee \) consists only of integral curves with at most one node. This is achieved in several steps.

**i.** Since \( \mathcal{O}_S(C) \) is ample, we can apply [BL, Theorem A] and conclude that each curve \( C' \in |\mathcal{O}_S(C)| \) is 2-connected, that is, it cannot be written as a sum of effective divisors \( C' = F + M \), where \( F \cdot M \leq 1 \). This implies that \( |\mathcal{O}_S(C)| \) does not contain any tree-like curves, that is, curves for which its irreducible components meet at a single point, which furthermore is a node.

**ii.** The essential step in our argument involves proving that \( \mathcal{P} \) contains no curves with singularities worse than nodes. Precisely, we show that \( |\mathcal{O}_S(C)| \) contains only finitely many non-nodal curves. Note first that the branch curve \( B \subseteq \mathbb{P}^2 \) of \( f \) is reduced, else we contradict the assumption that the pencil \( A \in W^1_3(C) \) on \( C \) has simple ramification. We introduce the discriminant curve

\[ J := \left\{ C' \in \left| \mathcal{O}_S(C) \right| : C' \text{ is singular} \right\}. \]

The dual curve \( B^\vee \) is contained in \( J \). Since \( B \) is reduced, the general tangent line to \( B \) is tangent at exactly one point \( p \in B \) and with multiplicity 2. A standard local calculation shows that \( f^* (\mathcal{T}_p(B)) \in \left| \mathcal{O}_S(C) \right| \) is a one-nodal curve, singular at exactly one point \( z \in f^{-1}(p) \) such that the differential \( df_z : T_z(S) \to T_p(\mathbb{P}^2) \) is not an isomorphism.

The complement \( J \setminus B^\vee \) is the (possibly empty) union of (some of) the pencils \( \mathcal{P}_b \), where \( b \in B_{\text{sing}} \) and \( \mathcal{P}_b \) is defined as the pull-back by \( f \) of the pencil of lines in \( \mathbb{P}^2 \) through \( b \). In view of the numerical situation at hand (that is, \( C^2 = 9 \)), the geometric possibilities for such a pencil

\[ \mathcal{P}_b \subseteq J \]

are quite constrained. Since \( f \) is finite, the pencil \( \mathcal{P}_b \) has no fixed components. Let \( Z := \text{Bs}(\mathcal{P}_b) \). Then a general \( C' \in \mathcal{P}_b \) is integral and smooth along \( C' \setminus Z \). Moreover, each
$C' \in \mathbb{P}_b$ is singular at a given point $z \in Z$ and a general such $C'$ has multiplicity $m \geq 2$ at $z$. Necessarily, the differential $df_z: T_z(S) \to T_p(\mathbb{P}^2)$ is zero. Since $m^2 \leq C^2 = (C')^2 = 9$, we find $m \in \{2,3\}$. Let 

$$\sigma: S' \to S$$

be the blow-up of $S$ at $z$ and denote by $E \subseteq S'$ the exceptional divisor. The pencil $|O_S(\sigma^*C - mE)|$ is the strict transform of $\mathbb{P}_b$. Observe that the restriction map 

$$r: H^0(S', O_S(\sigma^*C - mE)) \to H^0(E, O_E(m))$$

is not zero, hence $\text{Im}(r)$ defines a linear series $p_b$ on $E \cong \mathbb{P}^1$. Either $p_b$ is a pencil or a constant divisor of degree $m \in \{2,3\}$. We now list the possibilities for the pencil $p_b$.

**P1** If $m = 3$, then $\text{supp}(Z) = \{z\}$. Every curve $C' \in \mathbb{P}_b$ has a triple point at $z$.

**P2** If $m = 2$, then either each $C' \in \mathbb{P}_b$ has a node, or else, each $C' \in \mathbb{P}_b$ has a cusp at $z$. Indeed, if $p_b$ is a pencil on $E$, then each $C' \in \mathbb{P}_b$ is nodal at $z$. If $p_b = \{u_1 + u_2\}$ consists of a fixed divisor, then $\mathbb{P}_b$ contains a unique curve $C_z$ having multiplicity at least 3 at $z$. If $u_1 \neq u_2$, all other curves $C' \in \mathbb{P}_b \setminus \{C_z\}$ are nodal at $z$, whereas if $u_1 = u_2$, then all such $C'$ are cuspidal at $z$.

Both possibilities (P1) and (P2) can be ruled out by a parameter count that contradicts the generality of the pair $(C, A) \in H_{15,9}$ we started with. We first rule out (P1). Assume $C' \in \mathbb{P}_b$ has a triple point at $z$ and no further singularities and denote by $\nu: \tilde{C} \to C'$ the normalization. Set $\{z_1, z_2, z_3\} = \nu^{-1}(z)$ and $\tilde{A} := \nu^*(\mathcal{O}_{C'}(C')) \in W^3_3(\tilde{C})$. Since $\tilde{A}$ is induced from a pencil of curves with a triple point at $z$, it follows that $|\tilde{A}(-3z_1 - 3z_2 - 3z_3)| \neq \emptyset$, therefore for degree reason $\tilde{A} = \mathcal{O}_{\tilde{C}}(3z_1 + 3z_2 + 3z_3)$. We denote by $H^{\text{triple}}_{12,9}$ the Hurwitz space classifying degree 9 covers $\tilde{C} \to \mathbb{P}^1$ having a divisor of the form $3(z_1 + z_2 + z_3)$ in a fibre, where $\tilde{C}$ is of genus 12. Then $H^{\text{triple}}_{12,9}$ is pure of dimension $\dim(\mathcal{M}_{12}) - 1 = 32$. Let $Y_1$ be the parameter space of pairs $(S, C')$, where $S \subseteq \mathbb{P}^6$ is a smooth complete intersection of 4 quadrics and $C' \subseteq S$ is an integral curve of arithmetic genus 15 with a triple point as described by (P1). Let 

$$\begin{align*}
S & \leftarrow \pi_1^{-1} Y_1 \xrightarrow{\pi_2} H^{\text{triple}}_{12,9} \\
\text{be the projections given by } \pi_1([S, C']):=[S] \text{ and } \pi_2([S, C']):=[C, \tilde{A}] \text{ respectively. With the notation above, from the adjunction formula } \nu^*(\mathcal{O}_{C'}(1)) = \mathcal{O}_{\tilde{C}}(-z_1 - z_2 - z_3). \text{ The fibre } \pi_2^{-1}([S, C']) \text{ corresponds then to the choice of a 7-dimensional space of sections } V \subseteq H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(-z_1 - z_2 - z_3)) \text{ satisfying } \dim(V \cap H^0(\mathcal{O}_{\tilde{C}}(-2z_1 - 2z_2 - 2z_3))) \geq 6.
\end{align*}$$

Since $h^0(\mathcal{O}_{\tilde{C}}(-2z_1 - 2z_2 - 2z_3)) = 6$, it follows that 

$$\frac{V}{H^0(\mathcal{O}_{\tilde{C}}(-2z_1 - 2z_2 - 2z_3))} \subseteq \mathbb{P}^2.$$

Therefore $\dim(Y_1) = \dim(\mathcal{H}^{\text{triple}}_{12,9}) + 2 = 34 \leq 39$, so we can invoke Proposition 7 to conclude that $\overline{\pi_1(Y_1)} \neq S$ and rule out possibility (P1).

Next we rule out possibility (P2), focusing on the case when each $C' \in \mathbb{P}_b$ is cuspidal at $z$. Passing to the normalization $\nu: \tilde{C} \to C'$, setting $\tilde{z} := \nu^{-1}(z)$ we obtain that $\tilde{A} := \nu^*(\mathcal{O}_{C'}(C')) \in W^3_3(\tilde{C})$ verifies $h^0(\mathcal{O}_{\tilde{C}}(\tilde{A}(-4\tilde{z}))) \geq 1$. Let $H^{\text{four}}_{14,9}$ be the Hurwitz space classifying degree 9 covers $\tilde{C} \to \mathbb{P}^1$ containing a divisor of type $4\tilde{z}$ in one of its fibres.
and where $C$ has genus 14. Then $\mathcal{H}_{14,9}^\text{four}$ is irreducible of dimension $39 = \dim(\mathcal{M}_{14})$. Let $\mathcal{V}_2$ be the parameter space of pairs $(S, C')$, where $S \subseteq \mathbb{P}^6$ is a smooth complete intersection of 4 quadrics and $C' \subseteq S$ is an integral curve of arithmetic genus 15 with a cusp at $z$ as described by (P2). We consider the projections

$$S \xleftarrow{\pi_1} \mathcal{V}_2 \xrightarrow{\pi_2} \mathcal{H}_{14,9}^\text{four}$$

given by $\pi_1([S, C']) := [S]$ and $\pi_2([S, C']) := [\bar{C}, \bar{A}]$ respectively. Observe that $\pi_2$ is birational onto its image. Indeed, given $[\bar{C}, \bar{A}] \in \pi_2(\mathcal{V}_2)$, then we denote by $C'$ the image of the map $\varphi_{\omega_C(2y) \otimes \mathcal{A}^v}: \mathcal{C} \to \mathbb{P}^6$, in which case the canonical surface $S$ is recovered by (3). We conclude by Proposition 7 again that $\pi_1(\mathcal{V}_2)$ is not dense in $S$. The final case when all curves $C' \in \mathbb{P}_6$ are (at least) nodal at $z$ is ruled out analogously. 

Before stating our next result, recall that one sets $\delta_i := [\Delta_i] \in CH^1(\mathcal{M}_g)$ for $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$. We denote as usual by $\lambda \in CH^1(\mathcal{M}_g)$ the Hodge class. Recall also the formula [HM] for the canonical class of $\mathcal{M}_g$:

$$K_{\mathcal{M}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor \frac{g}{2} \rfloor} \in CH^1(\mathcal{M}_g).$$  

**Proposition 10.** The rational curve $\Gamma$ is a sweeping pencil for the boundary divisor $\Delta_0$. Its intersection numbers with the standard generators of $CH^1(\mathcal{M}_{16})$ are as follows:

$$\Gamma \cdot \lambda = 22, \quad \Gamma \cdot \delta_0 = 143, \quad \Gamma \cdot \delta_j = 0 \text{ for } j = 2, \ldots, 8.$$

**Proof.** First we construct a fibration whose moduli map is precisely the rational curve $m: \mathbb{P}^1 \to \mathcal{M}_{15,2}$ considered in (12). We consider the curve $C \subseteq S$ and observe that since $\mathcal{O}_C(C) \cong A \subset W_3^1(C)$, we have that $C^2 = 9$, that is, the pencil $|\mathcal{I}_{(x,y)}(C)|$ has precisely 9 base points, namely $x, y$, as well as the 7 further points lying in the same fibre of the pencil $|A|$ as $x$ and $y$. We consider the blow-up surface $\epsilon: \tilde{S} = Bl_9(S) \to S$ at these 9 points. It comes equipped with a fibration

$$\pi: \tilde{S} \to \mathbb{P}^1,$$

as well as with two sections $E_x, E_y \subseteq \tilde{S}$ corresponding to the exceptional divisors at $x$ and $y$ respectively.

In order to compute the intersection numbers of $R = m(\mathbb{P})$ with the tautological classes on $\mathcal{M}_{15,2}$, we use for instance [Tan]. The subscript indicates the moduli space on which the intersection number is computed.

$$\Gamma \cdot \lambda |_{\mathcal{M}_{15,2}} = \chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) + g - 1 = h^2(S, \mathcal{O}_S) + g = h^0(S, \mathcal{O}_S(1)) + 15 = 22.$$  

Here we have used $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = H^1(S, \mathcal{O}_S) = 0$, as well as the fact that $S$ is a canonical surface, hence $\omega_S = \mathcal{O}_S(1)$, therefore $h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}) = h^2(S, \mathcal{O}_S) = 7$. Furthermore, recalling that all curves in the fibres of $m$ are irreducible, we find via [Tan] that

$$\Gamma \cdot \delta_0 |_{\mathcal{M}_{15,2}} = c_2(\tilde{S}) + 4(g - 1) = c_2(\tilde{S}) + 56.$$  

From the Euler formula, $c_2(\tilde{S}) = 12\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) - K_{\tilde{S}}^2$. We have already computed that $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 8$, whereas $K_{\tilde{S}}^2 = K_S^2 - 9 - \deg(S) - 9 = 7$, for $S$ is an intersection of 4 quadrics. Thus $c_2(\tilde{S}) = 12 \cdot 8 - 7 = 89$, leading to $(R \cdot \delta_0)_{\mathcal{M}_{15,2}} = 89 + 4 \cdot 14 = 145$. 


If we denote by $\psi_x, \psi_y \in CH^1(\overline{M}_{15,2})$ the cotangent classes corresponding to the marked points labelled by $x$ and $y$ respectively, we compute furthermore

$$R \cdot \psi_x = -E_x^2 = 1 \quad \text{and} \quad R \cdot \psi_y = -E_y^2 = 1.$$ 

We now pass to the pencil $\xi : \mathbb{P}^1 \rightarrow \overline{M}_{16}$ obtained from $m$ by identifying pointwise the disjoint sections $E_x$ and $E_y$ on the surface $\overline{S}$. First, using (15) we observe that

$$\Gamma \cdot \lambda = \xi(P) \cdot \lambda = (R \cdot \lambda)_{\overline{M}_{15,2}} = 22.$$ 

Furthermore, using Proposition 9 we conclude that $\Gamma \cdot \delta_i = 0$ for $i = 1, \ldots, 8$. Finally, invoking for instance [CR3, page 271], we find that

$$\Gamma \cdot \delta_0 = (R \cdot \delta_0)_{\overline{M}_{15,2}} = (R \cdot \psi_x)_{\overline{M}_{15,2}} = (R \cdot \psi_y)_{\overline{M}_{15,2}} = 145 - 2 = 143.$$ 

Proof of Theorem 2. Since the image of $m$ passes through a general point of $\overline{M}_{15,2}$, the rational curve $\Gamma \subseteq \overline{M}_{16}$ constructed in Proposition 10 is a sweeping curve for the boundary divisor $\Delta_0$. Using the expression (14) for the canonical divisor of $\overline{M}_{16}$, we compute $\Gamma : K_{\overline{M}_{16}} = 13 \Gamma : \lambda - 2 \Gamma : \delta_0 = 13 \cdot 22 - 2 \cdot 143 = 0$. Also $\Gamma : \Delta_0 = 143 > 0$.

3. The slope of $\overline{M}_{16}$.

The slope of an effective divisor $D$ on the moduli space $\overline{M}_g$ not containing any boundary divisor $\Delta_i$ in its support is defined as the quantity $s(D) := \frac{a}{\min_{i} \lambda_i}$, where $[D] = a \lambda - \sum_{i=0}^{m} b_i \delta_i \in CH^1(\overline{M}_g)$, with $a, b_i \geq 0$. Then the slope $s(\overline{M}_g)$ of the moduli space $\overline{M}_g$ is defined as the infimum of the slopes $s(D)$ over such effective divisors $D$.

Corollary 11. We have that $s(\overline{M}_{16}) \geq \frac{13}{2}$.

Proof. For any effective divisor $D$ on $\overline{M}_{16}$ containing no boundary divisor in its support, we may assume that the curve $\Gamma$ constructed in Proposition 10 does not lie inside $D$, hence $\Gamma : D \geq 0$. Writing $[D] = a \lambda - \sum_{i=0}^{m} b_i \delta_i$, using Theorem 2 we obtain $\frac{a}{b_i} \geq \frac{13}{15 \lambda_1} = \frac{13}{2}$. Furthermore, using [FP, Theorem 1.4], we conclude that for this divisor $D$ we have $b_i \geq b_0$ for $i = 1, \ldots, 8$, that is, $s(D) = \frac{a}{b_0}$.

Final remarks: Our results establish that $\overline{M}_{16}$ is not of general type. Showing that the Kodaira dimension of $\overline{M}_{16}$ is non-negative amounts to constructing an effective divisor $D$ on $\overline{M}_{16}$ having slope $s(D) \leq s(K_{\overline{M}_{16}}) = \frac{13}{2}$. Currently the known effective divisor on $\overline{M}_{16}$ of smallest slope is the closure in $\overline{M}_{16}$ of the Kaszul divisor $Z_{16}$ consisting of curves $C$ having a linear system $L \in W_2^1(C)$ such that the image curve $\varphi_L : C \hookrightarrow \mathbb{P}^6$ is ideal-theoretically not cut out by quadrics. It is shown in [F1, Theorem 1.1] that $Z_{16}$ is an effective divisor on $\overline{M}_{16}$ and $s(Z_{16}) = \frac{407}{11} = 6.705$. In a related direction, it is shown in [F2] that the canonical class of the space of admissible covers $\overline{H}_{16,9}$ is effective. Note that one has a generically finite cover $\overline{H}_{16,9} \rightarrow \overline{M}_{16}$.

Soon after the appearance of the first version of this paper, it has been pointed out by Agostini and Barros [AB] that our proof of Theorem 2 yields in fact the bound $\kappa(\overline{M}_{16}) \leq \dim(\overline{M}_{16}) - 2$. Indeed, consider the parameter space $\mathcal{Z}$ of elements $[C, A, x, y]$, where $C$ is a genus 15 irreducible nodal curve, $A \in W_9^1(C)$ and $x, y \in C$ are points such
that $|A(-x - y)| \neq 0$. As we explain in this paper, $Z$ has the structure of a $\mathbb{P}^1$-bundle and one has a dominant morphism $v: Z \to \Delta_0$ given by $[C, A, x, y] \mapsto [C/x \sim y]$. In Proposition 10 we establish that the restriction of $v^* (K_{\overline{M}_{16}})$ to the general fibre of this fibration is trivial. Accordingly, $\kappa(\overline{M}_{16}) \leq \dim(Z) - 1 = \dim(\overline{M}_{16}) - 2$.

REFERENCES


[FP] G. Farkas and M. Popa, Effective divisors on $\overline{M}_g$, curves on K3 surfaces and the Slope Conjecture, Journal of Algebraic Geometry 14 (2005), 151–174.


HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, UNTER DEN LINDEN 6 10099 BERLIN, GERMANY
Email address: farkas@math.hu-berlin.de

UNIVERSITÀ ROMA TRE, DIPARTIMENTO DI MATEMATICA, LARGO SAN LEONARDO MURIALDO 1-00146 ROMA, ITALY
Email address: verra@mat.uniroma3.it