THE GEOMETRY OF THE MODULI SPACE OF ODD SPIN CURVES

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The set of odd theta-characteristics on a general curve C of genus g is in bijection with the set $\theta(C)$ of *theta hyperplanes* $H \in (\mathbf{P}^{g-1})^\vee$ everywhere tangent to the canonically embedded curve $C \stackrel{|K_C|}{\hookrightarrow} \mathbf{P}^{g-1}$. Even though the geometry and the intricate combinatorics of $\theta(C)$ have been studied classically, see [Dol], [DK] for a modern account, it was only recently proved in [CS] that one can reconstruct a general curve $[C] \in \mathcal{M}_g$ from the hyperplane configuration $\theta(C)$.

Odd theta-characteristics form a moduli space $\pi : \mathcal{S}_g^- \rightarrow \mathcal{M}_g$ which is an étale cover of degree $2^{g-1}(2^g-1)$. The normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{S}_g^-$ gives rise to a finite covering $\pi : \overline{\mathcal{S}}_g^- \rightarrow \overline{\mathcal{M}}_g$. Furthermore, $\overline{\mathcal{S}}_g^-$ has a modular meaning being isomorphic to the coarse moduli space of the Deligne-Mumford stack of odd stable spin curves, cf. [C], [CCC], [AJ]. The map π is branched along the boundary of $\overline{\mathcal{M}}_g$ and one expects $K_{\overline{\mathcal{S}}_g^-}$ to enjoy better positivity properties than $K_{\overline{\mathcal{M}}_g}$.

The aim of this paper is to describe the birational geometry of $\overline{\mathcal{S}}_g^-$ for all g. Our goals are (1) to understand the transition from rationality to maximal Kodaira dimension for $\overline{\mathcal{S}}_g^-$ as g increases, and (2) to use the existence of *Mukai models* of $\overline{\mathcal{M}}_g$ in order to construct explicit unirational parameterizations of $\overline{\mathcal{S}}_g^-$ for small genus. Remarkably, we end up having no gaps in the classification of $\overline{\mathcal{S}}_g^-$. First, we show that in the range where the general curve $[C] \in \mathcal{M}_g$ lies on a $K3$ surface, the existence of special *theta pencils* on $K3$ surfaces, provides an *explicit* uniruled parameterization of $\overline{\mathcal{S}}_g^-$:

Theorem 0.1. *The odd spin moduli space $\overline{\mathcal{S}}_g^-$ is uniruled for $g \leq 11$.*

When $g \leq 9$ or $g = 11$, a general spin curve $[C, \eta] \in \overline{\mathcal{S}}_g^-$ appears as a hyperplane section of a $K3$ surface $X \subset \mathbf{P}^g$, such that if $d := \mathrm{supp}(\eta)$ is the support of the theta-characteristic, then the linear span $\langle d \rangle \subset \mathbf{P}^g$ is a codimension 2 linear subspace. A rational curve $P \subset \overline{\mathcal{S}}_g^-$ is induced by the pencil of hyperplanes $\mathbf{P}H^0(X, \mathcal{I}_{d/X}(C))$ containing $\langle d \rangle$. We show in Section 3 that $P \subset \overline{\mathcal{S}}_g^-$ is a *covering* rational curve, satisfying

$$P \cdot K_{\overline{\mathcal{S}}_g^-} = 2g - 24 < 0.$$

Thus $P \cdot K_{\overline{\mathcal{S}}_g^-} < 0$ precisely when $g \leq 11$, which highlights the fact that the nature of $\overline{\mathcal{S}}_g^-$ is expected to change exactly when $g \geq 12$. This is something we shall achieve in the course of proving Theorem 0.3.

The previous argument no longer works for $\overline{\mathcal{S}}_{10}^-$, when the condition that a curve $[C] \in \overline{\mathcal{M}}_{10}$ lie on a $K3$ surface is divisorial [FP]. This case is in some sense a specialization of the genus 11 case. We use that a general 1-nodal irreducible curve $[C] \in \Delta_0 \subset \overline{\mathcal{M}}_{11}$ of arithmetic genus 11, lies on a $K3$ surface $X \subset \mathbf{P}^{11}$. By a degeneration argument, we show that this construction can be also carried out in such a way, that if $\nu : C' \to C$ denotes the normalization of C, then the points $x, y \in C'$ with $\nu(x) = \nu(y)$ (that is, mapping to the node of C), lie in the support of one of the odd-theta characteristics of $[C'] \in \mathcal{M}_{10}$. Ultimately, this produces a rational curve $P \subset \overline{\mathcal{S}}_{10}^-$ through a general point, which shows that $\overline{\mathcal{S}}_{10}^-$ is uniruled as well.

In the range in which a *Mukai model* of $\overline{\mathcal{M}}_g$ exists, our results are more precise:

Theorem 0.2. *$\overline{\mathcal{S}}_g^-$ is unirational for $g \leq 8$.*

The proof relies on the existence, in this range, of *Mukai varieties* $V_g \subset \mathbf{P}^{n_g+g-2}$, where $n_g = \dim(V_g)$, which have the property that general 1-dimensional linear sections of V_g are canonical curves $[C] \in \mathcal{M}_g$ with general moduli. We fix an integer $1 \leq \delta \leq g-1$ and consider the correspondence

$$\mathcal{P}_{g,\delta}^o := \{(C, \Gamma, Z) : Z \subset C \cap \Gamma \subset V_g, \ \ |\mathrm{sing}(\Gamma)| = \delta, \ \mathrm{sing}(\Gamma) \subset Z\},$$

where $Z \subset V_g$ is a 0-dimensional subscheme of V_g of length $2g-2$, supported at $g-1$ points and such that $\dim\langle Z\rangle = g-2$ (see Section 4 for a precise definition), $\Gamma \subset V_g$ is an irreducible δ-nodal curve section of V_g whose nodes are among the points in the support of Z, and $C \subset V_g$ is an arbitrary curve linear section of V_g containing Z as a subscheme. Thus if C is smooth, then $Z \subset C$ is a divisor of even degree at each point in its support, and $\mathcal{O}_C(Z/2)$ can be viewed as a theta-characteristic. The variety $\mathcal{P}_{g,\delta}^o$ comes equipped with two projections

$$\overline{\mathcal{S}}_g^- \stackrel{\alpha}{\longleftarrow} \mathcal{P}_{g,\delta}^o \stackrel{\beta}{\longrightarrow} B_{g,\delta}^-,$$

where $B_{g,\delta}^- \subset \overline{\mathcal{S}}_g^-$ denotes the moduli space of irreducible δ-nodal curves of arithmetic genus g together with an odd theta- characteristic on the normalization. It is easy to see that $\mathcal{P}_{g,\delta}^o$ is birational to a projective bundle over the irreducible variety $B_{g,\delta}^-$. Thus the unirationality of $\overline{\mathcal{S}}_g^-$ follows once we prove that (i) α is dominant, and (ii) $B_{g,\delta}^-$ itself is unirational. We carry out this program when $g \leq 8$. In the process of proving Theorem 0.2, we establish some facts of independent interest concerning the Mukai models

$$\mathfrak{M}_g := G(g, n_g + g - 1)^{\mathrm{ss}} // \mathrm{Aut}(V_g).$$

These are birational models of $\overline{\mathcal{M}}_g$ having $\mathrm{Pic}(\mathfrak{M}_g) = \mathbb{Z}$ and appearing as GIT quotients of Grassmannians; they can be viewed as log-minimal models of $\overline{\mathcal{M}}_g$ emerging from the constructions carried out in [M1], [M2], [M3].

Theorem 0.1 is sharp and the remaining moduli spaces $\overline{\mathcal{S}}_g^-$ are of general type:

Theorem 0.3. *The space $\overline{\mathcal{S}}_g^-$ is a variety of general type for $g > 11$.*

The border case of $\overline{\mathcal{S}}_{12}^-$ is particularly challenging and takes up the entire Section 6. We remark that in the range $11 < g < 17$, of the two moduli spaces $\overline{\mathcal{S}}_g^-$ and $\overline{\mathcal{M}}_g$, one is of general type whereas the other has negative Kodaira dimension. More strikingly, Theorems 0.3 and 0.1 coupled with results from [F3], show that for $9 \leq g \leq 11$, the space $\overline{\mathcal{S}}_g^-$ is uniruled while $\overline{\mathcal{S}}_g^+$ is of general type! Finally, we note that $\overline{\mathcal{S}}_8^-$ is unirational whereas $\overline{\mathcal{S}}_8^+$ is of Calabi-Yau type [FV].

We describe the main steps in the proof of Theorem 0.3. First, we use that for all $g \geq 4$ and $l \geq 0$, if $\epsilon : \widehat{\mathcal{S}}_g \to \overline{\mathcal{S}}_g^-$ denotes a resolution of singularities, then there is an induced isomorphism, see [Lud]

$$\epsilon^* : H^0(\overline{\mathcal{S}}^-_{g,\mathrm{reg}}, K_{\overline{\mathcal{S}}_g^-}^{\otimes l}) \xrightarrow{\sim} H^0(\widehat{\mathcal{S}}_g, K_{\widehat{\mathcal{S}}_g}^{\otimes l}).$$

Thus to conclude that $\overline{\mathcal{S}}_g^-$ is of general type, it suffices to exhibit an effective divisor D on $\overline{\mathcal{S}}_g^-$ such that for appropriately chosen rational constants $\alpha, \beta > 0$, a relation of the type $K_{\overline{\mathcal{S}}_g^-} \equiv \alpha\, \lambda + \beta\, D + E \in \mathrm{Pic}(\overline{\mathcal{S}}_g^-)$ holds, where $\lambda \in \mathrm{Pic}(\overline{\mathcal{S}}_g^-)$ is the pull-back to $\overline{\mathcal{S}}_g^-$ of the Hodge class, and E is an effective $\mathbb{Q}$-class which is typically a combination of boundary divisors. It is essential to pick D so that (1) its class can be explicitly computed, that is, points in D have good geometric characterization, and (2) $[D] \in \mathrm{Pic}(\overline{\mathcal{S}}_g^-)$ is in some way an extremal point of the effective cone of divisors so that the coefficients α, β stand a chance of being positive. In the case of $\overline{\mathcal{S}}_g^+$, the role of D is played by the divisor $\overline{\Theta}_{\mathrm{null}}$ of vanishing theta-nulls, see [F3]. In the case of $\overline{\mathcal{S}}_g^-$ we compute the class of *degenerate theta-characteristics*, that is, curves carrying a non-reduced odd theta-characteristic.

Theorem 0.4. *We fix $g \geq 3$. The locus consisting of odd spin curves*

$$\mathcal{Z}_g := \{[C, \eta] \in \mathcal{S}_g^- : \eta = \mathcal{O}_C(2x_1 + x_2 + \cdots + x_{g-2}) \text{ where } x_i \in C \text{ for } i = 1, \ldots, g-2\}$$

is a divisor on $\mathcal{S}_g^-$. The class of its compactification inside $\overline{\mathcal{S}}_g^-$ equals

$$\overline{\mathcal{Z}}_g \equiv (g+8)\lambda - \frac{g+2}{4}\alpha_0 - 2\beta_0 - \sum_{i=1}^{[g/2]} 2(g-i)\, \alpha_i - \sum_{i=1}^{[g/2]} 2i\, \beta_i \in \mathrm{Pic}(\overline{\mathcal{S}}_g^-),$$

where $\lambda, \alpha_0, \beta_0, \ldots, \alpha_{[g/2]}, \beta_{[g/2]}$ are the standard generators of $\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$.

For low genus, $\mathcal{Z}_g$ specializes to well-known geometric loci. For instance $\overline{\mathcal{Z}}_3$ is the divisor of hyperflexes on plane quartics. In particular, Theorem 0.4 yields the formula

$$\pi_*(\overline{\mathcal{Z}}_3) = 308\lambda - 32\delta_0 - 76\delta_1 \in \mathrm{Pic}(\overline{\mathcal{M}}_3),$$

for the class of quartic curves having a hyperflex. This matches [Cu] formula (5.5). Moreover, one has the following relation in $\mathrm{Pic}(\overline{\mathcal{M}}_3)$

$$\big[\{[C] \in \mathcal{M}_3 : \exists x \in C \text{ with } 4x \equiv K_C\}\big]^- \equiv 8 \cdot \overline{\mathcal{M}}^1_{3,2} + \pi_*(\overline{\mathcal{Z}}_3),$$

where $\overline{\mathcal{M}}^1_{3,2} \equiv 9\lambda - \delta_0 - 3\delta_1$ is the hyperelliptic class and the multiplicity 8 accounts for the number of hyperelliptic Weierstrass points.

We briefly explain how Theorem 0.4 implies that $\overline{\mathcal{S}}_g^-$ is of general type for $g > 11$. We choose an effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ of small slope; for composite $g+1$ one can take $D = \overline{\mathcal{M}}^r_{g,d}$ the closure of the Brill-Noether divisor of curves with a $\mathfrak{g}^r_d$, where $\rho(g, r, d) = -1$; there exists a constant $c_{g,d,r} > 0$ such that [EH2],

$$\overline{\mathcal{M}}^r_{g,d} \equiv c_{g,d,r}\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i\Big) \in \mathrm{Pic}(\overline{\mathcal{M}}_g).$$

We form the linear combination of divisors on $\overline{\mathcal{S}}_g^-$

$$\frac{2}{g-2}\overline{\mathcal{Z}}_g+\frac{3(3g-10)}{c_{g,d,r}(g-2)(g+1)}\pi^*(\overline{\mathcal{M}}^r_{g,d})=\frac{11g+37}{g+1}\lambda-2\alpha_0-3\beta_0-\sum_{i=1}^{[g/2]}(a_i\cdot\alpha_i+b_i\cdot\beta_i),$$

where $a_i, b_i \geq 2$ for $i \neq 1$ and $a_1, b_1 > 3$ are explicitly known rational constants. The canonical class of $\overline{\mathcal{S}}_g^-$ is given by the Riemann-Hurwitz formula

$$K_{\overline{\mathcal{S}}_g^-}\equiv\pi^*(K_{\overline{\mathcal{M}}_g})+\beta_0\equiv 13\lambda-2\alpha_0-3\beta_0-2\sum_{i=1}^{[g/2]}(\alpha_i+\beta_i)-(\alpha_1+\beta_1),$$

and by comparison, it follows that for $g > 12$ one can find a constant $\mu_g \in \mathbb{Q}_{>0}$ such that

$$K_{\overline{\mathcal{S}}_g^-}-\mu_g\cdot\lambda\in\mathbb{Q}_{\geq 0}\langle[\overline{\mathcal{Z}}_g],\alpha_1,\ \beta_1,\ldots,\alpha_{[g/2]},\beta_{[g/2]}\rangle,$$

which shows that $K_{\overline{\mathcal{S}}_g^-}$ is big and thus proves Theorem 0.3.

For $g = 12$, there is no Brill-Noether divisor, and the reasoning above shows that in order to conclude that $\overline{\mathcal{S}}_{12}^-$ is of general type, one needs an effective divisor $\overline{\mathfrak{D}}_{12}$ of slope $s(\overline{\mathfrak{D}}_{12}) < 6 + 12/13$, that is, a counterexample to the Slope Conjecture. We define

$$\mathfrak{D}_{12}:=\{[C]\in\mathcal{M}_{12}:\exists L\in W^4_{14}(C)\text{ with }\operatorname{Sym}^2H^0(C,L)\stackrel{\mu_0(L)}{\longrightarrow}H^0(C,L^{\otimes 2})\text{ not injective}\},$$

that is, points in $\mathfrak{D}_{12}$ correspond to curves that admit an embedding $C \subset \mathbf{P}^4$ with $\deg(C) = 14$, such that $H^0(\mathbf{P}^4, \mathcal{I}_{C/\mathbf{P}^4}(2)) \neq 0$. The computation of the class of $\overline{\mathfrak{D}}_{12} \subset \overline{\mathcal{M}}_{12}$ is carried out in Section 6 and it turns out that $s(\overline{\mathfrak{D}}_{12}) = \frac{4415}{642} < 6 + \frac{12}{13}$. In particular $\mathfrak{D}_{12}$ violates the Slope Conjecture on $\overline{\mathcal{M}}_{12}$, and as such, it contains the locus $\mathcal{K}_{12} := \{[C] \in \mathcal{M}_{12} : C \text{ lies on a } K3 \text{ surface}\}$.

1. Families of stable spin curves

We briefly review some relevant facts about the moduli space $\overline{\mathcal{S}}_g^-$ that will be used throughout the paper, see also [C], [F3], [Lud] for details. As a matter of notation, we follow the convention set in [FL]; if **M** is a Deligne-Mumford stack, then we denote by $\mathcal{M}$ its associated coarse moduli space.

Following [C], a *spin curve* of genus g consists of a triple (X, η, β), where X is a genus g quasi-stable curve, $\eta \in \operatorname{Pic}^{g-1}(X)$ is a line bundle of degree $g - 1$ such that $\eta_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$, and $\beta : \eta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X.

It follows from the definition that if (X, η, β) is a spin curve with exceptional components $E_1, \ldots, E_r$ and $\{p_i, q_i\} = E_i \cap \overline{X - E_i}$ for $i = 1, \ldots, r$, then $\beta_{|E_i} = 0$. Moreover, if $\tilde{X} := \overline{X - \bigcup_{i=1}^r E_i}$ (viewed as a subcurve of X), then we have an isomorphism of sheaves $\eta_{\tilde{X}}^{\otimes 2} \stackrel{\sim}{\to} \omega_{\tilde{X}}$.

We denote by $\overline{\mathbf{S}}_g$ the non-singular Deligne-Mumford stack of spin curves of genus g, which obviously splits into two connected components $\overline{\mathbf{S}}_g^+$ and $\overline{\mathbf{S}}_g^-$ of relative degree $2^{g-1}(2^g + 1)$ and $2^{g-1}(2^g - 1)$ respectively. It is proved in [C] that the coarse moduli space of $\overline{\mathbf{S}}_g$ is isomorphic to the normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{S}_g$. There is a proper morphism $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ given by $\pi([X,\eta,\beta]) := [\mathrm{st}(X)]$, where $\mathrm{st}(X)$ denotes the stable model of the nodal curve X.

1.1. **Spin curves of compact type.** We recall the description of the pull-back divisors $\pi^*(\Delta_i)$. We choose a spin curve $[X,\eta,\beta] \in \pi^{-1}([C\cup_y D])$ where $[C,y]\in\mathcal{M}_{i,1}$ and $[D,y]\in\mathcal{M}_{g-i,1}$. Then necessarily $X := C\cup_{y_1} E\cup_{y_2} D$, where E is an exceptional component such that $C\cap E=\{y_1\}$ and $D\cap E=\{y_2\}$. Moreover $\eta = \big(\eta_C,\eta_D,\eta_E = \mathcal{O}_E(1)\big)\in \mathrm{Pic}^{g-1}(X)$, and since $\beta_E=0$, it follows that $\eta_C^{\otimes 2}=K_C, \eta_D^{\otimes 2}=K_D$, that is, η_C and η_D are "honest" theta-characteristics on C and D respectively. The condition $h^0(X,\eta)\equiv 1 \bmod 2$ implies that η_C and η_D must have opposite parities. We denote by $A_i\subset\overline{\mathcal{S}}_g^-$ the closure in $\overline{\mathcal{S}}_g^-$ of the locus corresponding to pairs

$$([C,\eta_C,y],[D,\eta_D,y])\in\mathcal{S}_{i,1}^-\times\mathcal{S}_{g-i,1}^+,$$

and by $B_i\subset\overline{\mathcal{S}}_g^-$ the closure in $\overline{\mathcal{S}}_g^-$ of the locus corresponding to pairs

$$([C,\eta_C,y],[D,\eta_D,y])\in\mathcal{S}_{i,1}^+\times\mathcal{S}_{g-i,1}^-.$$

One has the relation $\pi^*(\Delta_i)=A_i+B_i$ and clearly $\deg(A_i/\Delta_i)=2^{g-2}(2^i-1)(2^{g-i}+1)$ and $\deg(B_i/\Delta_i)=2^{g-2}(2^i+1)(2^{g-i}-1)$. One denotes $\alpha_i:=[A_i],\beta_i:=[B_i]\in\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$.

1.2. **Spin curves with an irreducible stable model.** In order to describe $\pi^*(\Delta_0)$ we pick a point $[X,\eta,\beta]$ such that $\mathrm{st}(X)=C_{yq}:=C/y\sim q$, where $[C,y,q]\in\mathcal{M}_{g-1,2}$ is a general point of Δ_0. Unlike the case of curves of compact type, here there are two possibilities depending on whether X possesses an exceptional component or not. If $X=C_{yq}$ and $\eta_C:=\nu^*(\eta)$ where $\nu: C\to X$ denotes the normalization map, then $\eta_C^{\otimes 2}=K_C(y+q)$. For each choice of $\eta_C\in\mathrm{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X,\eta)\equiv 1 \bmod 2$. We denote by A_0 the closure in $\overline{\mathcal{S}}_g^-$ of the locus of those points $[C_{yq},\eta_C\in\sqrt{K_C(y+q)}]$ with $\eta_C(y)$ and $\eta_C(q)$ glued as above. One has that $\deg(A_0/\Delta_0)=2^{2g-2}$.

If $X=C\cup_{\{y,q\}}E$ where E is an exceptional component, then since $\beta_{|E}=0$ it follows that $\beta_{|C}\in H^0(C,\omega_{X|C}\otimes\eta_C^{\otimes(-2)})$ must vanish at both y and q and then for degree reasons $\eta_C:=\eta\otimes\mathcal{O}_C$ is a theta-characteristic on C. The condition $H^0(X,\omega)\cong H^0(C,\omega_C)\equiv 1 \bmod 2$ implies that $[C,\eta_C]\in\mathcal{S}_{g-1}^-$. In an étale neighborhood of a point $[X,\eta,\beta]$, the covering π is given by

$$(\tau_1,\tau_2,\ldots,\tau_{3g-3})\mapsto(\tau_1^2,\tau_2,\ldots,\tau_{3g-3}),$$

where one identifies $\mathbb{C}_\tau^{3g-3}$ with the versal deformation space of (X,η,β) and the hyperplane $(\tau_1=0)\subset\mathbb{C}_\tau^{3g-3}$ denotes the locus of spin curves where the exceptional component E persists. This discussion shows that π is simply branched over Δ_0 and we denote the ramification divisor by $B_0\subset\overline{\mathcal{S}}_g^-$, that is, the closure of the locus of spin curves $[C\cup_{\{y,q\}}E,(C,\eta_C)\in\mathcal{S}_{g-1}^-,\ \eta_E=\mathcal{O}_E(1)]$. If $\alpha_0=[A_0]\in\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$ and $\beta_0=[B_0]\in\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$, we then have the relation

$$\pi^*(\delta_0)=\alpha_0+2\beta_0. \tag{1}$$

We define several test curves in the boundary of $\overline{\mathcal{S}}_g^-$ which will be later used to compute divisor classes on the moduli space.

1.3. **The family F_i.** We fix $1 \leq i \leq [g/2]$ and construct a covering family for the boundary divisor A_i. We fix general curves $[C] \in \mathcal{M}_i$ and $[D,q] \in \mathcal{M}_{g-i,1}$ as well as an odd theta-characteristic η_C^- on C and an even theta-characteristic η_D^+ on D. If $E \cong \mathbf{P}^1$ is a fixed exceptional component, we define the family of spin curves

$$F_i := \bigl\{[C \cup_y \cup E \cup_q D, \eta] : \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^+, E \cap C = \{y\}, E \cap D = \{q\}\bigr\}_{y \in C}.$$

One has that $F_i \cdot \beta_i = 0$ and then $F_i \cdot \alpha_i = -2i + 2$; furthermore F_i has intersection number zero with the remaining generators of $\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$.

1.4. **The family G_i.** As above, we fix $1 \leq i \leq [g/2]$ and curves $[C] \in \mathcal{M}_i, [D,q] \in \mathcal{M}_{g-i,1}$. This time we choose an even theta-characteristic η_C^+ on C and an odd theta-characteristic η_D^- on D. The following family covers the divisor B_i:

$$G_i := \bigl\{[C \cup_y \cup E \cup_q D, \eta] : \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^-, E \cap C = \{y\}, E \cap D = \{q\}\bigr\}_{y \in C}.$$

Clearly $G_i \cdot \alpha_i = 0$, $G_i \cdot \beta_i = 2 - 2i$ and $G_i \cdot \lambda = G_i \cdot \alpha_j = G_i \cdot \beta_j = 0$ for $j \neq i$.

1.5. **Two elliptic pencils.** The boundary divisor $\Delta_1 \subset \overline{\mathcal{M}}_g$ is covered by a standard elliptic pencil R obtained by attaching to a fixed general pointed curve $[C, y] \in \mathcal{M}_{g-1,1}$ a pencil of plane cubic curves $\{E_\lambda = f^{-1}(\lambda)\}_{\lambda \in \mathbf{P}^1}$ where $f : \mathrm{Bl}_9(\mathbf{P}^2) \to \mathbf{P}^1$. The points of attachment on the elliptic pencil are given by a section $\sigma : \mathbf{P}^1 \to \mathrm{Bl}_9(\mathbf{P}^2)$ given by one of the base points of the pencil of cubics. We lift this pencil in two possible ways to the space $\overline{\mathcal{S}}_g^-$, depending on the parity of the theta-characteristic on the varying elliptic tail. We fix an even theta-characteristic $\eta_C^+ \in \mathrm{Pic}^{g-2}(C)$ and $E \cong \mathbf{P}^1$ will again denote an exceptional component. We define the family

$$F_0 := \{[C \cup_q E \cup_{\sigma(\lambda)} f^{-1}(\lambda),\ \eta_C = \eta_C^+,\ \eta_E = \mathcal{O}_E(1),\ \eta_{f^{-1}(\lambda)} = \mathcal{O}_{f^{-1}(\lambda)}] : \lambda \in \mathbf{P}^1\} \subset \overline{\mathcal{S}}_g^-.$$

Since $F_0 \cap B_1 = \emptyset$, we find that $F_0 \cdot \alpha_1 = \pi_*(F_0) \cdot \delta_1 = -1$. Similarly, $F_0 \cdot \lambda = \pi_*(F_0) \cdot \lambda = 1$ and obviously $F_0 \cdot \alpha_i = F_0 \cdot \beta_i = 0$ for $2 \leq i \leq [g/2]$. For each of the 12 points $\lambda_\infty \in \mathbf{P}^1$ corresponding to singular fibres of R, the associated $\eta_{\lambda_\infty} \in \overline{\mathrm{Pic}}^{g-1}(C \cup E \cup f^{-1}(\lambda_\infty))$ are actual line bundles on $C \cup E \cup f^{-1}(\lambda_\infty)$, that is, we do not have to blow-up the extra node. Thus we obtain that $F_0 \cdot \beta_0 = 0$ and then $F_0 \cdot \alpha_0 = \pi_*(F_0) \cdot \delta_0 = 12$.

A second lift of the elliptic pencil to $\overline{\mathcal{S}}_g^-$ is obtained by choosing an odd theta-characteristic $\eta_C^- \in \mathrm{Pic}^{g-2}(C)$ whereas on E_λ one takes each of the 3 possible even theta-characteristics, that is,

$$G_0 := \bigl\{\bigl[C \cup_q E \cup_{\sigma(\lambda)} f^{-1}(\lambda),\ \eta_C = \eta_C^-,\ \eta_E = \mathcal{O}_E(1), \eta_{f^{-1}(\lambda)} \in \gamma^{-1}[f^{-1}(\lambda)]\bigr] : \lambda \in \mathbf{P}^1\bigr\} \subset \overline{\mathcal{S}}_g^-,$$

where $\gamma : \overline{\mathcal{S}}_{1,1}^+ \to \overline{\mathcal{M}}_{1,1}$ is the projection of degree 3. Since $\pi_*(G_0) = 3R \subset \Delta_1$, we obtain that $G_0 \cdot \lambda = 3$. Obviously $G_0 \cdot \alpha_1 = 0$, hence $G_0 \cdot \beta_1 = \pi_*(G_0) \cdot \delta_1 = -3$. The map $\gamma : \overline{\mathcal{S}}_{1,1}^+ \to \overline{\mathcal{M}}_{1,1}$ is simply ramified over the point corresponding to j-invariant ∞. Hence, $G_0 \cdot \alpha_0 = 12$ and $G_0 \cdot \beta_0 = 12$.

1.6. **A covering family in B_0.** We start with a general pointed spin curve $[C, q, \eta_C^-] \in \mathcal{S}_{g-1,1}^-$ and as usual $E \cong \mathbf{P}^1$ denotes an exceptional component. We construct a family of spin curves $H_0 \subset B_0$ with general member

$$\bigl[C \cup_{\{y,q\}} E,\ \eta_C = \eta_C^-,\ \eta_E = \mathcal{O}_E(1)\bigr]_{y \in C} \subset \overline{\mathcal{S}}_g^-$$

and with special fibre corresponding to $y = q$ being the odd spin curve with support

$$C \cup_q E' \cup_{q'} E_2 \cup_{\{y_2, q_2\}} E,$$

where E' and E_2 are both smooth rational curves and $y_2, q_2 \in E$, $E_2 \cap E = \{y_2, q_2\}$, while $E_2 \cap E' = \{q'\}$. The stable model of this curve is $C \cup_q \left(\frac{E_2}{y_2 \sim q_2}\right)$, having an elliptic tail of j-invariant ∞. The underlying line bundle $\eta \in \mathrm{Pic}^{g-1}(C \cup E' \cup E_2 \cup E)$ satisfies $\eta_C = \eta_C^-, \eta_{E'} = \mathcal{O}_{E'}(1), \eta_E = \mathcal{O}_E(1)$ and, for degree reasons, $\eta_{E_2} = \mathcal{O}_{E_2}(-1)$. We have the following relations for the numerical parameters of H_0:

$$H_0 \cdot \lambda = 0,\ H_0 \cdot \beta_0 = 1 - g,\ H_0 \cdot \alpha_0 = 0,\ H_0 \cdot \beta_1 = 1,\ H_0 \cdot \alpha_1 = 0.$$

(The only non-trivial calculation here uses that $H_0 \cdot \beta_0 = \pi_*(H_0) \cdot \delta_0 / 2 = 1 - g$, cf. [HM]).

2. The Scorza curve

Here we study in detail the correspondence $T_\eta \subset C \times C$ associated to each (non-vanishing) theta-characteristic $[C, \eta] \in \mathcal{S}_g^+ - \Theta_{\mathrm{null}}$. This correspondence was used by G. Scorza [Sc] to provide a birational isomorphism between $\mathcal{M}_3$ and $\mathcal{S}_3^+$ (see also [DK]), and recently in [TZ] where several conditional statements of Scorza's have been rigourously established.

For a fixed theta-characteristic $[C, \eta] \in \mathcal{S}_g^+ - \Theta_{\mathrm{null}}$, we define the curve

$$T_\eta := \{(x, y) \in C \times C : H^0(C, \eta \otimes \mathcal{O}_C(x - y)) \neq 0\}.$$

By Riemann-Roch, it follows that T_η is a symmetric correspondence which misses the diagonal $\Delta \subset C \times C$. The curve T_η has a natural fixed point free involution and we denote by $f : T_\eta \to \Gamma_\eta$ the associated étale double covering. Under the assumption that T_η is a reduced curve, its class is computed in [DK] Proposition 7.1.5:

$$T_\eta \equiv (g - 1)F_1 + (g - 1)F_2 + \Delta.$$

Theorem 2.1. *For a general theta-characteristic $[C, \eta] \in \mathcal{S}_g^+$, the Scorza curve T_η is a smooth curve of genus $g(T_\eta) = 3g(g - 1) + 1$.*

Proof. It is straightforward to show that a point $(x, y) \in T_\eta$ is singular if and only if

$$H^0(C, \eta \otimes \mathcal{O}_C(x - 2y)) \neq 0 \text{ and } H^0(C, \eta \otimes \mathcal{O}_C(y - 2x)) \neq 0. \tag{2}$$

By induction on g, we show that for a general even spin curve such a pair (x, y) cannot exist. We assume the result holds for a general $[C, \eta_C] \in \mathcal{S}_{g-1}^+$. We fix a general point $q \in C$, an elliptic curve D together with $\eta_D \in \mathrm{Pic}^0(D) - \{\mathcal{O}_D\}$ with $\eta_D^{\otimes 2} = \mathcal{O}_D$ and consider the spin curve $t := [C \cup E \cup D, \eta_{|C} = \eta_C,\ \eta_{|E} = \mathcal{O}_E(1),\ \eta_{|D} = \eta_D] \in \overline{\mathcal{S}}_g^+$, obtained from $C \cup_q D$ by inserting an exceptional component E. Since the exceptional component plays no further role in the proof, we are going to suppress it.

We assume by contradiction that $t \in \overline{\mathcal{S}}_g^+$ lies in the closure of the locus of spin curves with singular Scorza curve. Then there exists a nodal curve $C \cup_q D'$ semistably equivalent to $C \cup_q D$ obtained by inserting a possibly empty chain on $\mathbf{P}^1$'s at the node q (therefore, $p_a(D') = 1$ and we may regard D as a subcurve of D'), as well as smooth points $x, y \in C \cup D'$ together with two limit linear series $\sigma = \{\sigma_C, \sigma_{D'}\}$ and $\tau = \{\tau_C, \tau_{D'}\}$ of type $\mathfrak{g}_{g-2}^0$ on $C \cup D'$ such that the underlying line bundles corresponding to σ (resp. τ) are uniquely determined twists at the nodes of the line bundle $\eta \otimes \mathcal{O}_{C \cup D'}(x - 2y)$ (resp. $\eta \otimes \mathcal{O}_{C \cup D'}(y - 2x)$). The precise twists are determined by the limit linear series condition that each aspect of a limit $\mathfrak{g}^0_{g-2}$ have degree $g-2$. We distinguish three cases depending on which components of $C\cup D'$ the points x and y specialize.

(i) $x, y\in C$. Then $\sigma_C\in H^0(C, \eta_C\otimes \mathcal{O}_C(x-2y+q)), \tau_C\in H^0(C, \eta_C\otimes \mathcal{O}_C(y-2x+q))$, while $\sigma_D, \tau_D\in H^0\big(D, \eta_D\otimes \mathcal{O}_D((g-2)q)\big)$. Denoting by $\{q'\}\in D\cap \overline{(C\cup D')-D}$ the point where D meets the rest of the curve, one has the compatibility conditions

$$\mathrm{ord}_q(\sigma_C)+\mathrm{ord}_{q'}(\sigma_D)\geq g-2 \ \text{ and } \ \mathrm{ord}_q(\tau_C)+\mathrm{ord}_{q'}(\tau_D)\geq g-2,$$

which leads to $\mathrm{ord}_q(\sigma_C)\geq 1$ and $\mathrm{ord}_q(\tau_C)\geq 1$, that is, we have found two points $x, y\in C$ such that $H^0(C, \eta_C(x-2y))\neq 0$ and $H^0(C, \eta_C(y-2x))\neq 0$, which contradicts the inductive assumption on C.

(ii) $x, y\in D'$. This case does not appear if we choose η_C such that $H^0(C, \eta_C)=0$. Indeed, for degree reason, both non-zero sections σ_C, τ_C must lie in the space $H^0(C, \eta_C)$.

(iii) $x\in C, y\in D'$. For simplicity, we assume first that $y\in D$. We find that

$$\sigma_C\in H^0(C, \eta_C\otimes \mathcal{O}_C(x-q)),\ \sigma_D\in H^0(D, \eta_D\otimes \mathcal{O}_D(g\cdot q'-2y)) \ \text{ and}$$

$$\tau_C\in H^0(C, \eta_C\otimes \mathcal{O}_C(2q-2x)),\ \tau_D\in H^0(D, \eta_D\otimes \mathcal{O}_C(y+(g-3)\cdot q')).$$

We claim that $\mathrm{ord}_q(\sigma_C)=\mathrm{ord}_q(\tau_C)=0$ which can be achieved by a generic choice of $q\in C$. Then $\mathrm{ord}_{q'}(\sigma_D)\geq g-2$, which implies that $\eta_D=\mathcal{O}_D(2y-2q)$. Similarly, $\mathrm{ord}_q(\tau_D)\geq g-2$ which yields that $\eta_D=\mathcal{O}_D(q-y)$, that is, $\eta_D^{\otimes 3}=\mathcal{O}_D$. Since η_D was assumed to be a non-trivial point of order 2 this leads to a contradiction. Finally, the case $y\in D'-D$, that is, when y lies on an exceptional subcurve $E'\subset D'$ is dealt with similarly: Since $\mathrm{ord}_q(\sigma_C)=\mathrm{ord}_q(\tau_C)=0$, by compatibility, after passing through the component E', one obtains that $\mathrm{ord}_{q'}(\sigma_D)\geq g-2$. Since $\sigma_D\in H^0(D, \eta_D\otimes \mathcal{O}_D((g-2)q'))$ and $\eta_D\neq \mathcal{O}_D$, we obtain a contradiction □

3. THETA PENCILS ON K3 SURFACES.

In this section we prove Theorem 0.1. As usual, we denote by $\mathcal{F}_g$ the moduli space of polarized $K3$ surfaces $[X, H]$, where X is a $K3$ surface and $H\in \mathrm{Pic}(X)$ is a (primitive) polarization of degree $H^2=2g-2$. For integers $0\leq \delta\leq g$, we introduce the universal *Severi variety* of pairs

$$\mathcal{V}_{g,\delta}:=\big\{\big([X,H], C\big): [X,H]\in \mathcal{F}_g \text{ and } C\in |\mathcal{O}_X(H)| \text{ is an integral } \delta-\text{nodal curve}\big\}.$$

If $\sigma:\mathcal{V}_{g,\delta}\rightarrow \mathcal{F}_g$ is the obvious projection, we set $V_{g,\delta}(|H|):=\sigma^{-1}([X,H])$. It is known that every irreducible component of $\mathcal{V}_{g,\delta}$ has dimension $19+g-\delta$ and maps dominantly onto $\mathcal{F}_g$. It is in general not known whether $\mathcal{V}_{g,\delta}$ is irreducible, see [De] for interesting work in this direction.

For a point $[X,H]\in \mathcal{F}_g$, we consider a pencil of curves $P\subset |H|$, and denote by Z the base locus of P. We assume that a general member $C\in P$ is a nodal integral curve. It follows that $C-Z$ is smooth and that $S:=\mathrm{sing}(C)$ is a, possibly empty, subset of Z. Let $\epsilon: X':=\mathrm{Bl}_S(X)\rightarrow X$ be the blow-up of X along the locus S of nodes, and denote by E the exceptional divisor of ϵ. Let

$$P'\subset |\epsilon^*H\otimes \mathcal{O}_{X'}(-2E)|$$

be the strict transform of P by ϵ, and Z' its base locus. Since a general member $C\in P$ is nodal precisely along S, a general curve $C'\in P'$ is smooth. We view $h':=Z'+E\cdot C'$ as a divisor on the smooth curve C'. By the adjunction formula, $h'\in |\omega_{C'}|$.

Definition 3.1. We say that P is a *theta pencil*, if h' has even multiplicity at each of its points, that is, $\mathcal{O}_{C'}(\frac{1}{2}h')$ is an odd theta-characteristic for every smooth curve $C' \in P'$.

The definition implies that the intersection multiplicity of two curves in P is even at each point $p \in \mathrm{supp}(Z)$. For every pair $[X, H] \in \mathcal{F}_g$ we have that:

Proposition 3.2. *Every smooth curve $C \in |H|$ belongs to a theta pencil.*

Proof. Let $d \in C_{g-1}$ be the support of a theta-characteristic on C such that $h^0(C, \mathcal{O}_C(d)) = 1$. Then $\mathbf{P}H^0\big(X, \mathcal{I}_{d/X}(H)\big)$ is a theta pencil. □

We can reverse the construction of a theta pencil, starting instead with the normalization of a nodal section of a $K3$ surface. Suppose

$$t := [C', x_1, y_1, \ldots, x_\delta, y_\delta,\ \eta] \in \mathcal{M}_{g-\delta, 2\delta} \times_{\mathcal{M}_{g-\delta}} \mathcal{S}^-_{g-\delta}$$

is a 2δ-pointed curve together with an isolated odd theta-characteristic η, such that:
(i) $h^0\big(C', \eta \otimes \mathcal{O}_{C'}(-\sum_{i=1}^{\delta}(x_i + y_i))\big) \geq 1$; we write $\mathrm{supp}(\eta) = \sum_{i=1}^{\delta}(x_i + y_i) + d$, where $d \in C'_{g-3\delta-1}$ is the residual divisor.
(ii) There exists a polarized $K3$ surface $[X, H] \in \mathcal{F}_g$ and a map $f : C' \to X$, such that $f(x_i) = f(y_i) = p_i$ for all $i = 1, \ldots, \delta$, $f_*(C') \in |H|$, and moreover $f : C' \to C$ is the normalization map of the δ-nodal curve $C := f(C')$.

If $\epsilon : X' \to X$ is the blow-up of X at the points $p_1, \ldots, p_\delta$ and $E := \sum_{i=1}^{\delta} E_{p_i} \subset X'$ denotes the exceptional divisor, we may view $C' \subset X$, where $C' \equiv \epsilon^* H - 2E$. Then

$$|\mathcal{I}_{d/X'}(C')| = |\mathcal{I}_{2d/X'}(C')| = |\mathcal{I}_{2d + \sum_{i=1}^{\delta}(x_i+y_i)/X'}(C')|$$

is a theta pencil of δ-nodal curves on X.

If $\mathcal{K}^-_{g-\delta,\delta} \subset \mathcal{M}_{g-\delta,2\delta} \times_{\mathcal{M}_{g-\delta}} \mathcal{S}^-_{g-\delta}$ is the locus of elements $[C, (x_i, y_i)_{i=1,\ldots,\delta},\ \eta]$ satisfying conditions (i) and (ii), the previous discussion proves the following:

Proposition 3.3. *Every irreducible component of $\mathcal{K}^-_{g-\delta,\delta}$ is uniruled.*

This implies the following consequence of Proposition 4.4 to be established in the next section:

Theorem 3.4. *We set $g \leq 9$ and $0 \leq \delta \leq (g+1)/3$. Then the variety $\mathcal{K}^-_{g-\delta,\delta}$ is non-empty, uniruled and dominates the spin moduli space $\mathcal{S}^-_{g-\delta}$.*

Definition 3.5. We say that a theta pencil P is *δ-nodal* if $|S| = \delta$. We say that P is *regular* if $\mathrm{supp}(Z)$ consists of $g-1$ distinct points.

If P is a δ-nodal theta pencil, we have an induced map

$$m' : P' \cong \mathbf{P}^1 \to \overline{\mathcal{S}}^-_{g-\delta},$$

obtained by sending a general $C' \in P'$ to the moduli point $\big[C', \mathcal{O}_{C'}\big(\frac{1}{2}h'\big)\big] \in \overline{\mathcal{S}}^-_{g-\delta}$. We note in passing that a theta pencil also induces a map $m : P' \to \overline{\mathcal{S}}^-_g$ defined as follows. Consider the pencil $E + P'$ having fixed component E. The general member is a quasi-stable curve $D \in (E + P')$ of arithmetic genus g, with exceptional components $\{E_i\}_{i=1,\ldots,\delta}$ corresponding to the exceptional divisors of the blow-up $\epsilon : X' \to X$. Then

$$m(C) := \big[C \cup \big(\cup_{i=1}^{\delta} E_i\big),\ \eta_{E_i} = \mathcal{O}_{E_i}(1),\ \eta_{C'} = \mathcal{O}_{C'}\big(\frac{1}{2}h'\big)\big] \in \overline{\mathcal{S}}^-_g.$$

These pencils will be used extensively in the proof of Theorem 0.2.

Assume that $[X, H] \in \mathcal{F}_g$ is a general point, in particular $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Then every smooth curve $C \in |H|$ is Brill-Noether general, [La], which implies that $h^0(C, \eta) = 1$, for every odd theta-characteristic η on C. Theta pencils with smooth general member define a locally closed subset in the Grassmannian $G(2, H^0(S, \mathcal{O}_S(H))$ of lines in $|H|$. Let $\Theta^-(X, H)$ be its Zariski closure in $G(2, H^0(S, \mathcal{O}_S(H))$.

Proposition 3.6. *$\Theta^-(X, H)$ is pure of dimension $g - 1$.*

Proof. Let $f : P^-(X, H) \to |H|$ be the projection map from the projectivized universal bundle over $\Theta^-(X, H)$, and $V_{g,0}(|H|) \subset |H|$ be the open locus of smooth curves. Under our assumptions f has finite fibres over $V_{g,0}(|H|)$. Thus $P^-(X, H)$ has pure dimension g, and $\Theta^-(X, H)$ has pure dimension $g - 1$. □

For a general (thus necessarily regular) theta pencil $P \in \Theta^-(X, H)$, we study in more detail the map $m : P' \to \overline{\mathcal{S}}_g^-$. Let $\Delta(X, H) \subset |H|$ be the discriminant locus. Since $[X, H] \in \mathcal{F}_g$ is general, $\Delta(X, H)$ is an integral hypersurface parameterizing the singular elements of $|H|$. It is well-known that $\deg \Delta(X, H) = 6g + 18$.

Proposition 3.7. *Let $P \in \Theta^-(X, H)$ be a general theta pencil with base locus Z. Then every singular curve $C \in P$ is nodal. Furthermore,*

$$P \cdot \Delta(X, H) = 2(a_1 + \cdots + a_{g-1}) + b_1 + \cdots + b_{4g+20},$$

where a_i is the parameter point of a curve $A_i \in P$ having a point of Z as its only singularity, and b_j is the parameter point of a curve $B_j \in P$ such that $\mathrm{sing}(B_j) \subset X - Z$. Accordingly,

$$P \cdot \alpha_0 = 4g + 20 \ \text{and} \ P \cdot \beta_0 = g - 1.$$

Proof. We set $\mathrm{supp}(Z) = \{p_1, \ldots, p_{g-1}\}$. Since P is regular, for $i = 1, \ldots, g - 1$, there exists a unique curve $A_i \in P$ singular at p_i. Moreover, for degree reasons, p_i is the unique double point of A_i. Each pencil $T \subset |H|$ having p_i in its base locus is a tangent line to $\Delta(X, H)$ at A_i. Hence the intersection multiplicity $\big(P \cdot \Delta(X, H)\big)_{A_i}$ is at least 2. It follows that the assertion to prove is open on any family of pairs $(P, [X, H])$ such that $P \in \Theta^-(X, H)$. Since $\mathcal{F}_g$ is irreducible, it suffices to produce one polarized $K3$ surface (X, H) satisfying this condition.

For this purpose, we use *hyperelliptic* polarized $K3$ surfaces (X, H). Consider a rational normal scroll $\mathbb{F} := \mathbb{F}_a \subset \mathbf{P}^g$, where $a \in \{0, 1\}$ and $g = 2n + 1 - a$. A general section $R \in |O_{\mathbb{F}}(1)|$ is a rational normal curve of degree $g - 1$. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}}(-2K_{\mathbb{F}} - R) \to \mathcal{O}_{\mathbb{F}}(-2K_{\mathbb{F}}) \to \mathcal{O}_R(-2K_{\mathbb{F}}) \to 0,$$

one finds that there exist a smooth curve $B \in |-2K_{\mathbb{F}}|$ and distinct points $o_1, \ldots, o_{g-1} \in B$ such that the pencil $Q \subset |\mathcal{O}_{\mathbb{F}}(R)|$ of hyperplane sections through $o_1, \ldots, o_{g-1}$ cuts out a pencil with simple ramification on B.

Let $\rho : X \to \mathbb{F}$ be the double covering of $\mathbb{F}$ branched along B. Then X is a K3 surface and $|H| := |\mathcal{O}_X(\rho^* R)|$ is a hyperelliptic linear system on X of genus g. Then $\rho^*(Q)$ is a regular theta pencil on X with the required properties. □

Since theta pencils cover $\overline{\mathcal{S}}_g^-$ when $g \leq 11$ and $g \neq 10$, the following consequence of Proposition 3.7 is very suggestive concerning the variation of $\kappa(\overline{\mathcal{S}}_g^-)$ as g increases, in particular, in highlighting the significance of the case $g = 12$.

Corollary 3.8. *With the same notation as above, we have that $P\cdot K_{\overline{\mathcal{S}}_g^-}=2g-24$. In particular general theta pencils of genus $g<12$ are $K_{\overline{\mathcal{S}}_g^-}$-negative.*

Proof. Use that $(P\cdot\lambda)_{\overline{\mathcal{S}}_g^-}=(\pi_*(P)\cdot\lambda)_{\overline{\mathcal{M}}_g}=g+1$, $P\cdot\alpha_0=4g+20$ and $P\cdot\beta_0=g-1$. $\square$

Proposition 3.9. *The locally closed set of nodal theta pencils in $\Theta^-(X,H)$ is non empty. If P is a general nodal theta pencil, then a general curve $C\in P$ has one node as its only singularity.*

Proof. We keep the notation from the previous proof and construct a smooth curve $B\in|-2K_{\mathbb{F}}|$ and choose general points $o,o_1,\ldots,o_{g-3}\in B$, such that the pencil $Q\subset|\mathcal{O}_{\mathbb{F}}(R)|$ of the hyperplane sections through $o_1+\cdots+o_{g-3}+2o$ cuts out a pencil with simple ramification on B. Then $\rho^*(Q)$ is a nodal theta pencil with the required properties. $\square$

Theorem 3.10. *$\overline{\mathcal{S}}_g^-$ is uniruled for $g\leq 11$.*

Proof. By [M1-4], a general curve $[C]\in\overline{\mathcal{M}}_g$ is embedded in a K3 surface X precisely when $g\leq 9$ or $g=11$. By Proposition 3.7, C belongs to a theta pencil $P\subset|\mathcal{O}_X(C)|$ (which moreover, is $K_{\overline{\mathcal{S}}_g^-}$-negative). Thus the statement follows for $g\leq 9$ and $g=11$. To settle the case of $\overline{\mathcal{S}}_{10}^-$, we show that $\mathcal{K}_{10,1}^-$ is non-empty and irreducible. Indeed, then by Proposition 3.3 it follows that $\mathcal{K}_{10,1}^-$ is uniruled, and since the projection map $\mathcal{K}_{10,1}^-\to\mathcal{S}_{10}^-$ is finite, $\mathcal{K}_{10,1}^-$ dominates $\mathcal{S}_{10}^-$. This implies that $\overline{\mathcal{S}}_{10}^-$ is uniruled.

The variety $\mathcal{K}_{10,1}^-$ is an open subvariety of the irreducible locus

$$\mathcal{U}:=\big\{\big([C,x,y],\eta\big)\in\mathcal{M}_{10,2}\times_{\mathcal{M}_{10}}\mathcal{S}_{10}^-:h^0(C,\eta\otimes\mathcal{O}_C(-x-y))\geq 1\big\},$$

hence it is irreducible as well. To establish its non-emptiness, it suffices to produce an example of an element $\big([C,x,y],\eta]\big)\in\mathcal{U}$, such that the curve C_{xy} can be embedded in a $K3$ surface. We specialize to the case when C is hyperelliptic and $x,y\in C$ are distinct Weierstrass points, in which case one can choose $\eta=\mathcal{O}_C(x+y+w_1+\cdots+w_7)$, where w_i are distinct Weierstrass points in $C-\{x,y\}$. Again we let $\rho:X\to\mathbb{F}\subset\mathbf{P}^{11}$ be a hyperelliptic $K3$ surface branched along $B\in|-2K_{\mathbb{F}}|$, with polarization $H:=\rho^*\mathcal{O}_{\mathbb{F}}(1)$, so that $[X,H]\in\mathcal{F}_{11}$. We set $C:=\rho^*(R)$, where $R\in|\mathcal{O}_{\mathbb{F}}(1)|$ is a rational normal curve of degree 10. We need to ensure that C is 1-nodal, with its node $p\in C$ such that if $f:C'\to C$ denotes the normalization map, then both points in $f^{-1}(p)$ are Weierstrass points. This is satisfied once we choose R in such a way that $B\cdot R\geq 2\rho(p)$. $\square$

4. Unirationality of $\overline{\mathcal{S}}_g^-$ for $g\leq 8$

To prove the claimed unirationality results, we use that a general curve $[C]\in\overline{\mathcal{M}}_g$ has a sextic plane model when $g\leq 6$, or is a linear section of a Mukai variety, when $7\leq g\leq 9$. We start with the easy case of small genus, before moving on to the more substantial study of Mukai models.

Theorem 4.1. *$\overline{\mathcal{S}}_g^-$ is unirational for $g\leq 6$.*

Proof. A general odd spin curve $[C,\eta]\in\overline{\mathcal{S}}_g^-$ of genus $3\leq g\leq 6$, is birational to a pair (Γ,η), where $\Gamma\subset\mathbf{P}^2$ is an integral nodal sextic. One can assume that $d:=\mathrm{supp}(\eta)$ is a reduced divisor contained in Γ_{reg}. Note that there exists a unique plane cubic E such that $E\cdot\Gamma=2e$, where e is an effective divisor of degree 9 on E, supported on

$\mathrm{sing}(\Gamma) \cup d$. We denote by $\mathcal{U} \subset (\mathbf{P}^2)^9$ the open set parameterizing general 9-tuples $(\bar{x}, \bar{y}) := (x_1, \ldots, x_\delta, y_1, \ldots, y_{g-1})$, where $g = 10 - \delta$. Over $\mathcal{U}$ lies a projective bundle $\mathcal{P}$ whose fibre at $(\bar{x}, \bar{y})$ is the linear system of plane sextics Γ which are singular along $\bar{x}$ and totally tangent to $E_{\bar{x},\bar{y}}$ along $\bar{y}$. Here $E_{\bar{x},\bar{y}} \in |\mathcal{O}_{\mathbf{P}^2}(3)|$ denotes the unique plane cubic through the points $x_1, \ldots, x_\delta, y_1, \ldots, y_{g-1}$. Then $\mathcal{P}$ is a rational variety, and by the previous remark, it dominates $\overline{\mathcal{S}}_g^-$. Thus $\overline{\mathcal{S}}_g^-$ is unirational. □

We assume now that $7 \leq g \leq 10$ and denote by $V_g \subset \mathbf{P}^{N_g}$ the rational homogeneous space V_g defined as follows [M1], [M2], [M3]:

- V_{10}: *the* 5*-dimensional variety* $G_2/P \subset \boldsymbol{P}^{17}$ *corresponding to the Lie group* G_2*,*
- V_9: *the Plücker embedding of the symplectic Grassmannian* $SG(3,6) \subset \boldsymbol{P}^{13}$*,*
- V_8: *the Plücker embedding of the Grassmannian* $G(2,6) \subset \boldsymbol{P}^{14}$*,*
- V_7: *the Plücker embedding of the orthogonal Grassmannian* $OG(5,10) \subset \boldsymbol{P}^{15}$*,*

Note that $N_g = g + \dim(V_g) - 2$. Inside the Hilbert scheme $\mathrm{Hilb}(V_g)$ of curvilinear sections of V_g, we consider the open set $\mathcal{U}_g$ classifying curves $C \subset V_g$ such that

- *C is a nodal integral section of V_g by a linear space of dimension $g-1$,*
- *the residue map $\rho : H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \mathcal{O}_{\mathrm{sing}(C)})$ is surjective.*

A general point $[C \hookrightarrow \mathbf{P}^{g-1}] \in \mathcal{U}_g$ is a smooth, canonical curve of genus g. Moreover C has general moduli if $g \leq 9$. For each $0 \leq \delta \leq g-1$, we define the locally closed sets of δ-nodal curvilinear sections of V_g

$$\mathcal{U}_{g,\delta} := \{[C \hookrightarrow \mathbf{P}^{g-1}] \in \mathcal{U}_g : \ |\mathrm{sing}(C)| = \delta\}.$$

Proposition 4.2. *$\mathcal{U}_{g,\delta}$ is smooth of pure codimension δ in $\mathcal{U}_g$.*

Proof. A general 2-dimensional linear section of V_g is a polarized K3 surface $(S, H) \in \mathcal{F}_g$ with general moduli. It is known [Ta], that δ-nodal hyperplane sections of S form a pure $(g-\delta)$-dimensional family $V_{g,\delta}(|H|) \subset |H|$. In particular $\mathcal{U}_{g,\delta} \neq \emptyset$ and $\mathrm{codim}(\mathcal{U}_{g,\delta}, \mathcal{U}_g) \leq \delta$. We fix a curve $[C] \in \mathcal{U}_{g,\delta}$, then consider the normal bundle N_C of C in V_g and the map $r : H^0(C, N_C) \to \mathcal{O}_{\mathrm{sing}(C)}$ induced by the exact sequence

$$0 \to T_C \to T_{V_g} \otimes \mathcal{O}_C \to N_C \xrightarrow{r} T_C^1 \to 0, \tag{3}$$

where $T_C^1 = \mathcal{O}_{\mathrm{sing}(C)}$ is the Lichtenbaum-Schlessinger sheaf of C. Using the identification $T_{[C]}(\mathcal{U}_g) = H^0(C, N_C)$, it is known that $\mathrm{Ker}(r)$ is isomorphic to $T_{[C]}(\mathcal{U}_{g,\delta})$. We have that $N_C \cong \omega_C^{\oplus(N_g - g+1)}$ and $r = \rho^{\oplus(N_g - g+1)}$, where $\rho : H^0(C, \omega_C) \to H^0(C, \mathcal{O}_{\mathrm{sing}(C)})$ is the map given by the residues at the nodes. Since ρ is surjective, $\mathrm{Ker}(r)$ has codimension δ inside $T_{[C]}(\mathcal{U}_g)$ and the statement follows. □

The automorphism group $\mathrm{Aut}(V_g)$ acts in the natural way on $\mathrm{Hilb}(V_g)$. Since the locus of singular curvilinear sections $[C] \in \mathcal{U}_g$ is an $\mathrm{Aut}(V_g)$-invariant divisor which misses a general point of $\mathcal{U}_g$, it follows that $\mathcal{U}_g^{\mathrm{ss}} := \mathcal{U}_g \cap \mathrm{Hilb}(V_g)^{\mathrm{ss}} \neq \emptyset$. Note that since $\rho(V_g) = 1$, the notion of stability is independent of the polarization. The (quasi-projective) GIT-quotient

$$\mathfrak{M}_g := \mathcal{U}_g^{\mathrm{ss}} // \mathrm{Aut}(V_g)$$

is said to be the *Mukai model* of $\overline{\mathcal{M}}_g$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{U}_g^{\rm ss} & \longrightarrow & \mathcal{U}_g \\ {\scriptstyle u_g}\downarrow & & {\scriptstyle m_g}\downarrow \\ \mathfrak{M}_g & \xrightarrow{\phi_g} & \overline{\mathcal{M}}_g \end{array}$$

where $u_g : \mathcal{U}_g^{\rm ss} \to \mathfrak{M}_g$ is the quotient map and $m_g : \mathcal{U}_g \to \overline{\mathcal{M}}_g$ is the moduli map. The general fibre of m_g is an $\mathrm{Aut}(V_g)$-orbit. Summarizing results from [M1], [M2], [M3], we state the following:

Theorem 4.3. *For $7 \leq g \leq 9$, the map $\phi_g : \mathfrak{M}_g \to \overline{\mathcal{M}}_g$ is a birational isomorphism. The inverse map ϕ_g^{-1} contracts the (unique) Brill-Noether divisor $\overline{\mathcal{M}}^r_{g,d} \subset \overline{\mathcal{M}}_g$ of curves with a $\mathfrak{g}^r_d$, as well as the boundary divisors Δ_i with $1 \leq i \leq [g/2]$.*

Next, let $\Delta_g^\delta \subset \Delta_0 \subset \overline{\mathcal{M}}_g$ be the locus of integral stable curves of arithmetic genus g with δ nodes. Then Δ_g^δ is irreducible of codimension δ in $\overline{\mathcal{M}}_g$.

Lemma 4.4. *Set $g \leq 9$ and let D be any irreducible component of $\mathcal{U}_{g,\delta}$. Then the restriction morphism $m_{g|D} : D \to \Delta_g^\delta$ is dominant. In particular, a general δ-nodal curve $[C] \in \Delta_g^\delta$ lies on a smooth $K3$ surface.*

Proof. Since $\mathcal{U}_{g,\delta}$ is smooth, D is a connected component of $\mathcal{U}_{g,\delta}$, that is, for $[C] \in D$, the tangent spaces to D and to $\mathcal{U}_{g,\delta}$ coincide. We consider again the sequence (3):

$$0 \to T_C \to T_{V_g} \otimes \mathcal{O}_C \to N'_C \to 0,$$

where $N'_C := \mathrm{Im}\,\{T_{V_g} \otimes \mathcal{O}_C \to N_C\}$ is the *equisingular sheaf* of C. We have that $H^0(C, N'_C) = \mathrm{Ker}(r)$. As remarked in the proof of Proposition 4.2, $H^0(C, N'_C)$ is the tangent space $T_{[C]}(\mathcal{U}_{g,\delta})$ and its codimension in $H^0(C, N_C)$ equals δ. Consider the coboundary map $\partial : H^0(C, N'_C) \to H^1(C, T_C)$. Since $H^1(C, T_C)$ classifies topologically trivial deformations of the nodal curve C, the image $\mathrm{Im}(\partial)$ is isomorphic to the image of the tangent map $dm_{g|\mathcal{U}_{g,\delta}}$ at $[C]$. On the other hand $H^0(C, T_{V_g} \otimes \mathcal{O}_C)$ is the tangent space to the orbit of C under the action of $\mathrm{Aut}(V_g)$. This is reduced and the stabilizer of C, being a subgroup of $\mathrm{Aut}(C)$, is finite, hence we obtain:

$$\dim\ \mathrm{Im}(\partial) = h^0(C, N_C) - \delta - \dim \mathrm{Aut}(V_g) = 3g - 3 - \delta.$$

Since Δ_g^δ has codimension δ in $\overline{\mathcal{M}}_g$, it follows that $m_{g|D}$ is dominant. □

Proposition 4.5. *Fix $0 \leq \delta \leq g-1$ and D an irreducible component of $\mathcal{U}_{g,\delta}$. Then $D^{\rm ss} \neq \emptyset$.*

Proof. It suffices to construct an $\mathrm{Aut}(V_g)$-invariant divisor which does not contain D. We carry out the construction when $g = 8$, the remaining cases being largely similar.

We fix a complex vector space $V \cong \mathbb{C}^6$, and then $V_8 := G(2, V) \subset \mathbf{P}(\wedge^2 V)$ and $\mathcal{U}_8 \subset G(8, \wedge^2 V)$. For a projective 7-plane $\Lambda \in G(8, \wedge^2 V)$, we denote the set of containing hyperplanes $F_\Lambda := \{H \in \mathbf{P}(\wedge^2 V)^\vee : H \supset \Lambda\}$, and define the $\mathrm{Aut}(V_8)$-invariant divisor

$$Z := \{\Lambda \in \mathcal{U}_8 : F_\Lambda \cap G(2, V^\vee) \subset \mathbf{P}(\wedge^2 V)^\vee \text{ is not a transverse intersection}\}.$$

We claim that $D \not\subseteq Z$. Indeed, let us fix a general point $[C \hookrightarrow \Lambda] \in D$, where $\Lambda = \langle C \rangle$, corresponding to a general curve $[C] \in \Delta_g^\delta$. In particular, we may assume that C lies outside the closure in $\overline{\mathcal{M}}_g$ of curves violating the Petri theorem. Thus C possesses no generalized $\mathfrak{g}^2_7$'s, that is, $\overline{W}^2_7(C)=\emptyset$, whereas $\overline{W}^1_5(C)\subset \mathrm{Pic}(C)$ consists of locally free pencils satisfying the Petri condition. We recall from [M2] the construction of $\phi_g^{-1}[C]$, which generalizes to irreducible Petri general nodal curves: There exists a unique rank two vector bundle E on C with $\det(E)=\omega_C$ and $h^0(C,E)=6$. This appears as an extension

$$0\to A\to E\to \omega_C\otimes A^{\vee}\to 0,$$

for every $A\in \overline{W}^1_5(C)$. Then one sets $\phi_g^{-1}([C]):=[C\hookrightarrow G(2,H^0(C,E)^{\vee})]$. Moreover,

$$F_{\Lambda}=\mathbf{P}\big(\mathrm{Ker}\{\wedge^2 H^0(C,E)\to H^0(C,\omega_C)\}\big).$$

In particular, the intersection $F_{\Lambda}\cap G(2,H^0(C,E))$ corresponds to the pencils $A\in \overline{W}^1_5(C)$. Since C is Petri general, $\overline{W}^1_5(C)$ is a smooth scheme, thus $[C\hookrightarrow \Lambda]\notin Z$. $\Box$

We consider the quotient $\mathfrak{M}_{g,\delta}:=\mathcal{U}_{g,\delta}^{\mathrm{ss}}//\mathrm{Aut}(V_g)$ and the induced map

$$\phi_{g,\delta}:\mathfrak{M}_{g,\delta}\to \Delta_g^{\delta}.$$

Theorem 4.6. *The variety $\mathfrak{M}_{g,\delta}$ is irreducible and $\phi_{g,\delta}$ is a birational isomorphism.*

Proof. By Lemma 4.4, any irreducible component Y of $\mathfrak{M}_{g,\delta}$ dominates Δ_g^{δ}. On the other hand, $\phi_g:\mathfrak{M}_g\to\overline{\mathcal{M}}_g$ is a birational morphism and $\phi_{g,\delta}=\phi_{g|\mathfrak{M}_{g,\delta}}$. Since $\overline{\mathcal{M}}_g$ is normal, each fibre of ϕ_g is connected, thus $\mathfrak{M}_{g,\delta}$ is irreducible and $\deg(\phi_{g,\delta})=1$. $\Box$

We lift our construction to the space of odd spin curves. Keeping $g\le 9$, we consider the Hilbert scheme $\mathrm{Hilb}_{2g-2}(V_g)$ of 0-dimensional subschemes of V_g having length $2g-2$.

Definition 4.7. Let $\mathfrak{Z}_{g-1}\subset \mathrm{Hilb}_{2g-2}(V_g)$ be the parameter space of those 0-dimensional schemes $Z\subset V_g$ such that:

(1) Z is a hyperplane section of a smooth curve section $[C]\in\mathcal{U}_g$,
(2) Z has multiplicity two at each point of its support,
(3) $\mathrm{supp}(Z)$ consists of $g-1$ linearly independent points.

One thinks of $\mathfrak{Z}_{g-1}$ as classifying length $g-1$ *clusters* on V_g. A general point of $\mathfrak{Z}_{g-1}$ corresponds to a 0-cycle $x_1+\cdots+x_{g-1}\in \mathrm{Sym}^{g-1}(V_g)$ satisfying

$$\dim\langle x_1,\ldots,x_{g-1}\rangle\cap \mathbb{T}_{x_i}(V_g)\ge 1, \text{ for } i=1,\ldots,g-1.$$

Clearly $\dim(\mathfrak{Z}_{g-1})=(g-1)(N_g-g+1)$. Then we consider the incidence correspondence

$$\mathcal{U}_g^-:=\{(C,Z)\in\mathcal{U}_g\times\mathfrak{Z}_{g-1}:Z\subset C\}.$$

The first projection map $\pi_1:\mathcal{U}_g^-\to\mathcal{U}_g$ is finite of degree $2^{g-1}(2^g-1)$; its fibre at a general point $[C]\in\mathcal{U}_g$ is in bijective correspondence with the set of odd theta-characteristics of C. In particular, both $\mathcal{U}_g^-$ and $\mathfrak{Z}_{g-1}$ are irreducible varieties. The spin moduli map

$$m_g^-:\mathcal{U}_g^- \dashrightarrow \overline{\mathcal{S}}_g^-$$

is defined by $m_g^-(C,Z):=[C,\mathcal{O}_C(Z/2)]$, for each point $(C,Z)\in\mathcal{U}_g^-$ corresponding to a smooth curve C. Later we shall extend the rational map m_g^- to a regular map over $\mathcal{U}_g^-$. It is clear that m_g^- induces a map $\phi_g^-:Q_g^-\dashrightarrow\overline{\mathcal{S}}_g^-$ from the quotient

$$Q_g^-:=\pi_1^{-1}(\mathcal{U}_g^{\mathrm{ss}})//\mathrm{Aut}(V_g).$$

We may think of Q_g^- as being the *Mukai model* of $\overline{\mathcal{S}}_g^-$. If $\pi^- : Q_g^- \to \mathfrak{M}_g$ is the map induced by π at the level of Mukai models, we have a commutative diagram:

$$\begin{array}{ccc} Q_g^- & \xrightarrow{\phi_g^-} & \overline{\mathcal{S}}_g^- \\ {\scriptstyle \pi^-}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ \mathfrak{M}_g & \xrightarrow{\phi_g} & \overline{\mathcal{M}}_g \end{array}$$

Proposition 4.8. *The spin Mukai model Q_g^- is irreducible and $\phi_g^- : Q_g^- \to \overline{\mathcal{S}}_g^-$ is a birational isomorphism.*

One extends the rational map m_g^- (therefore ϕ_g^- as well) to a regular morphism as follows. Let $(C, Z) \in \mathcal{U}_g^-$ be an arbitrary point, and set $\mathrm{supp}(Z) := \{p_1, \ldots, p_{g-1}\}$. Assume that $\mathrm{sing}(C) \cap \mathrm{supp}(Z) = \{p_1, \ldots, p_\delta\}$, where $\delta \le g-1$. Consider the partial normalization $\nu : N \to C$ at the points $p_1, \ldots, p_\delta$. In particular, there exists an effective Cartier divisor e on C of degree $g - \delta - 1$, such that $2e = Z \cap (C - \mathrm{sing}(C))$, and set $\epsilon := \mathcal{O}_N(\nu^* e)$. Then $m_g^-(C, Z)$ is the spin curve $[X, \eta] \in \overline{\mathcal{S}}_g^-$ defined as follows:

Definition 4.9.

(1) $X := N \cup E_1 \cup \cdots \cup E_\delta$, *where* $E_i = \mathbf{P}^1$ *for* $i = 1, \ldots, \delta$.
(2) $E_i \cap N = \nu^{-1}(p_i)$, *for every node* $p_i \in \mathrm{sing}(C) \cap \mathrm{supp}(Z)$.
(3) $\eta \otimes \mathcal{O}_N \cong \epsilon$ *and* $\eta \otimes \mathcal{O}_{E_i} \cong \mathcal{O}_{\mathbf{P}^1}(1)$.

We note that N is smooth of genus $g - \delta$, precisely when $\mathrm{sing}(C) \subset \mathrm{supp}(Z)$. In this case $\epsilon \in \mathrm{Pic}^{g-1-\delta}(N)$ is a theta characteristic and $h^0(N, \epsilon) = 1$. Since we are specially interested in this case, for $1 \le \delta \le g-1$ we introduce the locally closed sets

$$\mathcal{U}_{g,\delta}^- := \{(C, Z) \in \mathcal{U}_g^- : \mathrm{sing}(C) \subset \mathrm{supp}(Z), \ \ |\mathrm{sing}(C)| = \delta\}.$$

We denote by $B_{g,\delta}^-$ the closure of $m_g^-(\mathcal{U}_{g,\delta}^-)$ inside $\overline{\mathcal{S}}_g^-$; this is the closure in $\overline{\mathcal{S}}_g^-$ of the locus of δ-nodal spin curves having δ exceptional components. Clearly $B_{g,\delta}^-$ is an irreducible component of $\pi^{-1}(\Delta_g^\delta)$. We set

$$Q_{g,\delta}^- := \mathcal{U}_{g,\delta}^- \cap \pi_1^{-1}(\mathcal{U}_g^{\mathrm{ss}})//\mathrm{Aut}(V_g),$$

and let $u_g^- : \mathcal{U}_{g,\delta}^- \dashrightarrow Q_{g,\delta}^-$ denote the quotient map. Keeping all previous notation, we have a further commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_{g,\delta}^- & \xrightarrow{u_g^-} & Q_{g,\delta}^- & \xrightarrow{\phi_{g,\delta}^-} & B_{g,\delta}^- \\ \big\downarrow & & {\scriptstyle \pi^-}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ \mathcal{U}_{g,\delta} & \xrightarrow{u_g} & \mathfrak{M}_{g,\delta} & \xrightarrow{\phi_{g,\delta}} & \Delta_g^\delta \end{array}$$

where $\phi_{g,\delta}^-$ is the morphism induced on $Q_{g,\delta}^-$ by m_g^-.

Theorem 4.10. *We fix $7 \le g \le 9$ and $1 \le \delta \le g-1$. Then $\phi_{g,\delta}^- : Q_{g,\delta}^- \to B_{g,\delta}^-$ is a birational isomorphism.*

Proof. It suffices to note that $\phi_{g,\delta}$ is birational, and the vertical arrows of the diagram are finite morphisms of the same degree, namely the number of odd theta-characteristics on a curve of genus $g - \delta$. $\square$

We construct a projective bundle over $B^-_{g,\delta}$, then show that for certain values $\delta \leq g-1$, the locus $B^-_{g,\delta}$ itself is unirational, whereas the above mentioned bundle dominates $\mathcal{S}^-_g$. Let $\mathcal{C}_{g,\delta} \subset \mathcal{U}^-_{g,\delta} \times V_g$ be the universal curve, endowed with its two projection maps

$$\mathcal{U}^-_{g,\delta} \xleftarrow{\ p\ } \mathcal{C}_{g,\delta} \xrightarrow{\ q\ } V_g.$$

We fix an arbitrary point $(\Gamma, Z) \in \mathcal{U}^-_{g,\delta}$ and let $\nu : N \to \Gamma$ be the normalization map. Recall that $\mathrm{sing}(\Gamma)$ consists of δ linearly independent points and that $h^0(N, \mathcal{O}_N(\nu^* e)) = 1$, where e is the effective divisor on Γ characterized by $Z_{|\Gamma_{\mathrm{reg}}} = 2e$. Thus the restriction map $H^0(\Gamma, \omega_\Gamma) \to H^0(\omega_\Gamma \otimes \mathcal{O}_Z)$ has 1-dimensional kernel. In particular the relative cotangent sheaf ω_p admits a global section s inducing an exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}_{g,\delta}} \to \omega_p \to \mathcal{O}_W \otimes \omega_p \to 0,$$

which defines a subscheme $W \subset \mathcal{C}_{g,\delta}$, whose fibre at the point $(\Gamma, Z) \in \mathcal{U}^-_{g,\delta}$ is Z itself. We set

$$\mathcal{A} := p_*\big(\mathcal{I}_{W/\mathcal{C}_{g,\delta}} \otimes q^*\mathcal{O}_{V_g}(1)\big),$$

which is a vector bundle on $\mathcal{U}^-_{g,\delta}$ of rank $N_g - g + 2$. The fibre of $\mathcal{A}(\Gamma, Z)$ is identified with $H^0(V_g, \mathcal{I}_{Z/V_g}(1))$. One has a natural identification

$$\mathbf{P}H^0(\mathcal{I}_{Z/V_g}(1))^\vee = \{\textit{1-dimensional linear sections of } V_g \textit{ containing } Z\}.$$

Definition 4.11. *$\mathcal{P}_{g,\delta}$ is the projectivized dual of $\mathcal{A}$.*

From the definitions and the previous remark it follows:

Proposition 4.12. *$\mathcal{P}_{g,\delta}$ is the Zariski closure of the incidence correspondence*

$$\mathcal{P}^o_{g,\delta} := \{(C, (\Gamma, Z)) \in \mathcal{U}_g \times \mathcal{U}^-_{g,\delta} \ : \ Z \subset C\}.$$

Consider the projection maps

$$\mathcal{U}^-_g \xleftarrow{\ \alpha\ } \mathcal{P}^o_{g,\delta} \xrightarrow{\ \beta\ } \mathcal{U}^-_{g,\delta}.$$

We wish to know when α is a dominant map. For $1 \leq \delta < g \leq 9$, we have the following:

Proposition 4.13. *The map α is dominant if and only if $\delta \leq N_g + 1 - g = \dim(V_g) - 1$.*

Proof. By definition, the morphism β is surjective. Let $(\Gamma, Z) \in \mathcal{U}^-_{g,\delta}$ be an arbitrary point, and set $\mathrm{sing}(\Gamma) := \{p_1, \ldots, p_\delta\} \subset Z$. We define P_Z to be the locus of 1-dimensional linear sections of V_g containing Z. Inside P_Z we consider the space

$$P_{\Gamma,Z} = \{\Gamma' \in P_Z : \mathrm{sing}(\Gamma') \cap Z \supseteq \mathrm{sing}(\Gamma) \cap Z\},$$

First note that for $p \in \mathrm{sing}(\Gamma)$, the locus $H_p := \{\Gamma' \in P_Z : p \in \mathrm{sing}(\Gamma')\}$ is a hyperplane in P_Z. Indeed, we identify P_Z with the family of linear spaces $L \in G(g, N_g + 1)$ such that $\langle Z \rangle \subset L$. By the definition of the cluster Z, it follows that $l := \mathbb{T}_p(V_g) \cap \langle Z \rangle$ is a line. For $L \in P_Z$, the intersection $L \cap V_g$ is singular at p if and only if $\dim L \cap \mathbb{T}_p(V_g) \geq 2$. This is obviously a codimension 1 condition in P_Z. Therefore, if for $1 \leq i \leq \delta$ we define the hyperplane $H_i := \{L = \langle \Gamma' \rangle \in P_Z : \dim L \cap \mathbb{T}_{p_i}(V_g) \geq 2\}$, then

$$P_{\Gamma,Z} = H_1 \cap \cdots \cap H_\delta.$$

This shows that the general point in $\beta^{-1}(C, Z)$ corresponds to a smooth curve $C \supset Z$. We now fix a general point $(\Gamma, Z) \in \mathcal{U}^-_{g,\delta}$, corresponding to a general cluster $Z \in \mathfrak{Z}_{g-1}$.

Claim: $P_{\Gamma,Z}$ *has codimension* δ *in* P_Z*; its general element is a nodal curve with* δ *nodes.*
Proof of claim: Indeed P_Z is a general fibre of the projective bundle $\mathcal{U}_g^- \to \mathfrak{Z}_{g-1}$. The claim follows since $\mathrm{codim}(\mathcal{U}_{g,\delta}^-, \mathcal{U}_g^-) = \delta$.

The fibre $\alpha^{-1}\big((C,Z)\big)$ over a general point $(C,Z) \in \mathcal{U}_g^-$, is the union of $\binom{g-1}{\delta}$ linear spaces $H_1 \cap \cdots \cap H_\delta \subset P_Z$ as above. By the claim above, when $Z \in \mathfrak{Z}_{g-1}$ is a general cluster, this is a union of linear spaces $P_{\Gamma,Z}$ as before, having codimension δ in P_Z. Hence $\alpha^{-1}\big((C,Z)\big)$ is not empty if and only if $\delta \leq \dim P_Z$, that is, $\delta \leq N_g - g + 1$. □

Let us fix the following notation:

Definition 4.14.
(1) $\overline{\mathbb{P}}_{g,\delta} := \big(\mathcal{P}_{g,\delta}^o\big)^{\mathrm{ss}} // \mathrm{Aut}(V_g)$.
(2) $\overline{\beta} : \overline{\mathbb{P}}_{g,\delta} \to \overline{\mathcal{S}}_g^-$ *is the morphism induced by* β *at the level of quotients.*

Note that $\beta : \mathcal{P}_{g,\delta} \to \mathcal{U}_{g,\delta}^-$ is a projective bundle and $\mathrm{Aut}(V_g)$ acts linearly on its fibres, therefore β descends to a projective bundle on $B_{g,\delta}^-$. Then it follows from the previous remark that $\mathcal{P}_{g,\delta}$ is birationally isomorphic to $\mathbf{P}^{N_g-g+1} \times B_{g,\delta}^-$. To finish the proof of the unirationality of $\mathcal{S}_g^-$, we proceed as follows:

Theorem 4.15. *Let* $7 \leq g \leq 9$ *and assume that (i)* $B_{g,\delta}^-$ *is unirational and (ii)* $\delta \leq N_g - g + 1$. *Then* $\overline{\mathcal{S}}_g^-$ *is unirational.*

Proof. By assumption (ii), $\beta : \mathcal{P}_{g,\delta}^o \to \mathcal{U}_g^-$ is dominant, Hence the same is true for the induced morphism $\overline{\beta} : \overline{\mathbb{P}}_{g,\delta} \to \overline{\mathcal{S}}_g^-$. By (i) and the above remark, $\overline{\mathbb{P}}_{g,\delta}$ is unirational. Therefore $\overline{\mathcal{S}}_g^-$ is unirational as well. □

Theorem 4.15 has some straightforward applications. The case $\delta = g-1$ is particularly convenient, since $B_{g,g-1}^-$ is isomorphic to the moduli space of integral curves of geometric genus 1 with $g-1$ nodes. For $\delta = g-1$, the assumptions of Theorem 4.15 hold when $g \leq 8$. In this range, the unirationality of $\mathcal{S}_g^-$ follows from that of $B_{g,g-1}^-$.

Theorem 4.16. $B_{g,g-1}^-$ *is unirational for* $g \leq 10$.

Proof. Let $I \subset \mathbf{P}^2 \times (\mathbf{P}^2)^\vee$ be the natural incidence correspondence consisting of pairs (x,l) such that x is a point on the line l. For $\delta \leq 9$, we define

$$\Pi_\delta := \{(x_1, l_1, \ldots, x_\delta, l_\delta, E) \in I^\delta \times \mathbf{P}H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(3)) : \ x_1, \ldots, x_\delta \in E\}.$$

Then there exists a rational map $f_\delta : \Pi_\delta \dashrightarrow B_{\delta+1,\delta}^-$ sending $(x_1, l_1, \ldots, x_\delta, l_\delta, E)$ to the moduli point of the δ-nodal, integral curve C obtained from the elliptic curve E, by identifying the pairs of points in $E \cap l_i - \{x_i\}$ for $1 \leq i \leq \delta$. It is easy to see that Π_δ is rational if $\delta \leq 9$. Clearly f_δ is dominant, just because every elliptic curve can be realized as a plane cubic. It follows that $B_{\delta+1,\delta}^-$ is unirational when $\delta \leq 9$. □

Unfortunately one cannot apply Theorem 4.16 to the case $g = 9$, since the assumptions of Theorem 4.15 are satisfied only if $\delta \leq 5$.

5. THE STACK OF DEGENERATE ODD THETA-CHARACTERISTICS

In this section we define a Deligne-Mumford stack $\mathbf{X}_g \to \overline{\mathbf{S}}_g^-$ parameterizing limit linear series $\mathfrak{g}^0_{g-1}$ which appear as limits of degenerate theta-characteristics on smooth curves. The push-forward of $[\mathbf{X}_g]$ is going to be precisely our divisor $\overline{\mathcal{Z}}_g$. Having a good description of $\mathbf{X}_g$ over the boundary will enable us to determine all the coefficients in the expression of $[\overline{\mathcal{Z}}_g]$ in $\mathrm{Pic}(\overline{\mathcal{S}}_g^-)$.

We first define a partial compactification $\widetilde{\mathbf{M}}_g := \mathbf{M}_g \cup \widetilde{\Delta}_0 \cup \ldots \cup \widetilde{\Delta}_{[g/2]}$ of $\overline{\mathbf{M}}_g$, obtaining by adding to $\mathbf{M}_g$ the open sub-stack $\widetilde{\Delta}_0 \subset \Delta_0$ of one-nodal irreducible curves $[C_{yq} := C/y \sim q]$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$ is a Brill-Noether general curve together with their degenerations $[C \cup D_\infty]$ where D_∞ is an elliptic curve with $j(D_\infty) = \infty$, as well as the open substacks $\widetilde{\Delta}_j \subset \Delta_j$ for $1 \le j \le [g/2]$ classifying curves $[C \cup_y D]$ where $[C] \in \mathcal{M}_j$ and $[D] \in \mathcal{M}_{g-j}$ are Brill-Noether general curves in the respective moduli spaces. Let $p : \widetilde{\mathbf{M}}_{g,1} \to \widetilde{\mathbf{M}}_g$ be the universal curve. We denote $\widetilde{\mathbf{S}}_g^- := \pi^{-1}(\widetilde{\mathbf{M}}_g) \subset \overline{\mathbf{S}}_g^-$ and note that for all $0 \le j \le [g/2]$ the boundary divisors $A'_j := A_j \cap \widetilde{\mathcal{S}}_g^-,\ B'_j := B_j \cap \widetilde{\mathcal{S}}_g^-$ are mutually disjoint inside $\widetilde{\mathcal{S}}_g^-$. Finally, we consider $\mathcal{Z} := \widetilde{\mathbf{S}}_g^- \times_{\widetilde{\mathbf{M}}_g} \widetilde{\mathbf{M}}_{g,1}$ and denote by $p_1 : \mathcal{Z} \to \widetilde{\mathbf{S}}_g^-$ the projection.

Following the local description of the projection $\overline{\mathbf{S}}_g^- \to \overline{\mathbf{M}}_g$ carried out in [C], in order to obtain the universal spin curve over $\widetilde{\mathbf{S}}_g^-$ one has first to blow-up the codimension 2 locus $V \subset \mathcal{Z}$ corresponding to points

$$v = \Bigl(\bigl[C \cup_{\{y,q\}} E, \eta_C^{\otimes 2} = K_C, h^0(\eta_C) \equiv 1 \bmod 2,\ \eta_E = \mathcal{O}_E(1)\bigr],\ \nu(y) = \nu(q)\Bigr) \in B'_0 \times_{\widetilde{\mathbf{M}}_g} \widetilde{\mathbf{M}}_{g,1}$$

(recall that $\nu : C \to C_{yq}$ denotes the normalization map, so v corresponds to the marked point specializing to the node of the curve C_{yq}). Suppose that $(\tau_1, \ldots, \tau_{3g-3})$ are local coordinates in an étale neighbourhood of $[C \cup_{\{y,q\}} E, \eta_C, \eta_E] \in \widetilde{\mathcal{S}}_g^-$ such that the local equation of the divisor B'_0 is $(\tau_1 = 0)$. Then $\mathcal{Z}$ around v admits local coordinates $(x, y, \tau_1, \ldots, \tau_{3g-3})$ verifying the equation $xy = \tau_1^2$, in particular, $\mathcal{Z}$ is singular along V. Next, for $1 \le j \le [g/2]$ one blows-up the codimension 2 loci $V_j \subset \mathcal{Z}$ consisting of points

$$\Bigl(\bigl[C \cup_q D, \eta_C, \eta_D\bigr],\ q \in C \cap D\Bigr) \in (A'_j \cup B'_j) \times_{\widetilde{\mathbf{M}}_g} \widetilde{\mathbf{M}}_{g,1}.$$

This corresponds to inserting an exceptional component in each spin curve in $\pi^*(\widetilde{\Delta}_j)$. We denote by

$$\mathcal{C} := \mathrm{Bl}_{V \cup V_1 \cup \ldots \cup V_{[g/2]}}(\mathcal{Z})$$

and by $f : \mathcal{C} \to \widetilde{\mathbf{S}}_g^-$ the induced family of spin curves. Then for every $[X, \eta, \beta] \in \widetilde{\mathcal{S}}_g^-$ we have an isomorphism between $f^{-1}([X, \eta, \beta])$ and the quasi-stable curve X.

There exists a spin line bundle $\mathcal{P} \in \mathrm{Pic}(\mathcal{C})$ of relative degree $g-1$ as well as a morphism of $\mathcal{O}_{\mathcal{C}}$-modules $B : \mathcal{P}^{\otimes 2} \to \omega_f$ having the property that $\mathcal{P}_{|f^{-1}([X,\eta,\beta])} = \eta$ and $B_{|f^{-1}([X,\eta,\beta])} = \beta : \eta^{\otimes 2} \to \omega_X$, for all spin curves $[X, \eta, \beta] \in \widetilde{\mathcal{S}}_g^-$. We note that for the even moduli space $\widetilde{\mathcal{S}}_g^+$ one has an analogous construction of the universal spin curve.

Next we define the stack $\tau : \mathbf{X}_g \to \widetilde{\mathbf{S}}_g^-$ classifying limit $\mathfrak{g}^0_{g-1}$ which are twists of degenerate odd-spin curves. For a tree-like curve X we denote by $\overline{G}^r_d(X)$ the scheme of limit linear series $\mathfrak{g}^r_d$. The fibres of the morphism τ have the following description:

• $\tau^{-1}(\mathbf{S}_g^-)$ parameterizes triples $\bigl([C,\eta],\sigma,x\bigr)$, where $[C,\eta]\in\mathcal{S}_g^-,\ x\in C$ is a point and $\sigma\in\mathbf{P}H^0(C,\eta)$ is a section such that $\mathrm{div}(\sigma)\geq 2x$.
• For $1\leq j\leq [g/2]$ the inverse image $\tau^{-1}(A_j'\cup B_j')$ parameterizes elements of the form

$$\Bigl(X,\sigma\in\overline{G}^0_{g-1}(X),\ x\in X_{\mathrm{reg}}\Bigr),$$

where (X,x) is a 1-pointed quasi-stable curve semistably equivalent to the underlying curve of a spin curve $[C\cup_q E\cup_{q'} D,\ \eta_C,\eta_E,\eta_D]\in A_j'\cup B_j'$, with E denoting the exceptional component, $g(C)=j,\ g(D)=g-j, \{q\}=C\cap E, \{q'\}=E\cap D$ and

$$\sigma_C\in\mathbf{P}H^0\bigl(C,\eta_C\otimes\mathcal{O}_C((g-j)q)\bigr),\ \sigma_D\in\mathbf{P}H^0\bigl(D,\eta_D\otimes\mathcal{O}_D(jq')\bigr),\sigma_E\in\mathbf{P}H^0(E,\mathcal{O}_E(g-1))$$

are aspects of the limit linear series σ on X. Moreover, we require that $\mathrm{ord}_x(\sigma)\geq 2$.
• $\tau^{-1}(B_0')$ parameterizes elements $\bigl(X,\ \eta\in\mathrm{Pic}^{g-1}(X),\ \sigma\in\mathbf{P}H^0(X,\eta),\ x\in X_{\mathrm{reg}}\bigr)$, where (X,x) is a 1-pointed quasi-stable curve equivalent to the curve underlying a point $[C\cup_{\{y,q\}}E,\eta_C,\ \eta_E]\in B_0'$, the line bundle η on X satisfies $\eta_{|C}=\eta_C$ and $\eta_{|E}=\eta_E$ and $\eta_{|Z}=\mathcal{O}_Z$ for the remaining components of X. Finally, we require $\mathrm{ord}_x(\sigma)\geq 2$.
• $\tau^{-1}(A_0')$ corresponds to points $\bigl(X,\eta\in\mathrm{Pic}^{g-1}(X),\sigma\in\mathbf{P}H^0(X,\eta),\ x\in X_{\mathrm{reg}}\bigr)$, where (X,x) is a 1-pointed quasi-stable curve equivalent to the curve underlying a point $[C_{yq},\eta_{C_{yq}}]\in A_0'$, and if $\mu: X\to C_{yq}$ is the map contracting all exceptional components, then $\mu^*(\eta_{C_{yq}})=\eta$ (in particular η is trivial along exceptional components), and finally $\mathrm{ord}_x(\sigma)\geq 2$.

Using general constructions of stacks of limit linear series cf. [EH1], [F2], it is clear that $\mathbf{X}_g$ is a Deligne-Mumford stack. There exists a proper morphism

$$\tau:\mathbf{X}_g\to\widetilde{\mathbf{S}}_g^-$$

that factors through the universal curve and we denote by $\chi:\mathbf{X}_g\to\mathcal{C}$ the induced morphism, hence $\tau=f\circ\chi$. The push-forward of the coarse moduli space $\tau_*([\mathcal{X}_g])$ equals scheme-theoretically $\overline{\mathcal{Z}}_g\cap\widetilde{\mathcal{S}}_g^-$. It appears possible to extend $\mathbf{X}_g$ over the entire $\overline{\mathbf{S}}_g^-$ but this is not necessary in order to prove Theorem 0.3 and we skip the details.

We are now in a position to calculate the class of the divisor $\overline{\mathcal{Z}}_g$ and we expand its class in the Picard group of $\overline{\mathcal{S}}_g^-$

$$\overline{\mathcal{Z}}_g\equiv\bar{\lambda}\cdot\lambda-\bar{\alpha_0}\cdot\alpha_0-\bar{\beta}_0\cdot\beta_0-\sum_{i=1}^{[g/2]}\bar{\alpha}_i\cdot\alpha_i-\sum_{i=1}^{[g/2]}\bar{\beta}_i\cdot\beta_i\in\mathrm{Pic}(\overline{\mathcal{S}}_g^-),\tag{4}$$

where $\bar{\lambda},\bar{\alpha}_i,\bar{\beta}_i\in\mathbb{Q}$ for $i=0,\ldots,[g/2]$. We start by determining the coefficients of the divisors α_i and β_i for $1\leq i\leq[g/2]$.

Proposition 5.1. *For $1\leq i\leq[g/2]$ we have that $F_i\cdot\overline{\mathcal{Z}}_g=4(g-i)(i-1)$ and the intersection is everywhere transverse. It follows that $\bar{\alpha}_i=2(g-i)$.*

Proof. We recall from the definition of F_i that we have fixed theta-characteristics of opposite parity $\eta_C^-\in\mathrm{Pic}^{i-1}(C)$ and $\eta_D^+\in\mathrm{Pic}^{g-i-1}(D)$. We choose a point $t=(X,\eta,\sigma,x)\in\tau^{-1}(F_i)$. It is a simple exercise to show that the "double" point x of $\sigma\in\overline{G}^0_{g-1}(X)$ cannot specialize to the exceptional component, therefore one has only two cases to consider depending on whether x lies on C or on D. Assume first that $x\in C$ and then $\sigma_C\in\mathbf{P}H^0(C,\eta_C^-\otimes\mathcal{O}_C((g-i)q))$ and $\sigma_D\in\mathbf{P}H^0(D,\eta_D^+\otimes\mathcal{O}_D(iq))$, where $\{q\}=C\cap D$ is a point which moves on C but is fixed on D. Then $\mathrm{ord}_q(\sigma_D)\leq i-1$, therefore

$\text{ord}_q(\sigma_C) \geq g-i$ and then $\sigma_C(-(g-i)q) \in \mathbf{P}H^0(C, \eta_C^-)$. In particular, if we choose $[C, \eta_C^-] \in \mathcal{S}_i - \mathcal{Z}_i$, then the section $\sigma_C(-(g-i)q)$ has only simple zeros, which shows that x cannot lie on C, so this case does not occur.

We are left with the possibility $x \in D - \{q\}$. One quickly concludes that the only possibility is $\text{ord}_q(\sigma_C) = g-i+1$ and $\text{ord}_q(\sigma_D) = i-2$. In particular, $q \in \text{supp}(\eta_C^-)$ which gives $i-1$ choices for the moving point $q \in C$. Furthermore $\sigma_D(-(i-2)q) \in H^0(D, \eta_D^+ \otimes \mathcal{O}_D(2q-2x))$, that is, x specializes to one of the ramification points of the pencil $\eta_D^+ \otimes \mathcal{O}_D(2q) \in W^1_{g-i+1}(D)$. We note that because of the generality of $[D, \eta_D^+] \in \mathcal{S}^+_{g-i}$ as well as that of $q \in D$, the pencil is base point free and complete. From the Hurwitz-Zeuthen formula one finds $4(g-i)$ ramification points of $|\eta_D^+ \otimes \mathcal{O}_D(2q)|$, which leads to the formula $F_i \cdot \overline{\mathcal{Z}}_g = 4(g-i)(i-1)$. The fact that $\tau_*(\mathbf{X}_g)$ is transverse to F_i follows because the formation of $\mathbf{X}_g$ commutes with restriction to B_0' and then one can easily show in a way similar to [EH2] Lemma 3.4 or by direct calculation that $\mathbf{X}_g \times_{\widetilde{\mathbf{S}}_g^-} B_0'$ is smooth at any of the points in $\tau^{-1}(F_i)$. ☐

Proposition 5.2. *For $1 \leq i \leq [g/2]$ we have that $G_i \cdot \overline{\mathcal{Z}}_g = 4i(i-1)$ and the intersection is transversal. In particular $\bar{\beta}_i = 2i$.*

Proof. This time we fix general points $[C, \eta_C^+] \in \mathcal{S}_i^+$ and $[D, \eta_D^-] \in \mathcal{S}^-_{g-i}$ and $q \in C \cap D$ which is a fixed general point on D but an arbitrary point on C. Again, it is easy to see that if $t = (X, \sigma, x) \in \tau^{-1}(G_i)$ then x must lie either on C or on D. Assume first that $x \in C - \{q\}$. Then the aspects of σ are described as follows

$$\sigma_C \in \mathbf{P}H^0(C, \eta_C^+ \otimes \mathcal{O}_C((g-i)q)), \ \ \sigma_D \in \mathbf{P}H^0(D, \eta_D^- \otimes \mathcal{O}_D(iq))$$

and moreover $\text{ord}_x(\sigma_C) \geq 2$. The point $q \in D$ can be chosen so that it does not lie in $\text{supp}(\eta_D^-)$, hence $\text{ord}_q(\sigma_D) \leq i$ and then $\text{ord}_q(\sigma_C) \geq g-i-1$. This leads to the conclusion $H^0(C, \eta_C^+ \otimes \mathcal{O}_C(y-2x)) \neq 0$, or equivalently $(x,y) \in C \times C$ is a ramification point of the degree i covering $p_1 : T_{\eta_C^+} \to C$ from the associated Scorza curve. We have shown that $T_{\eta_C^+}$ is smooth of genus $1+3i(i-1)$ (cf. Theorem 2.1) and moreover all the ramification points of p_1 are ordinary, therefore we find

$$\deg \text{Ram}(p_1) = 2g(T_{C_{\eta_C^+}}) - 2 - \deg(p_1)(2i-2) = 4i(i-1)$$

choices when $x \in C$. Next possibility is $x \in D - \{q\}$. The same reasoning as above shows that $\text{ord}_q(\sigma_C) \leq g-i-1$, therefore $\text{ord}_q(\sigma_D) \geq i$ as well as $\text{ord}_x(\sigma_D) \geq 2$. Since $\sigma_D(-iq) \in \mathbf{P}H^0(D, \eta_D^-)$, this case does not occur if $[D, \eta_D^-] \in \mathcal{S}^-_{g-i} - \mathcal{Z}_{g-i}$. ☐

Next we prove that $\overline{\mathcal{Z}}_g$ is disjoint from both elliptic pencils F_0 and G_0:

Proposition 5.3. *We have that $F_0 \cdot \overline{\mathcal{Z}}_g = 0$ and $G_0 \cdot \overline{\mathcal{Z}}_g = 0$. The equalities $\bar{\alpha} - 12\bar{\alpha}_0 + \bar{\alpha}_1 = 0$ and $3\bar{\alpha} - 12\bar{\alpha}_0 - 12\bar{\beta}_0 + 3\bar{\beta}_1 = 0$ follow.*

Proof. We first show that $F_0 \cap \overline{\mathcal{Z}}_g = \emptyset$ and we assume by contradiction that there exists $t = (X, \sigma, x) \in \tau^{-1}(F_0)$. Let us deal first with the case when $st(X) = C \cap E_\lambda$, with E_λ being a smooth curve of genus 1. The key point is that the point of attachment $q \in C \cap E_\lambda$ being general, we can assume that $(x,q) \notin \text{Ram}\{p_1 : T_{\eta_C^+} \to C\}$, for all $x \in C$. This implies that $H^0(C, \eta_C^+ \otimes \mathcal{O}_C(q-2x)) = 0$ for all $x \in C$, therefore a section $\sigma_C \in \mathbf{P}H^0(C, \eta_C^+ \otimes \mathcal{O}_C(q))$ cannot vanish twice anywhere. Thus either $x \in E_\lambda - \{q\}$

or x lies on some exceptional component of X. In the former case, since $\mathrm{ord}_q(\sigma_C) = 0$, it follows that $\mathrm{ord}_q(\sigma_{E_\lambda}) \geq g-1$, that is, σ_{E_λ} has no zeroes other than q (simple or otherwise). In the latter case, when $x \in E$, with E being an exceptional component, we denote by $q' \in E$ the point of intersection of E with the connected subcurve of X containing C as a subcomponent. Since as above, $\mathrm{ord}_q(\sigma_C) = 0$, by compatibility it follows that $\mathrm{ord}_{q'}(\sigma_E) = g-1$. But $\sigma_E \in \mathbf{P}H^0(E, \mathcal{O}_E(g-1))$, that is, σ_E does not vanish at x, a contradiction. The proof that $G_0 \cap \overline{\mathcal{Z}}_g = \emptyset$ is similar and we omit the details. $\square$

The trickiest part in the calculation of $[\overline{\mathcal{Z}}_f]$ is the computation of the following intersection number:

Proposition 5.4. *If $H_0 \subset B_0$ is the covering family lying in the ramification divisor of $\overline{\mathcal{S}}_g^-$, then one has that $H_0 \cdot \overline{\mathcal{Z}}_g = 2(g-2)$ and the intersection consists of $g-2$ points each counted with multiplicity 2. Therefore the relation $(g-1)\bar{\beta}_0 - \bar{\beta}_1 = 2(g-2)$ holds.*

Proof. We first determine the set-theoretic intersection $\tau_*(\mathcal{X}_g) \cap H_0$. We recall that we have fixed $[C, q, \eta_C^-] \in \mathcal{S}_{g-1,1}^-$ and start by choosing an arbitrary point $t = (X, \eta, \sigma, x) \in \tau^{-1}(H_0)$. Assume first that $X = C \cup_{\{y,q\}} E$, where $y \in C$, that is, x does not specialize to one of the nodes of $C \cup E$. Suppose first that $x \in C - \{y, q\}$. From the Mayer-Vietoris sequence on X we write

$$0 \neq \sigma \in H^0(X, \eta \otimes \mathcal{O}_X(-2x)) = \mathrm{Ker}\big\{H^0(C, \eta_C^- \otimes \mathcal{O}_C(-2x)) \oplus H^0(E, \mathcal{O}_E(1)) \stackrel{ev_{y,q}}{\longrightarrow} \mathbb{C}^2_{\{y,q\}}\big\},$$

we obtain that $H^0(C, \eta_C^- \otimes \mathcal{O}_C(-2x)) \neq 0$. This case can be avoided by choosing $[C, \eta_C^-] \in \mathcal{S}_{g-1}^- - \mathcal{Z}_{g-1}$.

Next we consider the possibility $x \in E - \{y, q\}$. The same Mayer-Vietoris argument reads in this case $0 \neq \mathrm{Ker}\big\{H^0(C, \eta_C^-) \oplus H^0(E, \mathcal{O}_E(-1)) \stackrel{ev_{y,q}}{\longrightarrow} \mathbb{C}^2_{\{y,q\}}\big\}$, that is, $y + q \in \mathrm{supp}(\eta_C^-)$. This case can be avoided as well by starting with a general point $q \in C - \mathrm{supp}(\eta_C^-)$. Thus the only possibility is that x specializes to one of the nodes y or q.

We deal first with the case when x and q coalesce and there is no loss of generality in assuming that $X = C \cup E \cup E'$, where both E and E' are copies of $\mathbf{P}^1$ and $C \cap E = \{y\}, C \cap E' = \{q\}, E \cap E' = \{y'\}$ and moreover $x \in E' - \{y', q\}$. The restrictions of the line bundle $\eta \in \mathrm{Pic}^{g-1}(X)$ are such that $\eta_{|C} = \eta_C^-, \eta_E = \mathcal{O}_E(1)$ and $\eta_{E'} = \mathcal{O}_{E'}$. We write

$$0 \neq \sigma = (\sigma_C, \sigma_E, \sigma_{E'}) \in \mathrm{Ker}\big\{H^0(C, \eta_C^-) \oplus H^0(E, \mathcal{O}_E(1)) \oplus H^0(E', \mathcal{O}_{E'}(1)) \stackrel{ev_{y,y',q}}{\longrightarrow} \mathbb{C}_{y,y',q}\big\},$$

hence $\sigma_{E'} = 0$, and then by compatibility $\sigma_C(q) = 0$, that is, $q \in \mathrm{supp}(\eta_C^-)$ and again this case can be ruled out by a suitable choice of q. The last possible situation is when x and the moving point $y \in C$ coalesce, in which case $X = C \cup E \cup E'$, where this time $C \cap E = \{q\}, C \cap E' = \{y\}, E \cap E' = \{y'\}$ and again $x \in E' - \{y', q\}$. Writing one last time the Mayer-Vietoris sequence we find that $\sigma_{E'} = 0$ and then $\sigma_E(y') = 0$ and $\sigma_C(y) = 0$, that is, $y \in \mathrm{supp}(\eta_C^-)$ and then σ_C is uniquely determined up to a constant. Finally $\sigma_E \in H^0(E, \mathcal{O}_E(1)(-y'))$ is also uniquely specified by the gluing condition $\sigma_E(q) = \sigma_C(q)$. All in all, $H_0 \cap \overline{\mathcal{Z}}_g = \#\mathrm{supp}(\eta_C^-) = g-2$.

This discussion singles out an irreducible component $\Xi \subset \chi_*(\mathcal{X}_g) \subset \mathcal{C}$ of the intersection $\chi(\mathcal{X}_g) \cap f^{-1}(B_0')$, namely

$$\Xi = \{\big([C \cup_{\{y,q\}} E, \eta_C, \eta_E], x\big) : y \in \mathrm{supp}(\eta_C^-),\ x = y \in X_{\mathrm{sing}}\},$$

where we recall that $f : \mathcal{C} \to \widetilde{\mathbf{S}}_g^-$ is the universal spin curve. Since $\Xi \subset \mathrm{Sing}\big(\chi_*(\mathcal{X}_g)\big)$, it follows after a simple local analysis that each point in $\tau^{-1}(H_0)$ should be counted with multiplicity 2. □

Remark 5.5. An partial independent check of Theorem 0.4 is obtained by computing using the Porteous formula the coefficient $\bar{\lambda}$ in the expression of $[\overline{\mathcal{Z}}_g]$. By an abuse of notation we still denote by $f : \mathcal{C} \to \mathbf{S}_g^-$ the restriction of the universal spin curve to the locus of smooth curves and $\eta \in \mathrm{Pic}(\mathcal{C})$ the spin bundle of relative degree $g-1$. Then $\mathcal{Z}_g$ is the push-forward via $f : \mathcal{C} \to \mathbf{S}_g^-$ of the degeneration locus of the sheaf morphism $\phi : f_*(\eta) \to J_1(\eta)$ (both these sheaves are locally free away a subset of codimension 3 in $\mathbf{S}_g^-$ and throwing away this locus has no influence on divisor class calculations). Since $\det(f_*\eta) = (f_*\eta)^{\otimes 2}$, it follows that $c_1\big(f_*(\eta)\big) = -\lambda/4$, whereas the Chern classes of the first jet bundle $J_1(\eta)$ are calculated using the standard exact sequence on $\mathcal{C}$

$$0 \longrightarrow \eta \otimes \omega_f \longrightarrow J_1(\eta) \longrightarrow \eta \longrightarrow 0.$$

Remembering Mumford's formula $f_*(c_1^2(\omega_f)) = 12\lambda$, one finally writes that

$$[\mathcal{Z}_g] = f_*c_2\Big(J_1(\eta) - f_*(\eta)\Big) = f_*\Big(\frac{3}{4}c_1(\omega_f)^2 - 2c_1(\omega_f)\cdot c_1(f_*(\eta))\Big) = (g+8)\,\lambda \in \mathrm{Pic}(\mathbf{S}_g^-).$$

6. A divisor of small slope on $\overline{\mathcal{M}}_{12}$

The aim of this section is to construct an effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_{12})$ of slope $s(D) < 6 + 12/13$, that is, violating the Slope Conjecture. As pointed out in the proof of Theorem 0.3, this is precisely what is needed to show that $\overline{\mathcal{S}}_{12}^-$ is of general type.

Theorem 6.1. *The following locus consisting of curves of genus* 12

$\mathfrak{D}_{12} := \{[C] \in \mathcal{M}_{12} : \exists L \in W^4_{14}(C)$ *with* $\mu_0(L) : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ *not injective*$\}$

is a divisor on $\mathcal{M}_{12}$. *The class of its compactification inside* $\overline{\mathcal{M}}_{12}$ *equals*

$$\overline{\mathfrak{D}}_{12} \equiv 13245\ \lambda - 1926\ \delta_0 - 9867\ \delta_1 - \sum_{j=2}^{6} b_j\ \delta_j \in \mathrm{Pic}(\overline{\mathcal{M}}_{12}),$$

where $b_j \geq b_1$ *for* $j \geq 2$. *In particular,* $s(\overline{\mathfrak{D}}_{12}) = \frac{4415}{642} < 6 + \frac{12}{13}$.

This implies the following upper bound for the slope $s(\overline{\mathcal{M}}_{12})$ of the moduli space:

Corollary 6.2.

$$6 + \frac{10}{12} \leq s(\overline{\mathcal{M}}_{12}) := \inf_{D \in \mathrm{Eff}(\overline{\mathcal{M}}_{12})} s(D) \leq \frac{4415}{642}\ \big(= 6 + \frac{10}{12} + \frac{14}{321}\big).$$

Another immediate application, via [Log], [F1], concerns the birational type of $\overline{\mathcal{M}}_{g,n}$:

Theorem 6.3. *The moduli space of n-pointed curves $\overline{\mathcal{M}}_{12,n}$ is of general type for $n \geq 11$.*

The divisor $\mathfrak{D}_{12}$ is constructed as the push-forward of a codimension 3 cycle in the stack $\mathfrak{G}^4_{14} \to \mathbf{M}_{12}$ classifying linear series $\mathfrak{g}^4_{14}$. We describe the construction of this cycle, then extend this determinantal structure over a partial compactification of $\mathcal{M}_{12}$. This will be essential to understand the intersection of $\overline{\mathfrak{D}}_{12}$ with the boundary divisors Δ_0 and Δ_1 of $\overline{\mathcal{M}}_{12}$. We denote by $\mathbf{M}^p_{12}$ the open substack of $\mathbf{M}_{12}$ consisting of curves

$[C] \in \mathcal{M}_{12}$ such that $W^4_{13}(C) = \emptyset$ and $W^5_{14}(C) = \emptyset$. Results in Brill-Noether theory guarantee that $\mathrm{codim}(\mathcal{M}_{12} - \mathcal{M}^p_{12}, \mathcal{M}_{12}) \geq 3$. If $\mathfrak{Pic}^{14}_{12}$ denotes the Picard stack of degree 14 over $\mathbf{M}^p_{12}$, then we consider the smooth Deligne-Mumford substack $\mathfrak{G}^4_{14} \subset \mathfrak{Pic}^{14}_{12}$ parameterizing pairs $[C, L]$, where $[C] \in \mathcal{M}^p_{12}$ and $L \in W^4_{14}(C)$ is a (necessarily complete and base point free) linear series. We denote by $\sigma : \mathfrak{G}^4_{14} \to \mathbf{M}^p_{12}$ the forgetful morphism. For a general $[C] \in \mathcal{M}^p_{12}$, the fibre $\sigma^{-1}([C]) = W^4_{14}(C)$ is a smooth surface.

Let $\pi : \mathbf{M}^p_{12,1} \to \mathbf{M}^p_{12}$ be the universal curve and then $p_2 : \mathbf{M}^p_{12,1} \times_{\mathbf{M}^p_{12}} \mathfrak{G}^4_{14} \to \mathfrak{G}^4_{14}$ denotes the natural projection. If $\mathcal{L}$ is a Poincaré bundle over $\mathbf{M}^p_{12,1} \times_{\mathbf{M}^p_{12}} \mathfrak{G}^4_{14}$ (or over an étale cover), then by Grauert's Theorem, both $\mathcal{E} := (p_2)_*(\mathcal{L})$ and $\mathcal{F} := (p_2)_*(\mathcal{L}^{\otimes 2})$ are vector bundles over $\mathfrak{G}^4_{14}$, with $\mathrm{rank}(\mathcal{E}) = 5$ and $\mathrm{rank}(\mathcal{F}) = h^0(C, L^{\otimes 2}) = 17$ respectively. There is a natural vector bundle morphism over $\mathfrak{G}^4_{14}$ given by multiplication of sections,

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F},$$

and we denote by $\mathcal{U}_{12} \subset \mathfrak{G}^4_{14}$ its first degeneracy locus. We set $\mathfrak{D}_{12} := \sigma_*(\mathcal{U}_{12})$. Since the degeneracy locus $\mathcal{U}_{12}$ has expected codimension 3 inside $\mathfrak{G}^4_{14}$, the locus $\mathfrak{D}_{12}$ is a virtual divisor on $\mathcal{M}^p_{12}$.

We extend the vector bundles $\mathcal{E}$ and $\mathcal{F}$ over a partial compactification of $\mathfrak{G}^4_{14}$ given by limit $\mathfrak{g}^4_{14}$. We denote by $\Delta_1^p \subset \Delta_1 \subset \overline{\mathcal{M}}_{12}$ the locus of curves $[C \cup_y E]$, where E is an arbitrary elliptic curve, $[C] \in \mathcal{M}_{11}$ is a Brill-Noether general curve and $y \in C$ is an arbitrary point. We then denote by $\Delta_0^p \subset \Delta_0 \subset \overline{\mathcal{M}}_{12}$ the locus consisting of curves $[C_{yq}] \in \Delta_0$, where $[C, q] \in \mathcal{M}_{11,1}$ is Brill-Noether general and $y \in C$ is arbitrary, as well as their degenerations $[C \cup_q E_\infty]$ where E_∞ is a rational nodal curve. Once we set

$$\overline{\mathbf{M}}^p_{12} := \mathbf{M}^p_{12} \cup \Delta_0^p \cup \Delta_1^p \subset \overline{\mathbf{M}}_{12},$$

we can extend the morphism σ to a proper morphism

$$\sigma : \widetilde{\mathfrak{G}}^4_{14} \to \overline{\mathbf{M}}^p_{12},$$

from the stack $\widetilde{\mathfrak{G}}^4_{14}$ of limit linear series $\mathfrak{g}^4_{14}$ over the partial compactification $\overline{\mathbf{M}}^p_{12}$ of $\mathbf{M}_{12}$.

We extend the vector bundles $\mathcal{E}$ and $\mathcal{F}$ over the stack $\widetilde{\mathfrak{G}}^4_{14}$. The proof of the following result proceeds along the lines of the proof of Proposition 3.9 in [F1]:

Proposition 6.4. *There exist two vector bundles $\mathcal{E}$ and $\mathcal{F}$ defined over $\widetilde{\mathfrak{G}}^4_{14}$ with $\mathrm{rank}(\mathcal{E}) = 5$ and $\mathrm{rank}(\mathcal{F}) = 17$, together with a vector bundle morphism $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$, such that the following statements hold:*

- *For $[C, L] \in \mathfrak{G}^4_{14}$, with $[C] \in \mathcal{M}^p_{12}$, we have that*

$$\mathcal{E}(C, L) = H^0(C, L) \ \text{ and } \ \mathcal{F}(C, L) = H^0(C, L^{\otimes 2}).$$

- *For $t = (C \cup_y E, l_C, l_E) \in \sigma^{-1}(\Delta_1^p)$, where $g(C) = 11, g(E) = 1$ and $l_C = |L_C|$ is such that $L_C \in W^4_{14}(C)$ has a cusp at $y \in C$, then $\mathcal{E}(t) = H^0(C, L_C)$ and*

$$\mathcal{F}(t) = H^0(C, L_C^{\otimes 2}(-2y)) \oplus \mathbb{C} \cdot u^2,$$

where $u \in H^0(C, L_C)$ is any section such that $\mathrm{ord}_y(u) = 0$. If L_C has a base point at y, then $\mathcal{E}(t) = H^0(C, L_C) = H^0(C, L_C \otimes \mathcal{O}_C(-y))$ and the image of a natural map $\mathcal{F}(t) \to H^0(C, L_C^{\otimes 2})$ is the subspace $H^0(C, L_C^{\otimes 2} \otimes \mathcal{O}_C(-2y))$.

- *Fix $t = [C_{yq} := C/y \sim q, L] \in \sigma^{-1}(\Delta_0^p)$, with $q, y \in C$ and $L \in \overline{W}^4_{14}(C_{yq})$ such that $h^0(C, \nu^* L \otimes \mathcal{O}_C(-y-q)) = 4$, where $\nu : C \to C_{yq}$ is the normalization map. In the case when L is locally free we have that*
  $$\mathcal{E}(t) = H^0(C, \nu^* L) \ \text{and} \ \mathcal{F}(t) = H^0(C, \nu^* L^{\otimes 2} \otimes \mathcal{O}_C(-y-q)) \oplus \mathbb{C} \cdot u^2,$$
  where $u \in H^0(C, \nu^* L)$ is any section not vanishing at y and q. In the case when L is not locally free, that is, $L \in \overline{W}^4_{14}(C_{yq}) - W^4_{14}(C_{yq})$, then $L = \nu_*(A)$, where $A \in W^4_{13}(C)$ and the image of the natural map $\mathcal{F}(t) \to H^0(C, \nu^* L^{\otimes 2})$ is the subspace $H^0(C, A^{\otimes 2})$.

To determine the push-forward $[\overline{\mathfrak{D}}_{12}]^{virt} = \sigma_*\big(c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E})\big) \in A^1(\mathcal{M}^p_{12})$, we study the restriction of the morphism ϕ along the pull-backs of two curves sitting in the boundary of $\overline{\mathcal{M}}_{12}$ and which are defined as follows: We fix a general pointed curve $[C, q] \in \mathcal{M}_{11,1}$ and a general elliptic curve $[E, y] \in \mathcal{M}_{1,1}$. Then we consider the families

$$C_0 := \{C/y \sim q : y \in C\} \subset \Delta_0^p \subset \overline{\mathcal{M}}_{12} \text{ and } C_1 := \{C \cup_y E : y \in C\} \subset \Delta_1^p \subset \overline{\mathcal{M}}_{12}.$$

These curves intersect the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{12})$ as follows:

$$C_0 \cdot \lambda = 0, \ C_0 \cdot \delta_0 = \deg(\omega_{C_{yq}}) = -22, \ C_0 \cdot \delta_1 = 1 \text{ and } C_0 \cdot \delta_j = 0 \text{ for } 2 \le j \le 6, \text{ and}$$
$$C_1 \cdot \lambda = 0, \ C_1 \cdot \delta_0 = 0, \ C_1 \cdot \delta_1 = -\deg(K_C) = -20 \text{ and } C_1 \cdot \delta_j = 0 \text{ for } 2 \le j \le 6.$$

Next, we fix a general pointed curve $[C, q] \in \mathcal{M}_{11,1}$ and describe the geometry of the pull-back $\sigma^*(C_0) \subset \widetilde{\mathfrak{G}}^4_{14}$. We consider the determinantal 3-fold

$$Y := \{(y, L) \in C \times W^4_{14}(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = 4\}$$

together with the projection $\pi_1 : Y \to C$. Inside Y we consider the following divisors

$$\Gamma_1 := \{(y, A \otimes \mathcal{O}_C(y)) : y \in C, \ A \in W^4_{13}(C)\} \ \text{and}$$
$$\Gamma_2 := \{(y, A \otimes \mathcal{O}_C(q)) : y \in C, \ A \in W^4_{13}(C)\}$$

intersecting transversally along the curve $\Gamma := \{(q, A \otimes \mathcal{O}_C(q)) : A \in W^4_{13}(C)\} \cong W^4_{13}(C)$. We introduce the blow-up $Y' \to Y$ of Y along Γ and denote by $E_\Gamma \subset Y'$ the exceptional divisor and by $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2 \subset Y'$ the strict transforms of Γ_1 and Γ_2 respectively. We then define $\widetilde{Y} := Y'/\widetilde{\Gamma}_1 \cong \widetilde{\Gamma}_2$, to be the variety obtained from Y' by identifying the divisors $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ over each $(y, A) \in C \times W^4_{13}(C)$. Let $\epsilon : \widetilde{Y} \to Y$ be the projection map.

Proposition 6.5. *With notation as above, one has a birational morphism of 3-folds*

$$f : \sigma^*(C_0) \to \widetilde{Y},$$

which is an isomorphism outside a curve contained in $\epsilon^{-1}(\pi_1^{-1}(q))$. The map $f_{|(\pi_1 \epsilon f)^{-1}(q)}$ corresponds to forgetting the E_∞-aspect of each limit linear series. Accordingly, the vector bundles $\mathcal{E}_{|\sigma^*(C_0)}$ and $\mathcal{F}_{|\sigma^*(C_0)}$ are pull-backs under $\epsilon \circ f$ of vector bundles on Y.

Proof. We fix a point $y \in C - \{q\}$ and denote by $\nu : C \to C_{yq}$ the normalization map, with $\nu(y) = \nu(q)$. We investigate the variety $\overline{W}^4_{14}(C_{yq}) \subset \overline{\mathrm{Pic}}^{14}(C_{yq})$ of torsion-free sheaves L on C_{yq} with $\deg(L) = 14$ and $h^0(C_{yq}, L) \ge 5$. A locally free $L \in \overline{W}^4_{14}(C_{yq})$ is determined by $\nu^*(L) \in W^4_{14}(C)$, which has the property $h^0(C, \nu^* L \otimes \mathcal{O}_C(-y-q)) = 4$ (use that since $W^4_{12}(C) = \emptyset$, there exists a section of L that does not vanish simultaneously at both y and q). However, the line bundles of type $A \otimes \mathcal{O}_C(y)$ or $A \otimes \mathcal{O}_C(q)$ with $A \in W^4_{13}(C)$, do not appear in this association, though $(y, A \otimes \mathcal{O}_C(y)), (y, A \otimes \mathcal{O}_C(q)) \in$

Y. In fact, they correspond to the situation when $L \in \overline{W}^4_{14}(C_{yq})$ is not locally free, in which case necessarily $L = \nu_*(A)$ for some $A \in W^4_{13}(C)$. Thus, for a point $y \in C - \{q\}$, there is a birational morphism $\pi_1^{-1}(y) \to \overline{W}^4_{14}(C_{yq})$ which is an isomorphism over the locus of locally free sheaves. More precisely, $\overline{W}^4_{14}(C_{yq})$ is obtained from $\pi_1^{-1}(y)$ by identifying the disjoint divisors $\Gamma_1 \cap \pi_1^{-1}(y)$ and $\Gamma_2 \cap \pi_1^{-1}(y)$.

A special analysis is required when $y = q$, when C_{yq} degenerates to $C \cup_q E_\infty$, where E_∞ is a rational nodal cubic. If $\{l_C, l_{E_\infty}\} \in \sigma^{-1}([C \cup_q E_\infty])$, then the corresponding Brill-Noether numbers with respect to q satisfy $\rho(l_C, q) \geq 0$ and $\rho(l_{E_\infty}, q) \leq 2$. The statement about the restrictions $\mathcal{E}_{|\sigma^*(C_0)}$ and $\mathcal{F}_{|\sigma^*(C_0)}$ follows, because both restrictions are defined by dropping the information coming from the elliptic tail. □

To describe $\sigma^*(C_1) \subset \widetilde{\mathfrak{G}}^4_{14}$, where $[C] \in \mathcal{M}_{11}$, we define the determinantal 3-fold

$$X := \{(y, L) \in C \times W^4_{14}(C) : h^0(L \otimes \mathcal{O}_C(-2y)) = 4\}.$$

In what follows we use notation from [EH1], to denote vanishing sequences of limit linear series:

Proposition 6.6. *With notation as above, the 3-fold X is an irreducible component of $\sigma^*(C_1)$. Moreover one has that $c_3\big((\mathcal{F} - \mathrm{Sym}^2\mathcal{E})_{|\sigma^*(C_1)}\big) = c_3\big((\mathcal{F} - \mathrm{Sym}^2\mathcal{E})_{|X}\big)$.*

Proof. By the additivity of the Brill-Noether number, if $\{l_C, l_E\} \in \sigma^{-1}([C \cup_y E])$, we have that $2 = \rho(12, 4, 14) \geq \rho(l_C, y) + \rho(l_E, y)$. Since $\rho(l_E, y) \geq 0$, we obtain that $\rho(l_C, y) \leq 2$. If $\rho(l_E, y) = 0$, then $l_E = 9y + |\mathcal{O}_E(5y)|$, that is, l_E is uniquely determined, while the aspect $l_C \in G^4_{14}(C)$ is a complete $\mathfrak{g}^4_{14}$ with a cusp at the variable point $y \in C$. This gives rise to an element from X. The remaining components of $\sigma^*(C_1)$ are indexed by Schubert indices $\bar{\alpha} := (0 \leq \alpha_0 \leq \cdots \leq \alpha_4 \leq 10)$ such that $\bar{\alpha} > (0, 1, 1, 1, 1)$ and $5 \leq \sum_{j=0}^4 \alpha_j \leq 7$. For such $\bar{\alpha}$, we set $\bar{\alpha}^c := (10 - \alpha_4, \ldots, 10 - \alpha_0)$ to be the complementary Schubert index, then define

$$X_{\bar{\alpha}} := \{(y, l_C) \in C \times G^4_{14}(C) : \alpha^{l_C}(y) \geq \bar{\alpha}\} \text{ and } Z_{\bar{\alpha}} := \{l_E \in G^4_{14}(E) : \alpha^{l_E}(y) \geq \bar{\alpha}^c\}.$$

Then $\sigma^*(C_1) = X + \sum_{\bar{\alpha}} X_{\bar{\alpha}} \times Z_{\bar{\alpha}}$. The last claim follows by dimension reasons. Since $\dim X_{\bar{\alpha}} = 1 + \rho(11, 4, 14) - \sum_{j=0}^4 \alpha_j < 3$, for every $\bar{\alpha} > (0, 1, 1, 1, 1)$ and the restrictions of both $\mathcal{E}$ and $\mathcal{F}$ are pulled-back from $X_{\bar{\alpha}}$, one obtains that $c_3(\mathcal{F} - \mathrm{Sym}^2\mathcal{E})_{|X_{\bar{\alpha}} \times Z_{\bar{\alpha}}} = 0$. □

We also recall standard facts about intersection theory on Jacobians. For a Brill-Noether general curve $[C] \in \mathcal{M}_g$, we denote by $\mathcal{P}$ a Poincaré bundle on $C \times \mathrm{Pic}^d(C)$ and by $\pi_1 : C \times \mathrm{Pic}^d(C) \to C$ and $\pi_2 : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ the projections. We define the cohomology class $\eta = \pi_1^*([\mathrm{point}]) \in H^2(C \times \mathrm{Pic}^d(C))$, and if $\delta_1, \ldots, \delta_{2g} \in H^1(C, \mathbb{Z}) \cong H^1(\mathrm{Pic}^d(C), \mathbb{Z})$ is a symplectic basis, then we set

$$\gamma := -\sum_{\alpha=1}^{g} \Big(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Big) \in H^2(C \times \mathrm{Pic}^d(C)).$$

One has the formula $c_1(\mathcal{P}) = d\eta + \gamma$, corresponding to the Hodge decomposition of $c_1(\mathcal{P})$, as well as the relations $\gamma^3 = 0,\ \gamma\eta = 0,\ \eta^2 = 0$ and $\gamma^2 = -2\eta\pi_2^*(\theta)$. On $W^r_d(C)$ there is a tautological rank $r+1$ vector bundle $\mathcal{M} := (\pi_2)_*(\mathcal{P}_{|C \times W^r_d(C)})$. To compute the Chern numbers of $\mathcal{M}$ we employ the Harris-Tu formula [HT]. We write $\sum_{i=0}^r c_i(\mathcal{M}^\vee) =$

$(1+x_1)\cdots(1+x_{r+1})$, and then for every class $\zeta \in H^*(\mathrm{Pic}^d(C),\mathbb{Z})$ one has the following formula:

$$x_1^{i_1}\cdots x_{r+1}^{i_{r+1}}\,\zeta = \det\Big(\frac{\theta^{g+r-d+i_j-j+l}}{(g+r-d+i_j-j+l)!}\Big)_{1\le j,l\le r+1}\,\zeta. \tag{5}$$

We compute the classes of the 3-folds that appear in Propositions 6.5 and 6.6:

Proposition 6.7. *Let $[C,q]\in\mathcal{M}_{11,1}$ be a Brill-Noether general pointed curve. If $\mathcal{M}$ denotes the tautological rank 5 vector bundle over $W^4_{14}(C)$ and $c_i := c_i(\mathcal{M}^\vee)\in H^{2i}(W^4_{14}(C),\mathbb{C})$, then one has the following relations:*

(i) $[X]=\pi_2^*(c_4)-6\eta\theta\pi_2^*(c_2)+(48\eta+2\gamma)\pi_2^*(c_3)\in H^8(C\times W^4_{14}(C),\mathbb{C})$.
(ii) $[Y]=\pi_2^*(c_4)-2\eta\theta\pi_2^*(c_2)+(13\eta+\gamma)\pi_2^*(c_3)\in H^8(C\times W^4_{14}(C),\mathbb{C})$.

Proof. We start by noting that $W^4_{14}(C)$ is a smooth 6-fold isomorphic to the symmetric product C_6. We realize X as the degeneracy locus of a vector bundle morphism defined over $C\times W^4_{14}(C)$. For each pair $(y,L)\in C\times W^4_{14}(C)$, there is a natural map

$$H^0(C,L\otimes\mathcal{O}_{2y})^\vee\to H^0(C,L)^\vee$$

which globalizes to a vector bundle morphism $\zeta: J_1(\mathcal{P})^\vee\to\pi_2^*(\mathcal{M})^\vee$ over $C\times W^4_{14}(C)$. Then we have the identification $X=Z_1(\zeta)$ and the Thom-Porteous formula gives that $[X]=c_4\big(\pi_2^*(\mathcal{M})-J_1(\mathcal{P}^\vee)\big)$. From the usual exact sequence over $C\times\mathrm{Pic}^{14}(C)$

$$0\longrightarrow\pi_1^*(K_C)\otimes\mathcal{P}\longrightarrow J_1(\mathcal{P})\longrightarrow\mathcal{P}\longrightarrow 0,$$

we can compute the total Chern class of the jet bundle

$$c_t(J_1(\mathcal{P})^\vee)^{-1}=\Big(\sum_{j\ge 0}(d(L)\eta+\gamma)^j\Big)\cdot\Big(\sum_{j\ge 0}\big((2g(C)-2+d(L))\eta+\gamma\big)^j\Big)=1-6\eta\theta+48\eta+2\gamma,$$

which quickly leads to the formula for $[X]$. To compute $[Y]$ we proceed in a similar way. We denote by $\mu,\nu: C\times C\times\mathrm{Pic}^{14}(C)\to C\times\mathrm{Pic}^{14}(C)$ the two projections, by $\Delta\subset C\times C\times\mathrm{Pic}^{14}(C)$ the diagonal and we set $\Gamma_q:=\{q\}\times\mathrm{Pic}^{14}(C)$. We introduce the rank 2 vector bundle $\mathcal{B}:=(\mu)_*\big(\nu^*(\mathcal{P})\otimes\mathcal{O}_{\Delta+\nu^*(\Gamma_q)}\big)$ defined over $C\times W^4_{14}(C)$. We note that there is a bundle morphism $\chi:\mathcal{B}^\vee\to(\pi_2)^*(\mathcal{M})^\vee$, such that $Y=Z_1(\chi)$. Since we also have that

$$c_t(\mathcal{B}^\vee)^{-1}=\big(1+(d(L)\eta+\gamma)+(d(L)\eta+\gamma)^2+\cdots\big)\big(1-\eta\big),$$

we immediately obtained the stated expression for $[Y]$. □

Proposition 6.8. *Let $[C]\in\mathcal{M}_{11}$ and denote by $\mu,\nu: C\times C\times\mathrm{Pic}^{14}(C)\to C\times\mathrm{Pic}^{14}(C)$ the natural projections. We define the vector bundles $\mathcal{A}_2$ and $\mathcal{B}_2$ on $C\times\mathrm{Pic}^{14}(\mathrm{C})$ having fibres*

$$\mathcal{A}_2(y,L)=H^0(C,L^{\otimes 2}\otimes\mathcal{O}_C(-2y))\ \text{ and }\ \mathcal{B}_2(y,L)=H^0(C,L^{\otimes 2}\otimes\mathcal{O}_C(-y-q)),$$

respectively. One has the following formulas:

$$c_1(\mathcal{A}_2)=-4\theta-4\gamma-76\eta\quad c_1(\mathcal{B}_2)=-4\theta-2\gamma-27\eta,$$

$$c_2(\mathcal{A}_2)=8\theta^2+280\eta\theta+16\gamma\theta,\quad c_2(\mathcal{B}_2)=8\theta^2+100\eta\theta+8\theta\gamma,$$

$$c_3(\mathcal{A}_2)=-\frac{32}{3}\theta^3-512\eta\theta^2-32\theta^2\gamma\ \text{ and }\ c_3(\mathcal{B}_2)=-\frac{32}{3}\theta^3-184\eta\theta^2-16\theta^2\gamma.$$

Proof. Immediate application of Grothendieck-Riemann-Roch with respect to ν. □

Before our next result, we recall that if $\mathcal{V}$ is a vector bundle of rank $r+1$ on a variety X, we have the formulas:

(i) $c_1(\mathrm{Sym}^2(\mathcal{V})) = (r+2)c_1(\mathcal{V})$.

(ii) $c_2(\mathrm{Sym}^2(\mathcal{V})) = \frac{r(r+3)}{2}c_1^2(\mathcal{V}) + (r+3)c_2(\mathcal{V})$.

(iii) $c_3(\mathrm{Sym}^2(\mathcal{V})) = \frac{r(r+4)(r-1)}{6}c_1^3(\mathcal{V}) + (r+5)c_3(\mathcal{V}) + (r^2+4r-1)c_1(\mathcal{V})c_2(\mathcal{V})$.

We expand $\sigma_*\big(c_3(\mathcal{F} - \mathrm{Sym}^2\mathcal{E})\big) \equiv a\lambda - b_0\delta_0 - b_1\delta_1 \in A^1(\mathcal{M}_{12}^p)$ and determine the coefficients a, b_0 and b_1. This will suffice in order to compute $s(\overline{\mathfrak{D}}_{12})$.

Theorem 6.9. *Let $[C] \in \mathcal{M}_{11}$ be a Brill-Noether general curve and denote by $C_1 \subset \Delta_1 \subset \overline{\mathcal{M}}_{12}$ the associated test curve. Then the coefficient of δ_1 in the expansion of $\overline{\mathfrak{D}}_{22}$ is equal to*

$$b_1 = \frac{1}{2g(C)-2}\sigma^*(C_1)\cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}\big) = 9867.$$

Proof. We intersect the degeneracy locus of the map $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the 3-fold $\sigma^*(C_1) = X + \sum_{\bar{\alpha}} X_{\bar{\alpha}} \times Z_{\bar{\alpha}}$. As already explained in Proposition 6.6, it is enough to estimate the contribution coming from X and we can write

$$\sigma^*(C_1)\cdot c_3(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}) = c_3(\mathcal{F}_{|X}) - c_3(\mathrm{Sym}^2\mathcal{E}_{|X}) - c_1(\mathcal{F}_{|X})c_2(\mathrm{Sym}^2\mathcal{E}_{|X}) +$$

$$+2c_1(\mathrm{Sym}^2\mathcal{E}_{|X})c_2(\mathrm{Sym}^2\mathcal{E}_{|X}) - c_1(\mathrm{Sym}^2\mathcal{E}_{|X})c_2(\mathcal{F}_{|X}) + c_1^2(\mathrm{Sym}^2\mathcal{E}_{|X})c_1(\mathcal{F}_{|X}) - c_1^3(\mathrm{Sym}^2\mathcal{E}_{|X}).$$

We are going to compute each term in the right-hand-side of this expression.

Recall that we have constructed in Proposition 6.7 a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$. We consider the kernel line bundle $\mathrm{Ker}(\zeta)$. If U is the line bundle on X with fibre

$$U(y, L) = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-2y))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{2y})$$

over a point $(y, L) \in X$, then one has an exact sequence over X

$$0 \to U \to J_1(\mathcal{P}) \to \mathrm{Ker}(\zeta)^\vee \to 0.$$

In particular, $c_1(U) = 2\gamma + 48\eta - c_1(\mathrm{Ker}(\zeta))^\vee$. The products of the Chern class of $\mathrm{Ker}(\zeta)^\vee$ with other classes on $C \times W^4_{14}(C)$ can be computed from the Harris-Tu formula [HT]:

(6)

$$c_1(\mathrm{Ker}(\zeta)^\vee)\cdot\xi_{|X} = -c_5(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee)\cdot\xi_{|X} = -\big(\pi_2^*(c_5) - 6\eta\theta\pi_2^*(c_3) + (48\eta + 2\gamma)\pi_2^*(c_4)\big)\cdot\xi_{|X},$$

for any class $\xi \in H^2(C \times W^4_{14}(C), \mathbb{C})$.

If $\mathcal{A}_3$ denotes the rank 18 vector bundle on X having fibres $\mathcal{A}_3(y, L) = H^0(C, L^{\otimes 2})$, then there is an injective morphism $U^{\otimes 2} \hookrightarrow \mathcal{A}_3/\mathcal{A}_2$, and we consider the quotient sheaf

$$\mathcal{G} := \frac{\mathcal{A}_3/\mathcal{A}_2}{U^{\otimes 2}}.$$

Since the morphism $U^{\otimes 2} \to \mathcal{A}_3/\mathcal{A}_2$ vanishes along the locus of pairs (y, L) where L has a base point, $\mathcal{G}$ has torsion along $\Gamma \subset X$. A straightforward local analysis now shows that $\mathcal{F}_{|X}$ can be identified as a subsheaf of $\mathcal{A}_3$ with the kernel of the map $\mathcal{A}_3 \to \mathcal{G}$. Therefore, there is an exact sequence of vector bundles on X

$$0 \to \mathcal{A}_{2|X} \to \mathcal{F}_{|X} \to U^{\otimes 2} \to 0,$$

which over a general point of X corresponds to the decomposition

$$\mathcal{F}(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-2y)) \oplus \mathbb{C}\cdot u^2,$$

where $u \in H^0(C, L)$ is such that $\mathrm{ord}_y(u) = 1$. The analysis above, shows that the sequence stays exact over the curve Γ as well. Hence

$$c_1(\mathcal{F}_{|X}) = c_1(\mathcal{A}_{2|X}) + 2c_1(U),\ c_2(\mathcal{F}_{|X}) = c_2(\mathcal{A}_{2|X}) + 2c_1(\mathcal{A}_{2|X})c_1(U) \quad \text{and}$$
$$c_3(\mathcal{F}_{|X}) = c_3(\mathcal{A}_2) + 2c_2(\mathcal{A}_{2|X})c_1(U).$$

Furthermore, since $\mathcal{E}_{|X} = \pi_2^*(\mathcal{M})_{|X}$, we obtain that:

$$\sigma^*(C_1) \cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}\big) = c_3(\mathcal{A}_{2|X}) + c_2(\mathcal{A}_{2|X})c_1(U^{\otimes 2}) - c_3(\mathrm{Sym}^2\pi_2^*\mathcal{M}_{|X}) -$$
$$-\Big(\frac{r(r+3)}{2}c_1(\pi_2^*\mathcal{M}_{|X}) + (r+3)c_2(\pi_2^*\mathcal{M}_{|X})\Big)\cdot\Big(c_1(\mathcal{A}_{2|X}) + c_1(U^{\otimes 2}) - 2(r+2)c_1(\pi_2^*\mathcal{M}_{|X})\Big) -$$
$$-(r+2)c_1(\pi_2^*\mathcal{M}_{|X})c_2(\mathcal{A}_{2|X}) - (r+2)c_1(\pi_2^*\mathcal{M}_{|X})c_1(\mathcal{A}_{2|X})c_1(U^{\otimes 2}) +$$
$$+(r+2)^2c_1^2(\pi_2^*\mathcal{M}_{|X})c_1(\mathcal{A}_{2|X}) + (r+2)^2c_1^2(\pi_2^*\mathcal{M}_{|X})c_1(U^{\otimes 2}) - (r+2)^3c_1^3(\pi_2^*\mathcal{M}_{|X}).$$

As before, $c_i(\pi_2^*\mathcal{M}_{|X}^\vee) = \pi_2^*(c_i) \in H^{2i}(X, \mathbb{C})$. The coefficient of $c_1(\mathrm{Ker}(\zeta)^\vee)$ in the product $\sigma^*(C_1) \cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}\big)$ is evaluated via (6). The part of this product that does not contain $c_1(\mathrm{Ker}(\zeta)^\vee)$ equals

$$28\pi_2^*(c_2)\theta - 88\pi_2^*(c_1^2)\theta + 440\eta\pi_2^*(c_1^2) - 53\pi_2^*(c_1c_2) - \frac{32}{3}\theta^3 + 128\eta\theta^2 - 432\eta\theta\pi_2^*(c_1)$$
$$+64\pi_2^*(c_1^3) - 140\eta\pi_2^*(c_2) + 48\theta^2\pi_2^*(c_1) + 9\pi_2^*(c_3) \in H^6(C \times W^4_{14}(C), \mathbb{C}).$$

Multiplying this quantity by the class $[X]$ obtained in Proposition 6.7 and then adding to it the contribution coming from $c_1(\mathrm{Ker}(\zeta)^\vee)$, one obtains a homogeneous polynomial of degree 7 in η, θ and $\pi_2^*(c_i)$ for $1 \le i \le 4$. The only non-zero monomials are those containing η. After retaining only these monomials, the resulting degree 6 polynomial in $\theta, c_i \in H^*(W^4_{14}(C), \mathbb{Z})$ can be brought to a manageable form, by noting that, since $h^1(C, L) = 1$, the classes c_i are not independent. Precisely, if one fixes a divisor $D \in C_e$ of large degree, there is an exact sequence

$$0 \to \mathcal{M} \to (\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi^*D)\big) \to (\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\big) \to R^1\pi_{2*}\big(\mathcal{P}_{|C\times W^4_{14}(C)}\big) \to 0,$$

from which, via the well-known fact $c_t\big((\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D))\big) = e^{-\theta}$, it follows that

$$c_tR^1\pi_{2*}\big(\mathcal{P}_{|C\times W^4_{14}(C)}\big) \cdot e^{-\theta} = \sum_{i=0}^{4}(-1)^ic_i.$$

Hence $c_{i+1} = \theta^ic_i/i! - i\theta^{i+1}/(i+1)!$, for all $i \ge 2$. After routine manipulations, one finds that $b_1 = \sigma^*(C_1) \cdot c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))/20 = 9867$. □

Theorem 6.10. *Let $[C, q] \in \mathcal{M}_{11,1}$ be a Brill-Noether general pointed curve and we denote by $C_0 \subset \Delta_0 \subset \overline{\mathcal{M}}_{12}$ the associated test curve. Then $\sigma^*(C_0)\cdot c_3(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}) = 22b_0 - b_1 = 32505$. It follows that $b_0 = 1926$.*

Proof. As already noted in Proposition 6.5, the vector bundles $\mathcal{E}_{|\sigma^*(C_0)}$ and $\mathcal{F}_{|\sigma^*(C_0)}$ are both pull-backs of vector bundles on Y and we denote these vector bundles $\mathcal{E}$ and $\mathcal{F}$ as well, that is, $\mathcal{E}_{|\sigma^*(C^0)} = (\epsilon \circ f)^*(\mathcal{E}_{|Y})$ and $\mathcal{F}_{|\sigma^*(C_0)} = (\epsilon \circ f)^*(\mathcal{F}_{|Y})$. Like in the proof of Theorem 6.9, we evaluate each term appearing in $\sigma^*(C_0) \cdot c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))$.

Let V be the line bundle on Y with fibre

$$V(y, L) = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-y-q))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{y+q})$$

over a point $(y, L) \in Y$. There is an exact sequence of vector bundles over Y

$$0 \longrightarrow V \longrightarrow \mathcal{B} \longrightarrow \mathrm{Ker}(\chi)^\vee \longrightarrow 0,$$

where $\chi : \mathcal{B}^\vee \to \pi_2^*(\mathcal{M})^\vee$ is the bundle morphism defined in the second part of Proposition 6.7. In particular, $c_1(V) = 13\eta + \gamma - c_1(\mathrm{Ker}(\chi^\vee)$. By using again [HT], we find the following formulas for the Chern numbers of $\mathrm{Ker}(\chi)^\vee$:

$$c_1(\mathrm{Ker}(\chi)^\vee)\cdot\xi_{|Y} = -c_5\big(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee\big)\cdot\xi_{|Y} = -(\pi_2^*(c_5)+\pi_2^*(c_4)(13\eta+\gamma)-2\pi_2^*(c_3)\eta\theta)\cdot\xi_{|Y},$$

for any class $\xi \in H^2(C \times W^4_{14}(C), \mathbb{C})$. Recall that we introduced the vector bundle $\mathcal{B}_2$ over $C \times W^4_{14}(C)$ with fibre $\mathcal{B}_2(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-y - q))$. We claim that one has an exact sequence of bundles over Y

$$0 \longrightarrow \mathcal{B}_{2|Y} \longrightarrow \mathcal{F}_{|Y} \longrightarrow V^{\otimes 2} \longrightarrow 0. \tag{7}$$

If $\mathcal{B}_3$ is the vector bundle on Y with fibres $\mathcal{B}_3(y, L) = H^0(C, L^{\otimes 2})$, we have an injective morphism of sheaves $V^{\otimes 2} \hookrightarrow \mathcal{B}_3/\mathcal{B}_2$ locally given by

$$v^{\otimes 2} \mapsto v^2 \bmod H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-y - q)),$$

where $v \in H^0(C, L)$ is any section not vanishing at q and y. Then $\mathcal{F}_{|Y}$ is canonically identified with the kernel of the projection morphism

$$\mathcal{B}_3 \to \frac{\mathcal{B}_3/\mathcal{B}_2}{V^{\otimes 2}}$$

and the exact sequence (7) now becomes clear. Therefore $c_1(\mathcal{F}_{|Y}) = c_1(\mathcal{B}_{2|Y}) + 2c_1(V)$, $c_2(\mathcal{F}_{|Y}) = c_2(\mathcal{B}_{2|Y}) + 2c_1(\mathcal{B}_{2|Y})c_1(V)$ and $c_3(\mathcal{F}_{|Y}) = c_3(\mathcal{B}_{2|Y}) + 2c_2(\mathcal{B}_{2|Y})c_1(V)$. The part of the total intersection number $\sigma^*(C_0) \cdot c_3(\mathcal{F} - \mathrm{Sym}^2(\mathcal{E}))$ that does not contain $c_1(\mathrm{Ker}(\chi^\vee))$ equals

$$28\pi_2^*(c_2)\theta - 88\pi_2^*(c_1^2)\theta - 22\eta\pi_2^*(c_1^2) - 53\pi_2^*(c_1c_2) - \frac{32}{3}\theta^3 +$$

$$-8\eta\theta^2 + 24\eta\theta\pi_2^*(c_1) + 64\pi_2^*(c_1^3) + 7\eta\pi_2^*(c_2) + 48\theta^2\pi_2^*(c_1) + 9\pi_2^*(c_3) \in H^6(C \times W^4_{14}(C), \mathbb{C})$$

and this gets multiplied with the class $[Y]$ from Proposition 6.7. The coefficient of $c_1(\mathrm{Ker}(\zeta)^\vee)$ in $\sigma^*(C_0) \cdot c_3\big(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}\big)$ equals

$$-2c_2(\mathcal{B}_{2|Y}) - 2(r+2)^2\pi_2^*(c_1^2) - 2(r+2)c_1(\mathcal{B}_{2|Y})\pi_2^*(c_1) + r(r+3)\pi_2^*(c_1^2) + 2(r+3)\pi_2^*(c_2).$$

All in all, $22b_0 - b_1 = \sigma^*(C_0) \cdot c_3(\mathcal{F} - \mathrm{Sym}^2\mathcal{E})$ and we evaluate this using (6). □

The following result follows from the definition of the vector bundles $\mathcal{E}$ and $\mathcal{F}$ given in Proposition 6.4:

Theorem 6.11. *Let $[C, q] \in \mathcal{M}_{11,1}$ be a Brill-Noether general pointed curve and $R \subset \overline{\mathcal{M}}_{12}$ the pencil obtained by attaching at the fixed point $q \in C$ a pencil of plane cubics. Then*

$$a - 12b_0 + b_1 = \sigma_* c_3\big(\mathcal{F} - \mathrm{Sym}^2\mathcal{E}\big) \cdot R = 0.$$

End of the proof of Theorem 6.1. The fact that the virtual divisor $\mathfrak{D}_{12}$ is a genuine divisor on $\mathcal{M}_{12}$ follows from [T]. Assuming by contradiction that for every curve $[C] \in \mathcal{M}_{12}$, there exists $L \in W^4_{14}(C)$ such that $\mu_0(L)$ is not-injective, one can construct a stable vector bundle E of rank 2 sitting in an extension

$$0 \longrightarrow K_C \otimes L^\vee \longrightarrow E \longrightarrow L \longrightarrow 0,$$

such that $h^0(C,E) = h^0(C,L) + h^1(C,L) = 7$, and for which the Mukai-Petri map $\mathrm{Sym}^2 H^0(C,E) \to H^0(C, \mathrm{Sym}^2 E)$ is not injective. This is a contradiction. To determine the slope of the divisor $\overline{\mathcal{D}}_{12}$, we write $\overline{\mathfrak{D}}_{12} \equiv a\lambda - \sum_{j=0}^{6} b_j \delta_j \in \mathrm{Pic}(\overline{\mathcal{M}}_{12})$. Since $a/b_0 = 4415/642 \leq 71/10$, we are in a position to apply Corollary 1.2 from [FP], which gives the inequalities $b_j \geq b_0$ for $1 \leq j \leq 6$. Therefore $s(\overline{\mathfrak{D}}_{12}) = a/b_0 < 13/2$. $\square$

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