BMS Algebraic Geometry 2008, Problem Set Nr. 2

1. Prove that the curve (affine variety) given by the equation $xy = 1$ in $\mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$.

2. Show that the set X of points $\{(t, t^2, t^3) : t \in k\} \subset \mathbb{A}^3$ is an irreducible affine variety and compute the ideal $I(X) \subset k[x, y, z]$ by describing its generators. Is X isomorphic to $\mathbb{A}^1$?

3. Let X be the algebraic set in $\mathbb{A}^3$ defined by the polynomials $x^2 - yz$ and $xz - x$. Show that X is the union of three irreducible components. Describe them and find their prime ideals.

4. Show that a k-algebra R is isomorphic to the affine coordinate ring of some algebraic set $X \subset \mathbb{A}^n$ if and only if R is finitely generated as a k-algebra and has no nilpotent elements.

5. Show that any open subset of an irreducible topological space is dense and irreducible.

6. Suppose that $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. If $\mathcal{H}$ is defined by the collection of data $\mathcal{H}(U) := \mathcal{G}(U)/\phi(\mathcal{F}(U))$ for all open sets $U \subset X$, show that $\mathcal{H}$ is a presheaf. What are the stalks of its sheafification?

7. We fix $X := \mathbb{C}$ and denote by $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ the exponential map $f \mapsto e^{2\pi i f}$. Prove that when $U \subset X$ is simply connected then $\exp_U$ is surjective. Give an example ofan open subset $U \subset X$ when this is no longer true. Describe the image sheaf of the morhism exp.

8. Let $\phi : \mathcal{F} \to \mathcal{G}$ be any morphism of presheaves. Prove that there exists an induced morphisms between the sheafifications $\phi^{sh} : \mathcal{F}^{sh} \to \mathcal{G}^{sh}$.