BMS Algebraic Geometry 2008, Problem Set Nr. 3

1. Suppose that $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Show that there are natural isomorphisms of sheaves $\mathrm{Im}(\phi) \cong \mathcal{F}/\mathrm{Ker}(\phi)$ and $\mathrm{Coker}(\phi) \cong \mathcal{G}/\mathrm{Im}(\phi)$.

2. Suppose we have an exact sequence of sheaves of abelian groups on a space X

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}.$$

Show that if $U \subset X$ is any open set, then

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact.

3. Show that a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is surjective (injective) if and only if it is surjective (injective) on all stalks.

4. Let $I \subset k[x_0, \ldots, k_n]$ be a homogeneous ideal. Prove that its radical $\sqrt{I}$ is also a homogeneous ideal.

5. Prove that the image of the map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$

$$t \mapsto (t^2, t^3, t^6)$$

is given by $Z(x^3 - y^2, y^2 - z)$. Show that this map is an isomorphism of sets but not an isomorphism of affine varieties.

6. Check that the the map $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by

$$[x_0, x_1] \mapsto [x_0^2, x_0x_1, x_1^2]$$

is a morphism of prevarieties. Is this map an isomorphism of prevarieties?

7. Check that every prevariety $(X, \mathcal{O}_X)$ is an irreducible topological space.