BMS Algebraic Geometry 2008, Problem Set Nr. 6

1. Show that the intersection of two varieties need not be an (irreducible) variety: Let $Q_1, Q_2 \subset \mathbb{P}^3$ be quadric surfaces given by the equations $x^2 - yw = 0$ and $xy - zw = 0$. Show that $Q_1 \cap Q_2$ consists of the union of a line and a twisted cubic.

2. A variety X is said to be rational if it is birationally equivalent to $\mathbb{P}^n$. Show that any conic in $\mathbb{P}^2$ is rational. Show that the cuspidal cubic $y^2 = x^3$ is a rational curve. Finally, let $Y \subset \mathbb{P}^2$ be the nodal cubic curve $y^2 z = x^2(x+z)$. Find the explicit equation for the projection ϕ of Y from the point $p = [0,0,1]$ to the line $z = 0$ and show that ϕ induces a birational isomorphism from Y to $\mathbb{P}^1$. Conclude that Y is rational.

3. Let X and Y be algebraic varieties and denote by $X \times Y$ their product in the category of algebraic varieties. Prove that

$$\dim(X \times Y) = \dim(X) + \dim(Y).$$

4. Let $X, Y \subset \mathbb{A}^n$ be irreducible affine varieties such that $X \cap Y \neq \emptyset$. Show that for any irreducible component Z of $X \cap Y$ one has that

$$\dim(X \cap Y) \geq \dim(X) + \dim(Y) - n.$$

5. Prove that any variety X of dimension r is birational to a hypersurface $Y \subset \mathbb{P}^{r+1}$.

6. Prove that through any $r+2$ points in $\mathbb{P}^r$ in general linear position, there passes a unique rational normal (Veronese) curve $Y \subset \mathbb{P}^r$.

7. Let $Q \subset \mathbb{P}^3$ be a quadric surface, e.g. the surface $xw - yz = 0$ and let $p \in Q$ be any point, e.g. $p = [0,0,0,1]$.

- Write down the formula for the projection from p

$$\pi_p : Q - \{p\} \to \mathbb{P}^2.$$

- Prove that π_p is a birational isomorphism and write down explicitly its inverse rational map.

8. Let $C \subset \mathbb{P}^3$ be the twisted cubic curve. Show that any point $p \in \mathbb{P}^3 - C$ lies on a unique line secant line L to C, that is, of the form $L = \overline{xy}$ with $x, y \in C$.