BMS Algebraic Geometry 2008, Problem Set Nr. 7

1. Let $\rho_2 : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese map and we denote by $S := \mathrm{Im}(\rho_2) \subset \mathbb{P}^5$ the Veronese surface. We define the chordal variety of S to be the union (inside $\mathbb{P}^5$) of all secant lines to S, that is,

$$Ch(S) := \bigcup_{x,y \in S} \overline{x,y} \subset \mathbb{P}^5.$$

Show that $\dim(Ch(S)) = 4$. (Hint: Using that the ideal of S is generated by 6 quadrics in $\mathbb{P}^5$, show that the Veronese surface is the locus of 3×3 symmetric matrices of rank 1. Then use that the linear combination of two rank 1 matrices can have rank at most 2, hence $Ch(S)$ is equal to a cubic hypersurface.)

2. Let $\Sigma = \sigma(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ be the Segre variety of dimension 4. By arguments similar to Exercise 1, show that the chordal variety of Σ has dimension 7.

3. Let V be a vector space of dimension $n+1$ over k. Show that an element $\omega \in \wedge^2 V$ is decomposable if and only if $\omega \wedge \omega = 0$. Conclude once more that the Grassmannian $G(2,V) \subset \mathbb{P}(\wedge^2 V)$ is a projective variety cut out by quadratic relations.

4. Let $p \in \mathbb{P}^3$ be a point and $H \subset \mathbb{P}^3$ a plane containing p. We denote by $\Sigma \subset \mathbb{G}(1,3)$ the subvariety of the Grassmannian of consisting of those lines $l \subset H$ such that $p \in l$. Show that under the Plücker embedding $\mathbb{G}(1,3) \subset \mathbb{P}^5$, the subvariety Σ is carried to a line. Conversely, show that every line in $\mathbb{P}^5$ lying on $\mathbb{G}(1,3)$ appears in this way.

5. Let $l_1, l_2 \subset \mathbb{P}^3$ be skew lines. Show that the locus $Q \subset \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ meeting both l_1 and l_2 is the intersection of $\mathbb{G}(1,3)$ with a suitable 3-plane $\mathbb{P}^3 \subset \mathbb{P}^5$. Deduce once more that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. What happens if the lines l_1 and l_2 meet?

6. Let Y be the cuspidal curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $p = (0,0)$, and denote by $\pi : X = Bl_p(\mathbb{A}^2) \to \mathbb{A}^2$ the resulting map and then denote by $E = \pi^{-1}(0)$ the exceptional divisor. We set $\tilde{Y} := \overline{\pi^{-1}(Y - p)} \subset X$ for the blow-up of Y at p. Show that E meets $\tilde{Y}$ in one point and that $\tilde{Y}$ is isomorphic to $\mathbb{A}^1$. Is the morphism $\pi : \tilde{Y} \to Y$ an isomorphism?

7. The same problem as above when Y is replaced by the curve $y^3 = x^5$. Describe the intersection of $\tilde{Y}$ with the exceptional divisor. Is $\tilde{Y}$ a smooth curve?