

BMS Algebraic Geometry 2008, Problem Set Nr. 8

These problems are due on July 10th.

1. Find the Hilbert function of a hypersurface $X_d \subset \mathbb{P}^n$ given by a degree d polynomial $f \in k[x_0, \dots, x_n]$.
2. Find the Hilbert polynomial of the Segre variety $\Sigma_{m,n} := \sigma(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^{mn+m+n}$.
3. Let $X \subset \mathbb{P}^n$ be a set of d points. Show that for sufficiently large values of m relative to d (precisely for $m \geq d - 1$), the Hilbert function is $h_X(m) = d$.
4. Consider a plane curve $X \subset \mathbb{P}^2$ of degree d and its image $Y := \rho_2(X) \subset \mathbb{P}^5$ under the Veronese embedding $\rho_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5$. Compare the Hilbert polynomials of X and Y .
5. Determine the Hilbert polynomial of a pair of (1) skew lines in \mathbb{P}^3 and (2) incident lines in \mathbb{P}^3 .
6. Let $X \subset \mathbb{A}^n$ be an affine variety. Prove that the dimension of the Zariski tangent space $T_p(X)$ is an upper-semicontinuous function on p in the Zariski topology of X .
7. Prove that two smooth projective curves X and Y which are birationally isomorphic, are in fact isomorphic.
8. Let $p_1, \dots, p_r, q_1, \dots, q_s$ be distinct points of \mathbb{A}^1 . If $\mathbb{A}^1 - \{p_1, \dots, p_r\}$ is isomorphic to $\mathbb{A}^1 - \{q_1, \dots, q_s\}$ show that $r = s$. Is the converse true?
9. Prove that if a hypersurface $X \subset \mathbb{P}^n$ contains a linear subspace of dimension $\geq n/2$, then it has to be singular.
10. Find the normalization of the affine curve $y^3 = x^3 + x^4$, that is, find additional elements of the function field of the curve that satisfy integral equations over the coordinate ring of the original curve. Then show that the curve thus obtained is non-singular.