

# Algebraic Geometry 2014, Problem Set Nr. 2, Prof. G. Farkas, HU Berlin

1. Let  $\mathcal{F}$  be a presheaf on a topological space  $X$ . Show that the assignment

$$\tilde{\mathcal{F}}(U) := \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x : \text{for every } x \in U, \exists x \in V \subset U, t \in \mathcal{F}(V) \text{ with } s_y = t_y, \text{ for } y \in V \right\},$$

gives rise to a sheaf on  $X$  and to a morphism  $i_{\mathcal{F}} : \mathcal{F} \rightarrow \tilde{\mathcal{F}}$  of presheaves. The sheaf  $\tilde{\mathcal{F}}$  is called the sheafification of  $\mathcal{F}$ .

- (a) Show that the map induced at the level of stalks  $i_x : \mathcal{F}_x \rightarrow \tilde{\mathcal{F}}_x$  is an isomorphism.  
 (b) Show that if  $\mathcal{G}$  is a sheaf on  $X$  and  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of presheaves, there exists a unique morphism of sheaves  $\tilde{\phi} : \tilde{\mathcal{F}} \rightarrow \mathcal{G}$  such that  $\tilde{\phi} \circ i_{\mathcal{F}} = \phi$ .

2. Let  $\mathcal{F}$  be a sheaf of abelian groups on  $X$  and  $\mathcal{G}$  a subsheaf of  $\mathcal{F}$ . We define the quotient sheaf  $\mathcal{F}/\mathcal{G}$  as the sheafification of the presheaf

$$X \supset U \mapsto \mathcal{F}(U)/\mathcal{G}(U).$$

Show that there is a surjective morphism of sheaves  $\pi : \mathcal{F} \rightarrow \mathcal{F}/\mathcal{G}$  and determine its kernel.

2. Suppose that  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of sheaves. Show that there are natural isomorphisms of sheaves  $\text{Im}(\phi) \cong \mathcal{F}/\text{Ker}(\phi)$  and  $\text{Coker}(\phi) \cong \mathcal{G}/\text{Im}(\phi)$ .

3. Suppose we have an exact sequence of sheaves of abelian groups on a space  $X$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}.$$

Show that if  $U \subset X$  is any open set, then

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact.

4. Show that a morphism of sheaves  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is surjective (injective) if and only if it is surjective (injective) on all stalks.

5. Consider the Veronese embedding  $\nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ . Show that if  $Y \subset \mathbb{P}^2$  is a projective variety, then there exists a hypersurface  $Z \subset \mathbb{P}^5$  such that

$$\nu_2(\mathbb{P}^2) \cap Z = \nu_2(Y).$$