

Algebraic Geometry 2014, Problem Set

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1. Let \mathcal{F} be a presheaf on a topological space X . Show that the assignment

$$\tilde{\mathcal{F}}(U) := \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x : \text{for every } x \in U, \exists x \in V \subset U, t \in \mathcal{F}(V) \text{ with } s_y = t_y, \text{ for } y \in V \right\},$$

gives rise to a sheaf on X and to a morphism $i_{\mathcal{F}} : \mathcal{F} \rightarrow \tilde{\mathcal{F}}$ of presheaves. The sheaf $\tilde{\mathcal{F}}$ is called the sheafification of \mathcal{F} .

- (a) Show that the map induced at the level of stalks $i_x : \mathcal{F}_x \rightarrow \tilde{\mathcal{F}}_x$ is an isomorphism.
- (b) Show that if \mathcal{G} is a sheaf on X and $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\tilde{\phi} : \tilde{\mathcal{F}} \rightarrow \mathcal{G}$ such that $\tilde{\phi} \circ i_{\mathcal{F}} = \phi$.

2. Let \mathcal{F} be a sheaf of abelian groups on X and \mathcal{G} a subsheaf of \mathcal{F} . We define the quotient sheaf \mathcal{F}/\mathcal{G} as the sheafification of the presheaf

$$X \supset U \mapsto \mathcal{F}(U)/\mathcal{G}(U).$$

Show that there is a surjective morphism of sheaves $\pi : \mathcal{F} \rightarrow \mathcal{F}/\mathcal{G}$ and determine its kernel.

2. Suppose that $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves. Show that there are natural isomorphisms of sheaves $\text{Im}(\phi) \cong \mathcal{F}/\text{Ker}(\phi)$ and $\text{Coker}(\phi) \cong \mathcal{G}/\text{Im}(\phi)$.

3. Suppose we have an exact sequence of sheaves of abelian groups on a space X

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}.$$

Show that if $U \subset X$ is any open set, then

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact.

4. Show that a morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is surjective (injective) if and only if it is surjective (injective) on all stalks.

5. Consider the Veronese embedding $\nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5$. Show that if $Y \subset \mathbb{P}^2$ is a projective variety, then there exists a hypersurface $Z \subset \mathbb{P}^5$ such that

$$\nu_2(\mathbb{P}^2) \cap Z = \nu_2(Y).$$