Algebraic Geometry 2014, Problem Set Nr. 4, Prof. G. Farkas, HU Berlin

1. Suppose that X and Y are varieties over k and $f : Y \dashrightarrow X$ is a rational map. Show that there exists a largest open set $U \subset Y$ on which f can be represented by a morphism $f_U : U \to X$.

2. Show that the projective varieties $X, Y \subset \mathbb{P}^3$ defined by the equations $xw = yz$ and $x^2 + y^2 + z^2 = w^2$ respectively, are isomorphic.

3. We consider the Segre map $\sigma : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^5$ and denote by $\Sigma_{21} := \mathrm{Im}(\sigma) \subset \mathbb{P}^5$ the Segre 3-fold. Prove that the twisted cubic curve $C \subset \mathbb{P}^3$ can be realized as the intesection of Σ_{21} with a suitable 3-plane $\mathbb{P}^3 \subset \mathbb{P}^5$.

4. Show that any line $\ell \subset \Sigma_{2,1} \subset \mathbb{P}^5$ is contained in the image of a fiber of $\mathbb{P}^2 \times \mathbb{P}^2$ over $\mathbb{P}^2$ or $\mathbb{P}^1$.

5. Show that the image of the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ under the Segre map is isomorphic to the Veronese subvariety $\nu_2(\mathbb{P}^n)$ lying in a subspace of $\mathbb{P}^{n^2+2n}$. Deduce from this that the product of any projective variety with itself is a subvariety of that product.

6. Let f be the rational function on $\mathbb{P}^2$ defined by $f = x_0/x_1$. Find the set of points where f is defined and describe the regular function which represents f. If you think of f as being a function from $\mathbb{P}^2$ to $\mathbb{P}^1$ obtained by embedding the target $\mathbb{A}^1$ into $\mathbb{P}^1$, find the points where f is defined and describe the corresponding morphism.