Algebraic Geometry 2014, Problem Set Nr. 1, Prof. G. Farkas, HU Berlin

1. Prove that the curve (affine variety) given by the equation $xy = 1$ in $\mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$.

2. Show that the set X of points $\{(t, t^2, t^3) : t \in k\} \subset \mathbb{A}^3$ is an irreducible affine variety and compute the ideal $I(X) \subset k[x, y, z]$ by describing its generators. Is X isomorphic to $\mathbb{A}^1$?

3. Let X be the algebraic set in $\mathbb{A}^3$ defined by the polynomials $x^2 - yz$ and $xz - x$. Show that X is the union of three irreducible components. Describe them and find their prime ideals.

4. Show that a k-algebra R is isomorphic to the affine coordinate ring of some algebraic set $X \subset \mathbb{A}^n$ if and only if R is finitely generated as a k-algebra and has no nilpotent elements.

5. Show that any open subset of an irreducible topological space is dense and irreducible.

6. Compute the resultant of $X^5 - 3X^4 - 2X^3 + 3X^2 + 7X + 6$ and $X^4 + X^2 + 1$. Do these polynomials have a common factor in $\mathbb{Q}[X]$?

7. We fix $X := \mathbb{C}$ and denote by $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ the exponential map $f \mapsto e^{2\pi i f}$. Prove that when $U \subset X$ is simply connected then $\exp_U$ is surjective. Give an example ofan open subset $U \subset X$ when this is no longer true. Describe the image sheaf of the morhism exp.

8. If $f = a_n X^n + \cdots + a_0 \in K[X]$, where $a_n \neq 0$, then the discriminant of f is defined as the quantity

$$\mathrm{disc}(f) := \frac{(-1)^{\binom{n}{2}}}{a_n} \mathrm{Res}(f, f').$$

Show that f has a multiple factor if and only if $\mathrm{disc}(f) = 0$.