Algebraic Geometry 2018, Problem Set Nr. 2, Prof. G. Farkas, HU Berlin

1. Let $\mathcal{F}$ be a presheaf on a topological space X. Show that the assigment

$$\tilde{\mathcal{F}}(U) := \Big\{(s_x) \in \prod_{x\in U} \mathcal{F}_x : \text{for every } x \in U, \exists x \in V \subset U, t \in \mathcal{F}(V) \text{ with } s_y = t_y, \text{ for } y \in V\Big\},$$

gives rise to a sheaf on X and to a morphism $i_{\mathcal{F}} : \mathcal{F} \to \tilde{\mathcal{F}}$ of presheaves. The sheaf $\tilde{\mathcal{F}}$ is called the sheafification of $\mathcal{F}$.

(a) Show that the map induced at the level of stalks $i_x : \mathcal{F}_x \to \tilde{\mathcal{F}}_x$ is an isomorphism.

(b) Show that if $\mathcal{G}$ is a sheaf on X and $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\tilde{\phi} : \tilde{\mathcal{F}} \to \mathcal{G}$ such that $\tilde{\phi} \circ i_{\mathcal{F}} = \phi$.

2. Let $\mathcal{F}$ be a sheaf of abelian groups on X and $\mathcal{G}$ a subsheaf of $\mathcal{F}$. We define the quotient sheaf $\mathcal{F}/\mathcal{G}$ as the sheafification of the presheaf

$$X \supset U \mapsto \mathcal{F}(U)/\mathcal{G}(U).$$

Show that there is a surjective morphism of sheaves $\pi : \mathcal{F} \to \mathcal{F}/\mathcal{G}$ and determine its kernel.

3. Suppose that $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Show that there are natural isomorphisms of sheaves $\mathrm{Im}(\phi) \cong \mathcal{F}/\mathrm{Ker}(\phi)$ and $\mathrm{Coker}(\phi) \cong \mathcal{G}/\mathrm{Im}(\phi)$. Here $\mathrm{Im}(\phi)$ denotes the sheafification of the preseaf $U \mapsto \mathrm{Im}(\phi_U) \subseteq \mathcal{G}(U)$.

4. Suppose we have an exact sequence of sheaves of abelian groups on a space X

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}.$$

Show that if $U \subset X$ is any open set, then

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact.

5. Let $\mathcal{F}$ be a sheaf of groups on a topological space X and let $s \in \mathcal{F}(U)$ be a section. Show that the set $\{p \in U : s_p \neq 0\}$ is closed in U. Show that the set $\{p \in X : \mathcal{F}_p \neq 0\}$ need not be closed in X.

6. Consider the Veronese embedding $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$. Show that if $Y \subset \mathbb{P}^2$ is a projective variety, then there exists a hypersurface $Z \subset \mathbb{P}^5$ such that

$$\nu_2(\mathbb{P}^2) \cap Z = \nu_2(Y).$$