Algebraic Geometry 2018, Problem Set Nr. 3, Prof. Gavril Farkas, HU Berlin

1. Let H_i and H_j be hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant.

2. Let $\mathbb{P}^n$ be a hyperplane of $\mathbb{P}^{n+1}$ and we fix a point $p \in \mathbb{P}^{n+1} - \mathbb{P}^n$. We define a map $\phi : \mathbb{P}^{n+1} - \{p\} \to \mathbb{P}^n$ by $\phi(x) :=$ the point of intersection of the line containing p and x with the hyperplane $\mathbb{P}^n$.

- Show that ϕ is a morphism of prevarieties.
- Let $Y \subseteq \mathbb{P}^3$ be the twisted cubic curve given by points

$$[x_0, x_1, x_2, x_3] = [s^3, s^2 t, st^2, t^3],$$

  where $[s, t] \in \mathbb{P}^1$. Assume that $p = [0, 0, 1, 0] \in \mathbb{P}^3$ and let $\mathbb{P}^2$ be the hyperplane $x_2 = 0$. Find the equations in the plane of the curve $\phi(Y)$.

3. Let X be any prevariety and $p \in X$. Show that there is a $1 : 1$ correspondence between the prime ideals of the local ring $\mathcal{O}_{X,p}$ and the closed subvarieties of X containing p.

4. We fix $n, d > 0$ and let $M_0, M_1, \ldots, M_N$ be all monomials of degree d in the variables $x_0, \ldots, x_d$, where $N = \binom{n+d}{n} - 1$. We define the map

$$\rho_d : \mathbb{P}^n \to \mathbb{P}^N$$

obtained by sending a point $p = [a_0, \ldots, a_n]$ to the point $\rho_d(p) = [M_0(p), \ldots, M_N(p)]$ obtained by evaluating all the monomials M_j at the point$(a_0, \ldots, a_n)$. This is called the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$.

- Describe this map in the case $n = 1, d = 2$. What is the image of ρ_2?
- Prove that the image $\rho_d(\mathbb{P}^n)$ is always a projective subvariety of $\mathbb{P}^N$ given by some homogeneous ideal $I \subset k[X_0, \ldots, X_n]$.
- Show that the 3-uple embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$ has as image the twisted cubic curve in $\mathbb{P}^3$. Show that any finite set of points on a twisted cubic in $\mathbb{P}^3$ are in general linear position, that is, any four of them span the space $\mathbb{P}^3$.

- Show that through any $d+3$ points in general position in $\mathbb{P}^d$ (that is, each $d+1$ of them spans $\mathbb{P}^d$), there passes a *unique* rational normal curve.

5. Show that the quadric surface $Q : xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$.

6. Let X and Y prevarieties and suppose there exist points $p \in X$ and $q \in Y$ such that the local rings $\mathcal{Q}_{X,p}$ and $\mathcal{Q}_{Y,q}$ are isomorphic as k-algebras. Show that there exists open sets $p \in U \subseteq X$ and $q \in V \subseteq Y$ and an isomorphism $f : U \xrightarrow{\cong} V$, with $f(p) = q$.