Algebraic Geometry 2018, Problem Set Nr. 5, Prof. Gavril Farkas, HU Berlin

1. Let $\rho_2 : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese map and we denote by $S := \mathrm{Im}(\rho_2) \subset \mathbb{P}^5$ the Veronese surface. We define the chordal variety of S to be the union (inside $\mathbb{P}^5$) of all secant lines to S, that is,

$$Ch(S) := \bigcup_{x,y \in S} \overline{x, y} \subset \mathbb{P}^5.$$

Show that $\dim(Ch(S)) = 4$. (Hint: Using that the ideal of S is generated by 6 quadrics in $\mathbb{P}^5$, show that the Veronese surface is the locus of 3×3 symmetric matrices of rank 1. Then use that the linear combination of two rank 1 matrices can have rank at most 2, hence $Ch(S)$ is equal to a cubic hypersurface.)

2. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\dim R$ is the smallest number d such that there exist d elements $x_1, \ldots, x_d \in \mathfrak{m}$, with $\mathfrak{m}^n \subset (x_1, \ldots, x_d)$, for sufficiently high n.

3. If $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a map of local rings sending $\mathfrak{m}$ to $\mathfrak{n}$, then

$$\dim S \leq \dim R + \dim S/\mathfrak{m}S,$$

with equality if S is flat as an R-module.

4. Let R be a noetherian ring and let x be an indeterminate. Show that

$$\dim R[x, x^{-1}] = 1 + \dim R.$$

5. Let $\ell_1, l_2 \subset \mathbb{P}^3$ be skew lines. Show that the locus $Q \subset \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ meeting both ℓ_1 and ℓ_2 is the intersection of $\mathbb{G}(1,3)$ with a suitable 3-plane $\mathbb{P}^3 \subset \mathbb{P}^5$. Deduce once more that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. What happens if the lines ℓ_1 and ℓ_2 meet?

6. Let Y be the cuspidal curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $p = (0,0)$, and denote by $\pi : X = Bl_p(\mathbb{A}^2) \to \mathbb{A}^2$ the resulting map and then denote by $E = \pi^{-1}(0)$ the exceptional divisor. We set $\tilde{Y} := \overline{\pi^{-1}(Y - p)} \subset X$ for the blow-up of Y at p. Show that E meets $\tilde{Y}$ in one point and that $\tilde{Y}$ is isomorphic to $\mathbb{A}^1$. Is the morphism $\pi : \tilde{Y} \to Y$ an isomorphism?

7. The same problem as above when Y is replaced by the curve $y^3 = x^5$. Describe the intersection of $\tilde{Y}$ with the exceptional divisor. Is $\tilde{Y}$ a smooth curve?