Blatt 10

Aufgabe 1

1. $\phi_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad \phi_A(x) = Ax$

$$A = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1+\sqrt{2} & -1+\sqrt{2} \\ -\sqrt{2} & -1+\sqrt{2} & 1+\sqrt{2} \end{pmatrix} \in M(3,3:\mathbb{R})$$

$$AA^t = I$$

$$\det A = \left(\frac{1}{2\sqrt{2}}\right)^3 \cdot 16\sqrt{2} = 1 \quad \Rightarrow \quad A \in SO(3)$$

Drehachse: $\quad Av = v \quad \Rightarrow \quad v \in Eig(1, A)$

$$\Rightarrow \quad v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Drehwinkel: $\quad \cos\varphi = \frac{1}{2}(\operatorname{Tr} A - 1)$

$$= \frac{1}{2} \cdot \frac{4}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \varphi = \frac{\pi}{4}$$

2. $\phi, \zeta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\phi(e_1) = \frac{1}{\sqrt{2}}(e_1 + e_2)$$
$$\phi(e_2) = \frac{1}{\sqrt{2}}(e_1 - e_2)$$
$$\phi(e_3) = e_3$$

$$\zeta(e_1) = e_1, \quad \zeta(e_2) = \frac{1}{\sqrt{2}}(e_2 + e_3), \quad \zeta(e_3) = \frac{1}{\sqrt{2}}(e_2 - e_3)$$

$\tau = \zeta \circ \phi$

$$\tau(e_1) = \zeta(\phi(e_1)) = \zeta\left(\tfrac{1}{\sqrt{2}}(e_1 + e_2)\right) = \tfrac{1}{\sqrt{2}}\left(\zeta(e_1) + \zeta(e_2)\right)$$

$$= \tfrac{1}{\sqrt{2}}\left(e_1 + \tfrac{1}{\sqrt{2}}(e_2 + e_3)\right) = \tfrac{1}{\sqrt{2}} e_1 + \tfrac{1}{2} e_2 + \tfrac{1}{2} e_3$$

$$\tau(e_2) = \zeta(\phi(e_2)) = \zeta\left(\tfrac{1}{\sqrt{2}}(e_1 - e_2)\right) = \tfrac{1}{\sqrt{2}}\left(\zeta(e_1) - \zeta(e_2)\right)$$

$$= \tfrac{1}{\sqrt{2}}\left(e_1 - \tfrac{1}{\sqrt{2}}(e_2 + e_3)\right) = \tfrac{1}{\sqrt{2}} e_1 - \tfrac{1}{2} e_2 - \tfrac{1}{2} e_3$$

$$\tau(e_3) = \zeta(\phi(e_3)) = \zeta(e_3) = \tfrac{1}{\sqrt{2}} e_2 - \tfrac{1}{\sqrt{2}} e_3$$

$$\Rightarrow \quad A_\tau = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\det A_\tau = \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}}\right) - \tfrac{1}{\sqrt{2}}\left(-\tfrac{1}{2\sqrt{2}} - \tfrac{1}{2\sqrt{2}}\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$A_\tau^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = A_\tau^t$$

Drehachse: $A_\tau v = v \Rightarrow v \in \mathrm{Eig}(1, A_\tau)$

$$\Rightarrow v = \begin{pmatrix} 3 + 2\sqrt{2} \\ 1 + \sqrt{2} \\ 1 \end{pmatrix}$$

Drehwinkel: $\cos\varphi = \tfrac{1}{2}(\mathrm{Tr}\, A_\tau - 1) = \tfrac{1}{2}\left(-\tfrac{1}{2} - 1\right) = \tfrac{1}{2} \cdot \tfrac{-3}{2} = \tfrac{-3}{4}$

Aufgabe 2 $\mathbb{R}^3$

$$g(x,y,z) = 2x^2 - 2y^2 + z^2 + 8xy - 2xz$$

1. Diagonalisierung:

$$(x \;\; y \;\; z) \underbrace{\begin{pmatrix} 2 & 4 & -1 \\ 4 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = g(x,y,z)$$

Eigenwerten vom A: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & -1 \\ 4 & -2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix}$

$$= -\lambda^3 + \lambda^2 + 21\lambda - 18$$

$$\Delta_1 = |2| = 2$$

$$\Delta_2 = \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} = -4 - 16 = -20$$

$$\Delta_3 = \det A = -18$$

Jacobi $\Rightarrow$ Satz $\exists$ Basis so dass $g(x,y,z) = \dfrac{x^2}{\Delta_1} + \dfrac{\Delta_1}{\Delta_2} y^2 + \dfrac{\Delta_2}{\Delta_3} z^2$

$$= \frac{1}{2}x^2 - \frac{1}{10}y^2 + \frac{10}{9}z^2$$

2. $V_1 = \{x - 2y = 0\}$ $\qquad V_2 = \{x + y = 0,\; x - z = 0\}$

$$g|_{V_1} = 2 \cdot 4y^2 - 2y^2 + z^2 + 8 \cdot 2y \cdot y - 2xz$$

$$= 22y^2 + z^2 - 4yz \quad \Rightarrow \quad A = \begin{pmatrix} 22 & -2 \\ -2 & 1 \end{pmatrix}$$

Jacobi $\Rightarrow$ auf V_1

$\Delta_1 = 22$

$\Delta_2 = 22 + 4 = 26$

$\Rightarrow 22y^2 + \frac{11}{13}z^2 \Rightarrow$ positiv definit

$2) V_2 = 2x^2 - 2x^2 + x^2 - 8x^2 + 2x^2 = -5x^2$

$\Rightarrow$ negativ definit

3. $\mathbb{R}^3 = V_1 \oplus V_2$

$v \in V_1 \cap V_2 \Rightarrow \quad v = (x, y, z)^t$ und

$$\begin{cases} x = 2y \\ x = -y \\ x = z \end{cases} \Rightarrow 2y = -y \Rightarrow y = 0$$

$\Rightarrow x = y = z = 0$

$\Rightarrow \quad V_1 \cap V_2 = \{0\}.$

z.z. $v \in \mathbb{R}^3 \Rightarrow \quad v = v_1 + v_2$ mit $v_1 \in V_1$, $v_2 \in V_2$

$V_1 = \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$V_2 = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

und $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ lin. unab

$\Rightarrow \mathbb{R}^3 = \left\langle \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{= w_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= w_2}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{= w_3} \right\rangle$

$\Rightarrow \forall \; v \in \mathbb{R}^3, \quad v = \underbrace{\underbrace{a_1 w_1}_{\in V_1} + \underbrace{a_2 w_2}_{\in V_1}}_{\in V_1} + \underbrace{a_3 w_3}_{\in V_2} \Rightarrow v \in V_1 + V_2$

$\Rightarrow \mathbb{R}^3 = V_1 \oplus V_2$