Blatt 11

Aufgabe I

a) $(\mathcal{A}, V)$ reeller affiner Raum, $\lambda \in \mathbb{R} - \{\pm 1\}$.

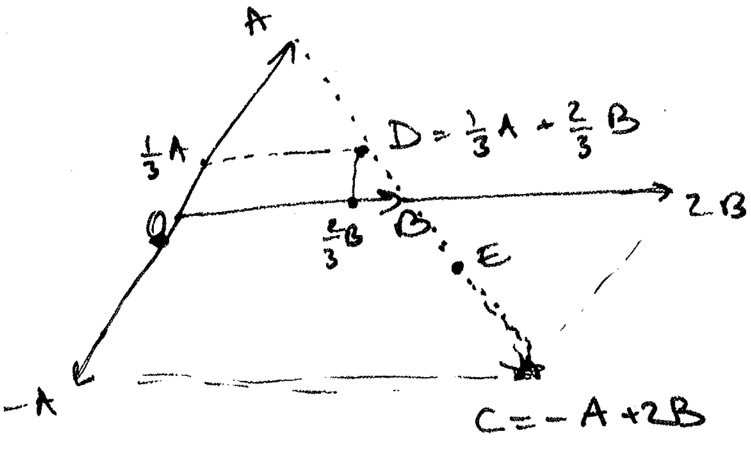
Falls $A, B \in \mathcal{A}$,

$$C := \frac{1}{1-\lambda} A + \frac{\lambda}{\lambda - 1} B \qquad D := \frac{1}{1+\lambda} A + \frac{\lambda}{1+\lambda} B$$

$$E := \frac{1}{2} C + \frac{1}{2} D$$

z.z. $\overrightarrow{EA} = \lambda^2 \overrightarrow{EB}$.

z.B für $\lambda = 2$:



$$C = \frac{1}{1-\lambda} A + \frac{\lambda}{\lambda - 1} B \quad \text{und} \quad \frac{1}{1-\lambda} + \frac{\lambda}{\lambda - 1} = 1 \Rightarrow$$

$$\overrightarrow{0C} = \frac{1}{1-\lambda} \overrightarrow{0A} + \frac{\lambda}{\lambda - 1} \overrightarrow{0B} \qquad / + \overrightarrow{C0}$$

$$0 = \frac{1}{1-\lambda} \overrightarrow{CA} + \frac{\lambda}{\lambda - 1} \overrightarrow{CB} \Rightarrow \overrightarrow{CA} = \lambda \overrightarrow{CB}$$

$$D = \frac{1}{1+\lambda} A + \frac{\lambda}{1+\lambda} B \Rightarrow \overrightarrow{DA} = -\lambda \overrightarrow{DB}.$$

$$E = \frac{1}{2} C + \frac{1}{2} D \Rightarrow \overrightarrow{CE} = \frac{1}{2} \overrightarrow{CD}$$

$$\overrightarrow{EA} = \frac{1}{2} \overrightarrow{CA} + \frac{1}{2} \overrightarrow{DA} = \frac{\lambda}{2} (\overrightarrow{CB} - \overrightarrow{DB}) = \frac{\lambda}{2} \overrightarrow{CD} = \lambda \overrightarrow{CE}.$$

$$C = \frac{1}{1-\lambda} A + \frac{\lambda}{\lambda-1} B \Rightarrow \overrightarrow{CE} = \frac{1}{1-\lambda} \overrightarrow{AE} + \frac{\lambda}{\lambda-1} \overrightarrow{BE} \quad / \cdot \lambda$$

$$\lambda \overrightarrow{CE} = \frac{\lambda}{1-\lambda} \overrightarrow{AE} + \frac{\lambda^2}{\lambda-1} \overrightarrow{BE}$$

$$\overrightarrow{EA} = \frac{\lambda}{\lambda-1} \overrightarrow{EA} + \frac{\lambda^2}{\lambda-1} \overrightarrow{BE}$$

$$\frac{1}{\lambda-1} \overrightarrow{EA} = \frac{\lambda^2}{\lambda-1} \overrightarrow{BE} \Rightarrow \overrightarrow{EA} = \lambda^2 \overrightarrow{BE}$$

b) $A, B, C \in \mathcal{A}$, $M, N, P \in \mathcal{A}$ so dass

$\overrightarrow{AP} = \lambda \overrightarrow{PB}$, $\overrightarrow{BM} = \lambda \overrightarrow{MC}$, $\overrightarrow{CN} = \lambda \overrightarrow{NA}$

z.z. $\frac{1}{3} A + \frac{1}{3} B + \frac{1}{3} C = \frac{1}{3} M + \frac{1}{3} N + \frac{1}{3} P$.

Von der Vorlesung: $\overrightarrow{AB} = \alpha \overrightarrow{AP} \Rightarrow B = (1-\alpha) A + \alpha P$

$$\Rightarrow \overrightarrow{PB} = (1-\alpha) \overrightarrow{PA} = (\alpha - 1) \overrightarrow{AP} \Rightarrow \alpha - 1 = \frac{1}{\lambda}$$

$$\Rightarrow \alpha = \frac{\lambda+1}{\lambda} \Rightarrow B = \left(1 - \frac{\lambda+1}{\lambda}\right) A + \frac{\lambda+1}{\lambda} P$$

$$= \frac{-1}{\lambda} A + \frac{\lambda+1}{\lambda} P$$

$$\Rightarrow \lambda B = -A + (\lambda+1)P \Rightarrow P = \frac{\lambda}{\lambda+1}B + \frac{1}{\lambda+1}A.$$

$$\overrightarrow{BM} = \lambda\overrightarrow{MC} \Rightarrow M = \frac{1}{\lambda+1}B + \frac{\lambda}{\lambda+1}C$$

$$\overrightarrow{CN} = \lambda\overrightarrow{NA} \Rightarrow N = \frac{1}{\lambda+1}C + \frac{\lambda}{\lambda+1}A$$

$$\Rightarrow \frac{1}{3}P + \frac{1}{3}M + \frac{1}{3}N = \frac{1}{3}(P+M+N) =$$

$$= \frac{1}{3}\Big(\frac{\lambda}{\lambda+1}B + \frac{1}{\lambda+1}A + \frac{1}{\lambda+1}B + \frac{\lambda}{\lambda+1}C +$$

$$+ \frac{1}{\lambda+1}C + \frac{\lambda}{\lambda+1}A\Big)$$

$$= \frac{1}{3}(A+B+C) = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C.$$

Aufgabe 11 (A, V) affiner Raum und $V' \subset V$

Auf A: $P \sim Q \Leftrightarrow \overrightarrow{PQ} \in V'$

a) $\sim$ Äquivalenzrelation:

$\sim$ symmetrisch: $P \sim Q \Rightarrow \overrightarrow{PQ} \in V' \Rightarrow -\overrightarrow{PQ} \in V'$

$\Rightarrow \overrightarrow{QP} \in V' \Rightarrow Q \sim P$

$\sim$ reflexiv: $\overrightarrow{PP} = 0 \in V' \Rightarrow P \sim P$

$\sim$ transitiv: $\left.\begin{matrix} P \sim Q \\ Q \sim R \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \overrightarrow{PQ} \in V' \\ \overrightarrow{QR} \in V' \end{matrix}\right\} \Rightarrow \overrightarrow{PQ} + \overrightarrow{QR} \in V'$

$\Rightarrow \overrightarrow{PR} \in V' \Rightarrow P \sim R.$

b) $(A/\sim, V/V')$ affiner Raum.

Sei $\pi_A : A \to A/\sim$
$\pi_V : V \to V/V'$ die kanonischen Projektionen

$(A/\sim, V/V')$ affiner Raum $\Rightarrow$ wir brauchen
$$\varphi : A/\sim \times A/\sim \to V/V'$$
mit den Eigenschaften von der Vorlesung.

Wir definieren $\varphi(\pi_A(P), \pi_A(Q)) = \pi_V(\overrightarrow{PQ})$
für $P, Q \in A$.

φ wohldefiniert: sei $P, P' \in A$ mit $\pi_A(P) = \pi_A(P')$
und $Q, Q' \in A$ mit $\pi_A(Q) = \pi_A(Q')$

$\pi_A(P) = \pi_A(P') \Rightarrow PP' \in V'$
$\pi_A(Q) = \pi_A(Q') \Rightarrow QQ' \in V'$.

Dann $\pi_V(\overrightarrow{PQ}) = \pi_V(\overrightarrow{PP'} + \overrightarrow{P'Q'} + \overrightarrow{Q'P'}) = \pi_V(\overrightarrow{P'Q'})$

$\Rightarrow \varphi$ wohldefiniert.

Jetzt für die Eigenschaften:

1) $\forall P, Q, R \in A \Rightarrow \pi_A(P), \pi_A(Q), \pi_A(R) \in A/\sim$
und $\varphi(\pi_A(P), \pi_A(Q)) + \varphi(\pi_A(Q), \pi_A(R)) =$
$= \pi_V(\overrightarrow{PQ}) + \pi_V(\overrightarrow{QR}) = \pi_V(\overrightarrow{PQ} + \overrightarrow{QR})$
$= \pi_V(\overrightarrow{PR}) = \varphi(\pi_A(P), \pi_A(R))$.

2) z.z $\forall \pi_A(P) \in A/\sim$, $\pi_V(v) \in V/V' \Rightarrow \exists!\, \pi_A(Q)$
im $A/\sim$ mit $\varphi(\pi_A(P), \pi_A(Q)) = \pi_V(v)$.

Existenz: $\forall \pi_V(v) \in V/V'$, $\exists$ $Q \in A$ so dass $\varphi(P,Q) = v$

(weil (A, V) affiner Raum) $\Rightarrow$ $v = \overrightarrow{PQ}$

$\Rightarrow$ $\pi_V(v) = \pi_V(\overrightarrow{PQ}) = \varphi(\pi_A(P), \pi_A(Q))$.

Eindeutigkeit: Nehmen wir an, dass $\exists$ $Q_1, Q_2 \in A$

mit $\varphi(\pi_A(P), \pi_A(Q_1)) = \varphi(\pi_A(P), \pi_A(Q_2))$

$\Rightarrow$ $\pi_V(\overrightarrow{PQ_1}) = \pi_V(\overrightarrow{PQ_2})$

$\Rightarrow$ $\pi_V(\overrightarrow{PQ_1} - \overrightarrow{PQ_2}) = 0$

$\Rightarrow$ $\pi_V(\overrightarrow{Q_2Q_1}) = 0$ $\Rightarrow$ $Q_2Q_1 \in V'$

$\Rightarrow Q_2 \sim Q_1$

$\Rightarrow \pi_A(Q_1) = \pi_A(Q_2)$.

Aufgabe III

$h_1(x,y,z) = 4xy$ auf $\mathbb{R}^3$

$h_2(x,y,z) = y^2 + z^2 + 2xy + 2xz + 2yz$

$h_1(x,y) = 4xy = x^2 + 2xy + y^2 - x^2 + 2xy - y^2$

$= (x+y)^2 - (x-y)^2$

$y_1 = x+y$ $\Rightarrow$ $h_1(y_1, y_2) = y_1^2 - y_2^2$ $\rightarrow$ $\operatorname{sgn}(h_1) = (1,1)$

$y_2 = x-y$

$h_2(x,y) = (x+y+z)^2 - x^2$

$y_3 = x+y+z$ $\Rightarrow$ $h_2(y_3, y_4) = y_3^2 - y_4^2$ $\Rightarrow$ $\operatorname{sgn}(h_2) = (1,1)$

$y_4 = x$

$\Rightarrow h_1 \sim h_2$.