Blatt 1

Aufgabe 1

a) $$\det A = \det\begin{pmatrix} -1 & 0 & 1 \\ 1-i & 0 & 2 \\ 1 & -2 & -1 \end{pmatrix} \overset{S_3+S_1}{=} \det\begin{pmatrix} -1 & 0 & 0 \\ 1-i & 0 & 3-i \\ 1 & -2 & 0 \end{pmatrix}$$

$$= -1\cdot\det\begin{pmatrix} 0 & 3-i \\ -2 & 0 \end{pmatrix} = -\left(0+2(3-i)\right) = -2(3-i)$$

b) $$\det B = \det\begin{pmatrix} 1 & 0 & 2i & 0 \\ 0 & 2i & 2i & 2i \\ 2i & 2 & -1 & 0 \\ 2i & 0 & 0 & 2i \end{pmatrix} \xlongequal{S_3 - 2i S_1} \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2i & 2i & 2i \\ 2i & 2 & 3 & 0 \\ 2i & 0 & 4 & 2i \end{pmatrix}$$

$$= \det\begin{pmatrix} 2i & 2i & 2i \\ 2 & 3 & 0 \\ 0 & 4 & 2i \end{pmatrix} = 2i\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 2i \end{pmatrix}$$

$$= 4i\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & i \end{pmatrix} \xlongequal[S_2-S_1]{S_3-S_1} 4i\det\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 2 & i \end{pmatrix}$$

$$= 4i\det\begin{pmatrix} 1 & -2 \\ 2 & i \end{pmatrix} = 4i\,(i+4)$$

c) $$C = \begin{pmatrix} \bar{2} & \bar{2} & \bar{3} \\ \bar{3} & \bar{1} & \bar{4} \\ \bar{2} & \bar{3} & \bar{2} \end{pmatrix}$$ über $\mathbb{Z}/5\mathbb{Z}$

$$\det C \underset{S_2+S_3}{=} \det\begin{pmatrix} \bar{2} & \bar{0} & \bar{3} \\ \bar{3} & \bar{0} & \bar{4} \\ \bar{2} & \bar{0} & \bar{2} \end{pmatrix} = \bar{0}$$

Aufgabe 2

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$n=2$: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det A = -1$

Ansatz: $\det A = (-1)^{\frac{n(n+1)}{2}}$ Fall $n=2$ ok

Induktion $n \to n+1$:

$$\det \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & & & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}}_{n+1}$$

$$= (-1)^{n+1} \det \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & & & & \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}}_{n} = (-1)^{n+1} \cdot (-1)^{\frac{n(n+1)}{2}}$$

$$= (-1)^{\frac{(n+1)(n+2)}{2}} \Rightarrow \text{ok.}$$

$$B_n = \underbrace{\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}}_{n} \Bigg\} n$$

Ansatz: $\det B = n+1, \quad n \geq 2$

$n=2$: $\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 4-1 = 3$ ✓

$n=3$: $\det\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} =$

$$= 1 \cdot \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - 1 \cdot \det\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$= 3 - 1 \cdot (-1) = 4$$ ✓

$n \longmapsto n+1$

$$\det B_{n+1} = \det \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & \vdots & & & & & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}}_{n} \Bigg\} n$$

$$= 1 \cdot \det B_n + (-1)^{n+1} \cdot (-1)^n = n+1 + 1 = (n+1) + 1$$ ✓

Aufgabe 3 A $n \times n$ Matrix über K

$$A = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

a) $B_{12} = 0$:

Falls $\det B_{11} = 0 \Rightarrow$ ihre Zeilen sind lin. abhängig

$\Rightarrow$ die Zeilen von A sind lin abhängig

$\Rightarrow \det A = 0$.

Falls $\det B_{11} \neq 0 \Rightarrow \exists B_{11}^{-1}$.

$$\Rightarrow \begin{pmatrix} I & 0 \\ -B_{21} B_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} I & 0 \\ -B_{21} B_{11}^{-1} & I \end{pmatrix} \det \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} = \det \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$$

$$1 \cdot \det A = \det \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$$

$$\det \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} = \det \begin{pmatrix} B_{11} & 0 \\ 0 & I \end{pmatrix} \det \begin{pmatrix} I & 0 \\ 0 & B_{22} \end{pmatrix}$$

~~det~~

Aber $\det \begin{pmatrix} B_{11} & 0 \\ 0 & I \end{pmatrix} = \det \begin{pmatrix} I & 0 \\ 0 & B_{11} \end{pmatrix} = \det B_{11}$

$$\Rightarrow \det \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} = \det B_{11} \det B_{22}$$

$$\Rightarrow \det A = \det B_{11} \det B_{22}$$

$$B_{21} = 0 \Rightarrow \quad \det A = \det \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

$$= \det \begin{pmatrix} B_{11}^t & 0 \\ B_{12}^t & B_{22}^t \end{pmatrix} = \det B_{11}^t \det B_{22}^t$$

$$= \det B_{11} \det B_{22}$$

b) $$\begin{pmatrix} I & 0 \\ -B_{21}B_{11}^{-1} & I \end{pmatrix} \underbrace{\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}}_{A} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{pmatrix}$$

$$\Rightarrow \quad 1 \cdot \det A = \det B_{11} \det \left(B_{22} - B_{21} B_{11}^{-1} B_{12}\right)$$