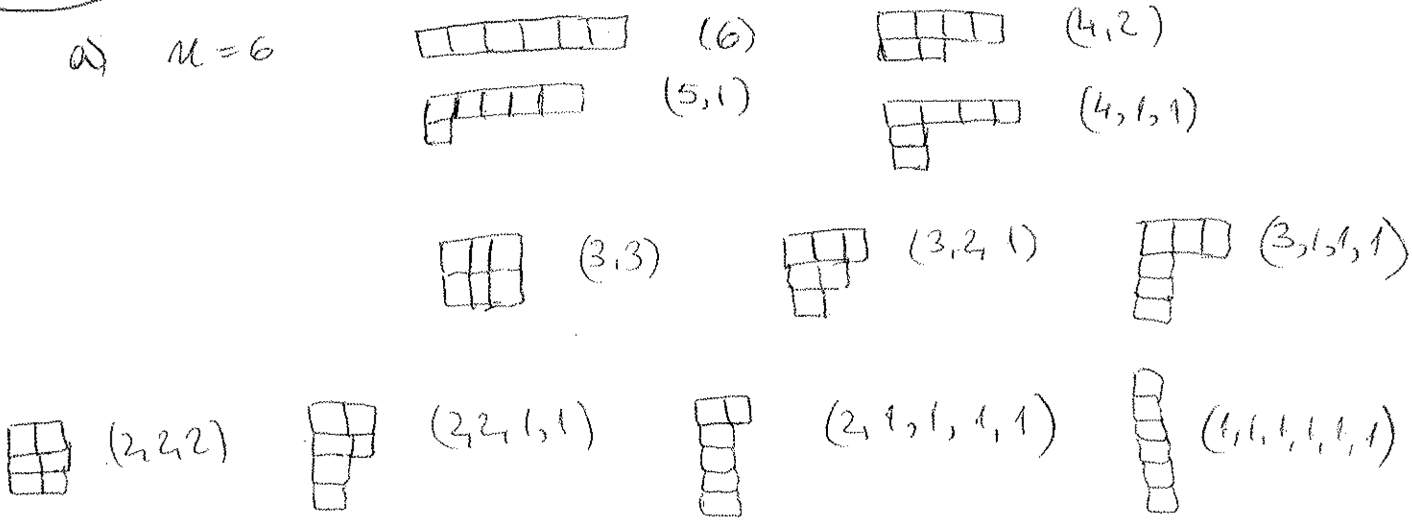
Blatt 5

Aufgabe 1

a) $n = 6$



(6), $(4,2)$, $(5,1)$, $(4,1,1)$, $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$, $(1,1,1,1,1,1)$

b) Nilpotente Endomorphismen $f: K^4 \longrightarrow K^4$

Nilpotent index 1 $\Rightarrow$ $A = 0$

index 2 $\Rightarrow$ $q = (3,1) \Rightarrow p = q^* = (2,1,1) \Rightarrow A = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$q = (2,2) \Rightarrow p = q^* = (2,2) \Rightarrow A = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

index 3 $\Rightarrow$ $q = \cancel{\ }(2,1,1) \Rightarrow p = q^* = (3,1) \Rightarrow A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

index 4 $\Rightarrow$ $q = (1,1,1,1) \Rightarrow p = q^* = (4) \Rightarrow A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} -2 & 4 & -5 & 4 \\ 4 & -2 & 4 & 4 \\ 4 & -4 & 6 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix}$

$A^2 = 0 \Rightarrow \operatorname{Im} A \subset \ker A$ und

$\operatorname{Rang} A = \dim \operatorname{Im} A \leq \frac{4}{2} = 2$

Rang A:

$$A=\begin{pmatrix}-2&4&-5&4\\4&-2&4&4\\4&-4&6&0\\0&-1&1&-2\end{pmatrix}\xrightarrow{z_3+z_1}\begin{pmatrix}-2&4&-5&4\\4&-2&4&4\\2&0&1&4\\0&-1&1&-2\end{pmatrix}\xrightarrow{z_1+z_3}\begin{pmatrix}0&4&-4&8\\4&-2&4&4\\2&0&1&4\\0&-1&1&-2\end{pmatrix}$$

$$\xrightarrow{z_2-2z_3}\begin{pmatrix}0&4&-4&8\\0&-2&2&-4\\2&0&1&4\\0&-1&1&-2\end{pmatrix}\longrightarrow\begin{pmatrix}0&0&0&0\\0&0&0&0\\2&0&1&4\\0&-1&1&-2\end{pmatrix}\Rightarrow \text{Rang } A=2$$

$\Rightarrow \dim\ker A = 4-2 = 2$

$\Rightarrow$ ~~Bz.~~ Nilpotenzindex $m=2$ $\Rightarrow$ $q=(q_1,q_2)$

$q_1=\dim\ker A = 2$

$q_2=\dim\ker A^2-\dim\ker A = 4-2=2$

$\Rightarrow q=(2,2) \Rightarrow p=q^*=(2,2)$

$\Rightarrow A\sim N((2,2))$.

Die Matrix S mit $SAS^{-1}=N((2,2))$:

$S_0=0$

$S_1=\ker A$: $\left(\begin{array}{cccc|c}2&0&1&4&0\\0&-1&1&-2&0\end{array}\right)\Rightarrow S_1 = \left\langle\begin{pmatrix}-2\\-2\\0\\1\end{pmatrix},\begin{pmatrix}-1\\2\\2\\0\end{pmatrix}\right\rangle$.

$v_{12}\in S_2\setminus S_1=\mathbb{R}^4\setminus S_1 \Rightarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix}=v_{12}$

$v_{11}=Av_{12}=\begin{pmatrix}-2\\4\\4\\0\end{pmatrix}$

$v_{22}\in S_2\setminus S_1 \Rightarrow v_{22}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; $v_{21}=Av_{22}=\begin{pmatrix}4\\-2\\-4\\-1\end{pmatrix}$

$$\Rightarrow S^{-1}=\begin{pmatrix}-2&1&4&0\\4&0&-2&1\\4&0&-4&0\\0&0&-1&0\end{pmatrix}\Rightarrow S=\begin{pmatrix}0&0&1/4&-1\\1&0&1/2&2\\0&0&0&-1\\0&1&-1&2\end{pmatrix}$$

Aufgabe 2 $A = (a_{ij})_{1 \le i,j \le n}$ obere Dreiecksmatrix

$a_{ii} = 0 \quad 1 \le i \le n$

$a_{i,i+1} \neq 0 \quad 1 \le i < n$

a) $\dim \ker(f_A) = 1$

Sei $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \ker(f_A) \Rightarrow \forall i, \sum_{j=1}^{n} a_{ij} x_j = 0 \quad (\text{d.h. } Av = 0)$

Falls $i = n-1$, $\sum_{j=1}^{n} a_{ij} x_j = \underbrace{a_{n-1\, n}}_{\neq 0} x_n = 0 \Rightarrow x_n = 0$

$i = n-2$, $\sum_{j=1}^{n} a_{ij} x_j = a_{n-2\, n-1} x_{n-1} + a_{n-2\, n} x_n$

$= a_{n-2\, n-1} x_{n-1} = 0 \Rightarrow x_{n-1} = 0$

$\vdots$

$\Rightarrow \quad x_2 = x_3 = \dots = x_{n-1} = x_n = 0$

$x_1 \neq 0, \ x_1 \in K \Rightarrow \dim \ker(f_A) = 1$

b) $q_1 = 1 \quad \& \quad q_{i+1} \le q_i \quad \Rightarrow \quad q_i = 1 \quad \forall\, i = 1, \dots n$

$\Rightarrow q = (1, 1, 1, \dots 1) \Rightarrow p = q^* = (n) \Rightarrow A \sim H((n))$

c) $B = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $v_{16} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{Q}^6 - (\ker B)$

$v_{16} \notin \ker B^2$

$v_{16} \notin \ker B^3$

$v_{16} \notin \ker B^4$

$v_{16} \notin \ker B^5$

$\Rightarrow$ Die Basis ist

$$\left(B^5 v_{16},\ B^4 v_{16},\ B^3 v_{16},\ B^2 v_{16},\ B v_{16},\ v_{16} \right)$$

$$\Rightarrow P^{-1} = \left(B^5 v_{16} \quad B^4 v_{16} \quad B^3 v_{16} \quad B^2 v_{16} \quad B v_{16} \quad v_{16} \right)$$

$$= \begin{pmatrix} 120 & 250 & 175 & 50 & 5 & 0 \\ 0 & 120 & 130 & 45 & 5 & 0 \\ 0 & 0 & 60 & 35 & 5 & 0 \\ 0 & 0 & 0 & 20 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$P = \ldots$